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TOPOLOGICAL FLUCTUATIONS AND FOAM IN SIMPLICIAL 2-D QUANTUM GRAVITY

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Abstract

A previous proposal for generating topological fluctuations in two-dimensional surfaces is reviewed and a semi-quantitative prediction is given for the most probable sizes of the wormholes. The Hausdorff dimension of the foam is numerically estimated and it turns out to be rather high. It is also shown how loss of quantum coherence might be a natural consequence of the model. Finally it is conjectured that matched loops (minimal cross-sections of wormholes) play a role in Euclidean gravity, similar to event horizons in Lorentzian gravity.

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1 Introduction

For a long time simplicial gravity has been considered a convenient tool for extracting non-perturbative results in classical and quantum gravity much in the same spirit that lattice gauge theories have recently been used to study QCD.

However, a fundamental difference separates these two methods: Whereas lattice gauge theory deals with fields defined on static space-time lattices, simplicial gravity deals with the dynamics of the space-time lattice itself. To cope with this extra complication, two different yet related lines of study were developed.

The first one started with the original work of Regge [1]. It identifies the lattice dynamical elements with the lengths of the links joining neighbour sites [2]. Interesting dynamical questions, such as fluctuations in euclidean volume or topology can be formulated and in principle answered, but they involve the study of limiting cases where for example the lengths of one or more links vanish. These are thus degenerate cases and form part of the boundary of the lattice configuration space.

The second line of research has been predominantly oriented towards the combinatorial aspects of lattice dynamics, sharing a great deal of results and methods with the statistical models of random surfaces [3] [4]. Briefly, the links loose their dynamical role, retaining only the information on the lattice connectivity, the metric information is encoded into the number of links along minimal paths and the counting of lattice configurations becomes a pure combinatorial problem. This time, questions on volume [5] and topology fluctuations [6] do not require considering any limiting cases.

Fundamental and extensive studies on lattice combinatorics, with a view to both simplicial gravity and to random lattice gauge theories, were made by Christ, Friedberg and Lee [7] [8] and by Bander and Itzykson [9], where the two methodologies described were used.

Recently [10] we suggested that the topological (and part of the volume) fluctuations in simplicial two-dimensional gravity could be described by a simple matching-sites mechanism, much in the spirit of the second methodology described above.

In that work we studied the simplest consequences of extending reparametrization invariance in models of two-dimensional surfaces, discretized

by regular simplices (equilateral triangles). Using purely combinatorial arguments (i.e. without assuming any action) we showed that the usual Einstein term (topological term in two dimensions) appears as a measure of the surface entropy.

The main idea was that in the class of theories -- which include General Relativity -- where the observer is free to make local choices of his coordinate frames, that freedom might be extended by allowing the observer to (ocasionally) choose identical coordinates for distinct points.

Hence, the observer becomes, in a sense, more local since now he is not obliged to carry along a list of the coordinates used when visiting the previous sites.

If that choice of the coordinates is done on a random basis -- as is assumed to be the case when the sum over metrics is constructed -- such repetitions are obviously very small in any discretized model with a sufficiently large number of points. Had we considered the continuum case from the start, the probability of any such coincidences would vanish.

However, if we insist on studying the continuum case, the topological fluctuations could still be generated but no longer through exact coincidence of coordinates at different points. Instead, they would now appear as a result of the effective identification of points¹ whose coordinates differ by an amount smaller than a fixed space-time resolution.

Although this is to a certain extent equivalent to a discretized situation, it shows that its classical limit (i.e., the limit where the topological fluctuations are frozen) would correspond to an infinite resolution, that is, the vanishing of the cut-off with respect to a typical length scale of fluctuations.

Conversely, the presence of a finite cut-off allows the symmetry to be enlarged. We will not discuss the continuum case any further.

Briefly, the proposal of [10] goes as follows:

One considers a large but fixed set of sites together with links between them, in such a way that the entire set of these geometrical elements forms a triangulated closed surface with spherical topology -- the Ur-lattice.

Different coordinated objects are then obtained from the Ur-lattice by randomly assigning labels to its sites, all labels having identical probability,

¹In the same sense that, in the discrete case, points with identical sets of coordinates are regarded as being coincident.

and then identifying those sites which happened to be given identical labels.

The vast majority of the resulting objects will not correspond to any triangulated surface – rather they form “foamy” structures.

A tiny minority of the resulting objects will correspond to triangulated surfaces, each with a definite topology.

In the thermodynamic limit of the lattice, the statistical distribution of the resulting individual surfaces among the different topologies, or the relative weight of the different topologies, should not depend on the details of the particular Ur-lattice chosen.

New (coordinated) configurations are obtained not from older ones but instead by relabelling the original Ur-lattice. In this way, the first-generation ensemble of triangulated surfaces with various topologies appears as a subset of all the possible the coordinated objects. The sum over the metrics on the 2-surfaces in the path-integral reduces, in the discrete case, to the sum over all triangulations.

There is no second- (or higher-) generation ensemble and this fact guarantees that particular topological details of some object will not be inherited by all the objects subsequently generated to form the ensemble. In this sense this procedure differs from the standard Monte-Carlo sequences, where each new configuration is obtained by acting on some older one.

The counting of the relative weights for the different topologies can be done easily since the handles (or wormholes) are formed by pairwise identification of corresponding points on two different loops.

To preserve the manifold character of the simplicial complexes obtained, such loops should not encircle any other points inside them (“thin loops”).

In this way, the total number of configuration with h handles is obtained:

$$D_h \approx 2^h N! \sum_{\wp=h\wp_{min}}^{h\wp_{max}} C(\wp, h) e^{2h \ln P} \quad (1)$$

where $\wp$ is the total number of ur-lattice sites and $C(\wp, h)$ is a combinatorial factor given by

$$C(\wp, h) = \frac{r^{2\wp} M(\wp, h)}{\wp!} \Omega_{h\mu}(\wp) \quad (2)$$

with

$$\begin{aligned}
M(\wp, h) &\approx M_0(h) + M_1(h)\wp^{h-1} \\
M_0(h) &= (-1)^{h-1}h \\
M_1(h) &= \begin{cases} \frac{h \sum_{k=0}^{h-2} \frac{(-1)^k (h-1-k)^{h-1}}{k!(h-k)!}}{0}, & \text{if } h \geq 0; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

and $\Omega_{h\mu}(\wp)$ is the constrained sum

$$\Omega_{h\mu}(\wp) = \underbrace{\sum_{P_1=\wp_{min}}^{\wp_{max}} \cdots \sum_{P_h=\wp_{min}}^{\wp_{max}}}_{P_1+\cdots+P_h=\wp} \prod_{k=1}^h P_k^{2\mu-1} \quad (3)$$

Due to the constraint in $\Omega_{h\mu}(\wp)$, it is not easy to find a simple analytic expression for $C(\wp, h)$ which might be useful in determining the asymptotic behaviour of $C(\wp, h)$ when $\wp \longrightarrow \infty$.

2 Numerical results

A sort of mean-field argument was used in [10] to estimate D_h , but a direct numerical evaluation of $C(\wp, h)$ can be done, and the result is shown in figures 1.a) and 1.b).

Qualitatively we see that the main conclusion of [10] regarding the apparent size of the wormholes remains valid, namely the wormholes seem not to blow up but rather to limit their size around or slightly (i.e., a few times) above the cut-off $\wp_{min}$.

Two objections could however be raised concerning this result:

(I) When summing over all triangulations, we have not identified the gauge equivalent ones. Thus the volume of the discrete form of the group of diffeomorphisms was not divided out and we are surely overcounting metrics.

(II) Whenever $\wp \approx h\wp_{min}$, the expression (1) might no longer be a good approximation for D_h . In fact (1) involves two approximations. The first one is the dilute-gas approximation for the wormholes. The total number of sites which are pairwise identified along the loops is much smaller than the total number of (Ur) lattice sites. Besides this comes a second approximation where one assumes that the number of sites forming each thin-loop (i.e.,

the minimal cross-sections of the wormholes) is much larger than the unity. It is this last approximation that allows the use of an asymptotic formula [11] [12] to count the thin loops [10]. The universal parameters μ and r enter through that formula.

Concerning (II), it is true that the detailed form of the curve for $\log C(\varphi, h)$ as shown in fig.1 should not be trusted for small values of φ . However, for φ large enough, the characteristics of $C(\varphi, h)$ should be the ones indicated in the figure.

Concerning (I), we unfortunately do not have much to say yet, since the counting of equivalent triangulations on surfaces with non-zero topology is a hard (and to my best knowledge unsolved) problem (in the spherical case, the solution to this problem can be found in [13]).

It seems nevertheless unlikely that, once corrected, the sum over triangulations will be able to invert the sign of the slope of $\log C(\varphi, h)$ for large φ , thus making the wormholes attain macroscopic sizes.

The next question we want to address is related to the “foamy” configurations. In [10] we have ignored them and limited our counting to the triangulated surfaces. But there might be no good reason for this exclusion. After all, field theories can also be defined over orbifolds, cone-folds, non-orientable surfaces and so on.

Since ultimately we will have to face our counting problems on configurations with an arbitrarily large number of sites (thermodynamic limit), it seems natural to ask what the aspect of those foamy configurations is when probed with resolutions much below their U.V. cut-offs. Traditionally this type of question involves the determination of the Hausdorff dimension of the structures.

The Hausdorff dimension (H) essentially relates two measures characterizing a given structure, namely, its “volume” V and its average linear dimension D ,

$$D \sim V^{\frac{1}{H}} \quad (4)$$

V is an intrinsic measure of the structure (e.g. it could represent the number of lattice sites visited), whereas D is typically a distance and as such it implies some sort of embedding of the structure in a metric space. Apart from this, H characterizes the structure but is otherwise independent of

which particular embedding was chosen – it just needs to be embedded (e.g., the standard random-walk has $H = 2$, no matter whether it is progressing on a line, a surface or a volume. Of course other statistical properties will depend on the selected embedding).

In our case the embedding corresponds to the identification of equally labelled sites, and the distance between any two sites is given by the minimal number of links joining them.

We simulated numerically the self-intersecting lattices using a line of sites and a regular triangular (honeycomb) lattice for the one- and two-dimensional cases respectively.

The choice of a regular, rather than a random, two-dimensional ur-lattice was made first on the basis of simplicity of the simulation algorithm and secondly, because its asymptotic properties (e.g. the Hausdorff dimension) should be independent of its original topology.

The results obtained, although with modest sized lattices, are shown in fig. 2. They indicate similar slopes for the one- and two-dimensional (labelled) lattices, reflecting similar Hausdorff dimensions. The slope remains finite (although small) even for the largest lattices considered.

By choosing two lattices with V_1 and V_2 sites and with linear sizes D_1 and D_2 , H can be approximated by

$$H \approx \frac{\text{Log} \frac{V_1}{V_2}}{\text{Log} \frac{D_1}{D_2}} \quad (5)$$

provided V_1 and V_2 are not too different. With substantial uncertainty (due to the limited lattice sizes) we estimate $H \approx 13.9$ and presumably this value will grow further with larger lattices.

The fact that any memory of the original dimensionality is totally washed out by the matchings seems to indicate the absence of any particular scale in the final lattice, which could otherwise be related to a typical “thickness” of the foam when seen from “far away”.

This is actually to be expected, given that matchings between all pairs of sites are equally probable. Moreover this relates to the fundamental assumption in the Coleman calculus [14], according to which the distances between the wormhole ends are arbitrary.

3 “Horizons” in euclidean gravity

Among the whole set of labelled configurations, only a tiny subset succeeded (by chance) not only in keeping the memory of the original lattice dimensionality but actually in preserving it. They are the triangulated surfaces considered in [10]. The complete absence of a scale referred to above might have curious consequences among those surfaces.

To see that, imagine a classical scalar field Φ defined on each ur-lattice site prior to the labelling (i.e., before any metric is chosen). Now comes the labelling, accidentally producing, say, a sphere with one wormhole. Its minimal cross-section corresponds to a pair of matching loops L_1 and L_2 (see fig.3.a). Consider now any path from any site P on L_1 , where the field has value $\Phi(P)$, to the matching site Q on L_2 , where Φ has value $\Phi(Q)$. Since, in general $\Phi(P) \neq \Phi(Q)$, a field ambiguity will be noticed on the matching sites of L_1 and L_2 after following along closed paths which start and end at those sites (fig.3.b). Moreover, any correlation the field Φ might have between nearest neighbours on the ur-lattice will effectively be lost – on the labelled lattice – when trying to go from one sheet Σ_1 to another Σ_2 . Everywhere else Φ will remain uniquely defined, with its short-distance correlations preserved ²

Hence, if Φ represents, for instance, a non-uniform vacuum configuration on the lattice, there will be a “discontinuity” across L and the vacuum before L is reached will differ from the one immediately beyond L .

Alternatively, if Φ represents a wave-function of a pure state evolving according to a discretized version of the Schrödinger equation in Σ_1 , Φ will abruptly change across L , to a (generally) different pure state in Σ_2 . However, for a given loop L_1 , the matching loop L_2 can in principle be found anywhere on the lattice, depending on the particular metric chosen. This means that each pure state Φ on Σ_1 will meet with an ensemble of pure states on Σ_2 and the entire set $\Sigma_1 \cup \Sigma_2$ will be associated with a mixed state.

²We should stress that, once we accept that a field function Φ can be defined on the original lattice before any metric is selected, there will be no way to impose a condition like $\Phi(P) = \Phi(Q)$ whenever P and Q match, unless Φ is trivially constant. The reason for this is again the fact that no scale is present and the same point P may (in principle) match with any other site Q , depending only on the metric chosen.

This is the way loss of quantum coherence would take place according to Hawking [15]. It is curious to see the same effect appearing again, now as a consequence of a mechanism generating topological fluctuations (and foam).

These last few points suggest that the (thin) loops which match for making higher topologies (and which become the minimal sections of wormholes) should somehow play, in Euclidean gravity, a role related to the role played by event-horizons in Lorentzian gravity. This may be interpreted as a conjecture.

Obviously, the absence of any causality structure in Euclidean gravity allows lattice world-lines to traverse these cross-sections³. Instead, it is the breakdown of short-distance field correlations across these sections which suggests that signals cannot pass through them. Otherwise how could one ignore what is on the other side of a wormhole ?

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³The same idea of site matchings could also be used in Lorentzian gravity

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Figure Captions

Fig.1

Logarithm of the combinatorial factor $C(\wp, h)$ given by eq.(2) versus the sum ($\wp$) of perimeters of h wormholes. The curves indicated correspond to different minimal perimeters or cut-offs ($\wp_{min}$) for the wormholes. The fact that the curves are monotonously decreasing indicates that the wormhole minimal sections do not tend to blow-up.

- a) Surface with two wormholes;
- b) Surface with four wormholes.

Fig.2

Logarithm of the maximal distance (D) attained after a sufficiently large number of sites (V) were labelled, versus $\log V$.

- a) Full triangles in the case of linear lattices;
- b) Full squares in the case of two-dimensional lattices with honeycomb structure.

Lines through the points were drawn to guide the eye. The dotted line has slope $\frac{1}{2}$ and represents the typical " $\sqrt{t}$ " dependence of standard random walks. The error bars measure standard deviations.

Fig.3

Path joining two matching sites on matching thin loops.

Fig.4

Two sheets (or two parts of a single sheet) Σ_1 and Σ_2 , joining at their boundary L , resulting from the identification of two matching thin loops.

Fig.1.a)

Fig.1.b)

Fig.2

Fig.3.a)

Fig.3.b)