THE ONSET OF INTERMITTENT BEHAVIOUR IN THE ISING MODEL

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Abstract:

We study the development of intermittent behaviour near the critical point of the two-dimensional Ising model for lattices of size 32×32 to 256×256 . Using Monte Carlo simulation, we calculate both standard and factorial moments. Our results approach the values of the intermittency indices calculated analytically for a infinite system.

1. Introduction

It was recently shown that the d -dimensional Ising model ($d \geq 2$) leads to intermittency at the critical point¹: the presence of correlation regions of all scales provides at $T = T_c$ the behaviour of a self-similar fractal system. Prior to this work, indications of strongly fluctuating behaviour had been observed in a numerical study of the two-dimensional Ising model on a 32×32 lattice². It is now suspected, however, that the results of this work are to a large extent due to the specific form of moments studied there³. The formulation used in ref. 2 explicitly breaks the Z_2 symmetry of the Ising model, and we shall in fact show that this is the reason for the "intermittent" behaviour found there. It therefore remains an open question how intermittency begins to manifest itself on a finite lattice near the critical point.

The aim of the present paper is to study in detail the onset of intermittency in the finite-size two-dimensional Ising model. We shall use a Z_2 symmetric formulation of the moments and then perform high statistics calculations on 32×32 , 64×64 and 128×128 lattices, both for standard and for factorial moments. At a few temperature values, we also have results for a 256×256 lattice.

The plan of the paper is the following. The general formalism is defined in section 2. In the following section, the results of ref. 2 are analysed; for this purpose, we have repeated, with higher statistics, the calculations carried out there. In section 4, we then present the results of our calculations of the Z_2 symmetric moments on lattices of different size, together with an analysis of the observed behaviour. Here we shall also compare the results obtained using standard and factorial moments. In section 5, we summarize our conclusions.

2. Intermittency in the Ising Model

Consider a d -dimensional lattice of R sites in each dimension. To each of the R^d sites, we associate a spin variable $s_i = \pm 1$. The relative weights for the different possible spin configurations are determined by the Boltzmann factor

$$\exp(-\beta \sum_{i,j} s_i s_j), \quad (1)$$

where $i \neq j$ runs over all nearest-neighbour pairs of the lattice; $\beta = T^{-1}$ denotes the inverse temperature. We subdivide the lattice into $n = (R/L)^d$ blocks of size L^d . For a given block scale L , the average moments are then defined as¹

$$f_l(L) = n^{l-1} \langle \sum_{\text{blocks}} [\sum_{i=1}^{L^d} s_i]^l / [\sum_{\text{blocks}} \sum_{i=1}^{L^d} s_i]^l \rangle. \quad (2)$$

The average over configurations is carried out with the weights (1); it thus depends on the temperature β^{-1} . In ref. 1 it was shown that these moments satisfy

$$\ln f_l(L) = -\lambda_l \ln L + \ln f_l(L=1), \quad (3)$$

where

$$f_l(L=1) = \begin{cases} \langle 1/s^l \rangle, & l \text{ even}, \\ \langle 1/s^{(l-1)} \rangle, & l \text{ odd}, \end{cases} \quad (4)$$

are the moments for one spin per block, and

$$s \equiv (1/R^2) \sum_{i=1}^{R^2} s_i. \quad (5)$$

is the spin averaged over the entire lattice, for a given configuration. Eq. (3) establishes intermittency for the Ising model; the intermittency indices were found to be

$$\lambda_l = (d-2+\eta) \times \begin{cases} (l/2), & l \text{ even}, \\ ((l-1)/2), & l \text{ odd}, \end{cases} \quad (6)$$

where η is the anomalous part of the scale dimension. In the classical theory of critical exponents, which neglects the effect of fluctuations, η vanishes. For the two-dimensional Ising model, it is thus the presence of fluctuations of all scales which provides the intermittent behaviour. For $d = 2$, the exact solution of the Ising model gives $\eta = 1/4$.

For a numerical study of the onset of intermittency on finite lattices, the form (2) turns out to be not so suitable. When the net spin of the whole system becomes zero, the denominator of equation (2) will vanish, but the numerator can remain finite. To avoid this difficulty, the relevant block spin variable was in ref. 2 defined to be the total number of spins pointing up. Such a cure, however, breaks the Z_2 symmetry of the Ising model; as we shall see, it leads to violent fluctuations when the system is in an ordered state with the overall spin pointing down (see also ref. 3). We shall instead define

$$r_i = [(1 + \text{sgn}(s_i)s_i)/2], \quad (7)$$

as the variables of our system: if for a given configuration, the average spin s is pointing up, we add one to each spin variable, and if s points down, we flip each s_i and then add one. This removes the vanishing denominator, but at the same time preserves the Z_2 symmetry of the system.

How are the intermittency relations changed if we use the variables r_i instead of the s_i ? The extension of eq. (3) to this case is straight-forward; the crucial difference is that

$$s_i^l = \begin{cases} 1 & l \text{ even,} \\ s_i, & l \text{ odd,} \end{cases} \quad (8)$$

while

$$r_i^l = r_i \quad \forall l. \quad (9)$$

As a result, we now obtain relation (3) with

$$f_l(L=1) = \langle 1/r^{(l-1)} \rangle \quad \forall l \quad (10)$$

and

$$\lambda_l = (d - 2 + \eta) \frac{(l-1)}{2} \quad \forall l \quad (11)$$

for the intermittency indices. Hence for the two-dimensional Ising model in terms of the variables r_i , the intermittency indices are $\lambda_l = (l-1)/8$.

For the onset of intermittency in finite systems, another problem is caused by statistical fluctuations. These depend here on both lattice and block size and can thus in certain regions lead to a behaviour similar to intermittency. To avoid such fluctuations, it has been suggested⁴ to consider *factorial* moments, defined as

$$F_l(L) = n^{l-1} \left\langle \sum_{m=1}^{(R/L)^2} \frac{k_m}{K} \times \frac{(k_m-1)}{(K-1)} \times \dots \times \frac{(k_m-l+1)}{(K-l+1)} \right\rangle, \quad (12)$$

with

$$k_m \equiv \sum_{i=1}^{L^2} r_i \quad (13)$$

or the corresponding form for the variables s_i , and

$$K = \sum_{m=1}^{(R/L)^2} k_m. \quad (14)$$

One of the aims of our study is to see how the behaviour of the forms (2) and (12) differs.

3. Wosiek's Formulation

The starting point of Wosiek's investigation² was eq.(12), i.e. the factorial form for the moments, but with k_m defined as the number of spins in the m -th block pointing up. A numerical study of this formulation on a 32×32 lattice led to logarithmic moments rising rapidly with decreasing block size when $\beta \simeq \beta_c$ (for an infinite lattice, $\beta_c = 0.4407$). In contrast, for higher or lower temperatures the moments became independent of the block size. The results of ref. 2 were based on 10.000 Monte Carlo iterations per temperature value, with the first 2000 discarded as pre-thermal.

We have repeated these calculations, in particular for $\beta = 0.45$; for this temperature, Wosiek obtained the most rapid block size variation of the moments. In fig. 1a, we show the result of 510.000 iterations, with the first 20.000 discarded to eliminate pre-thermal behaviour. The points of ref. 2 are also included in fig. 1a; they agree quite well with ours. Following ref. 2, we show the moments for all block sizes down to 2×2 sites; however, the definition (5) is really meaningful only if $L^2 \gg l$, since otherwise there can be many configurations which cause the numerator to vanish. For this reason, the logarithmic moments begin to drop below zero with increasing $\ln(1/L)$. By definition, all moments become unity when $L = R$, i.e., when we have just one block – the entire lattice.

Let us now study the situation at $\beta = 0.45$ in more detail. In fig 1b, the distribution of the average spin $\bar{s}$ (the average of s in eq. 6 over all configurations) is shown. There are clear peaks at $\bar{s} = \pm s_0$, $s_0 > 0$, with a minimum at $\bar{s} = 0$. Hence we are not yet at the critical point on a 32×32 lattice, but still in the ordered phase. However, because of the small lattice size and the proximity of β_c , there are many flips between the two degenerate ground states. Since the definition of k_m as the total number of spins up breaks the Z_2 symmetry of the model, we have to study separately for each of the two degenerate states the effect that blocking has on it. We have therefore picked out two ranges in the number of iterations, of 10.000 iterations each, such that in one range the average spin points up, in the other down. The corresponding spin distributions are shown in fig's. 2a and 3a. We now determine the block size dependence of the moments in each case; it is shown in fig's. 2b and 3b. It is evident that the variation of the logarithmic moments with block size is completely due to the fluctuations obtained by counting the number of up spins when the system is in a state in which the overall spin points down – i.e., it is a consequence of the Z_2 symmetry breaking of the definition of k_m . If we would define k_m instead as the number of spins up if $K > 0$ and as the number of spins down if

$K < 0$, the dependence of $\ln F_l$ on block size completely disappears. Thus fig. 1a has no direct connection to the intermittency of the Ising model at the critical point.

4. The Onset of Intermittency

We now want to look at the behaviour of the moments near T_c , using the Z_2 -symmetric formulation obtained with the variables (7) in place of the s_i in expressions (2) and (12). Since the precise value of the critical temperature depends on the lattice size, we have to determine T_c for each given value of R . We do so by defining T_c as that temperature for which the susceptibility measured on an $R \times R$ lattice attains its maximum. For the lattice sizes we have used here, the results are listed in Table 1. They are in accord with finite size corrections to β_c which vanish as $1/R$.

R	β_c
32	0.427
64	0.434
128	0.437
256	0.439
∞	0.441

Table 1: The size dependence of the critical temperature $T_c = \beta_c^{-1}$

On a 128×128 lattice, we now consider both standard and factorial moments around the critical point; they are shown in fig's. 4 and 5 for $R = 64$ and 128, in each case at T_c and at one temperature above and one below T_c . Here and in the following, we have always carried out at least 500.000 iterations, using the standard Metropolis Monte Carlo algorithm, and discarded the first 5.000 to reach thermalisation. We have checked that an increase of the number of discarded iterations does not change the results. In the mentioned figures, we plot $\ln F_l(R/L)$ vs. $\ln (R/L)$; in terms of R/L , all curves start at the origin, no matter which R we use.

Let us first compare the behaviour above and below the critical point. In both cases, there are increasing fluctuations with decreasing block size. Below T_c , in the ordered state, these are largely removed by using factorial moments; we just have to make sure that L is not too close to l , since otherwise we encounter the suppression already noted in the previous section. Above T_c , however, where we expect both f_l and F_l to vanish for an infinite system, there are still noticeable deviations even for the factorial moments; they increase with decreasing L . It is clear why this occurs: the average block spin, as well as the average overall spin, should vanish here for sufficiently large systems. In fig. 6 we see that the smaller L becomes, the less this is the case. These finite size deviations cannot be removed by using factorial moments; they are reduced only by going to larger L with correspondingly larger R . In any

case, it appears best to consider the factorial moments, keeping in mind, however, that they also still contain finite size fluctuations.

Do we get something like a linear increase of $\ln F_l$ in $\ln(R/L)$ at the critical point? In fig. 7, we show the factorial moments at the critical point (see table 1) for $R = 32, 64, 128$, and 256 . With increasing R , the behaviour indeed seems to be more and more in accord with what we would expect in case of intermittency.

The temperature variation of the moments is illustrated in fig. 8, where we show the factorial moment of order $l = 2$ at $L = 2$, for different lattice sizes R . We note that it has a peak around $\beta = \beta_c$; the peak position (i.e., β_c) is shifted to larger values of β with increasing R , in accord with the mentioned lattice size dependence of the critical temperature. Moreover, the peak becomes narrower with increasing R , as expected if we have intermittency only at the critical point.

To study the values reached by the intermittency indices for the lattice sizes considered here, we calculate the quantity

$$\lambda(R, L, \beta) \equiv [F_l(R, L; \beta) - F_l(R, 2L; \beta)] / [(l-1)\ln 2]. \quad (15)$$

For $R/L \rightarrow \infty$ at $\beta = \beta_c$, λ should converge for all l to $1/8$. In fig. 9, the results are shown for the factorial moment of order $l = 2$, calculated on 64×64 and 128×128 lattices. With decreasing block size, λ indeed seems to approach to the predicted $1/8$, although part of the values obtained here will still be due to finite size fluctuations. In fig. 10, we show the third order factorial moment for $R=64$ and 128 ; again there seems to be convergence towards $1/8$, with narrower peaks for larger R , as expected, but here, as well as for still higher order moments, the presence of finite size fluctuations leads to a slight overshooting for small L .

5. Conclusions

Let us summarize the results of our study of the 2-d Ising model.

(1) We have shown that the fluctuating behaviour found in a previous numerical investigation² on a 32×32 lattice is not due to the onset of intermittent behaviour, but is caused by a Z_2 breaking introduced through the definition of the moments used there.

(2) Using a Z_2 symmetric formulation, we have found indications for beginning intermittency at $T = T_c$, particularly on larger lattices (128×128 , 256×256).

(3) Even for the largest lattice studied here, there are still noticeable finite effects. Below T_c , in the ordered phase, the use of factorial moments removes such fluctuations to a large extent. Above T_c , however, in the disordered phase, they are due to the fact that the average block spin is quite different from zero for small values of L . These fluctuations contribute also to the factorial moments.

(4) The intermittency index $\lambda = \lambda_l/(l-1)$ approaches for decreasing block size the analytically calculated infinite volume value $1/8$. For moments of order $l > 2$, the presence of finite size effects causes a slight overshoot in the case of small block size L .

(5) We have not found a systematic way to deal quantitatively with the observed finite size effects. Unless such a method can be developed, studies on considerably larger lattices would be seem necessary to establish the functional finite size behaviour of the intermittency indices.

References

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Figure Captions

Fig. 1:

- (a) Factorial moments of order $l=2$ ($\circ$), 3 ($\triangle$), 4 ($\square$) and 5 (∇), on a 32×32 lattice at $\beta=0.45$, in Wosiek's formulation (ref. 2.); the solid symbols show his results.
- (b) Spin distribution for the configurations used in fig. 1a.

Fig. 2:

- (a) The spin distribution for the spin "up" sample.
- (b) The factorial moments for the spin "up" sample, with symbols as in fig. 1.

Fig. 3:

- (a) The spin distribution in the spin "down" sample.
- (b) The factorial moments for the spin "down" sample, with symbols as in fig. 1.

Fig. 4:

The Z_2 symmetric moments, with symbols as in fig. 1, for $R=64$ and $\beta=0.440$ (a), 0.434 (b) and 0.385 (c). Solid lines connect the factorial, dashed lines the standard moments.

Fig. 5:

The Z_2 symmetric moments, with symbols as in fig. 1, for $R=128$, and $\beta=0.446$ (a), 0.437 (b) and 0.355 (c). Solid lines connect the factorial, dashed lines the standard moments.

Fig. 6:

Average block spins $\langle |s| \rangle$, for $L=2$ (highest curve) to $L=128$ (whole lattice, lowest curve).

Fig. 7:

Factorial moments at $\beta = \beta_c$, for $r=32$ (a), 64 (b), 128 (c) and 256 (d), with symbols as in fig. 1.

Fig. 8:

Factorial moments of order $l=2$ for $L=8$, calculated on lattices of size $R=32$ ($\circ$), 64 ($\square$) and 128 ($\triangle$).

Fig. 9:

- (a) Intermittency index λ calculated from the factorial moments of order $l=2$, for $R=64$ and $L=8$ ($\triangle$), 16 ($\times$) and 32 (∇).
- (b) Intermittency index λ calculated from the factorial moments of order $l=2$, for $R=128$ and $L=8$ ($\triangle$), 16 ($\times$), 32 (∇) and 64 (+).

Fig. 10:

- (a) Intermittency index λ calculated from the factorial moments of order $l=3$, for $R=64$ and symbols as in fig. 9a.
- (b) Intermittency index λ calculated from the factorial moments of order $l=3$, for $R=128$ and symbols as in fig. 9b.

Fig. 1 a

Fig. 1 b

Fig. 2 a

Fig. 2 b

Fig. 3 a

Fig. 3 b

Fig. 4a

Fig. 4b

Fig. 4c

Fig. 5 a

Fig. 5 b

Fig. 5 c

Fig. 6

Fig. 7 a

Fig. 7 b

Fig. 7 c

Fig. 7 d

Fig. 8

Fig. 9 a

Fig. 9 b

Fig. 10 a

Fig. 10 b