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Gauging The Quantum Groups

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Abstract

A notion of local Hopf algebras is introduced, in which the co-algebra structure is intrinsically connected to the dynamics described on a phase space manifold. The concept is motivated by the introduction of an integrable dynamics in dimension $d = 3$. This leads to the notion of gauging of a quantum group. The connection of these results with gauge theories defined on a space of loops is also discussed.

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1. Quantum Groups

The concept of a quantum group was introduced by Drinfel'd in [1,2], in which he presented a Hopf algebra setting for the quantization prescription of classical dynamical systems. In particular in connection with exactly integrable systems a very beautiful structure arises. For these situations the classical system is described by a phase space which is a group (usually a Lie group), and a Poisson manifold in such a way that the group operation is a Poisson morphism [3]. In this description the notion of the classical r -matrix, which solves the classical limit of the Yang-Baxter equations (CYBE's), has a natural setting : it is basically the Poisson tensor that gives the proper Lie-Poisson structure compatible with the integrable dynamics, cf. also [4]. The CYBE's arise as the conditions imposed by the Jacobi identity for this Poisson structure. Drinfel'd shows that the Lie-Poisson structure on the group G reflects in a bi-algebra structure on its corresponding Lie-algebra g . This is a Lie co-algebra structure on g such that the dual g^* of g carries a natural Lie product compatible with the one on g itself. What is important for the transition to the quantum theory is that the bi-algebra structure can be extended to the universal enveloping algebra (UEA) of g which, in fact, turns this UEA into a Hopf co-Poisson algebra. It is this structure that can be made subject to a q - deformation ($q = e^h$ being the canonical deformation parameter). The resulting quantum "groups" or quantum algebras have recently become a subject of intensive study in the literature, cf. e.g., [5,6,7,8,9,10,11,12,13].

The purpose of this letter is to give a first account of the development of the notion of local Hopf algebra and of gauge theory associated to a quantum group. Our starting point will be a generalization of the QYBE's, given in [14,15], that turns this equation into a dynamical equation for matrices whose matrix entries are functions on space-time rather than being, as usual, space-time independent. The motivation for this generalization came from the consideration of the hierarchy of simplex equations, one of whose members is the QYBE, cf. [16,17] and also [18]. It is the purpose of this note to show that this generalized (local) Yang-Baxter equation, which we care to interpret as a *classical* equation governing integrable dynamics on a three-dimensional lattice, is amenable to a similar construction to the one underlying the quantum group as developed in [13]. Moreover, we will also show how these results lead in a natural way to the notion of gauging the quantum group on a space of functionals of loops, the corresponding gauge potentials being now associated to surface-like transformations, i.e., giving rise to translations on the loop-space, rather than translations on space-time as in usual gauge theories.

2. Local Yang-Baxter Equations

In a previous paper [15] we introduced the notion of classical integrability on a three-dimensional lattice, thereby generalizing the two-dimensional Lax (or more precisely the Zakharov-Shabat) equations

$$L_l^{l'} \cdot L_{l'} = L_{l'}^l \cdot L_l \quad (1)$$

The outstanding feature of this equation, which reads

$$\begin{aligned} K_{ll''}^{(12)} \cdot K_{ll''}^{(13)l'} \cdot K_{l'l''}^{(23)} &= K_{l'l''}^{(23)l} \cdot K_{ll''}^{(13)} \cdot K_{ll'}^{(12)l''}, \\ K_{ll'}^{(12)} \cdot K_{l'l}^{(21)} &= 1. \end{aligned} \quad (2)$$

is the close resemblance to the quantum Yang-Baxter equations (QYBE's). Let us explain the notation in (2) : $K_{ll'}^{(12)}$ is a matrix acting on a tensor product of two vector-spaces $V_1 \otimes V_2$, i.e. $K_{ll'}^{(12)} \in \text{End}(V_1 \otimes V_2)$ depending on parameters l and l' which label two different directions on a three-dimensional rectangular lattice (whose directions are denoted by l , l' and l'' respectively). The matrix entries in the $K_{ll'}^{(12)}$ are functions of the lattice sites, and any superscript l denotes a translation in the l -direction (and similarly for the directions l' and l''). The product in (2) is coupled in the same way as in the QYBE's, i.e. denoting the matrix entries by $(K_{ll'})_{ii'}^{jj'}$ we can write eq.(2) as

$$(K_{ll'})_{ii'}^{kk'} (K_{ll''}^{l'})_{ki''}^{jk''} (K_{l'l''})_{k'k''}^{j'j''} = (K_{l'l''}^l)_{i'i''}^{k'k''} (K_{ll''})_{ik''}^{kj''} (K_{ll'}^{l''})_{kk'}^{jj'},$$

where summation over the coinciding indices k, k', k'' is understood. It is clear that (2) is precisely the QYBE's if we ignore the lattice-translations, taking the $K_{ll'}$ to be numerical, constant, matrices. In the present case, however, we are interested in the structure arising from (2) when the entries of the matrices K depend non-trivially on the lattice sites. We argued in [15] that we can interpret (2) as a generating equation for *classical* integrable equations of motion on a three-dimensional lattice. It was shown in particular that this algebraic structure leads, as in the two-dimensional situation, to the construction of infinite series of conserved charges for such lattice models.

In [14,15] we obtain eqs.(2) as the compatibility equations of an overdetermined system, in much the same way as the Lax (or Zakharov-Shabat) equations arise from a linear spectral problem. In the present case the overdetermined system, however, is no longer linear but has a bilinear structure, namely

$$K_{ll'}^{(12)} (L_l^{(1)l'} \dot{\otimes} L_{l'}^{(2)}) = (\mathbf{1} \otimes L_{l'}^{(2)l}) \cdot (L_l^{(1)} \otimes \mathbf{1}), \quad (3)$$

where $L_l^{(1)}$ is an element of $V_1 \otimes \mathcal{A}$, $\mathcal{A}$ being an associative unital algebra, which we will specify later on, and V_1 and V_2 are vector-spaces on which the matrix K are acting (The dot in eq. (3) denotes the product in the algebra $\mathcal{A}$, and the tensor product is with respect to the matrix spaces associated to V_1 and V_2 , $\mathbf{1}$ being the unit matrix). So the vector elements of $L_l^{(1)}$ are $\mathcal{A}$ -valued functions of the lattice-sites. Assuming (3) to be valued for all three directions l, l', l'' on the lattice it is easily seen that (2) arises from the fact that there are two equivalent ways of inverting the product

$$L_l^{(1)l'l''} \dot{\otimes} L_{l'}^{(2)l''} \dot{\otimes} L_{l''}^{(3)}.$$

There is another (*adjoint*) overdetermined system of bilinear equations in terms of elements tL_l of $\mathcal{A} \otimes V^*$, V^* being the vector space dual to V , namely

$$\left({}^tL_l^{(1)} \dot{\otimes} {}^tL_{l'}^{(2)l} \right) K_{ll'}^{(12)} = \left(\mathbf{1} \otimes {}^tL_{l'}^{(2)} \right) \cdot \left({}^tL_l^{(1)l'} \otimes \mathbf{1} \right), \quad (4)$$

which leads to the *same* equation (2) for the matrices K . Finally, there is a third bilinear equation mixing the L_l and the ${}^tL_{l'}$

$$\left(\mathbf{1} \otimes {}^tL_{l'}^{(2)l} \right) \cdot K_{ll'}^{(12)} \cdot \left(L_l^{(1)l'} \otimes \mathbf{1} \right) = L_l^{(1)} \dot{\otimes} {}^tL_{l'}^{(2)}, \quad (5)$$

which again leads to (2) as compatibility equations. Note that in [15] we have shown that a *homogeneous* bilinear system for eq. (2) can be obtained by considering ordered products of factors $K_{ll'}$ along curves and along surfaces on the lattice. Thus, for example we can introduce the path-ordered objects J and I along an arbitrary curve C , defined starting from a point Q by the ordered set of lattice directions $(qq'q'' \dots)$, as

$$\begin{aligned} J_{l(qq'q'' \dots q^{(n)})}^{1(22'2'' \dots 2^{(n)})} &= K_{lq}^{(12)} K_{lq'}^{(12')q} K_{lq''}^{(12'')qq'} \dots K_{lq^{(n)}}^{(12^{(n)})qq' \dots q^{(n-1)}}, \\ I_{(q^{(n)} \dots q''q'q)l}^{(2^{(n)} \dots 2''2'2)1} &= K_{q^{(n)}l}^{(2^{(n)}1)q^{(n-1)} \dots q'q} \dots K_{q''l}^{(2''1)q'q} K_{q'l}^{(2'1)q} K_{ql}^{(21)}, \end{aligned} \quad (6)$$

where in (6) there is a matrix product in space (1) and a tensor product on the spaces $(2, 2', 2'', \dots)$. We will denote these quantities simply by $J_{l\hat{q}}^{(1\hat{2})}$ and $I_{\hat{q}l}^{(\hat{2}1)}$ respectively. In the same way we define the following surface-ordered object T

$$\begin{aligned} T_{(\dots p''p'p)(qq'q'' \dots)}^{(\dots 1''1'1)(22'2'' \dots)} &= I_{\hat{p}q}^{(\hat{1}2)} I_{\hat{p}q'}^{(\hat{1}2')q} I_{\hat{p}q''}^{(\hat{1}2'')qq'} \dots \\ &= \dots J_{p''\hat{q}}^{(1''\hat{2})pp'} J_{p'\hat{q}}^{(1'\hat{2})p} J_{p\hat{q}}^{(1\hat{2})}, \end{aligned} \quad (7)$$

which we will denote simply by $T_{\hat{p}\hat{q}}^{(\hat{1}\hat{2})}$, using the same compact notations as above. These

are nothing but the analogues of the transfer matrices for the usual QYBE's. The homogeneous bilinear equations that we can write down for J , I and T ([15])

$$\begin{aligned} K_{ll'}^{(12)} \cdot J_{l\hat{q}}^{(1\hat{3})l'} \cdot J_{l'\hat{q}}^{(2\hat{3})} &= J_{l'\hat{q}}^{(2\hat{3})l} \cdot J_{l\hat{q}}^{(1\hat{3})} \cdot K_{ll'}^{(12)\hat{q}}, \\ I_{\hat{p}l}^{(\hat{3}1)} \cdot I_{\hat{p}l'}^{(\hat{3}2)l} \cdot K_{ll'}^{(12)} &= K_{ll'}^{(12)\hat{p}} \cdot I_{\hat{p}l'}^{(\hat{3}2)} \cdot I_{\hat{p}l}^{(\hat{3}1)l'}, \end{aligned} \quad (8)$$

and

$$\begin{aligned} J_{l\hat{p}}^{(1\hat{2})} \cdot J_{l\hat{q}}^{(1\hat{3})\hat{p}} \cdot T_{\hat{p}\hat{q}}^{(2\hat{3})} &= T_{\hat{p}\hat{q}}^{(2\hat{3})l} \cdot J_{l\hat{q}}^{(1\hat{3})} \cdot J_{l\hat{p}}^{(1\hat{2})\hat{q}}, \\ T_{\hat{p}\hat{q}}^{(\hat{1}\hat{2})} \cdot I_{\hat{p}l}^{(\hat{1}3)\hat{q}} \cdot I_{\hat{q}l}^{(23)} &= I_{\hat{q}l}^{(23)\hat{p}} \cdot I_{\hat{p}l}^{(\hat{1}3)} \cdot T_{\hat{p}\hat{q}}^{(\hat{1}\hat{2})l}, \\ I_{\hat{p}l}^{(\hat{1}2)} \cdot T_{\hat{p}\hat{q}}^{(\hat{1}\hat{3})l} \cdot J_{l\hat{q}}^{(2\hat{3})} &= J_{l\hat{q}}^{(2\hat{3})\hat{p}} \cdot T_{\hat{p}\hat{q}}^{(\hat{1}\hat{3})} \cdot I_{\hat{p}l}^{(\hat{1}2)\hat{q}}. \end{aligned} \quad (9)$$

are direct consequences of the equation (2) for the matrices K . The superscripts $\hat{q}$ and $\hat{p}$ denote the combined shifts associated with the multi-indices $(qq'q''\dots)$ and $(\dots p''p'p)$ respectively. We would like to note at this point that all the above equations remain valid for the usual QYBE's case when we ignore the lattice shifts.

3. Gauging the Lattice Equations

In 1+1 dimensions the integrable lattice equations are given by the Lax equations (1). In that case is possible to absorb the shift operator in the definition of the link variable L_l as follows. Let σ_l denote the operator generating a shift or translation in the direction l on the lattice, i.e. $\sigma_l f = f^l \sigma_l$ for an arbitrary function f depending on the lattice sites, σ_l^{-1} being the shift in the opposite direction, and introduce the operator $\hat{L}_l = \sigma_l^{-1} L_l$, where the operator σ_l^{-1} is supposed to act on everything on its right. Now, in terms of the operator $\hat{L}_l$ the Lax equation (1) reduces to the commutativity condition $\hat{L}_l \cdot \hat{L}_{l'} = \hat{L}_{l'} \cdot \hat{L}_l$. Furthermore, L_l can be considered to be the lattice gauge field for the shift operator σ_l^{-1} , because it can be easily shown that by dressing the shift operator with L_l it renders it into a covariant translation operator $\hat{L}_l$ which generates objects covariant under gauge transformations. Indeed, the continuum limit of $\hat{L}_l$ leads to the usual covariant derivative associated to a gauge field A_l whose path-ordered exponential along an elementary link with direction l on the lattice gives the line element L_l . So the object $F_{l'l} = \hat{L}_{l'}^{-1} \cdot \hat{L}_l^{-1} \cdot \hat{L}_{l'} \cdot \hat{L}_l$ that transforms covariantly under gauge transformations may be considered as a lattice version of the curvature associated to this gauge field.

In comparison to the situation in 1+1 dimensions, it is natural to rewrite the local Yang-

Baxter equation (2), which we regard to be a generalization into dimension three of the Lax equation (1), also in operator form. In fact the loop-space reinterpretation of eq.(2) also suggests [15] that a 'gauging' procedure may be obtained as in the $d = 2$ case. In order to achieve this, we introduce an operator-valued tensor $\hat{K}_{ll'}$ given by

$$\hat{K}_{ll'}^{(12)} = \sigma_l^{(2)-1} \cdot K_{ll'}^{(12)} \cdot \sigma_{l'}^{(1)}, \quad (10)$$

such that eq.(2) takes the form of the usual QYBE

$$\begin{aligned} \hat{K}_{ll'}^{(12)} \cdot \hat{K}_{ll''}^{(13)} \cdot \hat{K}_{l'l''}^{(23)} &= \hat{K}_{l'l''}^{(23)} \cdot \hat{K}_{ll''}^{(13)} \cdot \hat{K}_{ll'}^{(12)}, \\ \hat{K}_{ll'}^{(12)} \cdot \hat{K}_{l'l}^{(21)} &= 1. \end{aligned} \quad (11)$$

where we have adopted the convention that the shift operators sitting inside the $\hat{K}$ act only on the factor in the tensor product corresponding to the one in which they are situated, which is indicated with a superscript. Thus, for example, we have the rule $(\sigma_l^{(1)} \otimes \sigma_l^{(2)}) K_{l'l''}^{(12)} = K_{l'l''}^{(12)l} (\sigma_l^{(1)} \otimes \sigma_l^{(2)})$. It is implicit in eq.(11) that the matrix product between the $\hat{K}$ goes in the same fashion as in eq.(2). Again we have to remember that the $\hat{K}_{ll'}$ are dressed tensorial operators acting on everything on their right. They can be considered to be the covariant operators generating translations no longer along a single direction on the lattice, but along plaquettes determined by the directions l and l' , see e.g. section 5. The matrices $K_{ll'}$ that transform under a local gauge transformations g_l , that leave equations (11) invariant, as follows

$$K_{ll'} \rightarrow (g_l \otimes g_{l'}) \cdot K_{ll'} \cdot (g_l'^{-1} \otimes g_{l'}^{-1}),$$

cf.[15], transform in a covariant way when they are replaced by the operators $\hat{K}_{ll'}$

$$\hat{K}_{ll'} \rightarrow (g_l \otimes g_{l'}) \cdot \hat{K}_{ll'} \cdot (g_l^{-1} \otimes g_{l'}^{-1}).$$

This means in particular that the object $G_{ll''}$ defined by

$$G_{ll''}^{(123)} = \hat{K}_{ll'}^{(12)} \cdot \hat{K}_{ll''}^{(13)} \cdot \hat{K}_{l'l''}^{(23)} \cdot \hat{K}_{l'l}^{(21)} \cdot \hat{K}_{l'l'}^{(31)} \cdot \hat{K}_{l'l}^{(32)},$$

transforms covariantly under the above local symmetry, and in fact has to be considered as a generalized lattice curvature associated to lattice gauge fields $K_{ll'}$ living on elementary plaquettes of the three-dimensional lattice. Note that if $\hat{K}$ obeys eq.(11), then this curvature trivializes, i.e., is equal to the unit matrix. Starting now from the operators $\hat{K}$

we can define line-ordered products (i.e. the transfer matrices) in the usual way, ignoring all the lattice shifts. Thus, we can introduce the operators $\hat{J}$ and their dual $\hat{I}$ as

$$\begin{aligned}\hat{J}_l^{(1)} &= \hat{J}_{l\hat{q}}^{(1\hat{2})} = \hat{K}_{lq}^{(12)} \cdot \hat{K}_{lq'}^{(12')} \cdot \hat{K}_{lq''}^{(12'')} \dots \hat{K}_{lq^{(n)}}^{(12^{(n)})}, \\ \hat{I}_l^{(3)} &= \hat{I}_{\hat{q}l}^{(\hat{2}3)} = \hat{K}_{q^{(n)}l}^{(2^{(n)}3)} \dots \hat{K}_{q'l}^{(2'3)} \cdot \hat{K}_{ql}^{(23)},\end{aligned}\quad (12)$$

The operators $\hat{J}_l$ and $\hat{I}_l$ in eq.(12) are related to the tensors J_l and I_l in eq.(6) by

$$\hat{J}_{l\hat{q}}^{(1\hat{2})} = \sigma_l^{(\hat{2})-1} \cdot \tilde{J}_{l\hat{q}}^{(1\hat{2})} \cdot \sigma_{\hat{q}}^{(1)}, \quad \hat{I}_{\hat{q}l}^{(\hat{2}3)} = \sigma_{\hat{q}}^{(3)-1} \cdot \tilde{I}_{\hat{q}l}^{(\hat{2}3)} \cdot \sigma_l^{(\hat{2})}, \quad (13)$$

where

$$J_{l\hat{q}}^{(1\hat{2})} = \tilde{\sigma}_{\hat{q}}^{(\hat{2})} \cdot \tilde{J}_{l\hat{q}}^{(1\hat{2})} \cdot \tilde{\sigma}_{\hat{q}}^{(\hat{2})-1},$$

and similarly

$$I_{\hat{q}l}^{(\hat{2}3)} = \tilde{\sigma}_{\hat{q}}^{(\hat{2})} \cdot \tilde{I}_{\hat{q}l}^{(\hat{2}3)} \cdot \tilde{\sigma}_{\hat{q}}^{(\hat{2})-1},$$

and we have defined

$$\tilde{\sigma}_{\hat{q}}^{(\hat{2})} = \sigma_q^{(2' \dots 2^{(n)})} \sigma_{q'}^{(2'' \dots 2^{(n)})} \dots \sigma_{q^{(n-1)}}^{(2^{(n)})}.$$

The operator $\sigma_{\hat{q}}^{(1)}$ is the shift operator for the combined shifts $(qq'q'' \dots)$ acting on the space (1), and $\sigma_l^{(\hat{2})}$ is the shift operator in direction l acting in the tensor product space $(\hat{2})$. Eq.(11) can then be viewed as the compatibility condition for the following equations, that are respectively satisfied by the operators $\hat{J}$ and $\hat{I}$ given by eq.(13),

$$\hat{K}_{l'l'}^{(12)} \cdot (\hat{J}_l^{(1)} \otimes \hat{J}_{l'}^{(2)}) = (\mathbf{1} \otimes \hat{J}_{l'}^{(2)}) \cdot (\hat{J}_l^{(1)} \otimes \mathbf{1}) \cdot \hat{K}_{l'l'}^{(12)}. \quad (14)$$

or equivalently (using the dual objects $\hat{I}$)

$$\hat{K}_{l'l}^{(21)} \cdot (\hat{I}_l^{(1)} \otimes \hat{I}_{l'}^{(2)}) = (\mathbf{1} \otimes \hat{I}_{l'}^{(2)}) \cdot (\hat{I}_l^{(1)} \otimes \mathbf{1}) \cdot \hat{K}_{l'l}^{(21)}. \quad (15)$$

Eqs.(14, 15) reduce to (8) when we extract the shift operators using (13). Hence, the procedure of gauging away the shift operators is consistent with the bilinear structure underlying the operator Yang-Baxter equation.

4. Local Hopf Algebra

In the case of the usual (i.e. static) QYBE's a solution R of the Yang-Baxter equation is a matrix, acting on a vector-space $V_1 \otimes V_2$ which may be viewed as a representation space

of a given Lie algebra g . As such the R -matrix is a numerical object, which has the property that it defines relations among generators of another object, namely an associative unital algebra $\mathcal{A}$. In fact, if V has finite dimension N , then the generators of $\mathcal{A}$ can be viewed to take values in $\mathcal{A} \otimes Mat_N$. This allows one to give $\mathcal{A}$ a Hopf algebra structure [20] where the coproduct is defined relative to the product in Mat_N . In fact, $\mathcal{A} \otimes Mat_N$ becomes a *double Hopf Algebra*, because the construction allows one to interchange the roles of $\mathcal{A}$ and Mat_N . Following [13] one can use the R -matrix to define the dual $\mathcal{A}^*$ of $\mathcal{A}$ which inherits from the latter the Hopf algebra structure. If $\mathcal{A}$ is interpreted as the space of functions on some formal group $\hat{G}$, we can identify the dual object $\mathcal{A}^*$ as the quantum group.

It is natural to extend the construction of [13] to the present case where the basic equation is no longer the usual QYBE, but the local Yang- Baxter equation (2). This leads to the construction of a local Hopf algebra structure associated to $\mathcal{A}$ where $\mathcal{A}$ is now an associative unital algebra of functions defined on a 3- d lattice. Its generating elements are algebra valued functions $\{(J_l)_i^j\}_{i,j=1,\dots,N}$ where the indices i, j will be interpreted as matrix indices in Mat_N . We will further adopt the compact notation $J_l^{(1)}$, the superscript (1) standing as usual for the matrix space associated to J_l . The (quadratic) local relations between the elements of $\mathcal{A}$ are given by the matrices $K_{ll'}$ as

$$K_{ll'}^{(12)} J_l^{(1)l'} \cdot J_{l'}^{(2)} = J_{l'}^{(2)l} \cdot J_l^{(1)} K_{ll'}^{(12)\hat{a}} \quad (16)$$

where the dot product in eq.(16) is the product in $\mathcal{A}$, and there is, between the J 's, a tensor product in the matrix spaces (1) and (2); the superscript $\hat{a}$ stands for an algebra-valued shift, the significance of which will be explain in the following. For that purpose let us note that if we choose any curve C on the lattice, the objects $J_{l\hat{q}}^{(1\hat{2})}$ of eq.(6) realize a representation of $\mathcal{A}$ associated to their multi-index $\hat{2}$, the algebra valued shift $\hat{a}$ being represented by the corresponding multi-shift $\hat{q}$. Hence, to construct a local structure for $\mathcal{A}$ we have to introduce the notion of algebra-valued shift operators acting, say, on a matrix space (1) that we will denote by $\sigma_{\hat{a}}^{(1)}$; so taking a representation of $\mathcal{A}$ will also mean that we choose a multi-shift representation for $\hat{a}$; therefore, we will denote by $\rho_{\hat{q}}^{(1)}$ a representation of $\mathcal{A}$ acting on a vector space V_1 , $\hat{q}$ being the shift representation index. Let us further note that the compatibility condition for eq.(16) is just (2), as expected. Now in order to achieve the construction of the local Hopf algebra structure for $\mathcal{A}$, the key idea is to use, instead of eq.(2), the gauged version (11) of this equation; following [13] we can then construct a Hopf algebra structure for the algebra $\mathcal{B} = (\mathcal{A} \otimes \Sigma(\mathcal{F}))$, where $\Sigma(\mathcal{F})$ is the commutative group of shift operators acting on the space $\mathcal{F}$ of functions on

the lattice, taking into account, however, that the objects $\hat{K}$ are in our case operator-valued solutions of the QYBE. The generating elements $\hat{J}_l$ of $\mathcal{B}$ are related to the one in $\mathcal{A}$ through the following shift-operator dressing

$$\hat{J}_l^{(1)} = \sigma_l^{(a)-1} \cdot J_l^{(1)} \cdot \sigma_a^{(1)}, \quad (17)$$

where $\sigma_l^{(a)}$ is a shift operator in direction l acting in $\mathcal{A}$ only (and not on the matrix space (1)), and $\sigma_a^{(1)}$ is an algebra-valued shift operator acting on space (1). Now the defining relations between the generating elements of $\mathcal{B}$ are given by eq.(14) (i.e. without any explicit shift superscript), where the product between the $\hat{J}$ is a product in $\mathcal{B}$ and there is a tensor product on the matrix indices (1) and (2). Note that eq.(14) is a gauged version of (16) from which it may be deduced using (17). The compatibility condition (associativity in $\mathcal{B}$) for this equation is just the gauged version of (2), namely (11) for the operators $\hat{K}$. The unital associative algebra $\mathcal{B}$ is a Hopf algebra with comultiplication (which is an algebra morphism) $\hat{\Delta} : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$

$$\hat{\Delta}(\hat{J}_l^{(1)}) = \hat{J}_l^{(1)} \otimes \hat{J}_l^{(1)}, \quad (18)$$

obeying the standard coassociativity rule. In eq.(18), there is a product on the matrix space (1) and a tensor product in $\mathcal{B}$. The co-unit ϵ and the antipode S are also defined in a standard way [13,20], remembering, however, the operator nature of the elements of $\mathcal{B}$. The next point now is the definition of the dual object to the algebra, which in the present case is $\mathcal{B}^* = (\mathcal{A} \otimes \Sigma(\mathcal{F}))^*$. The dual is defined via a bilinear form $(,)$ as follows. Let $\hat{I}_l^{(2)}$ be the generating elements of $\mathcal{B}^*$ that we define by their action on elements of $\mathcal{B}$ through the relations

$$\begin{aligned} (\hat{I}_k^{(2)}, 1) &= 1 \\ (\hat{I}_k^{(2)}, \hat{J}_l^{(1)}) &= \hat{K}_{lk}^{(12)} \\ (\hat{I}_k^{(3)}, \hat{J}_l^{(1)} \cdot \hat{J}_{l'}^{(2)}) &= \hat{K}_{lk}^{(13)} \cdot \hat{K}_{l'k}^{(23)} \\ &\dots \end{aligned} \quad (19)$$

These definitions make sense because of the relations (11) and (14) for the operators $\hat{K}$ and $\hat{J}$. More precisely we have

$$\hat{K}_{l'l}^{(12)} \cdot (\hat{I}_k^{(3)}, \hat{J}_l^{(1)} \cdot \hat{J}_{l'}^{(2)}) = (\hat{I}_k^{(3)}, \hat{J}_{l'}^{(2)} \cdot \hat{J}_l^{(1)}) \cdot \hat{K}_{l'l}^{(12)} \quad (20)$$

In terms of the operator language the coproduct (18) is the standard one, [1,13], hence we can define a multiplication on the dual algebra, e.g.,

$$\begin{aligned} (\hat{I}_k^{(2)} \cdot \hat{I}_{k'}^{(3)}, \hat{J}_l^{(1)}) &= (\hat{I}_k^{(2)} \otimes \hat{I}_{k'}^{(3)}, \hat{\Delta}(\hat{J}_l^{(1)})) \\ &= \hat{K}_{lk}^{(12)} \cdot \hat{K}_{lk'}^{(13)}, \end{aligned} \quad (21)$$

and so forth. Again this definition is consistent with the defining relations for the algebra $\mathcal{B}^*$ that we take to be given by eq.(15)). Of course, the product in the algebra $\mathcal{B}$ gives rise to a coproduct $\hat{\Delta}^*$ on $\mathcal{B}^*$, thereby defining also a (double) Hopf algebra structure on the dual, i.e.,

$$\hat{\Delta}^*(\hat{I}_l^{(1)}) = \hat{I}_l^{(1)} \otimes \hat{I}_l^{(1)}, \quad (22)$$

and we have the duality relation

$$\begin{aligned} (\hat{\Delta}^*(\hat{I}_k^{(3)}), \hat{J}_l^{(1)} \otimes \hat{J}_{l'}^{(2)}) &= (\hat{I}_k^{(3)}, \hat{J}_l^{(1)} \cdot \hat{J}_{l'}^{(2)}) \\ &= \hat{K}_{lk}^{(13)} \cdot \hat{K}_{l'k}^{(23)} \end{aligned} \quad (23)$$

Let us further note that we can also define here the universal $\hat{K}$ operator, which is the analogue of the universal R matrix [1,2,6], acting now in the tensor product $\mathcal{B}^* \otimes \mathcal{B}^*$ and defined by

$$\begin{aligned} (\hat{K} \hat{\Delta}^*(\hat{I}_k^{(3)}) \hat{K}^{-1}, \hat{J}_l^{(1)} \otimes \hat{J}_{l'}^{(2)}) &= \hat{K}_{ll'}^{(12)} (\hat{I}_k^{(3)}, \hat{J}_l^{(1)} \cdot \hat{J}_{l'}^{(2)}) \hat{K}_{ll'}^{(12)-1} \\ &= \hat{K}_{l'k}^{(23)} \cdot \hat{K}_{lk}^{(13)} \end{aligned} \quad (24)$$

Hence we have the relation

$$\hat{K} \hat{\Delta}^* = \mathcal{P} \hat{\Delta}^* \mathcal{P} \hat{K} \quad (25)$$

where $\mathcal{P}$ is the permutation operator in the tensor product $\mathcal{B}^* \otimes \mathcal{B}^*$. Note that the universal $\hat{K}$ operator no longer have lattice direction subscripts ll' . The reason for this is that we may recover the operator $\hat{K}_{ll'}^{(12)}$ acting in the tensor product of two matrix spaces $V_1 \otimes V_2$ by choosing a matrix representation for $\mathcal{B}^* \otimes \mathcal{B}^*$ in $V_1 \otimes V_2$ which means, as we have already noted, that we have also to choose representations for the shift operators. Hence such a representation of $\mathcal{B}^* \otimes \mathcal{B}^*$ has to be written as $\rho_l^{(1)} \otimes \rho_{l'}^{(2)}$ and we obtain

$$(\rho_l^{(1)} \otimes \rho_{l'}^{(2)}) \hat{K} = \hat{K}_{ll'}^{(12)} \quad (26)$$

What remains to be done now is to reconstruct the local Hopf algebra structure for $\mathcal{A}$ and its dual $\mathcal{A}^*$ from the one we just describe for $\mathcal{B}$ and $\mathcal{B}^*$ meaning that from the operator

language we have to go back to the formulation in terms of the original algebra generated by the elements J_l and I_l . In fact, the aim is to find a local coalgebra structure which by a gauge procedure reduces to the coalgebra structure in terms of the operators. In order to achieve this let us first define the generating elements of dual algebra $\mathcal{A}^*$ as algebra-valued functions defined on the 3-d lattice that we denote by $\{(I_l)_i^j\}_{i,j=1,\dots,N}$ where the indices i, j will be interpreted as matrix indices in Mat_N . We will further adopt the compact notation $I_l^{(1)}$, the superscript (1) standing as usual for the matrix space associated to I_l . At this point we would like to note that the action of the elements of $\mathcal{B}^*$, $\hat{I}_l^{(1)}$, on $\mathcal{B}$ define via the duality relations (19) a representation of $\mathcal{B}$ on a vector space V_1 , that we can denote by $\rho_l^{(1)}$ and similarly the action of the elements of $\mathcal{B}$, $\hat{J}_{l'}^{(2)}$, define, via the same duality relations, a representation of $\mathcal{B}^*$, that we can denote by $\rho_{l'}^{(2)}$. Hence the operators $\hat{J}$ and $\hat{I}$ in eqs.(12) and satisfying eqs.(14,15) define representations of $\mathcal{B}$ and $\mathcal{B}^*$ respectively, associated with an arbitrary curve C . Using this remark we can construct the elements of $\mathcal{A}^*$ and the associated duality relations for $\mathcal{A}$ and $\mathcal{A}^*$ from eq.(19,17) and the relation (see (17) for comparison)

$$\hat{I}_q^{(1)} = \sigma_a^{(1)-1} \cdot I_q^{(1)} \cdot \sigma_q^{(a)}, \quad (27)$$

In fact, using in addition the following bilinear form for the shift operators, i.e.,

$$(\sigma_a^{(1)}, \sigma_l^{(a)}) = \sigma_l^{(1)}, \quad (\sigma_q^{(a)}, \sigma_a^{(1)}) = \sigma_q^{(1)}$$

we obtain the duality relations

$$\begin{aligned} (I_k^{(3)}, \mathbf{1}) &= \mathbf{1} \\ (I_k^{(3)}, J_l^{(1)}) &= K_{lk}^{(13)} \\ (I_k^{(3)}, J_l^{(1)l'} \cdot J_{l'}^{(2)}) &= K_{lk}^{(13)l'} \cdot K_{l'k}^{(23)} \\ &\dots \end{aligned} \quad (28)$$

Again these definitions make sense because of the local equations (16, 2), namely we have

$$K_{ll'}^{(12)} \cdot (I_k^{(3)}, J_l^{(1)l'} \cdot J_{l'}^{(2)}) = (I_k^{(3)}, J_{l'}^{(2)l} \cdot J_l^{(1)}) \cdot K_{ll'}^{(12)k} \quad (29)$$

Therefore, we may define a *local* product $\mathcal{M}$ on $\mathcal{A}$ (the usual product in $\mathcal{A}$ being simply denoted by a dot) as $\mathcal{M}(J_l^{(1)}, J_{l'}^{(2)}) = J_l^{(1)l'} \cdot J_{l'}^{(2)}$ and a *local* coproduct $\Delta_{\hat{a}}: \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$, that carries an algebra-valued shift index $\hat{a}$ as $\Delta_{\hat{a}}(J_l^{(1)}(a')) = J_l^{(1)}(a) \otimes J_l^{(1)\hat{a}}(a')$.

Using the above duality relations we also define a *local* product and coproduct on $\mathcal{A}^*$, e.g., $\mathcal{M}^*(I_q^{(1)}, I_{q'}^{(2)}) = I_q^{(1)} \cdot I_{q'}^{(2)q}$ and $\Delta_{\hat{a}}^* : \mathcal{A}^* \longrightarrow \mathcal{A}^* \otimes \mathcal{A}^*$ by $\Delta_{\hat{a}}^*(I_q^{(1)}(a')) = I_q^{(1)\hat{a}}(a') \otimes I_q^{(1)}(a)$, where we have denoted by (a) and (a') the two spaces, upon which the algebra-valued objects are acting in the tensor product $\mathcal{A} \otimes \mathcal{A}$, respectively $\mathcal{A}^* \otimes \mathcal{A}^*$, (denoted by $(a) \otimes (a')$), and $\hat{a}$ is as before an algebra-valued shift, and we have the following duality relations

$$\begin{aligned} (\Delta_{\hat{a}}^*(I_k^{(3)}), J_l^{(1)} \otimes J_{l'}^{(2)}) &= (I_k^{(3)}, \mathcal{M}(J_l^{(1)}, J_{l'}^{(2)})) \\ &= K_{lk}^{(13)l'} \cdot K_{l'k}^{(23)} \end{aligned} \quad (30)$$

and we have a similar relation between $\mathcal{M}^*$ and $\Delta_{\hat{a}}$. Let us note that we also obtain local versions of the usual co-associativity relation and of the connecting axiom [10], i.e., that follow from the one's we have in $\mathcal{B}$ and in $\mathcal{B}^*$. In addition, it is possible to define here the universal $\hat{K}_{\hat{a}\hat{a}'}$ tensor, depending now on two algebra-valued shift indices, acting on $\mathcal{A}^* \otimes \mathcal{A}^*$, and with the property,

$$K_{\hat{a}\hat{a}'} \Delta_{\hat{a}'}^*(I_l^{(1)}(a)) = \Delta_{\hat{a}}^*(I_l^{(1)}(a')) K_{\hat{a}\hat{a}'}^l, \quad (31)$$

that may be deduced from (25). A more detailed description of this local Hopf algebra structure will be given in [19]. It is important to note that, although the above construction is very reminiscent of the one encountered in the structure of the Yangians [2,7], the setting here is altogether a different one, because the objects in the theory, apart from being elements in the algebra $\mathcal{A} \otimes \text{Mat}_N$, are also elements of $\mathcal{F}$, i.e. the ring of functions of the lattice variables. Remembering that we are dealing here with a *classical* theory; it is our conjecture that in the transition to a quantum theory (i.e. when we replace $\mathcal{F}$ also by a non-commutative algebra) we will obtain a structure that is related rather to the $d = 3$ -simplex equation (the tetrahedron equation [16,21,22]) rather than to the QYBE. In [18] we investigate the algebraic structure underlying these equations, cf. also [17].

5. Gauge Theory in Loop Space

We will now give a reinterpretation of the above construction to relate the notion of a local Hopf algebra to the procedure of gauging a quantum group on a space of functional of loops. The original reason for considering a loop-space rather than the usual space-time is related to the fact that the quadratic defining relations for $\mathcal{B}$ and for $\mathcal{B}^*$

can be linearized on a loop-space, as shown in [15]. Moreover, eq.(11) can then be reinterpreted as a zero curvature condition on such a space of functional of loops, the objects K playing the role of generalized gauge potentials. Hence, Integrable Systems in $d = 3$ are again related to a zero curvature condition and to *linear* difference (differential) equations, as in $d = 2$, but the relevant quantities are now functional of loops rather than being ordinary functions, and the associated gauge potentials are living on plaquettes of the $3-d$ lattice, rather than being line elements (given as the path-ordered exponential of a one-form gauge field).

The other reason, for considering a loop-space action for the quantum group, come from the fact that the local Hopf algebra structure, we constructed in the preceding sections, admits natural representations on the loop-space. Indeed, to any oriented curve C , drawn on the $3-d$ lattice, it is possible to associate the objects I of eq.(6) that realize representations of the (local) quantum group $\mathcal{A}^*$. So, we consider the space of oriented closed and non-intersecting loops C_Q drawn on a $3-d$ lattice. A loop C_Q is defined by giving a point Q on the lattice and an ordered string of directions $\{l_i\}$, $i = 1 \dots N$, N being the length of the curve C . To each loop of this type we associate functionals $\Phi[C]$ that are vectors in the representation space of the (local) quantum group $\mathcal{A}^*$ associated to C . This means, in particular, that such functionals of the loop C carry vector indices $\{\alpha_i\}$, $i = 1, \dots, N$ distributed all along the loop C . So $\Phi[C]$ is to be considered as a vector taking value in the representation space $\otimes_{i=1}^N V_i$ of $\mathcal{A}^*$. Hence, we can realize a (local) action of the quantum group $\mathcal{A}^*$ on such objects as,

$$\Phi[C]_{\{\alpha_i\}} \longrightarrow \Phi[C]^G = G[C]_{\{\alpha_i\}}^{\{\beta_i\}} \cdot \Phi[C]_{\{\beta_i\}},$$

where a sum is understood on the common repeated indices, and $G[C]$ is a matrix representing an arbitrary element of $\mathcal{A}^*$ in this C -representation. Note that $G[C]$ may be obtained from the generating elements of $\mathcal{A}^*$ in this representation, i.e., from the $I_l^{(1)}[C]$, constructed in eq.(6) as a path-ordered product along C of K 's. That means that the action of $\mathcal{A}^*$ is a *local* one on the loop-space, since the $I_l^{(1)}[C]$ are depending intrinsically on the loop C through the local potentials K . Hence the quantum group action and the operation of deforming (translating) a loop do not commute in general, which is a loop-space extension of the situation one encounter when making local a group action. Therefore, what needs to be defined here, is the analogue of the usual gauge field potential, but now on the loop-space. In order to achieve that, let us consider elementary displacement of the loop C given by the action of the shift operator $\sigma_i = \sigma_{l_{i+1}}^{(i)-1} \otimes \sigma_{l_i}^{(i+1)}$ in the vector spaces $V_i \otimes V_{i+1}$ which induce a local displacement of the curve C along

the plaquette $(l_i l_{i+1})$. We can associate to any shift of this type the operator $\hat{B}_i$ obtained from plaquette potentials $B_{l_i, l_{i+1}}^{(i, i+1)}$, in a similar way as $\hat{K}$ was constructed from K in eq.(10), and such that it reduces to the shift operator σ_i if the potential B is trivial, i.e., equal to the unit matrix, and with the property that its action on the functionals of loops as defined above transform covariantly under the local quantum group actions given by $G[C]$. Moreover, the associated curvature on this space of functionals of loops is given as expected by the object $G_{l'l''}$ defined as in section (3), i.e.,

$$G_{l'l''}^{(123)} = \hat{B}_{l'l'}^{(12)} \cdot \hat{B}_{l'l''}^{(13)} \cdot \hat{B}_{l''l'}^{(23)} \cdot \hat{B}_{l''l}^{(21)} \cdot \hat{B}_{l'l''}^{(31)} \cdot \hat{B}_{l'l}^{(32)},$$

that transform covariantly under the local loop-space action of $\mathcal{A}^*$. Let us further note that eq.(2) plays, in this context, the role of (generalized) Maurer-Cartan equations, now on the loop-space, i.e., they express in particular a zero curvature condition for K , e.g., $G[K] = \mathbf{1}$. A more detailed description of these results, together with applications to the possible definition of topological field theories in $d = 3$ will be given in [19].

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