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Wilson Lines in Chern-Simons Theory and Link Invariants

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Abstract

The vacuum expectation values of Wilson line operators $\langle W(L) \rangle$ in the Chern-Simons theory are computed to second order of perturbation theory. The meaning of the framing procedure for knots is analyzed in the context of the Chern-Simons field theory. The relation between $\langle W(L) \rangle$ and the link invariant polynomials is discussed. We derive an explicit analytic expression for the second coefficient of the Alexander-Conway polynomial, which is related to the Arf and Casson invariants. We present also some new relations between the HOMFLY coefficients.

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1. Introduction

There has recently been some interest [1,2,3,4,5,6] in applying quantum field theory for the study of the geometry of three- and four-dimensional manifolds. In particular, Schwarz [1] and Atiyah [2] have suggested that the pure Chern-Simons (C-S) action, associated with a non-Abelian gauge group G , could be relevant for knot theory and might shed new light on the link invariants. A considerable development of this idea has been accomplished by Witten [3,4]. He has shown that the physical states of the C-S model also describe the conformal blocks of the two-dimensional conformal invariant Wess-Zumino-Witten (WZW) model. Using this correspondence, he has argued that the vacuum expectation values of Wilson line operators $\langle W(L) \rangle$ in the fundamental representation of G are related to a recently discovered link invariant, the HOMFLY polynomial [7]. The existence of a simple recursive procedure for the computation of this link invariant provides in turn a quick and elegant way for computing $\langle W(L) \rangle$ in the fundamental representation. Summarizing, Witten has shown that there exists indeed an intrinsically three-dimensional definition of the new link invariants, the Jones [8] and HOMFLY polynomials, which was one of the open problems quoted in [2].

In the present paper we explore the consequences of the existence of a quantum field theory description of the link invariants. One of the main issues considered in our work is the calculability of $\langle W(L) \rangle$ in the three-dimensional field theory framework. This problem, which is uncovered in the present literature, plays a fundamental role in the program of connecting the C-S theory with the geometry of three-manifolds and knot theory. It turns out that $\langle W(L) \rangle$ has a meaningful perturbative expansion in powers of the coupling constant. It is worth stressing that, differently from what one could naively expect, *no* divergences appear in the computation of $\langle W(L) \rangle$ in the C-S theory. In this paper we report the explicit calculations at second order of perturbation theory. By using the conformal invariance of the C-S theory one can actually argue that the above features of $\langle W(L) \rangle$ persist at higher orders [9].

Another crucial problem that we solve in our paper is how to give an operative meaning to the framing procedure for knots [10] in the context of quantum field theory.

Contrary to a widespread belief, the framing has nothing to do with divergences of $\langle W(L) \rangle$. As already mentioned, there are no divergences in $\langle W(L) \rangle$ at all. In the C-S model, the necessity of framing is related to the self-linking problem. As is well known to mathematicians [11,12], the framing procedure assigns a topological meaning to the self-linking for knots in R^3 by fixing the ambiguities due to a certain direction-dependent but finite limit.

We also show below that the perturbative expansion of $\langle W(L) \rangle$ in the C-S model provides a systematic method for deriving explicit expressions of link and knot invariants. We give in particular an analytic expression of the second coefficient $a_2(C)$ of the Alexander-Conway polynomial [13] for a knot C . This coefficient is of particular interest because it is related to the Arf-invariant [14] of C and the Casson-invariant [15] associated with Dehn surgeries constructed with C .

Our analysis at second order in perturbation theory confirms that $\langle W(L) \rangle$, computed in the fundamental representation of G , coincides with the S_L -polynomial, namely the most general regular isotopy invariant satisfying a linear skein relation. The S_L -polynomial has been introduced in [16] and generalizes both the Kauffman [17] and HOMFLY polynomials.

The C-S field theory also makes manifest the existence of relations between the coefficients of the HOMFLY polynomial. We explain how these relations can be derived by means of the perturbative expansion of $\langle W(C) \rangle$. The relations following from the first two orders of this expansion were already known to mathematicians [18], whereas those following from the next orders are new. We present two examples of new relations between the HOMFLY coefficients for knots.

The paper is organized as follows. In sect.2 we recall some aspects of the covariant quantization of the C-S model and review some of its symmetry properties. The explicit computation of $\langle W(L) \rangle$ at second order in perturbation theory and for a generic link L with arbitrary representation of the gauge group G is described in sect.3. In sect.4 we recall the skein relations satisfied by the regular isotopy invariant S_L -polynomial and its relation with the HOMFLY polynomial. The connection between S_L and $\langle W(L) \rangle$ is established, at second order in perturbation theory, in sect.5. Here we derive (to this

order) also the values of the parameters entering the skein relation. In the last section we present two new relations between the coefficients of the HOMFLY polynomial. Some technical details of the computations are reported in the Appendices. In particular, in Appendix B we discuss the framing-independence of the knot invariant appearing in $\langle W(L) \rangle$ at second order in perturbation theory.

2. Chern-Simons Action, Symmetries and Perturbation Theory

In this section, we recall the basic properties of the Chern-Simons model [19] and summarize some of the results obtained in [20] in the standard perturbative approach to quantization in field theory. The expression of the classical action of the C-S model on a smooth three-manifold M^3 is

$$S_{CS} = \frac{k}{4\pi} \int_{M^3} d^3x \epsilon^{\mu\nu\rho} \text{Tr} (A_\mu \partial_\nu A_\rho + i \frac{2}{3} A_\mu A_\nu A_\rho) \quad , \quad (2.1)$$

where $A_\mu = A_\mu^a T^a$, T^a being the generators in the defining representation of a compact simple Lie group G . Consider for example $G = SU(N)$ with

$$\text{Tr} T^a T^b = \frac{1}{2} \delta^{ab} \quad . \quad (2.2)$$

Under a gauge transformation $\Omega(x) \in SU(N)$,

$$A_\mu(x) \mapsto A_\mu^\Omega(x) \equiv \Omega^{-1}(x) A_\mu(x) \Omega(x) - i \Omega^{-1}(x) \partial_\mu \Omega(x) \quad , \quad (2.3)$$

the action (2.1) transforms as

$$S_{CS}[A^\Omega] = S_{CS}[A] + 2\pi k S_{WZ}[\Omega] \quad , \quad (2.4)$$

where

$$S_{WZ}[\Omega] = \frac{1}{24\pi^2} \int_{M^3} d^3x \epsilon^{\mu\nu\rho} \text{Tr} [\Omega^{-1}(x) \partial_\mu \Omega(x) \Omega^{-1}(x) \partial_\nu \Omega(x) \Omega^{-1}(x) \partial_\rho \Omega(x)] \quad . \quad (2.5)$$

The Wess-Zumino functional S_{WZ} takes integer values and therefore for integer k , the C-S model is invariant also under large (not connected to the identity) gauge transformations. In general, for any compact simple Lie group G , the C-S action can be normalized in such a way that the complete gauge invariance is preserved for integer values of k .

Another important property of the action (2.1) is its general covariance; being the integral of a three-form on a three-dimensional manifold, S_{CS} is independent of the metric that one can introduce on M^3 . This property is essential in relating the C-S model with two-dimensional conformal field theory and knot theory [1,2,3].

Because of gauge invariance, in order to perform the quantization we adopt the standard Faddeev-Popov procedure. The total action is then

$$S = S_{CS} + S_{GF} + S_{FP} \quad , \quad (2.6)$$

where

$$S_{GF} = \frac{k}{4\pi} \int_{M^3} d^3x \sqrt{g} g^{\mu\nu} A_\mu^a \partial_\nu B^a \quad , \quad (2.7)$$

$$S_{FP} = - \int_{M^3} d^3x \sqrt{g} g^{\mu\nu} \partial_\mu \bar{c}^a (D_\nu c)^a \quad , \quad (2.8)$$

and

$$(D_\mu c)^a = \partial_\mu c^a - f^{abc} A_\mu^b c^c \quad , \quad (2.9)$$

f^{abc} being the structure constants of the group G . In eqs.(2.7,8) $g_{\mu\nu}$ is some metric on M^3 , B^a are Lagrange multipliers enforcing the gauge constraint $\partial_\mu(\sqrt{g}g^{\mu\nu}A_\nu^a) = 0$ and c and $\bar{c}$ are the usual Faddeev-Popov ghosts. The total action S is invariant under the BRS transformations [21]

$$\delta A_\mu^a = (D_\mu c)^a \quad , \quad \delta B^a = 0 \quad , \quad (2.10)$$

$$\delta c^a = \frac{1}{2} f^{abc} c^b c^c \quad , \quad \delta \bar{c}^a = \frac{k}{4\pi} B^a \quad . \quad (2.11)$$

The associated BRS charge operator Q is off-shell nilpotent at any order of perturbation theory because there are no gauge anomalies in three dimensions. The cohomology defined by Q determines in the usual way the physical space of the C-S model.

The term $S_{GF} + S_{FP}$ unavoidably spoils the general covariance. Since the metric dependence enters only in the gauge-fixing procedure however, general covariance is maintained on the physical space [22]. In fact, the symmetric energy-momentum tensor $\Theta_{\mu\nu}$ associated with (2.6) is

$$\Theta_{\mu\nu} = \frac{k}{4\pi} (A_{\{\mu}^a \partial_{\nu\}} B^a - g_{\mu\nu} A_\rho^a \partial^\rho B^a) - \partial_{\{\mu} \bar{c}^a (D_{\nu\}} c)^a + g_{\mu\nu} \partial_\rho \bar{c}^a (D^\rho c)^a \quad (2.12)$$

and can be rewritten in the form

$$\Theta_{\mu\nu} = [Q, \partial_{\{\mu} \bar{c}^a A_{\nu\}}^a] - g_{\mu\nu} \partial^\rho \bar{c}^a A_\rho^a \quad . \quad (2.13)$$

Since Q annihilates the physical states, the mean value of $\Theta_{\mu\nu}$ between physical states vanishes and general covariance is indeed preserved. Consequently, the expectation value of any gauge invariant and metric independent observable is a topological invariant in M^3 [1,2,3]. A typical example of such an observable is the Wilson line operator

$$W_R(C) = \text{Tr}_R \text{P} \exp i \oint_C A_\mu dx^\mu \quad , \quad (2.14)$$

where the path ordering is performed along a given oriented closed path (knot) C and the trace is computed in the R -representation of G . In what follows we will be mainly concerned with the computation of expectation values of the type

$$\langle W(L) \rangle \equiv \langle W_{R_1}(C_1) \cdots W_{R_m}(C_m) \rangle \quad , \quad (2.15)$$

where the link L is the union of non-intersecting knots C_i . As is well known, this problem is twofold. First of all, one has to construct well-defined correlation functions $\{ \langle A_{\mu_1}(x_1) \cdots A_{\mu_j}(x_j) \rangle \}$. Second, one should give a definite meaning to $W_R(C)$ as a composite operator preserving, as much as possible, its geometrical meaning. In dealing with these problems we shall use standard perturbation theory on $M^3 = R^3$ (or S^3) with the flat metric $g_{\mu\nu} = \delta_{\mu\nu}$. In this case the propagators following from the action (2.6) are [20]

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle = \frac{i}{k} \delta^{ab} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \quad , \quad (2.16)$$

$$\langle A_\mu^a(x) B^b(y) \rangle = -\frac{i}{k} \delta^{ab} \frac{(x-y)_\mu}{|x-y|^3} \quad , \quad (2.17)$$

$$\langle B^a(x) B^b(y) \rangle = 0 \quad , \quad (2.18)$$

$$\langle c^a(x) \bar{c}^b(y) \rangle = -\frac{i}{4\pi} \delta^{ab} \frac{1}{|x-y|} \quad . \quad (2.19)$$

In a recent paper [20] we have analyzed the properties of the perturbative expansion of the correlation functions $\{ \langle A_{\mu_1}(x_1) \cdots A_{\mu_j}(x_j) \rangle \}$ and we have studied the possible coupling constant and wave function renormalizations. We have shown that the C-S model has a meaningful perturbative expansion in $\frac{1}{k}$. The two-loop computation confirms that the theory is finite. The vanishing of the beta function to all orders of perturbation theory follows from the invariance under large gauge transformations, which

implies that k must be an integer. We have also shown that the one- and two-loop radiative corrections to $\langle A_\mu^a(x) A_\nu^b(y) \rangle$ and $\langle A_\mu^a(x) A_\nu^b(y) A_\rho^c(z) \rangle$ are not only finite but vanish identically. These results will be our starting point in the computation of $\langle W(L) \rangle$.

We conclude by displaying a peculiar discrete symmetry of the action (2.6). In addition to the standard C , P and T transformations defined for instance in [19], it is useful to introduce the inversion operator I acting as

$$\begin{aligned} A_\mu^a(x) &\mapsto -A_\mu^a(-x) \quad , \quad B(x) \mapsto -B(-x) \quad , \\ c(x) &\mapsto c(-x) \quad , \quad \bar{c}(x) \mapsto \bar{c}(-x) \quad , \\ k &\mapsto -k \quad . \end{aligned} \tag{2.20}$$

It is easily seen that the total action (2.6) is invariant under CPT and I transformations separately. The inversion operator I will be of great importance for our analysis and has actually a simple geometrical meaning. The I -invariance states that a change in the sign of the coupling constant k is equivalent to (or can be compensated by) reversing the orientation of the three manifold M^3 . As we show now, the I -symmetry implies that $\langle W(L) \rangle$ computed with the coupling constant k is equal to $\langle W(\tilde{L}) \rangle$ computed with the coupling constant $-k$, where the link $\tilde{L}$ is the mirror image of the link L . In fact, consider the transformation of the Wilson line operator (2.14) under (2.20). One has

$$A_\mu(x) dx^\mu \mapsto -A_\mu(-x) dx^\mu = A_\mu(y) dy^\mu \quad , \tag{2.21}$$

where

$$y^\mu = -x^\mu \quad . \tag{2.22}$$

Note that in three dimensions the transformation (2.22) is equivalent, up to a π -rotation, to the inversion of one of the axes. Therefore, the change of variables (2.22) in (2.14) implies

$$W_R(C) \mapsto W_R(\tilde{C}) \quad , \tag{2.23}$$

where $\tilde{C}$ is the mirror (and actually π -rotated) image of the knot C . Combining (2.23) with the rotational and I -invariance of the total action one finds

$$\langle W(L) \rangle|_k = \langle W(\tilde{L}) \rangle|_{-k} \quad . \tag{2.24}$$

Eq.(2.24) provides a non-trivial check on the computation of $\langle W(L) \rangle$ and gives some restrictions on the k -dependence of the variables entering in the S_L polynomials. We shall return to this point in sect.5.

It is worth mentioning that the C-S theory makes sense for both signs of k because the Hamiltonian vanishes on the physical space. In other words, with a given orientation of M^3 , both the positive and negative values of k are permitted. These two possibilities are simply related by the exchange of the links, associated with the Wilson line operators, with their mirror images. For this reason, it is sufficient to consider positive values of k .

3. Computing Wilson Line Expectation Values

As is well known, the trace of the holonomy operator along the knot C has the following expansion

$$\begin{aligned} W_R(C) = \text{Tr} \left[1 + i \oint_C dx^\mu A_\mu(x) - \oint_C dx^\mu \int^x dy^\nu A_\nu(y) A_\mu(x) \right. \\ \left. - i \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho A_\rho(z) A_\nu(y) A_\mu(x) \right. \\ \left. + \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma A_\sigma(w) A_\rho(z) A_\nu(y) A_\mu(x) + \dots \right] , \end{aligned} \quad (3.1)$$

where $A_\mu = A_\mu^a R^a$ and R^a are the generators of the R -representation of G . All line integrals are performed on the same contour C . With an explicit parametrization $\{x^\mu(t) : 0 \leq t \leq 1\}$ of C the multiple integrals take the form

$$\oint_C dx^\mu \int^x dy^\nu \equiv \int_0^1 ds \int_0^s dt \dot{x}^\mu(s) \dot{x}^\nu(t) , \quad (3.2)$$

and so on.

In computing $\langle W_{R_1}(C_1) \cdots W_{R_n}(C_n) \rangle$ one has to integrate correlation functions of the type $\langle A_{\mu_1}(x_1) \cdots A_{\mu_j}(x_j) \rangle$ over the paths $C_1, \dots, C_n$ in different combinations. As mentioned before, $\langle A_{\mu_1}(x_1) \cdots A_{\mu_j}(x_j) \rangle$ present no problems because the theory is renormalizable and, in our case, actually finite. The problems occur when some of the

fields appearing in the correlation functions have to be integrated over the same path. Differently from the case of the standard Yang-Mills theory, here the difficulties are not related to the appearance of divergences in the expression of $\langle W(L) \rangle$, but rather to some kind of ambiguities in finite terms! As we shall see, all these ambiguities can be eliminated on geometrical grounds. It should be noted that the finiteness of $\langle W(L) \rangle$ is of primary importance in both relating the C-S model with two-dimensional conformal field theory and interpreting $\langle W(L) \rangle$ as a link invariant. The connection of the C-S theory with quantum groups [4,23,24] also requires $\langle W(L) \rangle$ to be finite.

In this section we shall illustrate all the above features of the C-S model by computing explicitly $\langle W(L) \rangle$ at second order in perturbation theory.

3A. Self-Linking Number

Let us consider a single knot C . The expectation value $\langle W_R(C) \rangle$ has an expansion in powers of $\frac{2\pi}{k}$. Let us denote by $\langle W_R(C) \rangle^{(n)}$ the $(\frac{2\pi}{k})^n$ -contribution. Clearly,

$$\langle W_R(C) \rangle^{(0)} = \dim R \quad , \quad (3.3)$$

whereas the $\frac{2\pi}{k}$ -contribution to $\langle W_R(C) \rangle$ is

$$-\text{Tr}(R^b R^a) \oint_C dx^\mu \int^x dy^\nu \langle A_\nu^b(y) A_\mu^a(x) \rangle = -i \frac{2\pi}{k} \dim R c_2(R) \varphi(C) \quad , \quad (3.4)$$

where

$$c_2(R)1 = R^a R^a \quad (3.5)$$

and $\varphi(C)$ is given by

$$\begin{aligned} \varphi(C) &= \frac{1}{4\pi} \oint_C dx^\mu \oint_C dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \\ &= \frac{1}{4\pi} \int_0^1 ds \int_0^1 dt \epsilon_{\mu\nu\rho} \dot{x}^\mu(s) \dot{x}^\nu(t) \frac{(x(s)-x(t))^\rho}{|x(s)-x(t)|^3} \quad . \end{aligned} \quad (3.6)$$

It is well known that the integral (3.6) is *well defined* and *finite*; a detailed analysis of $\varphi(C)$ has been performed for instance by Calugareanu in [25]. It is perhaps instructive

to recall the main point of Calugareanu's argument. At first sight one could suspect that the integrand of (3.6) diverges at $t = s$. Anyone who has tried to compute (3.6) knows however that this is not the case. Indeed, expanding $x(t)$ around $t = s$ one finds

$$\epsilon_{\mu\nu\rho} \dot{x}^\mu(s) \dot{x}^\nu(t) \frac{(x(s) - x(t))^\rho}{|x(s) - x(t)|^3} = -\frac{1}{6} |s - t| \epsilon_{\mu\nu\rho} \frac{\dot{x}^\mu(s) \ddot{x}^\nu(s) \ddot{x}^\rho(s)}{|\dot{x}(s)|^3} + O(|s - t|^2) \quad , \quad (3.7)$$

which shows that the integrand actually vanishes at $t = s$.

The expression (3.6) is known as the cotorsion of C . Unfortunately, the cotorsion is metric dependent and is not invariant under deformations of C , which seems to contradict the claim that $\langle W_R(C) \rangle$ is a topological invariant. The point is that there exists a certain freedom in defining the composite operator

$$A_\nu(y) dy^\nu A_\mu(x) dx^\mu \quad (3.8)$$

at coincident points and different prescriptions lead in general to different finite values of $\langle W_R(C) \rangle^{(1)}$. We shall show now that one can use this freedom to attribute a geometrical meaning to $\langle W_R(C) \rangle^{(1)}$. In the context of knot theory, this problem, known as the self-linking problem, has been considered some time ago by mathematicians [11,12,25]. The solution consists in introducing a *framing* contour C_f defined by

$$x^\mu(t) \mapsto y^\mu(t) = x^\mu(t) + \epsilon n^\mu(t) \quad , \quad (\epsilon > 0 \quad , \quad |n(t)| = 1) \quad , \quad (3.9)$$

where n^μ is a vector field orthogonal to C (see Fig.1). Consider then $\varphi_f(C)$ defined by

$$\begin{aligned} \varphi_f(C) &= \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi} \int_0^1 ds \int_0^1 dt \epsilon_{\mu\nu\rho} \dot{x}^\mu(s) (\dot{x}^\nu(t) + \epsilon \dot{n}^\nu(t)) \frac{(x(s) - x(t) - \epsilon n(t))^\rho}{|x(s) - x(t) - \epsilon n(t)|^3} \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi} \oint_C dx^\mu \oint_{C_f} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x - y)^\rho}{|x - y|^3} = \lim_{\epsilon \rightarrow 0} \chi(C, C_f) \quad . \end{aligned} \quad (3.10)$$

In the expression $\chi(C, C_f)$ one recognizes the linking number for two non-intersecting curves C and C_f given by the Gauss integral

$$\chi(C, C_f) = \frac{1}{4\pi} \oint_C dx^\mu \oint_{C_f} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x - y)^\rho}{|x - y|^3} \quad . \quad (3.11)$$

The integral (3.11) is well defined, takes integer values and equals the number of windings of C_f around C . Since $\chi(C, C_f)$ is a topological invariant and, in particular, it is ϵ -independent, the $\epsilon \rightarrow 0$ limit in (3.10) is trivially performed and gives

$$\varphi_f(C) = \chi(C, C_f) \quad . \quad (3.12)$$

If the ϵ -limit and the integration in (3.10) were commuting, then $\varphi_f(C)$ would coincide with $\varphi(C)$. This is not the case however; in fact, one has for the self-linking number

$$\varphi_f(C) = \varphi(C) + \tau_f(C) \quad , \quad (3.13)$$

where

$$\tau_f(C) = \frac{1}{2\pi} \int_0^1 ds \epsilon_{\mu\nu\rho} \dot{n}^\mu(s) \dot{x}^\nu(s) n^\rho(s) |\dot{x}(s)|^{-1} \quad (3.14)$$

is known as the torsion of C . A proof of eq.(3.13) can be found in [11].

Summarizing, the ambiguities that one finds in the computation of $\langle W_R(C) \rangle$ are finite terms which are related to the behaviour of (3.8) at coincident points. The framing procedure is a way of defining (3.8) at coincident points which permits us to express $\langle W_R(C) \rangle^{(1)}$ in metric-independent terms:

$$\langle W_R(C) \rangle_f^{(1)} = -i \frac{2\pi}{k} \dim R c_2(R) \varphi_f(C) \quad . \quad (3.15)$$

The expression (3.15) is now a topological invariant but the price one has to pay is a dependence of $\langle W_R(C) \rangle_f^{(1)}$ on the choice of the framing contour C_f . It turns out that at the quantum level one cannot define a metric-independent $\langle W_R(C) \rangle$ exclusively in terms of C . Later we shall discuss this point in more detail. For notational simplicity, from now on we will omit the subscript f in $\langle W_R(C) \rangle_f$.

One may think that the framing dependence of $\langle W_R(C) \rangle$ leads to a loss of information. On the contrary, the behaviour of $\langle W_R(C) \rangle$ under a change of framing introduces a new physically meaningful variable related to the conformal weights of the associated two-dimensional conformal field theory.

At this stage one can imagine that the geometrical support of the Wilson line operator behaves like a band; this band is determined by the original path C and the

framing vector n^μ entering in (3.9). A twist of the band is equivalent to a change of framing in which $\varphi_f(C)$ gets modified by the unity (Fig.1). As we shall see later, the interpretation of the Wilson line as a band is useful in relating $\langle W(L) \rangle$ with the link invariants.

Now the question is if the framing procedure consistently extends to higher orders in perturbation theory. In the Abelian case this extension is trivial because $\langle W(L) \rangle$ can be entirely expressed in terms of linking and self-linking integrals. With a non-Abelian group, the problem becomes more complicated because of the field interactions and the non-commutativity of the group.

3B. The Next Knot Invariant

Let us analyse now the $\left(\frac{2\pi}{k}\right)^2$ part of $\langle W_R(C) \rangle$. As in the previous subsection, we first write down the expression before the framing has been performed. Since the one-loop radiative correction to $\langle A_\mu(x)A_\nu(y) \rangle$ vanishes [20], there are only two contributions, V_1 and V_2 . V_1 involves one interaction Lagrangian contracted with the A^3 -term of eq.(3.1) and reads

$$V_1 = \left(\frac{2\pi}{k}\right)^2 \dim R c_v c_2(R) \varrho_1(C) \quad , \quad (3.16)$$

where

$$\delta^{ab} c_v = f^{acd} f^{bcd} \quad (3.17)$$

and

$$\varrho_1(C) = -\frac{1}{32\pi^3} \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \epsilon^{\alpha\beta\gamma} \epsilon_{\mu\alpha\sigma} \epsilon_{\nu\beta\lambda} \epsilon_{\rho\gamma\tau} I^{\sigma\lambda\tau}(x, y, z) \quad , \quad (3.18)$$

with

$$I^{\sigma\lambda\tau}(x, y, z) = \int d^3w \frac{(w-x)^\sigma}{|w-x|^3} \frac{(w-y)^\lambda}{|w-y|^3} \frac{(w-z)^\tau}{|w-z|^3} \quad . \quad (3.19)$$

As shown in Appendix A, the integral (3.19) gives

$$I^{\sigma\lambda\tau}(x, y, z) = \partial_y^\lambda \partial_z^\tau I^\sigma(y-x, z-x) \quad , \quad (3.20)$$

where

$$I^\sigma(a, b) = \frac{2\pi(|a| + |b| - |a - b|)}{(|a||b| + a \cdot b)} \left[\frac{a^\sigma}{|a|} + \frac{b^\sigma}{|b|} \right] . \quad (3.21)$$

The explicit expression of the integrand of (3.18) is given by eq.(A.8) in Appendix A and one may verify that $\varrho_1(C)$ is finite.

Let us consider now the contribution V_2 , which is obtained by contracting the A^4 -term of the expansion (3.1); one has

$$\begin{aligned} V_2 = & -\frac{1}{k^2} \dim R c_2^2(R) \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \left[\epsilon_{\sigma\rho\alpha} \epsilon_{\nu\mu\beta} \frac{(w-z)^\alpha (y-x)^\beta}{|w-z|^3 |y-x|^3} \right. \\ & \left. + \epsilon_{\sigma\nu\alpha} \epsilon_{\rho\mu\beta} \frac{(w-y)^\alpha (z-x)^\beta}{|w-y|^3 |z-x|^3} + \epsilon_{\sigma\mu\alpha} \epsilon_{\rho\nu\beta} \frac{(w-x)^\alpha (z-y)^\beta}{|w-x|^3 |z-y|^3} \right] \\ & + \frac{1}{2k^2} \dim R c_v c_2(R) \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \epsilon_{\sigma\nu\alpha} \epsilon_{\rho\mu\beta} \frac{(w-y)^\alpha (z-x)^\beta}{|w-y|^3 |z-x|^3} . \end{aligned} \quad (3.22)$$

The integrand of the first integral in (3.22) is a symmetric function in the exchange of any two variables. Consequently, there the integration domain can be taken to be $D = C \times C \times C \times C$, provided one divides by $4!$. Afterwards, the three terms of the first integral in (3.22) give the same contribution when integrated over D . Therefore

$$V_2 = -\frac{1}{2} \left(\frac{2\pi}{k} \right)^2 \dim R c_2^2(R) \varphi^2(C) + \left(\frac{2\pi}{k} \right)^2 \dim R c_v c_2(R) \varrho_2(C) , \quad (3.23)$$

where $\varphi(C)$ is given by eq.(3.6) and

$$\varrho_2(C) = \frac{1}{8\pi^2} \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \epsilon_{\sigma\nu\alpha} \epsilon_{\rho\mu\beta} \frac{(w-y)^\alpha (z-x)^\beta}{|w-y|^3 |z-x|^3} . \quad (3.24)$$

The argument following eq.(3.6) implies that $\varrho_2(C)$ is well defined.

Summarizing, $V_1 + V_2$ is finite but unfortunately metric-dependent because of the $\varphi^2(C)$ -term. Note that even if the sum

$$\varrho(C) = \varrho_1(C) + \varrho_2(C) \quad (3.25)$$

were metric-dependent, it could never compensate the metric-dependence of $\varphi^2(C)$ because of the different Casimir factors. As discussed below, there is a strong evidence that $\varrho(C)$ is actually a topological invariant.

Let us examine now what happens after framing. First of all, we must extend the framing procedure to operator products of the form

$$A_{\mu_1}(x_1)dx_1^{\mu_1}A_{\mu_2}(x_2)dx_2^{\mu_2}\dots A_{\mu_n}(x_n)dx_n^{\mu_n} \quad , \quad (3.26)$$

which have to be integrated over the same contour C . Of course, in shifting the operators one must use the same vector field n^μ introduced in (3.9). The simplest recipe for framing consists of shifting the argument of the j -th factor of (3.26) according to

$$A_{\mu_j}(x_j)dx_j^{\mu_j} \mapsto A_{\mu_j}(x_j + (j-1)\epsilon n)d(x_j + (j-1)\epsilon n)^{\mu_j} \quad .$$

Clearly, for $\epsilon \neq 0$ all factors of the framed version of (3.26) always occur in different points. After performing all the Wick-contractions and integrations, one should take the $\epsilon \rightarrow 0$ limit. Two possibilities may occur: the limit depends on n^μ (see e.g. eqs.(3.13,14)) or the limit is framing-independent. The sum $V_1 + V_2$ illustrates both the situations. First of all, one finds that after framing, the $\varphi^2(C)$ -contribution in (3.23) is replaced by $\varphi_f^2(C)$. As expected, the $\varphi_f^2(C)$ -term appears with the right coefficient in order to exponentiate the $\frac{1}{k}$ -result, precisely like in the Abelian case.

Let us concentrate now on $\varrho(C)$. Our investigation shows that:

- i) $\varrho(C)$ is *framing-independent*, that is, the $\epsilon \rightarrow 0$ limit leads precisely to the same expression (3.25) obtained without framing. The details on this point are given in Appendix B;
- ii) $\varrho(C)$ represents in *analytic form* the second coefficient of the Alexander-Conway polynomial, which in turn is related to the Arf- and Casson-invariants [14,15]. This remarkable result just follows from the existence of a quantum field theory description of the link invariant polynomials. We shall discuss this point in sect.6.

Concerning the explicit form of $\varrho(C)$, some comments are in order. Let us recall that, apart from the Gauss integral (3.11), the known analytic and explicit expressions of link invariants include the "higher order linking coefficients" (HOLC) [26]. These link invariants $\{\chi_n(L)\}$ are labelled by a positive integer n in such a way that $\chi_n(L)$ makes sense only if $\chi_1(L) = \dots = \chi_{n-1}(L) = 0$. The HOLC can be expressed [26] in

terms of the one-forms

$$u(C_i) = u(C_i; x)_\mu dx^\mu = \frac{1}{4\pi} \left(\oint_{C_i} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \right) dx^\mu \quad , \quad C_i \in L \quad . \quad (3.27)$$

The form $u(C)$ is known as the Alexander dual of the knot C . The first linking coefficient χ_1 can be written in terms of the Gauss integral (3.11) as follows:

$$\chi_1(L) = \max_{i \neq j} |\chi(C_i, C_j)| \quad .$$

A simple example which illustrates the role of HOLC is represented by the Borromean rings B , see Fig.2. Exactly like the link U_2 shown in Fig.3, B has vanishing χ_1 . It is the second order linking coefficient which distinguishes between B and U_2 . Indeed, one has $\chi_2(U_2) = 0$ but $\chi_2(B) = 1$.

A natural question one may pose is whether $\varrho(C)$ is somehow related to the HOLC. Of course, being an invariant of a single knot, $\varrho(C)$ is *not* a HOLC. One could suspect, however, that $\varrho(C)$ can be obtained from the HOLC by performing an appropriate limit in which the different components C_i of a link L coincide. In fact, the expression of $\varrho_1(C)$, eqs.(3.18) and (3.19), looks similar to the integral over the space of the external product of three Alexander duals. However, the integration region of the multiple line integrals *cannot* be reduced to the product $C \times C \times C$; this is also the reason why $\varrho_1(C)$ is actually non-vanishing.

The expression of $\varrho_2(C)$, eq.(3.24), also looks very similar to a "double" Gauss integral. But here again the particular structure of the domain, in which the line integrals have to be performed, plays a crucial role in making $\varrho_2(C)$ substantially different from the product of two Gauss integrals. Note that ϱ_2 vanishes for closed non-intersecting curves on the plane.

Another important issue concerning $\varrho(C)$ is its relation with the coefficients of the HOMFLY polynomials [7]. As is well known, each of these coefficients represents an invariant of ambient isotopy but, apart from a few of them, their topological meaning is completely unknown. One of the reasons for this unpleasant situation is that even if one can compute the coefficients of the HOMFLY polynomials by means of the skein relation, there are no analytic expressions for them. We will show in sect.6 that $\varrho(C)$ is simply

related with the second coefficient $a_2(C)$ of the Alexander-Conway polynomial, which in turn is a particular case of the HOMFLY polynomial. It is known [14] that $a_2(C) \bmod 2$ is the Arf-invariant associated to C ; it is also known [15] that $a_2(C)$ gives the difference between the values of the Casson-invariant associated with two consecutive Dehn surgeries constructed with the knot C . We find it quite remarkable that the C-S field theory provides an analytic expression for $a_2(C)$.

Finally it is worth mentioning that since $\varrho(C)$ represents a $(\frac{1}{k^2})$ -contribution to $\langle W_R(C) \rangle$, the I -symmetry introduced in sect.2 implies $\varrho(C) = \varrho(\tilde{C})$, that is, $\varrho(C)$ is a knot invariant which does not distinguish a knot C from its mirror image $\tilde{C}$.

Because of the rather complicated form of $\varrho(C)$, a direct computation for a generic knot is not an easy matter. As an example, we have reported in Appendix C the complete analytic computation of the value of $\varrho(C)$ for the unknot U_0 . It turns out that $\varrho(U_0) = -\frac{1}{12}$.

Summarizing, by means of the framing procedure the contribution at second order to $\langle W_R(C) \rangle$ reads

$$\langle W_R(C) \rangle^{(2)} = \left(\frac{2\pi}{k} \right)^2 \dim R \left[-\frac{1}{2} c_2^2(R) \varphi_f^2(C) + c_v c_2(R) \varrho(C) \right] . \quad (3.28)$$

In conclusion, by adding (3.3), (3.15) and (3.28), one has for a single knot

$$\begin{aligned} \langle W_R(C) \rangle = \dim R \left[1 - i \left(\frac{2\pi}{k} \right) c_2(R) \varphi_f(C) \right. \\ \left. - \frac{1}{2} \left(\frac{2\pi}{k} \right)^2 c_2^2(R) \varphi_f^2(C) + \left(\frac{2\pi}{k} \right)^2 c_v c_2(R) \varrho(C) \right] + O \left(\frac{1}{k^3} \right) . \end{aligned} \quad (3.29)$$

3C. Interaction between Knots

In computing $\langle W(L) \rangle$ for a link L with m components $C_1, \dots, C_m$, in addition to the "self-energy" contributions (3.29), one has to take into account also the "interactions" between the different components of L . Consider first the case $m = 2$; the first

non-vanishing contribution to the interaction between the two knots C_1 and C_2 is of order $\left(\frac{2\pi}{k}\right)^2$ (see Fig.4) and reads

$$-2 \left(\frac{2\pi}{k}\right)^2 \frac{\dim R_1 \dim R_2}{\dim G} c_2(R_1) c_2(R_2) \chi^2(C_1, C_2) \quad , \quad (3.30)$$

where R_1 and R_2 are the representations of G associated with C_1 and C_2 . Clearly the expression (3.30) is a topological invariant. In the general case, combining (3.29) with (3.30), one finds

$$\begin{aligned} \langle W(L) \rangle = & \left(\prod_{i=1}^m \dim R_i \right) \left\{ 1 - i \left(\frac{2\pi}{k}\right) \sum_{i=1}^m c_2(R_i) \varphi_{f_i}(C_i) \right. \\ & - \left(\frac{2\pi}{k}\right)^2 \sum_{i=1}^m \left[\frac{1}{2} c_2^2(R_i) \varphi_{f_i}^2(C_i) - c_v c_2(R_i) \varrho(C_i) \right] \\ & \left. - \left(\frac{2\pi}{k}\right)^2 \sum_{i \neq j} c_2(R_i) c_2(R_j) \left[\varphi_{f_i}(C_i) \varphi_{f_j}(C_j) + \frac{\chi^2(C_i, C_j)}{\dim G} \right] \right\} + O\left(\frac{1}{k^3}\right) \quad . \quad (3.31) \end{aligned}$$

This equation gives the value of the framed $\langle W(L) \rangle$ computed in perturbation theory at second order in $\left(\frac{2\pi}{k}\right)$. Note that, at this stage, the framings of the different components C_i of L are arbitrary and a priori unrelated. We shall see later that, in connecting $\langle W(L) \rangle$ with the link invariants in R^3 , the above arbitrariness is naturally solved.

In the remaining part of the paper, we shall relate $\langle W(L) \rangle$ with the link invariants and we shall extract the values of the parameters (at order $\left(\frac{2\pi}{k}\right)^2$) entering the associated link polynomials.

4. Link Invariants

In this section, we recall some basic notions from knot theory [10,17]. Consider two oriented links L_1 and L_2 in R^3 (or S^3). They are called *ambient isotopic* ($L_1 \sim L_2$) if there exists a smooth deformation through embeddings from one to the other. A practical procedure for verifying ambient isotopy is the following one. One associates

with any link L a four-valent planar diagram, namely a diagram obtained by projecting L on a plane in such a way that the projection contains only simple crossing points where the choice of over/under crossing is given. It can be shown that $L_1 \sim L_2$ if and only if there exists a finite sequence of Reidemeister moves, shown in Fig.5, which transforms the diagram of L_1 into the diagram of L_2 .

It is useful to introduce also the concept of *regular isotopy for diagrams* [17]. Two diagrams are said to be regular isotopic if one can be obtained from the other by a sequence of Reidemeister moves of type II and type III only. Regular isotopy arises for example when, instead of a link we consider an oriented twisted band. Clearly, the Reidemeister moves of type I do not relate ambient isotopic bands, see Fig.6. It turns out that the ambient isotopy of twisted bands is equivalent to regular isotopy for link (or knot) diagrams. In order to clarify this statement, it is useful to introduce the *writhe* (or Tait number) $w(D_L)$ of a *link diagram* D_L . By definition

$$w(D_L) = \sum_p \epsilon(p) \quad , \quad (4.1)$$

where the sum is performed over all crossings of the link diagram and

$$\epsilon(L_{\pm}) = \pm 1 \quad , \quad (4.2)$$

$L_{\pm}$ being displayed on Fig.7. Note that the integer $w(D_L)$ is regular isotopy invariant, it is additive and can be computed by analysing locally the diagram D_L associated with L .

In order to simplify the notations, instead of D_L , from now on we shall simply write " L ", keeping in mind that for regular isotopy invariant quantities L actually indicates a link diagram.

The connection between regular isotopy for link diagrams and ambient isotopy of twisted bands can be understood as follows. As we have seen in sect.3, by framing a knot C with an orthonormal vector n^{μ} , one obtains an oriented band. The Gauss linking number $\varphi_f(C)$ of eq.(3.12) is an ambient isotopy invariant of the band. On the other hand, in passing from a knot C in the space to its projections on a plane, one finds a collection of diagrams. By using Reidemeister moves of type II and III only, these

diagrams fall into equivalence classes of regular isotopy. Each class is characterized by an integer number: the writhe w . As shown by Kauffmann [17], the framing defined by

$$\varphi_f(C) = w(C) \quad (4.3)$$

transmutes ambient isotopy of twisted bands into regular isotopy of knot diagrams. Some examples are illustrated in Fig.8. The *correspondence* (4.3) is called "*vertical framing*" in [4]. Actually, the connection (4.3) was already known to Tait in the early days of knot theory (see [27] p.308). In the context of the self-linking problem, Calugareanu [11] has shown that the vertical framing consists of identifying n^μ with the principal normal in the Frenet basis associated with C . As far as the computation of $\langle W(L) \rangle$ is concerned, the vertical framing procedure simply means that the framing of each component C_i of any link L is fixed by eq.(4.3). In what follows we shall see that $\langle W(L) \rangle$, defined by the vertical framing procedure, is directly related to the new link invariants, the S_L -polynomials, introduced in [16].

In the particular case of knots in R^3 , it is also convenient to introduce an alternative framing procedure, the so-called "*standard framing*"

$$\varphi_f(C) = 0 \quad . \quad (4.4)$$

We shall see that $\langle W_R(C) \rangle$, computed by standard framing, is simply related to the HOMFLY polynomials.

For discussing the properties of $\langle W(L) \rangle$, we have introduced in [16] the so-called S_L -polynomials. $S_L(\alpha, \beta, z)$ is a finite Laurent polynomial with integer coefficients, which is regular isotopy invariant. It is defined by the skein relations

$$S_{\widehat{L}_+} = \alpha S_{\widehat{L}_0} \quad , \quad (4.5)$$

$$S_{\widehat{L}_-} = \alpha^{-1} S_{\widehat{L}_0} \quad , \quad (4.6)$$

$$\beta S_{\widehat{L}_+} - \beta^{-1} S_{\widehat{L}_-} = z S_{\widehat{L}_0} \quad , \quad (4.7)$$

where the diagrams associated with $\{L_\pm, L_0\}$ and $\{\widehat{L}_\pm, \widehat{L}_0\}$ are shown in Fig.7 and Fig.9 respectively. By means of (4.5-7) and Reidemeister moves of type II and III only, one

can construct S_L recursively, starting from the polynomial S_{U_0} of the unknot with zero writhe, which is usually normalized as follows

$$S_{U_0} = 1 \quad . \quad (4.8)$$

As a simple exercise, the reader can derive the S_L polynomials associated with the knot and link diagrams shown in Figs.2,3,10,11. One finds

$$S_{U_n} = u^n \quad , \quad w(U_n) = 0 \quad , \quad (4.9)$$

$$S_T = \alpha^3 [z^2 t^{-2} + 2t^{-2} - t^{-4}] \quad , \quad w(T) = 3 \quad , \quad (4.10)$$

$$S_K = t^2 - 1 + t^{-2} - z^2 \quad , \quad w(K) = 0 \quad , \quad (4.11)$$

$$S_J = \alpha^{-5} t^4 [(1 + z^2)(z^2 + 3 - t^2) - t^2] \quad , \quad w(J) = -5 \quad , \quad (4.12)$$

$$S_H = \alpha^2 (u t^{-2} + z t^{-1}) \quad , \quad w(H) = 2 \quad , \quad (4.13)$$

$$S_Q = \alpha^{-4} [t^4 (1 + z^2) u - z^3 t^3 - 2z t^3] \quad , \quad w(Q) = -4 \quad , \quad (4.14)$$

$$S_B = (1 - z^4) u^2 + z^4 \quad , \quad w(B) = 0 \quad , \quad (4.15)$$

where

$$t = \alpha \beta \quad , \quad u = (t - t^{-1}) z^{-1} \quad . \quad (4.16)$$

For particular values of the parameters α, β and z in eq.(4.5-7), S_L gives rise to the Kauffman R_L -polynomial [17]

$$R_L(\alpha, z) = S_L(\alpha, \beta = 1, z) \quad (4.17)$$

and to the HOMFLY P_L -polynomial

$$P_L(t, z) = S_L(\alpha = 1, \beta = t, z) \quad . \quad (4.18)$$

Note that R_L is a regular isotopy invariant, while P_L is actually an ambient isotopy invariant because $\alpha = 1$ implies (see eqs.(4.5,6)) invariance under Reidemeister moves of

type I also. The S_L - and P_L -polynomials are related each to the other also for arbitrary values of the parameters. One can show [16] that

$$P_L(t = \alpha\beta, z) = \alpha^{-w(L)} S_L(\alpha, \beta, z) \quad . \quad (4.19)$$

As is well known, the link invariant $P_L(1, z)$ is the Alexander-Conway polynomial (or Conway potential function) [13], whereas $P_L(t, z = \sqrt{t} - \frac{1}{\sqrt{t}})$ is the Jones polynomial [8]. For applications to Wilson lines it is useful to recall also that the S_L -polynomials of a link diagram L and its mirror image $\bar{L}$ are related according to

$$S_L(\alpha, \beta, z) = S_{\bar{L}}(\alpha^{-1}, \beta^{-1}, -z) \quad . \quad (4.20)$$

Eq.(4.20) simply follows from the skein relations (4.5-7). Combining eq.(4.20) with the I -invariance of the C-S action, one can derive the following restriction on the k -dependence of the variables entering the link invariants obtained in the C-S theory:

$$\alpha(k) = \alpha^{-1}(-k) \quad , \quad \beta(k) = \beta^{-1}(-k) \quad , \quad z(k) = -z(-k) \quad . \quad (4.21)$$

We shall see in the next section that eqs.(4.21) are indeed satisfied.

5. Wilson Lines and Link Invariants

As mentioned in sect.2, $\langle W(L) \rangle = \langle W_{R_1}(C_1) \cdots W_{R_m}(C_m) \rangle$ computed in the Chern-Simons theory represents a link invariant for any choice of the representations $R_1, \dots, R_m$. This has been explicitly verified at second order in perturbation theory, eq.(3.31). Because of the framing dependence, it turns out that $\langle W(L) \rangle$ can be understood as a *regular* rather than an *ambient* isotopy invariant.

It is of some interest to consider the case when $R_1 = R_2 = \dots = R_m$ equal the fundamental representation of $SU(N)$. It was suggested by Witten that in this case $\langle W(L) \rangle$ is somehow related to the HOMFLY polynomials P_L , (4.18). Even more, assuming that the level ℓ of the associated two-dimensional WZW-model coincides with the coupling constant k of the C-S theory, Witten was able to compute the values of

the variables t and z of P_L . The precise relation between $\langle W(L) \rangle$ and the link invariants has been further clarified in [4,16]. First of all, it is clear that for a generic link L , $\langle W(L) \rangle \neq P_L$ because $\langle W(L) \rangle$ is only regular isotopy invariant. By using a variational method, based on the properties of the Wilson line operators in the three-dimensional theory, we have shown [16] that $\langle W(L) \rangle$ in the fundamental representation satisfies at the first non-trivial order in $\frac{1}{k}$ the skein relations (4.5-7) of a S_L -polynomial with the following values of the parameters:

$$\alpha = 1 - i \frac{2\pi}{k} \left(\frac{N^2 - 1}{2N} \right) + O\left(\frac{1}{k^2}\right) \quad , \quad (5.1)$$

$$\beta = 1 - i \frac{2\pi}{k} \frac{1}{2N} + O\left(\frac{1}{k^2}\right) \quad , \quad (5.2)$$

$$z = -i \frac{2\pi}{k} + O\left(\frac{1}{k^2}\right) \quad . \quad (5.3)$$

Unfortunately, in the variational approach the $\frac{1}{k^2}$ -corrections to α, β and z are not easy to compute. Our purpose in what follows will be to extract from eq.(3.31) the values of the parameters (5.1-3) at order $\frac{1}{k^2}$.

For Wilson lines in the fundamental representation of $SU(N)$, eq.(3.31) takes the form

$$\begin{aligned} \langle W(L) \rangle = N^m & \left\{ 1 - i \left(\frac{2\pi}{k} \right) \left(\frac{N^2 - 1}{2N} \right) \sum_{i=1}^m \varphi_{f_i}(C_i) \right. \\ & - \left(\frac{2\pi}{k} \right)^2 \frac{1}{2} \left(\frac{N^2 - 1}{2N} \right)^2 \sum_{i=1}^m \varphi_{f_i}^2(C_i) + \left(\frac{2\pi}{k} \right)^2 N \left(\frac{N^2 - 1}{2N} \right) \sum_{i=1}^m \varrho(C_i) \\ & - \left(\frac{2\pi}{k} \right)^2 \left(\frac{N^2 - 1}{2N} \right)^2 \sum_{i \neq j} \varphi_{f_i}(C_i) \varphi_{f_j}(C_j) - \left(\frac{2\pi}{k} \right)^2 \frac{1}{2N} \left(\frac{N^2 - 1}{2N} \right) \sum_{i \neq j} \chi^2(C_i, C_j) \Big\} \\ & + O\left(\frac{1}{k^3}\right) \quad . \end{aligned} \quad (5.4)$$

In this section we shall use the *vertical framing* procedure, i.e. $\varphi_{f_i}(C_i)$ are fixed by eq.(4.3) in terms of the writhe of the diagrams associated with the single components C_i of L . In this procedure, the skein relations (4.5-6) are associated with a change of the framing given by $\Delta \varphi_{f_i}(C_i) = \Delta w(C_i) = \pm 1$. Consequently, using (5.4) one finds

$$\alpha = 1 - i \left(\frac{2\pi}{k} \right) \left(\frac{N^2 - 1}{2N} \right) - \frac{1}{2} \left(\frac{2\pi}{k} \right)^2 \left(\frac{N^2 - 1}{2N} \right)^2 + O\left(\frac{1}{k^3}\right) \quad . \quad (5.5)$$

Since we do not know how $\varrho(C)$ behaves in passing from L_+ and L_- to L_0 , we cannot directly verify eq.(4.7) at order $\frac{1}{k^2}$. So, in order to determine β and z at order $\frac{1}{k^2}$, we use an indirect method, which consists of the following steps:

1) We assume that *in vertical framing* the equation

$$\frac{\langle W(L) \rangle}{\langle W(U_0) \rangle} = S_L \quad , \quad (5.6)$$

which holds at order $\frac{1}{k}$, extends to higher orders. In order to agree with the convention $S_{U_0} = 1$, we have divided $\langle W(L) \rangle$ by the value of $\langle W(U_0) \rangle$.

- 2) By means of eq.(5.4) and the explicit form of S_L for the link diagrams U_1 and H for example, we determine the $\frac{1}{k^2}$ -part of β and z .
- 3) By using (5.5) and the values of β and z found in point 2), one can then verify that eq.(5.6) holds for whatever link diagram L .

This last point provides a consistency check and shows a posteriori that $\langle W(L) \rangle$ satisfies the skein relations (4.5-7).

Let us now go through the above steps. From eq.(5.4) one obtains

$$\begin{aligned} \frac{\langle W(L) \rangle}{\langle W(U_0) \rangle} = & N^{m-1} \left\{ 1 - i \left(\frac{2\pi}{k} \right) \left(\frac{N^2 - 1}{2N} \right) \sum_{i=1}^m \varphi_{f_i}(C_i) \right. \\ & - \left(\frac{2\pi}{k} \right)^2 \frac{1}{2} \left(\frac{N^2 - 1}{2N} \right)^2 \sum_{i=1}^m \varphi_{f_i}^2(C_i) \\ & - \left(\frac{2\pi}{k} \right)^2 N \left(\frac{N^2 - 1}{2N} \right) \left[\varrho(U_0) - \sum_{i=1}^m \varrho(C_i) \right] \\ & - \left(\frac{2\pi}{k} \right)^2 \left(\frac{N^2 - 1}{2N} \right)^2 \sum_{i \neq j} \varphi_{f_i}(C_i) \varphi_{f_j}(C_j) - \left(\frac{2\pi}{k} \right)^2 \frac{1}{2N} \left(\frac{N^2 - 1}{2N} \right) \sum_{i \neq j} \chi^2(C_i, C_j) \Big\} \\ & + O \left(\frac{1}{k^3} \right) \quad , \end{aligned} \quad (5.7)$$

and therefore, using that $\varrho(U_0) = -\frac{1}{12}$, one gets

$$\frac{\langle W(U_1) \rangle}{\langle W(U_0) \rangle} = N - \frac{1}{24} \left(\frac{2\pi}{k} \right)^2 N(N^2 - 1) + O \left(\frac{1}{k^3} \right) \quad , \quad (5.8)$$

$$\frac{\langle W(H) \rangle}{\langle W(U_0) \rangle} = N - \frac{1}{24} \left(\frac{2\pi}{k} \right)^2 N(N^2 - 1) - \left(\frac{2\pi}{k} \right)^2 \left(\frac{N^2 - 1}{2N} \right) + O \left(\frac{1}{k^3} \right) \quad . \quad (5.9)$$

According to eq.(5.6), we now compare the explicit form of S_{U_1} and S_H , see eqs.(4.9) and (4.13), with the values (5.8) and (5.9). In this way one finds for β and z

$$\beta = 1 - i \left(\frac{2\pi}{k} \right) \frac{1}{2N} - \frac{1}{2} \left(\frac{2\pi}{k} \right)^2 \left(\frac{1}{2N} \right)^2 + O \left(\frac{1}{k^3} \right) \quad , \quad (5.10)$$

$$z = -i \left(\frac{2\pi}{k} \right) + O \left(\frac{1}{k^3} \right) \quad . \quad (5.11)$$

The values of α , β and z given by (5.5) and (5.10,11) satisfy eqs.(4.21), as they should, because of the invariance of the C-S action under the I -transformations (2.20). In addition to (5.10) and (5.11), one obtains the relation

$$2(\alpha_3 + \beta_3) - Nz_3 = \frac{i}{24} \frac{(N^2 - 1)(N^2 - 3)}{N} \quad , \quad (5.12)$$

where α_3, β_3 and z_3 are the $\left(\frac{2\pi}{k}\right)^3$ -contributions to α, β and z .

Let us concentrate now on point 3) above. We have not a general proof of eq.(5.6) for *any* link diagram. However, in all the particular cases we have examined, we found complete agreement with eq.(5.6). There are also wide classes of link diagrams, involving connected sums of links, for which it is not difficult to prove eq.(5.6). One of the problems in verifying eq.(5.6) for arbitrary link diagrams is that one has to know $\varrho(C)$ for a generic knot C . Using the explicit analytic expressions (3.18,19) and (3.24), we have performed numerical computations of $\varrho(C)$ for some simple knots. The results fit the predictions

$$\varrho(T) = \frac{23}{12} \quad , \quad \varrho(K) = -\frac{25}{12} \quad , \quad \varrho(J) = \frac{71}{12} \quad , \quad (5.13)$$

following from eq.(5.6).

At order $\frac{1}{k^2}$ the values of the parameters α , β and z , furnished by the C-S theory, are compatible with the expressions

$$\alpha = \exp \left[-i \frac{2\pi}{k} \left(\frac{N^2 - 1}{2N} \right) \right] \quad , \quad \beta = \exp \left(-i \frac{2\pi}{k} \frac{1}{2N} \right) \quad , \quad z = -2i \sin \frac{\pi}{k} \quad . \quad (5.14)$$

The values (5.14) obey the conditions (4.21) and moreover are consistent with eq.(5.12). Additional highly non-trivial checks on eqs.(5.6) and (5.14) will be performed in the next section.

In the framework of the two-dimensional WZW-model, the parameters α, β and z have been derived in [3]. The expressions (5.14) are in agreement with Witten's results for

$$k = \ell + N \quad , \quad (5.15)$$

where ℓ is the level of the WZW-model. Note that the relation between k and ℓ is not obvious a priori [3,4] and its complete understanding is an open problem [5]. Perturbation theory suggests eq.(5.15). Eq.(5.15) is also supported by a non-perturbative argument [24], based on the comparison of the quantum holonomies of A_μ with the monodromies associated with the correlation functions of primary fields in the WZW model, i.e. the Knizhnik-Zamolodchikov equations [28].

6. New Relations between the HOMFLY Coefficients

In this section we show that combining the explicit form of the perturbative expansion of $\langle W(C) \rangle$ with eqs.(5.6) and (5.14), one can extract new information about the coefficients of the HOMFLY polynomials. Let us consider first the relation between the values of $\langle W(C) \rangle$ computed by *vertical* and *standard* framings. Assuming (5.6), from eqs.(4.3-6) it follows that

$$\langle W(C) \rangle_{sf} = \alpha^{-w(C)} \langle W(C) \rangle_{vf} \quad . \quad (6.1)$$

Using eqs.(5.6) and (4.19), by means of (6.1) one finds

$$\frac{\langle W(C) \rangle_{sf}}{\langle W(U_0) \rangle} = \alpha^{-w(C)} S_C(\alpha, \beta, z) = P_C(t = \alpha\beta, z) \quad . \quad (6.2)$$

Therefore for knots in R^3 the normalized expectation value of the Wilson operator $W(C)$, computed in the fundamental representation of $SU(N)$ and with standard framing, equals the HOMFLY polynomial. This is in agreement with the results of [4]. Eq.(6.2) is the source of new relations between the HOMFLY coefficients. Indeed, for a single knot C one has [18]

$$P_C(t, z) = \sum_{i \geq 0} \sum_{j \in \mathbb{Z}} b_{ij} z^{2i} t^{2j} \quad , \quad (6.3)$$

where b_{ij} are integers. Inserting in eq.(6.3) the values of the parameters t and z obtained from eq.(5.14), one finds that P_C has an expansion in powers of $\frac{2\pi}{k}$. The first few terms are

$$\begin{aligned} P_C = & \sum_{j \in \mathbb{Z}} b_{0j} - ixN \sum_{j \in \mathbb{Z}} j b_{0j} - x^2 \sum_{j \in \mathbb{Z}} \left(\frac{1}{2} N^2 j^2 b_{0j} + b_{1j} \right) \\ & + ix^3 \sum_{j \in \mathbb{Z}} \left(\frac{N^3}{6} j^3 b_{0j} + N j b_{1j} \right) \\ & + x^4 \sum_{j \in \mathbb{Z}} \left(\frac{N^4}{24} j^4 b_{0j} + \frac{N^2}{2} j^2 b_{1j} + \frac{1}{12} b_{1j} + b_{2j} \right) + \dots \quad , \end{aligned} \quad (6.4)$$

where $x = \frac{2\pi}{k}$. According to (6.2), the expansion (6.4) should coincide (for any N) with the perturbative expansion of $\langle W(C) \rangle_{sf} \langle W(U_0) \rangle^{-1}$ in $\frac{2\pi}{k}$. Therefore from eq.(3.29) one finds

$$\sum_{j \in \mathbb{Z}} b_{0j} = 1 \quad , \quad (6.5)$$

$$\sum_{j \in \mathbb{Z}} j b_{0j} = 0 \quad , \quad (6.6)$$

$$\frac{1}{2} N^2 \sum_{j \in \mathbb{Z}} j^2 b_{0j} + \sum_{j \in \mathbb{Z}} b_{1j} = \frac{1 - N^2}{2} \left[\varrho(C) + \frac{1}{12} \right] \quad . \quad (6.7)$$

Let us analyze these equations. As mentioned in sect.4,

$$P_C(t=1, z) = \nabla_C(z) = a_0(C) + a_2(C)z^2 + \dots \quad , \quad (6.8)$$

where $\nabla_C(z)$ is the Conway potential function. The L.H.S. of eq.(6.5) is precisely the coefficient $a_0(C)$ of ∇_C and eq.(6.5) states that $a_0(C) = 1$, which is indeed known to be the case. The validity of eq.(6.6) has been proven recently by Lickorish and Millet [18]. Since eq.(6.7) holds for any N , one finds

$$\sum_{j \in \mathbb{Z}} b_{1j} = \frac{1}{2} \left[\varrho(C) + \frac{1}{12} \right] \quad , \quad (6.9)$$

$$\sum_{j \in \mathbb{Z}} j^2 b_{0j} = -\varrho(C) - \frac{1}{12} \quad . \quad (6.10)$$

The L.H.S. of eq.(6.9) coincides with the second coefficient of the Conway potential function and so

$$a_2(C) = \frac{1}{2} \left[\varrho(C) + \frac{1}{12} \right] . \quad (6.11)$$

The result (6.11) is remarkable because it provides an analytic expression for $a_2(C)$. This coefficient in turn plays a distinguished role: it is related to the Arf- and Casson-invariants by [14,15]

$$a_2(C) \bmod 2 = \text{Arf}(C) , \quad (6.12)$$

$$a_2(C) = \lambda(C_{n+1}) - \lambda(C_n) , \quad (6.13)$$

where C_n is the $\frac{1}{n}$ -th Dehn surgery constructed by C . It is also known [18] that $a_2(C)$ represents the *total twisting* of the knot C . Finally, eliminating $\varrho(C)$ from eqs.(6.9,10) one gets

$$\sum_{j \in \mathbb{Z}} (j^2 b_{0j} + 2b_{1j}) = 0 , \quad (6.14)$$

which has been proven in [18].

The existing derivation of eqs.(6.6) and (6.14) [18] is based on the use of the skein relation and is not quite straightforward. Moreover it is not obvious if there are new relations of this type and how to find them. The use of the C-S field theory sheds new light on this problem. Note that in obtaining eq.(6.14) from eq.(6.7) the particular form of $\varrho(C)$ is irrelevant; it is only the group structure $(N^2 - 1)$, appearing in the perturbative expansion, which is essential. This fact provides a systematic method for deriving new relations between the HOMFLY coefficients. For example, examining the $\frac{1}{k^3}$ - and $\frac{1}{k^4}$ -terms in eq.(6.2) one finds [29]

$$\sum_{j \in \mathbb{Z}} (j^3 b_{0j} + 6j b_{1j}) = 0 , \quad (6.15)$$

$$\sum_{j \in \mathbb{Z}} [j^4 b_{0j} + 2(6j^2 + 1) b_{1j} + 24 b_{2j}] = 0 . \quad (6.16)$$

Using the table given in [18] for the coefficients b_{ij} , we have verified eqs.(6.15,16) for all the knots with up to and including 9 crossings. The validity of eqs.(6.5,6,14-16) confirms once more eq.(5.6) and the values of the parameters α , β and z given by eq.(5.14).

7. Conclusions

The results of the present paper confirm that the application of quantum field theory for the study of the geometry of three-manifolds and link invariants is a reality. The Chern-Simons theory gives in fact new insights on the subject. The perturbative expansion of $\langle W(L) \rangle$ provides a set $\mathcal{I}$ of a priori new link and knot invariants. A nice feature is that for each element of $\mathcal{I}$ one can produce in principle an explicit analytic expression. As Lickorish recently stressed [30], the task of finding an analytic interpretation of the link/knot invariants is one of the chief open problems in knot theory. At any given order of perturbation theory there is a finite number of invariants belonging to $\mathcal{I}$. At lowest order one finds the Gauss linking number and the knot invariant $\varrho(C)$. The set $\mathcal{I}$ covers at least all the invariants represented by the coefficients of the HOMFLY polynomial.

A first interesting application of the invariants $\mathcal{I}$ is the discovery of new relations between the HOMFLY coefficients. Another application which comes to mind is the classification problem for knots and links. One may wonder if by means of $\mathcal{I}$ one could distinguish between different links with coinciding HOMFLY polynomials. This fascinating problem, together with the complete understanding of the role of quantum groups emerging as a unifying structure between integrable two-dimensional field theories and link invariants, deserves further investigation.

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Appendix A

We give here the main steps in the computation of the integral $I^{\sigma\lambda\tau}(x, y, z)$ given by eq.(3.19). Obviously one has

$$I^{\sigma\lambda\tau}(x, y, z) = \partial_y^\lambda \partial_z^\tau I^\sigma(y - x, z - x) \quad ,$$

where

$$I^\sigma(a, b) = \int d^3w \frac{w^\sigma}{|w|^3 |w - a| |w - b|} \quad . \quad (A.1)$$

Introducing the Feynman parameters s and t , one easily performs the integral in w and finds

$$I^\sigma(a, b) = 2 \int_0^1 ds \int_0^1 dt \frac{\sqrt{t}}{\sqrt{s(1-s)}} \frac{[sa^\sigma + (1-s)b^\sigma]}{\{s(1-s)(a-b)^2 + t[sa + (1-s)b]^2\}} \quad . \quad (A.2)$$

After the integration in t and the substitution

$$s = \frac{u^2}{1 + u^2} \quad ,$$

I^σ takes the form

$$I^\sigma(a, b) = 8 \int_0^\infty du \frac{(u^2 a^\sigma + b^\sigma)}{|u^2 a + b|^3} \left[|u^2 a + b| - u|a - b| \arctan \left(\frac{|u^2 a + b|}{u|a - b|} \right) \right] \quad . \quad (A.3)$$

At this stage it is convenient to consider the scalar $a_\sigma I^\sigma(a, b)$ and to perform the integration over u by means of Cauchy's theorem. After some algebra one gets

$$a_\sigma I^\sigma(a, b) = \frac{2\pi}{|b|} (|a| + |b| - |a - b|) \quad . \quad (A.4)$$

Using that $I^\sigma(a, b)$ is symmetric in the exchange $a^\mu \leftrightarrow b^\mu$, one deduces that

$$b_\sigma I^\sigma(a, b) = \frac{2\pi}{|a|} (|a| + |b| - |a - b|) \quad . \quad (A.5)$$

Finally, observing that I^σ has the general structure

$$I^\sigma(a, b) = a^\sigma A(a, b) + b^\sigma A(b, a) \quad , \quad (A.6)$$

one obtains from eqs.(A.4,5) the expression (3.21). Finally,

$$\varrho_1(C) = -\frac{1}{32\pi^3} \oint dx^\mu \int^x dy^\nu \int^y dz^\rho H_{\mu\nu\rho}(y-x, z-x) \quad , \quad (A.7)$$

where

$$\begin{aligned} H_{\mu\nu\rho}(a, b) = & C_1 C_2 C_3 [\delta_{\nu\rho}(a-b)_\mu + \delta_{\mu\rho}b_\nu - \delta_{\mu\nu}a_\rho] \\ & - C_1 C_2^2 C_3 \epsilon_{\lambda\sigma\tau} a_\sigma b_\tau \left[\epsilon_{\rho\mu\alpha} \delta_{\lambda\nu} \left(a_\alpha + b_\alpha \frac{|a|}{|b|} \right) + \epsilon_{\nu\mu\alpha} \delta_{\lambda\rho} \left(b_\alpha + a_\alpha \frac{|b|}{|a|} \right) \right] \\ & + C_1 C_2 \epsilon_{\lambda\sigma\tau} a_\sigma b_\tau \left\{ \epsilon_{\rho\mu\alpha} \delta_{\lambda\nu} \left[b_\alpha \frac{|a-b| - |a|}{b^2} + (a-b)_\alpha \frac{|a| + |b|}{(a-b)^2} \right] \right. \\ & \left. + \epsilon_{\nu\mu\alpha} \delta_{\lambda\rho} \left[a_\alpha \frac{|a-b| - |b|}{a^2} + (b-a)_\alpha \frac{|a| + |b|}{(a-b)^2} \right] \right\} \end{aligned} \quad (A.8)$$

and

$$\begin{aligned} C_1(a, b) &= \frac{2\pi}{|a||b||a-b|} \quad , \\ C_2(a, b) &= \frac{1}{|a||b| + a_\mu b_\mu} \quad , \\ C_3(a, b) &= |a| + |b| - |a-b| \quad . \end{aligned} \quad (A.9)$$

Appendix B

We show in this appendix the framing-independence of $\varrho(C)$. It comes out that $\varrho_1(C)$ and $\varrho_2(C)$ are separately framing independent. Let us start by analysing $\varrho_2(C)$, whose expression (3.24) is simpler. Using the framing procedure defined after eq.(3.26), one obtains

$$\varrho_2(C)_f = \lim_{\varepsilon \rightarrow 0} \frac{1}{8\pi^2} \int_0^1 ds \int_0^s dt \int_0^t du \int_0^u dv F(s, t, u, v) \quad , \quad (B.1)$$

where

$$F(s, t, u, v) = \epsilon_{\sigma\nu\alpha}\epsilon_{\rho\mu\beta} [\dot{x}(s) + 3\varepsilon\dot{n}(s)]^\mu [\dot{x}(t) + 2\varepsilon\dot{n}(t)]^\nu [\dot{x}(u) + \varepsilon\dot{n}(u)]^\rho \dot{x}(v)^\sigma \\ \times \frac{[x(t) + 2\varepsilon n(t) - x(v)]^\alpha [x(s) + 3\varepsilon n(s) - x(u) - \varepsilon n(u)]^\beta}{|x(t) + 2\varepsilon n(t) - x(v)|^3 |x(s) + 3\varepsilon n(s) - x(u) - \varepsilon n(u)|^3} \quad . \quad (B.2)$$

From eq.(B.2) it follows that the potentially dangerous region in the domain of the integral (B.1), which in principle could lead to framing dependence is $u \sim s$ and/or $v \sim t$. In the case when $u \rightarrow s$ (or $v \rightarrow t$), the specific domain of integration implies that also $t \rightarrow s$ (or $u \rightarrow t$) and consequently the ε -limit is easily seen to be n^μ -independent. Let us consider the case when $u \rightarrow s$ and $v \rightarrow t$ simultaneously. Now one has actually $t, u, v \rightarrow s$ and following Calugareanu's [11] analysis of $\varphi_f(C)$

$$\varrho_2(C)_f = \varrho_2(C) + \lim_{\delta \rightarrow 0} \left[\lim_{\varepsilon \rightarrow 0} \int_0^1 ds G(s; \varepsilon, \delta) \right] \quad . \quad (B.3)$$

The integrand of the second term in (B.3), which carries the potential framing dependence, reads

$$G(s; \varepsilon, \delta) = \int_{s-\delta}^s dt \int_{s-\delta}^t du \int_{s-\delta}^u dv \frac{N(s, t, u, v)}{D(s, t, u, v)} \quad (B.4)$$

where

$$N(s, t, u, v) = 4\varepsilon^2 \epsilon_{\sigma\nu\alpha}\epsilon_{\rho\mu\beta} n^\alpha(s) n^\beta(s) \\ \times [(v-s)\ddot{x}^\sigma(s)\dot{x}^\nu(s) + 2\varepsilon\dot{x}^\sigma(s)\dot{n}^\nu(s)][(u-s)\ddot{x}^\rho(s)\dot{x}^\mu(s) + 2\varepsilon\dot{x}^\rho(s)\dot{n}^\mu(s)] \quad , \quad (B.5)$$

$$D(s, t, u, v) = [(t-v)^2 \dot{x}^2(s) + 4\varepsilon^2]^{\frac{3}{2}} [(s-u)^2 \dot{x}^2(s) + 4\varepsilon^2]^{\frac{3}{2}} \quad . \quad (B.6)$$

In studying the limits in (B.3) one can use the following estimates for the integrals in u and v :

$$\left| \int_A^B d\alpha \frac{1}{(\alpha^2 + 4\varepsilon^2)^{\frac{3}{2}}} \right| \leq \frac{1}{2\varepsilon^2} \quad , \quad (B.7)$$

$$\left| \int_A^B d\alpha \frac{\alpha}{(\alpha^2 + 4\varepsilon^2)^{\frac{3}{2}}} \right| \leq \frac{\pi}{2\varepsilon} \quad . \quad (B.8)$$

The $\frac{1}{\varepsilon}$ -factors coming from (B.7,8) compensate exactly the ε -factors in the numerator (B.5). Therefore, after performing the t -integration in (B.4) one finds

$$|G(s; \varepsilon, \delta)| \leq \delta G_0(s) \quad , \quad (B.9)$$

where $G_0(s)$ is ε - and δ -independent. Combining (B.3) and (B.9) we see that $\varrho_2(C)$ is indeed framing-independent.

Consider now $\varrho_1(C)$ given by eqs.(A.7-9) and concentrate for example on the term proportional to $C_1 C_2 C_3$ in (A.8). After some algebra it turns out that the contribution, which is potentially framing-dependent, is proportional to

$$\lim_{\delta \rightarrow 0} \left\{ \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \int_0^1 ds \int_{s-\delta}^s dt \int_{s-\delta}^t du \frac{1}{[(s-t)^2 + \varepsilon^2]^{\frac{1}{2}} [(s-u)^2 + 4\varepsilon^2]^{\frac{1}{2}} (s-u)} \right\} \quad . \quad (B.10)$$

The integral in u can be estimated by

$$\left| \int_A^B d\alpha \frac{1}{\alpha(\alpha^2 + 4\varepsilon^2)^{\frac{1}{2}}} \right| \leq \frac{1}{\varepsilon} \quad . \quad (B.11)$$

Using (B.11) and

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_A^B d\alpha \frac{1}{(\alpha^2 + \varepsilon^2)^{\frac{1}{2}}} = 0 \quad , \quad (B.12)$$

one finds that the limit (B.10) vanishes. The terms in (A.8), which are proportional to $C_1 C_2^2 C_3$ and $C_1 C_2$, can be studied analogously and are also n^μ -independent. In conclusion, $\varrho(C)$ is framing-independent.

Appendix C

This appendix is devoted to the computation of $\varrho(U_0)$. As easily seen from eq.(3.22), for a general non-intersecting planar curve C_0 one has $\varrho_2(C_0) = 0$. Therefore, using eqs.(3.18-21) one finds

$$\begin{aligned} \varrho(C_0) &= \varrho_1(C_0) = \\ &= \frac{1}{16\pi^2} \int_0^1 ds \int_0^s dt \int_0^t du \frac{(|x(s) - x(t)| + |x(u) - x(s)| - |x(t) - x(u)|)}{|x(s) - x(t)||x(u) - x(t)||x(s) - x(u)|} \\ &\times \frac{\dot{x}_i(s) \dot{x}_j(t) \dot{x}_k(u) \{\delta_{ij}[x(t) - x(s)]_k + \delta_{ik}[x(s) - x(u)]_j + \delta_{jk}[x(u) - x(t)]_i\}}{\{|x(s) - x(t)||x(u) - x(s)| + [x(t) - x(s)]_i[x(u) - x(s)]_i\}} \quad . \quad (C.1) \end{aligned}$$

Consider now the circle

$$U_0 = \{x(s) = (\cos 2\pi s, \sin 2\pi s, 0) : 0 \leq s \leq 1\} \quad . \quad (C.2)$$

Inserting the parametrization (C.2) in (C.1), after some algebra one obtains

$$\begin{aligned} \varrho(U_0) &= \\ &= -\frac{1}{16\pi^2} \int_0^{2\pi} d\theta \int_0^\theta d\chi \int_0^\chi d\psi \left[\sin\left(\frac{\theta - \chi}{2}\right) + \sin\left(\frac{\theta - \psi}{2}\right) + \sin\left(\frac{\chi - \psi}{2}\right) \right]^{-1} \quad . \quad (C.3) \end{aligned}$$

The consecutive integrations in (C.3) give

$$\begin{aligned} \varrho(U_0) &= -\frac{1}{16\pi^2} \int_0^{2\pi} d\theta \int_0^\theta d\chi \cos^{-2}\left(\frac{\chi - \theta}{4}\right) \ln \left[1 - \coth\left(\frac{\chi - \theta}{4}\right) \tan \frac{\chi}{4} \right] = \\ &= -\frac{1}{8\pi^2} \int_0^{2\pi} d\theta \left[\theta + 4 \coth \frac{\theta}{4} \ln \left(\cos \frac{\theta}{4} \right) \right] = -\frac{1}{12} \quad . \quad (C.4) \end{aligned}$$

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FIGURE CAPTIONS

Fig.1 Two examples of framing. The thin line represents the framed contour defined by eq. (3.9) in terms of n^μ .

Fig.2 Link B : the Borromean Rings.

Fig.3 Link U_n : distant union of $(n + 1)$ unknots.

Fig.4 First non-vanishing contribution to the interaction between different knots.

Fig.5 Reidemeister moves of type I, II and III.

Fig.6 A twisted band diagram is not invariant under Reidemeister moves of type I.

Fig.7 Configurations connected by the skein relation.

Fig.8 Correspondence between oriented twisted bands diagrams and knot diagrams of regular isotopy.

Fig.9 Configurations related by eqs.(4.5) and (4.6).

Fig.10 Simple knots: (a) Right-handed knot 3_1 (trefoil), (b) knot 4_1 , (c) right-handed knot 5_1 .

Fig.11 Simple links: (a) link 2_1^2 (Hopf), (b) link 4_1^2 .

(1.a)

(1.b)

Fig.1

Fig.2

Fig. 3

Fig. 4

I)

II)

III)

Fig. 5

Fig.6

L_+

L_-

L_0

Fig. 7

Fig. 8

$\hat{L}_+$

$\hat{L}_-$

$\hat{L}_0$

Fig. 9

(10.a)

(10.b)

J

(10.c)

Fig. 10

H

(11.a)

Q

(11.b)

Fig. 11