MINIMAL MODELS ON RIEMANN SURFACES

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Abstract

We extend the Coulomb gas representation of minimal model correlation functions to higher genus Riemann surfaces. Specifically, we incorporate the "BRS" structure of this representation into the operator formalism. This allows traces over the spectrum of a minimal model to be computed from traces over Fock spaces of a free scalar field. Thus our results are obtained by standard sewing techniques. The formulation is general, but we present here in particular the two loop characters of the Ising model.

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There has recently been considerable interest in two-dimensional conformal field theories (CFT) [1] and their extension to Riemann surfaces of arbitrary genus [2]. This is partially motivated by the desire to better understand string perturbation theory, partially as a means of producing consistency conditions on the operator content of a given model [3], but also stands as an interesting problem in its own right. At the same time the operator formalism for strings in a flat background [4, 5] has been developed to the point that multiloop amplitudes are rather well understood.

In this letter we apply the approach of [5] to extend to higher genus the simplest CFT's: the minimal models of the Virasoro algebra [1, 6].

These models contain only a finite number of primary fields, corresponding to completely degenerate representations of the Virasoro algebra. For two given positive and coprime integers, p and p' , a minimal model is characterized by the central charge

$$c = 1 - \frac{6(p' - p)^2}{p'p} \quad (1)$$

and the spectrum of its primary fields, whose dimensions are given by the Kac formula [7]

$$h_{n',n} = \frac{(n'p - np')^2 - (p' - p)^2}{4p'p} = h_{p' - n', p - n} \quad (2)$$

with $1 \leq n' \leq p' - 1$ and $1 \leq n \leq p - 1$. For example, the Ising model has $p = 3$ and $p' = 4$, and three primary fields: $\phi_{3,1} = \phi_{3,2}$ (identity, I), $\phi_{2,1} = \phi_{2,2}$ (spin, σ) and $\phi_{3,1} = \phi_{3,2}$ (energy, ϵ). Their fusion rules are: $I \times I = I$, $I \times \sigma = \sigma$, $I \times \epsilon = \epsilon$, $\sigma \times \sigma = I$, $\sigma \times \epsilon = \epsilon$ and $\epsilon \times \epsilon = I$.

In general, a correlation function of k primary fields may be decomposed into a sum of factorized (analytic/antianalytic) contributions, and each factor - called a conformal block, F_{μ} - corresponds to a definite choice of intermediate representations $\mu = \{(m'_1, m_1), \dots, (m'_{k-1}, m_{k-1})\}$. By introducing a chiral vertex operator for each fusion rule, $\Phi_{n',n}^{l,l}$, we may write the formal expression

$$F_{\mu}(\{z_i\}) = \langle \prod_{i=1}^k \Phi_{n'_i, n_i}^{m'_{i-1}, m_{i-1}}(z_i) \rangle \quad (3)$$

where $m'_0 = p' - 1$, $m_0 = p - 1$ and $m'_k = m_k = 1$.

Using the work of Feigin and Fuchs [8], Dotsenko and Fateev [9] gave the complete solution for minimal model correlation functions on the sphere. They were able to write an integral representation for the conformal blocks of chiral vertices $\Phi_{n',n}^{l,l}$ in terms of correlation functions of vertex operators of a free bosonic scalar field φ , coupled to a background charge $Q = 2\sqrt{2}\alpha_0$. In the conventions of [5], the vertex operator with charge $\alpha_{n',n}$ corresponding to the weight $h_{n',n} = \alpha_{n',n}(\alpha_{n',n} - 2\alpha_0)$, is

$$V_{\alpha_{n',n}}(z) = : \exp[-\sqrt{2}\alpha_{n',n}\varphi(z)] : \quad (4)$$

where the field φ has the standard mode expansion

$$\varphi(z) = \hat{q} - \hat{p} \ln z + \sum_{n \neq 0} \frac{\hat{\alpha}_n}{n} z^{-n}$$

with $[\hat{\alpha}_n, \hat{\alpha}_m] = n\delta_{n+m,0}$, $[\hat{q}, \hat{p}] = 1$. Introducing $\alpha_{\pm} = \alpha_0 \pm \sqrt{1 + \alpha_0^2}$, we easily find that for $\alpha_0^2 = \frac{(p'-p)^2}{4p'p}$ (i.e. $\alpha_+ = \sqrt{\frac{p'}{p}}$ and $\alpha_- = -\sqrt{\frac{p'}{p}}$), the vertex operator $V_{\alpha_{n',n}}$ with charge

$$\alpha_{n',n} = \frac{1}{2}((1 - n')\alpha_- + \frac{1}{2}(1 - n)\alpha_+$$

has precisely the dimension $h_{n',n}$ given in (2). Moreover, $V_{\alpha_{\pm}}$ have dimension 1, and their integrals $\int dz V_{\alpha_{\pm}}(z)$ are just the screening operators used in [9] to balance the anomalous charge conservation in the explicit calculation of conformal blocks.

Several groups have recently considered the extension of the Dotsenko-Fateev construction to genus 1 [10, 11, 12, 13] and to hyperelliptic surfaces (genus 2) [14]; however a clear geometrical interpretation of the screening charges on higher genus is still missing and this is also part of our program.

For our purposes, the most relevant of these works is the one by Felder [10], which we briefly recall here to set notations. In direct correspondence with the chiral vertex $\Phi_{n',n}^{l,l}$, he introduced the screened vertex operator

$$\begin{aligned} V_{n',n}^{r',r}(z) &\equiv V_{\alpha_{n',n}}(z) S^{(r',r)}(z) \\ &= \int_z^z \prod_{i=1}^{r'} du_i \prod_{j=1}^r dv_j V_{\alpha_{n',n}}(z) V_{\alpha_-}(u_1) \dots V_{\alpha_-}(u_{r'}) V_{\alpha_+}(v_1) \dots V_{\alpha_+}(v_r) \end{aligned} \quad (5)$$

where $r' = n' + m' - 2r' - 1$, $l = n + m - 2r - 1$ and the variables u_i and v_j are integrated along the contours shown in Fig. 1. If we denote by $F_{n',n}$ the Fock space built by the oscillators of φ , on the vacuum $|n', n\rangle$ such that $\hat{p}|n', n\rangle = \sqrt{2}\alpha_{n',n}|n', n\rangle$, then $V_{n',n}^{r',r}$ is an operator of dimension $h_{n',n}$ mapping $F_{n',n}$ into $F_{l,l}$. Furthermore, because of the non-zero background charge, the space dual to $F_{n',n}$ is $F_{n',n}^* = F_{p'-n', p-n}$ (since $\alpha_{n',n}^* = 2\alpha_0 - \alpha_{n',n} = \alpha_{p'-n', p-n}$) and the normalization of the screened vertex

$$N_{n',n}^{l,l} \equiv \langle p' - l' - p - l | V_{n',n}^{r',r}(1) | m', m \rangle$$

can be computed, and is non vanishing only when consistent with the fusion rules.

Felder also introduced the operators $Q_m : F_{m',m} \rightarrow F_{m',-m}$

$$Q_m \equiv \left(\frac{e^{2\pi i m \alpha_+^2} - 1}{e^{2\pi i \alpha_+^2} - 1} \right) \frac{1}{m} \oint_{\gamma_0} dv_0 \int_{\gamma_0} \prod_{i=1}^{m-1} dv_i V_{\alpha_+}(v_0) \dots V_{\alpha_+}(v_{m-1}) \quad (6)$$

which commute with the Virasoro generators L_k , since the final v_0 integrand is single valued on $F_{m',m}$. These Q operators satisfy the following remarkable properties [10]:

i) "nilpotency"

$$Q_m Q_{p-m} = 0 \quad ; \quad Q_{p-m} Q_m = 0$$

ii) the complex

$$\dots F^{[-1]} = F_{m',-m+2p} \xrightarrow{Q^{[-1]}=Q_{p-m}} F^{[0]} = F_{m',m} \xrightarrow{Q^{[0]}=Q_m} F^{[1]} = F_{m',-m} \dots$$

¹The prefactor of "phases" was not present in [10], but it is essential for the identity (8). We thank G. Felder for a discussion of this point.

has non trivial cohomology only at $F^{[0]}$. In other words, if we define $H_{m',m}^{[j]} \equiv \frac{\text{Ker} Q^{[j]}}{\text{Im} Q^{[j-1]}}$, then

$$H_{m',m}^{[j]} = 0 \quad j \neq 0 \quad (7)$$

and

$$H_{m',m}^{[0]} \simeq \mathcal{H}_{m',m}$$

where $\mathcal{H}_{m',m}$ is the irreducible highest weight module with highest weight $h_{m',m}$ as in (2). Finally,

iii)

$$Q_l V_{n',n}^{r',r}(z) = e^{2\pi i l \alpha + \alpha_{n',n}} V_{n',n}^{r',n-r-1}(z) Q_m \quad (8)$$

with $l = n + m - 2r - 1$, and thus actually $V_{n',n}^{r',r}(z) : H_{m',m}^{[0]} \rightarrow H_{l',l}^{[0]}$.

With this, the conformal blocks (3) are given by

$$F_\mu(\{z_i\}) = \left(\prod_i N_{n'_i, n_i}^{m'_i, m_i - 1} \right)^{-1} < 0 | \prod_i V_{n'_i, n_i}^{r'_i, r_i}(z_i) | 0 > \quad (9)$$

Even more, (7) and (8) contain the information we need to make the sum over intermediate states explicit, since, for any operator $\mathcal{O}$, the Lefschetz principle gives

$$T\tau \Big|_{\mathcal{H}_{m',m}} \mathcal{O} = \sum_{j=-\infty}^{\infty} (-1)^j T\tau \Big|_{F^{[j]}} \mathcal{O}^{[j]} \quad (10)$$

where the $\mathcal{O}^{[j]}$'s are defined recursively by $\mathcal{O}^{[j]} Q^{[j-1]} = Q^{[j-1]} \mathcal{O}^{[j-1]}$, with $\mathcal{O}^{[0]} = \mathcal{O}$. Thus the trace over the irreducible representation $\mathcal{H}_{m',m}$ may be computed as the alternating sum of Fock space traces. This is the essential step required for our extension to higher genus.

For our purposes, we must give a more convenient representation of these ideas. Thus, following [5], we first introduce the 3-Reggeon Vertex in the Della Selva-Saito form [15]

$$\begin{aligned} W_A^{(n',n)} &= A < p' - n', p - n | W_A \\ &\equiv A < p' - n', p - n | \exp \left\{ -\frac{1}{2} \oint dz \partial \varphi^{(A)}(z) \log[\partial f_A(z)] (\tilde{p}^{(A)} - 2\sqrt{2}\alpha_0) \right\} \\ &\quad : \exp \left\{ \oint dz \varphi[f_A(z)] \partial \varphi^{(A)}(z) \right\} : \end{aligned} \quad (11)$$

which contains two independent free scalar fields defined on the Riemann sphere: φ , which will be called "internal", and $\varphi^{(A)}$, called "external". The state $A < p' - n', p - n |$ is the dual to $|n', n >_A$ which is the vacuum of the Fock space with $\tilde{p}^{(A)} |n', n >_A = \sqrt{2}\alpha_{n',n} |n', n >_A$. The function f_A is just the choice of local coordinate f_A^{-1} vanishing at the puncture z_A . Then one may show [5] that

$$W_A^{(n',n)} |n', n >_A (\partial f_A^{-1}(z_A))^{h_{n',n}} = V_{\alpha_{n',n}}(z_A) \quad (12)$$

$$W_A^{(n',n)} \hat{\alpha}_{-k}^{(A)} |n', n >_A (\partial f_A^{-1}(z_A))^{h_{n',n}} = \oint_{z_A} dz z^{-k} \partial \varphi(z) V_{\alpha_{n',n}}(z_A) \quad (13)$$

and hence the N -Point Vertex

$$V_{N,0}(\{n'_i, n_i\}) \equiv < 0 | W_1^{(n'_1, n_1)} \dots W_N^{(n'_N, n_N)} | 0 > \quad (14)$$

reproduces the correlation functions of N primary fields of charges $\alpha_{n'_1, n_1}, \dots, \alpha_{n'_N, n_N}$, $\sum_i \alpha_{n'_i, n_i} = 2\alpha_0$, when it is saturated with the corresponding highest weight states. Further, (13) implies that all secondary correlators are also reproduced by (14) using the appropriate external states.

To construct a loop amplitude (which will be used to build conformal blocks on higher genus Riemann surfaces) one simply has to identify two legs of $V_{N,0}$ and trace over the corresponding representation space with the insertion of a "propagator" $P(x)$, which generates the moduli of the loop. This procedure is called sewing. The explicit expression of $P(x)$ is determined for a given choice of local coordinates by requiring factorization in the tree level amplitudes as discussed in [5]. Less figuratively, to obtain the N -Point g -Loop Vertex corresponding to a genus g Riemann surface, with external states $(n'_1, n_1), \dots, (n'_N, n_N)$ and with representation (M'_μ, M_μ) in the μ -th loop, we proceed as follows. We take $V_{(N+2g),0}$ in which for convenience the first legs are labelled by an index i , running from 1 to N , and the remaining $2g$ legs are divided into "odd" and "even" labelled respectively by $2\mu - 1$ and 2μ , with $\mu = 1, \dots, g$. Then we assign to the external legs momenta corresponding to (n'_i, n_i) , to the odd legs momenta (M'_μ, M_μ) , and to the even legs momenta $(p' - M'_\mu, p - M_\mu)$. This last assignment is due to the fact that we will sew $2\mu - 1$ to 2μ , and so we have to take momenta dual to each other on these legs. Finally, we change

$$2\mu < l', l | \rightarrow | l', l >_{2\mu}, \quad \tilde{p}^{(2\mu)} \rightarrow \tilde{p}^{(2\mu)} t \equiv 2\sqrt{2}\alpha_0 - \tilde{p}^{(2\mu)}, \quad \alpha_n^{(2\mu)} \rightarrow \alpha_n^{(2\mu)} t \quad (15)$$

which defines $V_{(N+2g),0} \rightarrow V_{(N+2g),0}^\dagger$. Then the N -Point g -Loop Vertex is given by

$$V_{N,g}(\{n'_i, n_i\}, \{M'_\mu, M_\mu\}) = \prod_{\mu=1}^g T\tau_{(2\mu-1, 2\mu)} \left[V_{(N+2g),0}^\dagger \prod_{\mu=1}^g P(x_\mu) \right] \quad (16)$$

where, in contrast to [5], there is *no* sum over the zero modes, since the internal representation is part of the data specifying a conformal block. We stress that (16) contains a delta function for the zero modes, corresponding to the anomalous conservation of the charge. It is clear that since sewn legs $2\mu - 1$ and 2μ have dual momenta, the conservation equation on a genus g Riemann surface is $\delta(\sum \tilde{p}^{(i)} + (g-1)2\sqrt{2}\alpha_0)$.

It has been shown in [5], by explicit calculation, that this procedure gives the conformal blocks of the free "charged" scalar on a genus g Riemann surface in the Schottky parametrization [16]. The corresponding Schottky group generators are determined by the local coordinates and the propagator.

To incorporate the cohomology structure in this formalism, we must determine the commutation relations of the Q operators with the W vertices. These are summarized in the following equation

$$\begin{aligned} e^{\pi i(l-1)(1+n\alpha_0^2)} W_A^{(p'+n', p-n)} Q_n^{(A)} S^{(r', m-r-1)}(z_A) = \\ -e^{\pi i(l(1-n')+(n-1)\alpha_0^2)} Q_l W_A^{(n', m)} S^{(r', r)}(z_A) + W_A^{(n', n)} S^{(r', n-r-1)}(z_A) Q_m \end{aligned} \quad (17)$$

where $l = n + m - 2r - 1$, with the indices l, n and m in the range $\{1, \dots, p-1\}$. This relation is understood to hold between an arbitrary bra from $F_{l',l}$ and an arbitrary ket from $F_{m',m}$, both

Fock spaces of the “internal” field. In the Ising model the fusion rules allow only the following four possibilities for the values of (l, n, m) , namely: $(1, 1, 1)$, $(2, 1, 2)$, $(1, 2, 2)$ and $(2, 2, 1)$. The proof of (17) is rather simple for $l = n = m = 1$, the other cases can be explicitly checked by considering different examples. The general proof of (17) and its extension to arbitrary p and p' is deferred to [17]. We observe however, that if we saturate the “external” field with a highest weight state $|n', n >_A$ with $0 \leq n' \leq p' - 1$, then the left hand side vanishes, and, by using (12), the right hand side reduces to (8). When $l = n = m = 1$ we simply note that (if we take without loss of generality $z_A = 1$)

$$\begin{aligned} W_A^{(p'+n', 2)} Q_1^{(A)} &= -e^{-2\pi i \alpha_{n', 1} \alpha_{n', 1} \alpha_A} < -n', -1 | \oint_{|v_0| > 1} dz_0 e^{-\sqrt{2}\alpha + \varphi(w_0)} \mathcal{W}_A \\ &= -e^{-2\pi i \alpha_{n', 1} \alpha_{n', 1} \alpha_A} < -n', -1 | \left[\oint_{|v_0| > 1} dz_0 - \oint_{|v_0| < 1} dz_0 \right] e^{-\sqrt{2}\alpha + \varphi(w_0)} \mathcal{W}_A \\ &= -e^{-2\pi i \alpha_{n', 1} \alpha_{n', 1} \alpha_A} \left[Q_1 W_A^{(n', 1)} - W_A^{(n', 1)} Q_1 \right] e^{-i\pi \sqrt{2}\alpha + \oint_0 dz \partial \varphi^{(A)}(z)} \end{aligned}$$

and so acting on $F_{n', 1}^{(A)}$ we get the result.

We are now in a position to explain how the sewing procedure can be used to give the Feigin-Fuchs type integral representation for the conformal blocks of the minimal models on higher genus Riemann surfaces. The method we propose is completely general, and we will attempt to present the general result throughout. All details and the general proof will be given in [17]. For reason of simplicity and space limitations, in this letter we discuss in detail just the Ising model and present its two loop characters.

To pick out a particular N -Point conformal block on a genus g Riemann surface, we first assign the quantum numbers as shown on Fig. 2, namely $\{(n'_i, n_i), i = 1, \dots, N\}$ to the external legs, $\{(M'_\mu, M_\mu); (p' - M'_\mu, p - M_\mu), \mu = 1, \dots, g\}$ to the pair $(2\mu - 1, 2\mu)$ which are to be sewn, and $\{(m'_I, m_I), I = 1, \dots, N + 2g - 1\}$ to the intermediate legs chosen to be consistent with the fusion rules. This is conveniently summarized by chiral vertices as in (3), and may be translated into screened vertex operators as in (9). We should point out that in this regard the higher genus characters are somewhat special. This is easily seen in an example, say $Z_{II}^{(I)}$, the two loop Ising character as represented in Fig. 3. Applying the above prescription strictly leads to

$$< \Phi_{3,2,1,1}^{3,2}(\infty) \Phi_{3,2,3,2}^{1,1} \Phi_{3,2,3,2}^{3,2} \Phi_{3,2,1,1}^{1,1} > \rightarrow < 0 | V_{3,2,1,1}^{0,0}(\infty) V_{3,2,1,1}^{2,1} V_{3,2,1,1}^{0,0} V_{3,2,1,1}^{0,0} | 0 > \quad (18)$$

The point here is that a charge insertion via the identity operator is required, in order to have dual momenta on the sewn legs. For convenience we will often make this insertion *inside* the diagram, for example as

$$< \Phi_{3,2,1,1}^{3,2} \Phi_{3,2,1,1}^{1,1} \Phi_{3,2,3,2}^{1,1} \Phi_{3,2,1,1}^{3,2} > \rightarrow < 0 | V_{3,2,1,1}^{0,0} V_{3,2,1,1}^{0,0} V_{3,2,1,1}^{2,1}(\xi_1) V_{3,2,1,1}^{0,0} V_{3,2,1,1}^{0,0} | 0 > \quad (19)$$

Of course, since this insertion is just the identity operator, the dependence on ξ_1 in (19) actually drops *after* evaluating the integrals over all the screening variables.

Now, we may simply interpret this expression as a correlator of the “internal” field, and introduce an “external” field for each leg to be sewn, according to

$$V_{n', n}^{r', r}(z_A) \rightarrow W_A^{(n', n)} S^{(r', r)}(z_A)$$

where $S^{(r', r)}$ denotes the screening charges as in (5). With this, we are able to apply the sewing procedure of [5] as discussed above. When the Lefschetz sum is properly implemented, this reproduces the minimal model conformal block. Indeed, the decoupling of the “unphysical” states from higher loop conformal blocks reduces to the same problem at one loop, as we will see explicitly using (17). The one loop case can then be proven directly and is contained in the results of [10]. Here we simply demonstrate how this works in our example of $Z_{II}^{(I)}$.

To the configuration (19) we associate the 4-Point Vertex

$$< 0 | W_A^{(3,2)} W_{A'}^{(1,1)} V_{3,2}^{2,1}(\xi_1) W_B^{(3,2)} W_{B'}^{(1,1)} | 0 >$$

Then, we reverse legs A' and B' according to the rules (15), and insert a sewing operator $P(x_\mu)$ ($\mu = 1, 2$) on each of these legs; we symbolically write the result as an operator $\mathcal{O}_{A,B} : F_{3,2}^A \otimes F_{3,2}^B \rightarrow F_{3,2}^{A'} \otimes F_{3,2}^{B'}$, via

$$\mathcal{O}_{A,B} = < 0 | W_A^{(3,2)} W_{A'}^{(1,1)} V_{3,2}^{2,1}(\xi_1) W_B^{(3,2)} W_{B'}^{(1,1)} | 0 > \quad (20)$$

Thus, the two loop Ising character is given by

$$Z_{II}^{(I)} = T\tau_{(A,A')} \Big|_{\mathcal{H}_{3,2}} T\tau_{(B,B')} \Big|_{\mathcal{H}_{3,2}} \mathcal{O}_{A,B} \quad (21)$$

To calculate the traces over the irreducible representation, we have to apply (10). To this aim we define the operator $\mathcal{O}_A : F_{3,2}^A \rightarrow F_{3,2}^{A'}$, by

$$\mathcal{O}_A = T\tau_{(B,B')} \Big|_{\mathcal{H}_{3,2}} \mathcal{O}_{A,B} = < 0 | W_A^{(3,2)} W_{A'}^{(1,1)} | C | 0 > \quad (22)$$

$$C = V_{3,2}^{2,1}(\xi_1) T\tau_{(B,B')} \Big|_{\mathcal{H}_{3,2}} \left(W_B^{(3,2)} W_{B'}^{(1,1)} \right)^\dagger \quad (23)$$

Applying $Q_2^{(A)}$ and using (17) twice, we get

$$Q_2^{(A')} \mathcal{O}_A = \mathcal{O}_A^{[1]} Q_2^{(A)} + R \quad (24)$$

$$\text{where } \mathcal{O}_A^{[1]} = < 0 | W_A^{(3,-2)} W_{A'}^{(1,5)} | C | 0 > \quad (25)$$

and

$$R = e^{i\theta} < 0 | Q_1 W_A^{(3,2)} S^{(0,1)}(z_A) W_{A'}^{(1,5)} | C | 0 > + e^{i\theta'} < 0 | W_A^{(3,2)} W_{A'}^{(1,5)} S^{(0,1)}(z_A) Q_1 C | 0 > \quad (26)$$

The remainder term R vanishes because $< 0 | Q_1 = 0$ and $Q_1 C | 0 > = 0$. Indeed, using (8) and (23), we may write

$$Q_1 C | 0 > = e^{2\pi i \alpha + \alpha_{3,2} V_{3,2}^{2,0}(\xi_1) T\tau_{(B,B')} \Big|_{\mathcal{H}_{3,2}}} \left(Q_2 W_B^{(3,2)} W_{B'}^{(1,1)} \right)^\dagger$$

which of course vanishes since $\mathcal{H}_{3,2} \subset \text{Ker } Q_2$.

Iterating (24) and the analogous one

$$\mathcal{O}_A Q_1^{(A)} = Q_1^{(A')} \mathcal{O}_A^{[-1]}$$

one can define the operator

$$\mathcal{O}_A^{[j]} = \langle 0 | W_A^{(3,m,j)} W_{A'}^{(p'-3,p-m,j)^\dagger} C | 0 \rangle, \quad \begin{cases} m_{2l} = 2 - 2lp \\ m_{2l+1} = -2 - 2lp \end{cases}, \quad l \in \mathbb{Z} \quad (27)$$

which maps $F_A^{[j]}$ into $F_{A'}^{[j]}$ for any $j \in \mathbb{Z}$. Now we can apply (10) to calculate the trace on the loop built by sewing legs A and A' , giving

$$Z_{II}^{(l)} = \sum_{j=-\infty}^{\infty} (-1)^j T\tau \Big|_{F_A^{[j]}} \mathcal{O}_A^{[j]} \quad (28)$$

The trace over the other loop can be calculated in a similar way. The remainder terms which arise here vanish using the same argument as above in conjunction with the nilpotency property $Q_1 Q_2 = Q_2 Q_1 = 0$. Thus, $Z_{II}^{(l)}$ becomes a double sum of traces, in which each loop is treated independently

$$Z_{II}^{(l)} = \sum_{j_A, j_B = -\infty}^{\infty} (-1)^{j_A + j_B} T\tau \Big|_{F_A^{[j_A]}} T\tau \Big|_{F_B^{[j_B]}} \mathcal{O}_{AB}^{[j_A, j_B]} \quad (29)$$

where

$$\mathcal{O}_{A,B}^{[j_A, j_B]} = \langle 0 | W_A^{(3,m,j_A)} W_{B'}^{(p'-3,p-m,j_A)^\dagger} V_{3,2}^{2,1}(\xi_1) W_B^{(3,m,j_B)} W_{B'}^{(p'-3,p-m,j_B)^\dagger} | 0 \rangle \quad (30)$$

with m_{j_A} and m_{j_B} defined in (27).

The traces in (29) can be computed as in [5] by removing from (30) the “internal” field and writing the expectation value as an 8-Point Vertex saturated with the four states corresponding to the screened vertex $V_{3,2}^{2,1}(\xi_1)$. From the expression of $V_{N,g}$ with $N = 4$, $g = 2$ given in Eq. (3.7) of [5] one gets

$$Z_{II}^{(l)} = \sum_{\{\tilde{\gamma}; \gamma_\mu = 0,1\}} \phi(\tilde{\gamma}) \mathcal{Z}_{II}^{(j)}(\tilde{\gamma})$$

where

$$Z_{II}^{(l)}(\tilde{\gamma}) = \int \prod_{i=1}^4 d\xi_i \prod_{\substack{i,j=1 \\ i < j}}^4 E(\xi_i, \xi_j)^{2\alpha_i \alpha_j} \prod_{j=1}^4 \sigma(\xi_j)^{-\sqrt{2}Q\alpha_j} \\ \Theta \left[\begin{matrix} A_\mu + a_\mu \gamma_\mu \\ 0 \end{matrix} \right] \left((\sqrt{2pp'}) \frac{1}{2\pi i} \sum_{i=1}^4 \sqrt{2\alpha_i} \int_{\omega_\mu}^{\xi_i} (\omega_\mu + Q(\Delta_\mu + \frac{1}{2})) (2pp'\tau) \right) \quad (31)$$

In this equation ξ_i (for $i = 2, \dots, 4$) denote the variables of the screening operators which are integrated along the contours specified by the initial configuration (19), whereas ξ_1 denotes the position where the identity operator was inserted in order to balance the anomalous charge conservation². The corresponding charges are denoted by α_i and the background charge $Q = -\sqrt{2pp'}(1/p - 1/p')$. The geometrical objects appearing in (31) are the prime form E , the holomorphic differential without zeroes σ , the abelian differentials ω_μ , the vector of Riemann constants Δ_μ and the period matrix τ . The characteristics of the Riemann theta function Θ are related to the representations (M'_μ, M_μ) flowing in the loops as follows

$$A_\mu = \left(\frac{1 - M_\mu}{2p} - \frac{1 - M'_\mu}{2p'} \right) \quad ; \quad a_\mu = \frac{M_\mu}{p} \quad (32)$$

² As we have remarked the dependence on ξ_1 actually drops out.

Finally $\phi(\tilde{\gamma})$ are just the phases which arise when the Q operators are commuted through the diagram as we have found above, and in our two loop example are simply given by $\phi(0,0) = \phi(1,1) = 1$, $\phi(0,1) = \phi(1,0) = -1$.

In our example of $Z_{II}^{(l)}$ we have $M'_\mu = 3$ and $M_\mu = 2$ for $\mu = 1, 2$. The other two loop characters of the Ising model are also given by (31) with the following modifications: the characteristics of the theta function are changed consistently with (32) and the screening variables are integrated along contours dictated by the analogue of (19).

The procedure we have described above can also be applied to compute all conformal blocks on Riemann surfaces of arbitrary genus. The projection onto irreducible representations in the internal loops is realized as the appropriate combination of conformal blocks of a free charged scalar field compactified on a circle, as calculated in sect. 4 of [5] and also in [18]. To prove the decoupling of the unphysical states in a g -loop conformal block, we just apply the previous procedure, starting from the first loop on the left. This gives the Lefschetz sum which reduces to $T\tau|_{\mathcal{H}_{m',m}}$ plus extra terms which however, are always of the form of Q going into the remainder of the diagram. This Q will then propagate through the diagram, exactly as above, to cancel on the vacuum $|0\rangle >$ on the far right.

In our approach there is nothing which depends on the particular properties of the Ising model and actually it can be applied to all minimal models. The elaboration of this statement is the content of [17].

When completing this work, we received a preprint [19] which considers a similar problem to the one discussed in this paper, but in an entirely different way.

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Figure 1

Figure 2

Figure 3

Figure Captions
 Fig. 1: Integration contours for screening operators.
 Fig. 2: Typical configuration prepared for sewing.
 Fig. 3: Two loop character $Z_H^{(I)}$ in the Ising model.