**AN ENLARGED PHASE SPACE FOR FINITE-DIMENSIONAL
CONSTRAINED SYSTEMS, UNIFYING THEIR LAGRANGIAN, PHASE-
AND VELOCITY-SPACE DESCRIPTIONS**

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ABSTRACT

The theory of singular Lagrangians is developed by using the second Noether theorem. The Noether identities are also obtained in presence of second-class constraints and compared with the Dirac-Bergmann algorithm in phase space (T^*Q). The true gauge transformations are those Noether transformations which satisfy the Jacobi equation and an open gauge algebra everytime they are velocity dependent. The velocity-space (TQ) description is done: the extra gauge symmetries in TQ are connected with local dynamical symmetries of the Euler-Lagrange equations. Then the $T^*(TQ)$ description is developed and the associated path integral defined: it allows the measure for the TQ path integral to be found and is locally connected with the standard T^*Q one. The canonical quantization of first-class constraints is done without introducing gauge fixings by means of the 'multitemporal approach', which was developed for relativistic particle mechanics.

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CM-P00062194

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1. INTRODUCTION

Constrained systems play a dominant role in theoretical physics owing to the relevance of the development of gauge theories. The study of singular Lagrangians, having a degenerate Hessian matrix not allowing all the velocities to be expressed in terms of the coordinates and the momenta, and of the associated phase-space description, based on the theory of constraints, was initiated by Dirac [1a] and Bergmann [1b] in the fifties. It was widely used in general relativity and gauge theories, as is shown in Refs. [3–6]. A revival of interest in this theory arose with the dual-string model, which introduced a new starting point for the study of relativistic particle mechanics [7–9]. The development of the geometrical aspects started in the seventies [10–13] and interest is now concentrated on the problem of quantization with the geometrical quantization [14–18] and the group theoretical quantization [19] approaches. The issue of quantization of gauge theories was initiated by Faddeev and Popov [20] and Slavnov [21], and led to the BRS approach [22–23]. Its phase-space formulation led to the BFV approach [24] for arbitrary first-class constraints, with a related Lagrangian formulation [25], and to further insights into the theory of constraints [26] and its cohomological aspects [27]. Also, in this formulation, there is an extension to second-class constraints [28]. Since the Hamiltonian approach to field theory has still many unsolved problems in connection with locality and geometry [29], and moreover, since there is no proof of renormalizability of the Schrödinger representation of gauge theories, their canonical quantization has still to be developed. Further problems are connected with the anomalies and their topological aspects [30–32].

Notwithstanding all these results, there are still aspects of the classical theory which have not been sufficiently developed. The case of singular Lagrangians, whose Hessian matrix has varying rank due either to the Euler–Lagrange equations or to the existence of various states of motion of the system with different dynamics, have not yet been fully understood and have generated the so-called ‘pathological examples’ [33] in which the Dirac–Bergmann method does not seem to work. As a consequence, it is not completely clear when the ‘Dirac conjecture’ [1a] is valid [34–36], that is when all the secondary first-class constraints are generators of gauge transformations, and to which class the secondary constraints deriving from a primary first- (or second-) class constraint may belong [37]. In some of these papers [35, 36] a comparison was made between velocity space and phase space for singular systems, and then a unification of the two descriptions was attempted in other papers [38, 39]. Also the presymplectic approach [13] was reformulated in the velocity space [40] and the problem was raised of when a singular system has a second-order evolution vector field, i.e. when its dynamics may be described by second-order differential equations (see also Ref. [41]). This is a non-trivial problem because, in the generic case, the Euler–Lagrange equations of motion, deriving from the variational principle associated with the singular Lagrangian, are a mixed system of second-, first-, and zeroth- (when holonomic constraints are present) order differential equations. Particularly intriguing is the connection [4, 42–44] of the first-order Euler–Lagrange equations with the secondary constraints of the Dirac–Bergmann theory in phase space, and no convincing answer seems to exist. However, in the meantime, the study of the symmetries of constrained systems in phase space had been initiated [45] and extended also to velocity space [46], and some clarification was made of the connection between the canonical transformations generated by the first-class constraints and the Lagrangian gauge transformations [4, 34]. However, the role of the primary second-class constraints remained unclear until it was realized that they too are associated with null eigenvalues of the Hessian matrix and associated with generalized local invariance of the action at the Lagrangian level [47].

It becomes clear that there is a lack of a reformulation of the theory of singular Lagrangians by means of a generalization of the second Noether theorem [48–51] which could treat on the same level

all the null eigenvalues of the Hessian matrix, independently of their being associated with either first- or second-class primary constraints in the Dirac–Bergmann theory. The resulting Noether identities, which for Yang–Mills theory are the classical background of the Slavnov identities, should provide a Lagrangian formulation of the phase-space Dirac–Bergmann algorithm [1–4]. Partial results in this direction, in the absence of second-class constraints, are quoted in Ref. [52].

In this paper we shall try to answer some of these problems, restricting ourselves to singular systems with a finite number of degrees of freedom, so that we can use the full theory of symplectic geometry [13, 53, 54] without further complications. Many of our results can be directly transferred to field theory. We shall treat only the case of constant rank, assuming that all the secondary first-class constraints are generators of gauge transformations, postponing the case of variable rank to a future paper. We shall also treat explicitly only a restricted class of singular systems, having neither tertiary constraints nor bifurcations of the chains of secondary constraints, to simplify the exposition of the main points. However, this class of systems is sufficiently general to include many of the physically interesting cases. The general case will be treated elsewhere. Moreover, the constraints will be assumed to be irreducible, i.e. functionally independent, to avoid the complications of the reducible case [55, 26].

Given one singular system, let Q be its configuration space with coordinates q^i , and let $q^i(t)$ be one of its trajectories and $L(q(t), \dot{q}(t))$ its singular Lagrangian. The phase-space description will be based on the cotangent bundle T^*Q , while that of the velocity space (first-order formalism) will be based on the tangent bundle TQ . The other result of this paper, besides the Noether identities and their connection with the Dirac–Bergmann algorithm in T^*Q , is the proposal of a new kind of unification of the Lagrangian, TQ , and T^*Q , descriptions by means of the phase space over the velocity space, i.e. $T^*(TQ)$. This unification will allow the incorporation of a feature of the Euler–Lagrange equations in the singular case, which is usually neglected, that is their being invariant under local transformations of the accelerations as a consequence of the degeneracy of the Hessian matrix.

The unification via $T^*(TQ)$ is firstly defined in the regular case starting from its TQ description, and it is found that in $T^*(TQ)$ there are second-class constraints, some of which will become first-class for singular systems. For these latter systems, a subset of the first-class constraints is connected with the local dynamical symmetries of the Euler–Lagrange equations. This information, which is already present in the TQ description, is completely lost in phase space owing to the degeneracy of the Legendre transformation. Therefore, the $T^*(TQ)$ approach is more general than the T^*Q one.

Another aspect investigated in this paper is the characterization of the gauge transformations, i.e. of those local Noether transformations leaving the action quasi-invariant, which are generated by first-class constraints, as Jacobi fields. This is a consequence of the fact that the variation [56] of the Euler–Lagrange equations under the local Noether transformations gives the Jacobi equations [57] and that the true gauge transformations are deviations between neighbouring extremals of the action, so that they must satisfy such equations. By using the Noether identities inside the Jacobi equations, some properties of the Noether transformations associated with second-class constraints are displayed. Also sketched is how to study the multiple commutators of gauge transformations, assumed to form a gauge algebra [58] to ensure the global integrability of the extremals, notwithstanding the existence of the gauge transformations, owing to the Frobenius theorem [53]. It is shown that the gauge transformations satisfy an open algebra [59] when they are velocity dependent.

Turning to the quantization problem, we show how to define a path integral on $T^*(TQ)$ for regular systems so that, on the one hand, we can recover the standard T^*Q path integral and, on the

other hand, we can find the measure for the TQ path integral. Moreover, how to do the Gupta-Bleuler quantization of the second-class constraints in $T^*(TQ)$ is shown. In the singular case, we first describe the canonical quantization, in the absence of anomalies due to ordering problems, of the first-class constraints, either in T^*Q or $T^*(TQ)$, with the so-called multitemporal method [60], which does not require the introduction of gauge-fixing constraints, and we show how the physical scalar product should be worked out. Then, we define the path integral on $T^*(TQ)$, we find the measure for the TQ path integral, and we show that the $T^*(TQ)$ path integral is globally more general than the T^*Q one, which can be recovered only with local methods in general.

The plan of the paper is the following. In Section 2, the TQ, T^*Q , and $T^*(TQ)$ descriptions of regular systems are given, while in Section 3 the associated canonical and path-integral quantizations are treated. In Section 4, a brief review of the T^*Q approach to singular systems is given, with some emphasis on presymplectic geometry, which is relevant also to TQ. In Section 5, we give a T^*Q characterization of the special class of singular systems we are going to describe, and their Lagrangian description is developed together with a study of the Euler-Lagrange equations, the statement of the generalized second Noether theorem, and the evaluation of the Noether identities. Then, the Noether theorem and identities are reformulated in T^*Q and a comparison with the Dirac-Bergmann algorithm is made. The section ends with the characterization of the gauge transformations as Jacobi fields. Section 6 treats the properties of the gauge transformations and their commutators. In Section 7, the TQ description of singular systems is given and Section 8 is devoted to their treatment in $T^*(TQ)$. In Section 9, the canonical quantization of first-class constraints with the multitemporal approach is explained, and finally in Section 10 the path-integral quantization in $T^*(TQ)$ is defined and its implication for TQ and T^*Q explored. In the Appendix, the T^*Q formulation of the first Noether theorem is reported.

2. REGULAR SYSTEMS

Let us consider a system with an n-dimensional configuration space Q with local coordinates q^i , $i = 1, \dots, n$. Let it be described by a regular Lagrangian $L(q, \dot{q})$ [the time-dependent case $L(t, q, \dot{q})$ may be treated in the same way]. Its Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = A_{ji} \quad (2.1)$$

is non-singular, i.e. it has rank n. The Euler-Lagrange equations are

$$L_i = \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = -(A_{ij} \ddot{q}^j - \alpha_i) \stackrel{0}{=} 0 \quad (2.2)$$

$$\alpha_i = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^k \partial q^j} \dot{q}^j \quad (2.3)$$

Here $\stackrel{0}{=}$ means 'evaluated on the extremals' of the variational principle $\delta S = 0$, where $S = \int_{t_0}^{t_f} dt L$, for variations δq^i vanishing at $t = t_0, t_f$. Since $\det A \neq 0$, Eqs. (2.2) can be put in normal form:

$$\ddot{q}^i - \Lambda^i \stackrel{0}{=} 0 \quad (2.4)$$

$$\Lambda^i = B^{ij} \alpha_j \quad (2.5)$$

where Λ^i are the accelerations and $B = A^{-1}$.

The phase-space (T^*Q) approach is obtained by means of the Legendre transformation and employs the natural symplectic structure (closed non-degenerate 2-form [53]), $\bar{\omega} = dp_i \wedge dq^i = d\bar{\theta}$, where $\bar{\theta} = p_i dq^i$ is the Cartan-Liouville 1-form and $p_i = \partial L / \partial \dot{q}^i$, of T^*Q , allowing the introduction of Poisson brackets $\{q^i, p_j\} = \delta^i_j$ independent of the system under consideration. After having introduced the Hamiltonian $\bar{H} = \dot{q}^i p_i - L$, Hamilton's equations

$$\begin{cases} \dot{q}^i \stackrel{0}{=} \{q^i, \bar{H}\} = \frac{\partial \bar{H}}{\partial p_i} \\ \dot{p}_i \stackrel{0}{=} \{p_i, \bar{H}\} = -\frac{\partial \bar{H}}{\partial q^i} \end{cases} \quad (2.6)$$

where $\partial \bar{H} / \partial q^i = -\partial L / \partial q^i$ from Eqs. (2.2), are obtained from the variational principle associated with the action

$$\begin{cases} \bar{S} \Rightarrow \int_{t_0}^{t_f} dt \bar{L} \\ \bar{L} = \dot{q}^i p_i - \bar{H} \end{cases} \quad (2.7)$$

with $\delta q^i(t) = 0$ at $t = t_0, t_f$, $\delta p_i(t)$ arbitrary. We will use a bar to denote the functions on T^*Q : $\bar{f} = \bar{f}(q, p)$. If we introduce the Hamiltonian vector fields on T^*Q

$$\bar{X}_f = \frac{\partial \bar{f}}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial \bar{f}}{\partial q^i} \frac{\partial}{\partial p_i}$$

we get

$$\{\bar{f}, \bar{g}\} = \bar{X}_{\bar{g}} \bar{f} = i_{\bar{X}_{\bar{f}}} i_{\bar{X}_{\bar{g}}} \bar{\omega} = \bar{\omega}(\bar{X}_{\bar{f}}, \bar{X}_{\bar{g}}) \quad (2.8)$$

where $i_{\bar{X}}$ denotes the inner contraction. Thus $\bar{X}_{\bar{f}} = \{\cdot, \bar{f}\}$. The Poincaré-Cartan 1-form and the contact 2-form on the extended phase space $R \times T^*Q$, R being the time axis, are

$$\begin{cases} \tilde{\bar{\theta}} = \bar{L} dt = \bar{\theta} - \bar{H} dt \\ \tilde{\bar{\omega}} = d\tilde{\bar{\theta}} = \bar{\omega} - d\bar{H} \wedge dt \end{cases} \quad (2.9)$$

and with this formalism the role of Hamilton's equations is taken by the equations

$$i_{\bar{F}} \tilde{\bar{\omega}} = 0 \quad i_{\bar{F}} \tilde{\bar{\theta}} = 1 \quad (2.10)$$

which determine the evolution vector field on $R \times T^*Q$

$$\bar{F} = \frac{\partial}{\partial t} + \bar{X}_{\bar{H}} \quad (2.11)$$

The equations (2.6), $\dot{q}^i \stackrel{0}{=} \bar{\Gamma}^i q^i$, $\dot{p}_i \stackrel{0}{=} \bar{\Gamma}_i p_i$, are those for the integral curves of $\bar{F}$, so that they turn out to be the extremals of the variational principle $\delta \bar{S} = 0$. Equations (2.10), (2.11) imply that on T^*Q Eqs. (2.10) are replaced by $i_{\bar{F}_0} \bar{\omega} = -d\bar{H}$, $\bar{F}_0 = \bar{X}_{\bar{H}}$.

The velocity-space, i.e. tangent space TQ , description is based on the replacement of the second-order Euler-Lagrange differential equations (2.2) by pairs of first-order differential equations, as occurs in the phase-space description, by introducing independent coordinates v^i . These will be related to $\dot{q}^i(t)$ by half of the new equations of motion, similarly to the way in which the first half of Hamilton's equations (2.6) determine $\dot{q}^i$ as a function of q^i, p_i by realizing the inversion of the relations $p_i = \partial L / \partial \dot{q}^i$.

Let us denote by $\tilde{f}(q, v)$ a function on TQ which has been obtained from $f(q, \dot{q})$ with the replacement $\dot{q}^i \rightarrow v^i$. Then $L(q, \dot{q}) \rightarrow \tilde{L}(q, v)$. Instead of the Hamiltonian $\bar{H}(q, p)$, let us introduce the energy function

$$\tilde{E}(q, v) = v^i \frac{\partial \tilde{L}}{\partial v^i} - \tilde{L} \quad (2.12)$$

and the following action

$$\begin{cases} \tilde{S} = \int_{t_0}^{t_e} dt \tilde{L}_v \\ \tilde{L}_v = \dot{q}^i \frac{\partial \tilde{L}}{\partial v^i} - \tilde{E} \end{cases} \quad (2.13)$$

If we introduce the following antisymmetric matrix:

$$R_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} - \frac{\partial^2 L}{\partial \dot{q}^j \partial q^i} \quad R_{ij}(q, \dot{q}) \rightarrow \tilde{R}_{ij}(q, v) \quad (2.14)$$

the variation of $\tilde{S}$ is

$$\begin{aligned} \delta \tilde{S} &= \int_{t_0}^{t_e} dt \left\{ \frac{\partial \tilde{L}}{\partial v^i} \delta q^i + \dot{q}^i \left(\frac{\partial^2 \tilde{L}}{\partial v^i \partial q^j} \delta q^j + \frac{\partial^2 \tilde{L}}{\partial v^i \partial v^j} \delta v^j - \frac{\partial \tilde{E}}{\partial q^i} \delta q^i - \frac{\partial \tilde{E}}{\partial v^i} \delta v^i \right) \right\} = \\ &= \int_{t_0}^{t_e} dt \left\{ \frac{d}{dt} \left(\frac{\partial \tilde{L}}{\partial v^i} \delta q^i \right) - \delta q^i \left(\tilde{A}_{ij} \dot{v}^j + \tilde{R}_{ij} \dot{q}^j + \frac{\partial \tilde{E}}{\partial q^i} \right) + \right. \\ &\quad \left. + \delta v^i \left(\tilde{A}_{ij} \dot{q}^j - \frac{\partial \tilde{E}}{\partial v^i} \right) \right\} \end{aligned} \quad (2.15)$$

Here $\tilde{A}$ corresponds to the Hessian matrix A and $\tilde{B}$ will denote $\tilde{A}^{-1}$. As we have

$$\frac{\partial \tilde{E}}{\partial q^i} = v^j \frac{\partial^2 \tilde{L}}{\partial v^j \partial q^i} - \frac{\partial \tilde{L}}{\partial q^i}, \quad \frac{\partial \tilde{E}}{\partial v^i} = \tilde{A}_{ij} v^j$$

we can rewrite $\delta \tilde{S}$ in the following form:

$$\begin{aligned} \delta \tilde{S} &= \int_{t_0}^{t_e} dt \left\{ \frac{d}{dt} \left(\frac{\partial \tilde{L}}{\partial v^i} \delta q^i \right) - \delta q^i \tilde{A}_{ij} \left[\dot{v}^j + \tilde{B}^{jk} \left(\frac{\partial \tilde{E}}{\partial q^k} + \right. \right. \right. \\ &\quad \left. \left. + \tilde{R}_{kh} \tilde{B}^{hr} \frac{\partial \tilde{E}}{\partial v^r} \right) \right] + \left(\delta v^i \tilde{A}_{ij} - \delta q^i \tilde{R}_{ij} \right) \left(\dot{q}^j - \tilde{B}^{jk} \frac{\partial \tilde{E}}{\partial v^k} \right) \right\} = \\ &= \int_{t_0}^{t_e} dt \left\{ \frac{d}{dt} \left(\frac{\partial \tilde{L}}{\partial v^i} \delta q^i \right) - \delta q^i \tilde{A}_{ij} \left[\dot{v}^j + \tilde{B}^{jk} \left(\tilde{R}_{kh} v^h - \frac{\partial \tilde{L}}{\partial q^k} + \right. \right. \right. \\ &\quad \left. \left. + \frac{\partial^2 \tilde{L}}{\partial q^k \partial v^h} v^h \right) \right] + \left(\delta v^i \tilde{A}_{ij} - \delta q^i \tilde{R}_{ij} \right) \left(\dot{q}^j - v^j \right) \right\} \end{aligned} \quad (2.16)$$

When $\delta q^i(t) = 0$ at $t = t_0, t_e$, $\delta v^i(t)$ arbitrary, $\delta \tilde{S} = 0$ implies the first-order Euler-Lagrange differential equations

$$\begin{cases} \tilde{L}_{qi} = -\tilde{A}_{ij} [\dot{v}^j + \tilde{B}^{jk} (\frac{\partial \tilde{E}}{\partial q^k} + \tilde{R}_{kh} \tilde{B}^{hr} \frac{\partial \tilde{E}}{\partial v^r})] - \tilde{R}_{ij} (\dot{q}^j - v^j) \stackrel{=}{=} 0 \\ \tilde{L}_{vi} = \tilde{A}_{ij} (\dot{q}^j - v^j) \stackrel{=}{=} 0 \end{cases} \quad (2.17)$$

whose normal form is

$$\begin{cases} \dot{q}^i \stackrel{=}{=} v^i \\ \dot{v}^i \stackrel{=}{=} -\tilde{B}^{ij} (\frac{\partial \tilde{E}}{\partial q^j} + \tilde{R}_{jk} \tilde{B}^{kh} \frac{\partial \tilde{E}}{\partial v^h}) = \tilde{B}^{ij} \tilde{\alpha}_j = \tilde{\Lambda}^i \end{cases} \quad (2.18)$$

where $\tilde{\Lambda}^i$ is obtained from the Λ^i of Eqs. (2.5).

The normal form (2.18) can be rewritten in the form of Hamilton's equations by using suitably-defined Poisson brackets. In TQ, there is neither a natural Cartan-Liouville 1-form nor a symplectic 2-form independent of the system under consideration. Instead, TQ admits [61] only an intrinsic so-called 'vertical endomorphism' [a (1,1) tensor field S on TQ], which allows the definition of the vertical derivative d_v [an antiderivation of degree one on $\Lambda(TQ)$, the Grassmann algebra of the differential forms on TQ] with the following properties:

$$\begin{cases} d_v^2 = 0 & d d_v + d_v d = 0 \\ d_v \tilde{f} = \frac{\partial \tilde{f}}{\partial v^i} dq^i & d_v dq^i = 0 & d_v dv^i = dq^i \end{cases} \quad (2.19)$$

where d is the exterior derivation. $d_v^2 = 0$ is connected with the vanishing of the (1,2) Nijenhuis tensor field (the analogue of torsion in Riemannian geometry) associated with the vertical endomorphism. d_v allows the following definition of the Cartan-Liouville and symplectic forms on TQ for a system with the Lagrangian L :

$$\begin{cases} \vartheta_L = d_v \tilde{L} = \frac{\partial \tilde{L}}{\partial v^i} dq^i \\ \omega_L = d\vartheta_L = -\frac{1}{2} \tilde{R}_{ij} dq^i \wedge dq^j + \tilde{A}_{ij} dv^i \wedge dq^j \end{cases} \quad (2.20)$$

Equations (2.20) are also the pullback of the $\bar{\theta}$, $\bar{\omega}$ on T^*Q through the Legendre transformation $FL: TQ \rightarrow T^*Q$. Analogously, the Poincaré-Cartan and contact forms on the extended velocity space $R \times TQ$ [isomorphic to the first-jet bundle $J^1(R, Q)$] are

$$\begin{cases} \tilde{\vartheta}_L = \vartheta_L - \tilde{E} dt \\ \tilde{\omega}_L = d\tilde{\vartheta}_L = \omega_L + (\tilde{\alpha}_i + \tilde{R}_{ij} v^j) dq^i \wedge dt - \tilde{A}_{ij} v^j dv^i \wedge dt \end{cases} \quad (2.21)$$

ω_L induces the following Poisson brackets [3] on TQ:

$$\{\tilde{g}_i, \tilde{g}_j\}_L = \tilde{B}^{ij} \left(\frac{\partial \tilde{g}}{\partial q^i} \frac{\partial \tilde{g}}{\partial v^j} - \frac{\partial \tilde{g}}{\partial v^j} \frac{\partial \tilde{g}}{\partial q^i} \right) - \frac{\partial \tilde{g}}{\partial v^i} \tilde{B}^{ih} \tilde{R}_{hk} \tilde{B}^{kj} \frac{\partial \tilde{g}}{\partial v^k} \quad (2.22)$$

which imply

$$\left\{ \begin{array}{l} \{q^i, q^j\}_L = 0 \\ \{q^i, v^j\}_L = \tilde{B}^{ij} \\ \{v^i, v^j\}_L = \tilde{B}^{ih} \tilde{R}_{hk} \tilde{B}^{kj} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \{q^i, \frac{\partial \tilde{L}}{\partial v^j}\}_L = \delta_j^i \\ \{\frac{\partial \tilde{L}}{\partial v^i}, \frac{\partial \tilde{L}}{\partial v^j}\}_L = 0 \end{array} \right. \quad (2.23)$$

Therefore, a symplectic basis is not (q^i, v^i) but $(q^i, \tilde{p}_i = \partial \tilde{L} / \partial v^i)$, as in phase space. Equations (2.18) become Hamilton's equations with the energy $\tilde{E}$ as the Hamiltonian:

$$\left\{ \begin{array}{l} \dot{q}^i \stackrel{0}{=} \{q^i, \tilde{E}\}_L = v^i = \tilde{p}^i(q^i) \\ \dot{v}^i \stackrel{0}{=} \{v^i, \tilde{E}\}_L = \tilde{\lambda}^i = \tilde{p}^i(v^i) \end{array} \right. \quad (2.24)$$

where the evolution vector field $\tilde{\Gamma}$ on $R \times TQ$, satisfying $i_{\tilde{\Gamma}} \tilde{\omega}_L = 0$, $i_{\tilde{\Gamma}} dt = 1$ is

$$\tilde{\Gamma} = \frac{\partial}{\partial t} + \tilde{X}_{\tilde{E}} \quad \tilde{X}_{\tilde{E}} = \{ \cdot, \tilde{E} \}_L \quad (2.25)$$

It is a second-order vector field [40, 41, 61], i.e. its integral curves, solutions of Eqs. (2.24) and extremals of $\tilde{S}$ are the lift to TQ of the curves in Q solutions of the second-order equations (2.2). The content of this statement is that the first half of Eqs. (2.24) must be $\dot{q}^i \stackrel{0}{=} v^i$. For completeness, let us quote [61] that with the vertical endomorphism S and the second-order vector field $\tilde{\Gamma}$ on TQ may be endowed with an 'almost tangent structure', i.e. a (1,1) tensor field J on TQ with $J^2 = -1$. It turns out that the symplectic form ω_L is invariant under J, $\omega_L(J\tilde{X}, \tilde{Y}) + \omega_L(\tilde{X}, J\tilde{Y}) = 0$ for every vector field $\tilde{X}, \tilde{Y}$ on TQ, allowing the introduction of a Hermitian Riemannian metric on TQ, $g_L(\tilde{X}, \tilde{Y}) = -\omega_L(\tilde{X}, J\tilde{Y})$. This makes TQ an almost Kähler manifold with ω_L as a fundamental 2-form. TQ becomes a Kähler manifold when the (1,2) Nijenhuis tensor field associated with J vanishes, i.e. if J is a 'complex structure', which means that the vector fields $\tilde{X}_{\pm}$ satisfying $J\tilde{X}_{\pm} = \pm i\tilde{X}_{\pm}$ are closed under commutation. For instance, in T^*Q with

$$\bar{\omega} = dp_i \wedge dq^i = i da_i \wedge da_i^* \quad , \quad a_i = \frac{1}{\sqrt{2}} (p_i + i q^i),$$

we get the Kähler potential $K = \sum_i \bar{a}_i a_i$ so that $\bar{\omega} = -i \partial \bar{\partial} K$, since $d = \partial + \bar{\partial}$.

Let us now look at the phase space over the velocity space, i.e. $T^*(TQ)$, with the functions over $T^*(TQ)$ denoted by $\tilde{f}(q, \tilde{p}, v, \tilde{\pi})$. The momenta are

$$\begin{cases} \tilde{p}_i = \frac{\partial \tilde{L}_v}{\partial \dot{q}^i} = \frac{\partial \tilde{L}}{\partial v^i}(q, v) & \Rightarrow \tilde{\chi}_i = \tilde{p}_i - \frac{\partial \tilde{L}}{\partial v^i} \approx 0 \\ \tilde{\pi}_i = \frac{\partial \tilde{L}_v}{\partial \dot{v}^i} = 0 & \Rightarrow \tilde{\pi}_i \approx 0 \end{cases} \quad (2.26)$$

and the Hamiltonian is

$$\tilde{H} = \dot{q}^i \tilde{p}_i + \dot{v}^i \tilde{\pi}_i - \tilde{L}_v = \tilde{E} \quad (2.27)$$

The new Poisson brackets are

$$\{q^i, \tilde{p}_j\}' = \{v^i, \tilde{\pi}_j\}' = \delta_j^i \quad (2.28)$$

In $T^*(TQ)$ all the equations (2.26) are second-class constraints, because

$$\begin{cases} \{\tilde{\chi}_i, \tilde{\chi}_j\}' = -\tilde{R}_{ij} \\ \{\tilde{\chi}_i, \tilde{\pi}_j\}' = -\tilde{A}_{ij} \end{cases} \quad \det \tilde{C} = \det \begin{pmatrix} -\tilde{R}_{ij} & -\tilde{A}_{ik} \\ \tilde{A}_{hj} & 0 \end{pmatrix} = (\det \tilde{A})^2 \neq 0 \quad (2.29)$$

This is due to the fact that the Hessian matrix associated with $\tilde{L}_v$ vanishes identically. If we introduce the Dirac Hamiltonian

$$\tilde{H}_D = \tilde{E} + \lambda_1^i(t) \tilde{\chi}_i + \lambda_2^i(t) \tilde{\pi}_i \quad (2.30)$$

the preservation of these constraints gives the following conditions

$$\begin{cases} \dot{\tilde{\chi}}_i = \{\tilde{\chi}_i, \tilde{H}_D\}' \approx -\frac{\partial \tilde{E}}{\partial q^i} - \tilde{R}_{ij} \lambda_1^j - \tilde{A}_{ij} \lambda_2^j \approx 0 \\ \dot{\tilde{\pi}}_i = \{\tilde{\pi}_i, \tilde{H}_D\}' \approx \tilde{A}_{ij} (\lambda_1^j - v^j) \approx 0 \end{cases} \quad (2.31)$$

Their solution is $\lambda_1^i = v^i$, $\lambda_2^i = -\tilde{B}^{ij} [(\partial \tilde{E} / \partial q^j) + \tilde{R}_{jk} v^k] = \tilde{B}^{ij} \tilde{\alpha}_j = \tilde{A}^i$, so that $\tilde{H}_D$ becomes

$$\tilde{H}_D^F = \tilde{E} + v^i (\tilde{p}_i - \frac{\partial \tilde{L}}{\partial v^i}) + \tilde{A}^i \tilde{\pi}_i = v^i \tilde{p}_i - \tilde{L} + \tilde{A}^i \tilde{\pi}_i \quad (2.32)$$

These second-class constraints are due to the fact that $\tilde{L}_v$ describes a first-order dynamics. As a consequence, $\tilde{L}_v$ is weakly quasi-invariant under the following local variations (see Section 5 for the definition of weak quasi-invariance):

$$\begin{cases} \delta q^i = \eta^i(t) \Rightarrow \delta \hat{L}_V = \frac{d}{dt} \left(\frac{\partial \hat{L}}{\partial v^i} \eta^i \right) + \eta^i \left(\hat{L}_{q^i} + 2 \hat{R}_{ik} \tilde{B}^{kj} \tilde{L}_{vj} \right) \doteq \frac{d}{dt} \left(\frac{\partial \hat{L}}{\partial v^i} \eta^i \right) \\ \delta v^i = \varepsilon^i(t) \Rightarrow \delta \hat{L}_V = \varepsilon^i \hat{L}_{v^i} \doteq 0 \end{cases} \quad (2.33)$$

That is, we get quasi-invariance by using the first-order differential equations of motion which, as shown in Ref. [47], is the property distinguishing these local variations from the ones associated with first-class constraints, i.e. with the gauge transformations.

In $T^*(TQ)$ the action is

$$\tilde{S} = \int_{t_0}^{t_e} dt \left(\dot{q}^i \tilde{p}_i + \dot{v}^i \tilde{\pi}_i - \tilde{H}_0 \right) \quad (2.34)$$

and the symplectic and contact forms are

$$\begin{cases} \tilde{\omega}' = d\hat{p}_i \wedge dq^i + d\hat{\pi}_i \wedge dv^i \\ \tilde{\tilde{\omega}}' = \tilde{\omega}' - d\tilde{H}_0 \wedge dt \end{cases} \quad (2.35)$$

Hamilton's equations are

$$\begin{cases} \dot{q}^i \doteq \{q^i, \tilde{H}_0^F\}' = v^i \\ \dot{v}^i \doteq \{v^i, \tilde{H}_0^F\}' = \tilde{\chi}^i \\ \dot{\tilde{p}}_i \doteq \{\tilde{p}_i, \tilde{H}_0^F\}' \approx \frac{\partial \hat{L}}{\partial q^i} \\ \dot{\tilde{\pi}}_i \doteq \{\tilde{\pi}_i, \tilde{H}_0^F\}' = -\tilde{\chi}_i \approx 0 \end{cases} \quad (2.36)$$

As the inverse of the matrix $\tilde{C}$ of Eqs. (2.29) is

$$\tilde{C}^{-1} = \begin{pmatrix} 0 & \tilde{B} \\ -\tilde{B} & -\tilde{B} \tilde{R} \tilde{B} \end{pmatrix} \quad (2.37)$$

the Dirac brackets associated with the second-class constraints are

$$\begin{aligned} \{\hat{g}, \hat{g}\}'^* &= \{\hat{g}, \hat{g}\}' - \tilde{B}^{ij} [\{\hat{g}, \tilde{\chi}_i\}' \{\hat{\pi}_j, \hat{g}\}' - \{\hat{g}, \hat{\pi}_i\}' \{\tilde{\chi}_j, \hat{g}\}'] + \\ &+ \{\hat{g}, \hat{\pi}_i\}' \tilde{B}^{ih} \tilde{R}_{hk} \tilde{B}^{kj} \{\hat{\pi}_j, \hat{g}\}' \end{aligned} \quad (2.38)$$

As the second-class constraints (2.26) eliminate the variables v^i , $\tilde{\pi}_i$, we get

$$\left\{ \begin{array}{l} \{q^i, q^j\}^{I*} = \{\tilde{p}_i, \tilde{p}_j\}^{I*} = 0 \\ \{q^i, \tilde{p}_j\}^{I*} = \delta_i^j \\ \tilde{H}_0^F \equiv \tilde{E} \equiv v^i \tilde{p}_i - \tilde{L} = \tilde{H}(q, \tilde{p}) \end{array} \right. \quad (2.39)$$

where $\tilde{H}$ is the same function as H in T^*Q . Therefore, we get the same description as in the ordinary phase space with Hamilton's equations (2.6). On the other hand, if we consider the Lagrangian subspace [55] $TQ = (q^i, v^i)$ of $T^*(TQ)$, i.e. a subspace such that the restriction of $\tilde{\omega}'$ to it vanishes, then the Dirac brackets (2.38) coincide with the Poisson brackets (2.22) for the functions $\tilde{f}(q, v)$. Finally, let us remark that Eqs. (2.36) and (2.6) imply $v^i \stackrel{0}{=} \dot{q}^i \stackrel{0}{=} \partial \tilde{H}(q, \tilde{p}) / \partial \tilde{p}_i$: this is the resolution in the v^i of the constraints $\tilde{\chi}_i = \tilde{p}_i - (\partial \tilde{L} / \partial v^i) \approx 0$.

3. QUANTUM MECHANICS IN THE REGULAR CASE

The canonical quantization of the ordinary phase space T^*Q is realized by considering Hermitian operators $\hat{q}^i, \hat{p}_i$ acting on a Hilbert space and satisfying the commutation relations

$$[\hat{q}^i, \hat{p}_j] = i\hbar \delta_j^i \quad (3.1)$$

Modulo ordering problems one defines the Hermitian operator $\hat{H}$ and writes the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(q, t) = \hat{H} \Psi(q, t) \quad (3.2)$$

The scalar product is $\langle \psi_1 | \psi_2 \rangle = \int d^n q \psi_1^* \psi_2$.

To the evolution operator $\exp[-i/\hbar \hat{H}(t-t_0)]$ can be given the usual representation [62] by means of a path integral in T^*Q with the Liouville measure

$$K(q, q_0, t-t_0) = \int_{q_0, t_0}^{q, t} \mathcal{D}q \mathcal{D}p \ e^{-\frac{i}{\hbar} \int_{t_0}^t dt (\dot{q}^i p_i - \vec{H})} \quad (3.3)$$

always modulo ordering problems and canonical transformations [63]. When $L = \Sigma_i (m_i/2)(\dot{q}^i)^2 - V(q^i)$, by making the integration over the momenta, one also gets the following representation with a trivial measure:

$$K(q, q_0, t-t_0) = \int_{q_0, t_0}^{q, t} \mathcal{D}q \ e^{-\frac{i}{\hbar} \int_{t_0}^t dt L} \quad (3.4)$$

For more complicated L the measure in Eq. (3.4) becomes non trivial. The solution of Eq. (3.2) with initial data at $t = t_0$ has then the form

$$\Psi(q, t) = \int d^n q_0 \ K(q, q_0, t-t_0) \Psi(q_0, t_0) \quad (3.5)$$

Let us now make the canonical quantization of $T^*(TQ)$. If we quantize the reduced phase space defined by the Dirac brackets (2.39), we get the same results as Eqs. (3.1) to (3.5). Let us instead quantize the full $T^*(TQ)$ by imposing the commutation relations

$$[\hat{q}^i, \hat{p}_j] = [\hat{v}^i, \hat{\pi}_j] = i\hbar \delta_j^i \quad (3.6)$$

The Schrödinger equation will be

$$i\hbar \frac{\partial}{\partial t} \Phi(q, v, t) = \hat{\vec{H}}_D \Phi(q, v, t) \quad (3.7)$$

and the scalar product is now

$$\langle \Phi_1 | \Phi_2 \rangle = \int d^n q d^n v \ \Phi_1^* \Phi_2 \quad (3.8)$$

The second-class constraints are usually imposed by requiring that the physical states $|\phi_p\rangle$ satisfy

$$\langle \phi_p | \hat{\tilde{\chi}}_i | \phi_p \rangle = \langle \phi_p | \hat{\tilde{\pi}}_i | \phi_p \rangle = 0 \quad (3.9)$$

Another possibility is to rewrite the classical constraints $\tilde{\chi}_i \approx 0$ as $\tilde{q}^i = v^i - (\partial \tilde{H} / \partial \tilde{p}_i) \approx 0$ (see the end of the previous section). Then one performs the following canonical transformation [42]:

$$\begin{array}{cc} q^i & \tilde{p}_i \\ v^i & \tilde{\pi}_i \end{array} \longrightarrow \begin{array}{cc} \tilde{Q}^i(q, \tilde{p}, \tilde{\pi}) & \tilde{P}_i(q, \tilde{p}, \tilde{\pi}) \\ \tilde{q}^i & \tilde{\pi}_i \end{array} \quad (3.10)$$

For instance, if $L = (m_i/2)(\dot{q}^i)^2 - V(q^i)$, we get $\tilde{q}^i = v^i - \delta^{ij} \tilde{p}_j / m_j$, $\tilde{Q}^i = q^i - \delta^{ij} \tilde{\pi}_j / m_i$, $\tilde{P}_i = p_i$. If this canonical transformation is unitarily implementable with respect to the scalar product (3.8), we can do a Gupta-Bleuler quantization of the constraints $\tilde{q}^i \approx 0$, $\tilde{\pi}_i \approx 0$. By introducing the operators $\hat{a}_i^+ = (1/\sqrt{2})(\tilde{q}^i + i \tilde{\pi}_i)$, $\hat{a}_i = (1/\sqrt{2})(\tilde{q}^i - i \tilde{\pi}_i)$, we impose the conditions

$$\hat{a}_i^+ |\phi_p\rangle = 0 \quad \langle \phi_p | \hat{a}_i = 0 \quad (3.11)$$

which are a solution of Eqs. (3.9).

The solutions of Eqs. (3.11) are non normalizable with respect to the scalar product (3.8), except for the trivial solutions $|\psi(q_i)\rangle \propto |0\rangle$ corresponding to the origin of the vector subspace $\mathcal{H}_{\tilde{q}^i, \tilde{p}_i}$ of the Hilbert space $\mathcal{H}_{\tilde{q}^i, \tilde{p}_i, \tilde{\pi}_i}$. The vector $|0\rangle$ has zero norm and the physical Hilbert space can be defined by going to the quotient with respect to this state, recovering thus the Hilbert space associated with T^*Q . This is a limiting case of the Gupta-Bleuler method used in QED and string theory.

Let us now consider the following path integral with a Liouville measure over $T^*(TQ)$ modified to a Faddeev-Popov measure due to the second-class constraints [64]

$$K(q, q_0, t-t_0) = \int_{q_0, t_0}^{q, t} \mathcal{D}q \mathcal{D}\tilde{p} \mathcal{D}v \mathcal{D}\tilde{\pi} \prod_i \delta(\tilde{\chi}_i) \delta(\tilde{\pi}_i) \sqrt{\det \tilde{C}} \quad (3.12)$$

$$e^{-\frac{i}{\hbar} \int_{t_0}^t dt [\dot{q}^i \tilde{p}_i + v^i \tilde{\pi}_i - \tilde{H}_0]}$$

where $\tilde{C}$ is defined in Eqs. (2.29). As $\sqrt{\det \tilde{C}} = |\det \tilde{A}| = |\det(\{\tilde{\chi}_i, \tilde{\pi}_j\})| = |\det(\partial \tilde{\chi}_i / \partial v^j)|$, we get

$$\prod_i \delta(\tilde{\chi}_i) \sqrt{\det \tilde{C}} = \prod_i \delta(\tilde{p}_i - \frac{\partial \tilde{L}}{\partial v^i}) |\det(\frac{\partial \tilde{\chi}_i}{\partial v^j})| = \prod_i \delta(v^i - \frac{\partial \tilde{H}}{\partial \tilde{p}_i}) \quad (3.13)$$

Therefore, by doing the integrations over v^i , $\tilde{\pi}_i$ and by noting that this implies $\tilde{E}(q, v) \rightarrow \tilde{H}(q, \tilde{p})$, we recover Eq. (3.3).

On the other hand the constraints $\tilde{\chi}_i$ are resolved in the momenta $\tilde{p}_i$. Therefore, when one is allowed to interchange the order of the integrations over $\tilde{p}_i$ and v^i (this has to be checked for each system [62, 63]), we get the following measure for the path integral over TQ

$$K(q, q_0, t - t_0) = \int_{q_0 t_0}^{qt} \mathcal{D}q \mathcal{D}v |\det \tilde{A}| e^{-\frac{i}{\hbar} \int_{t_0}^t dt \left(\dot{q}^i \frac{\partial \tilde{L}}{\partial v^i} - \tilde{E} \right)} \quad (3.14)$$

with $\tilde{S}$ of Eqs. (2.13) as the effective action. The non-trivial measure is the remnant of the second-class constraints and has the remarkable property that the square root of Eqs. (3.12) has naturally disappeared owing to Eqs. (2.29). When Eq. (3.4) holds, it can be reached from Eq. (3.12) either via T^*Q [Eq. (3.3)] or via TQ [Eq. (3.14)]. The price to pay for the velocity-space description is the presence of second-class constraints, so that to define a BRS prescription for $\tilde{L}_v$ and/or $\dot{q}^i \tilde{p}_i + \dot{v}^i \tilde{\pi}_i - \tilde{E}$ requires finding an embedding of our system in a larger system $(q^i, v^i) \rightarrow (q^i, v^i, \xi^i)$, in which all the constraints become first class owing to the presence of the extra auxiliary variables ξ^i and of their momenta κ_i , along the lines of Ref. [28].

4. SINGULAR SYSTEMS: THE T*Q APPROACH

Let us now consider a system described by a singular Lagrangian $L(q, \dot{q})$, i.e. such that $\det A = 0$. In this case [1-4, 42] the relation $p_i = \partial L / \partial \dot{q}^i$ cannot be inverted to get all the velocities in terms of q, p . A certain number of functions $g(q, \dot{q})$, depending in an essential way on the velocities, will not be projectable to T^*Q , that is cannot be expressed as $\bar{g}(q, p)$ via the Legendre transformation, and an equal number of primary constraints $\bar{\Phi}_a(q, p) \approx 0$ will appear in T^*Q . Therefore, only the closed submanifold [13, 15, 46] $\bar{\gamma}_1 \subset T^*Q$ defined by these constraints is relevant for the system under consideration. However, $\bar{\gamma}_1$ is not a symplectic manifold, but only a presymplectic one [13]: that is the restriction $\bar{\omega}_1$ of the symplectic 2-form $\bar{\omega}$ of T^*Q to $\bar{\gamma}_1$ is closed but degenerate. This means that a set of vector fields $\bar{X}$ on $\bar{\gamma}_1$ exists, said to be the kernel of $\bar{\omega}_1$, $\ker \bar{\omega}_1$, such that $i_{\bar{X}} \bar{\omega}_1 = 0$, $\bar{X} \in \ker \bar{\omega}_1$. It turns out [13, 46] that $\ker \bar{\omega}_1$ is spanned by the vector fields $\bar{X}_{a'} = \{ \cdot, \bar{\Phi}_{a'} \}$ on T^*Q which are tangent to $\bar{\gamma}_1$ and such that $\{ \bar{\Phi}_{a'}, \bar{\Phi}_a \}|_{\bar{\gamma}_1} = 0$ for all a, a' , i.e. the $\bar{\Phi}_{a'}$ are first class with respect to all the primary constraints. The canonical Hamiltonian $\bar{H}_c = \dot{q}^i p_i - L = \bar{H}_c(q, p)$ on T^*Q has to be generalized to the Dirac Hamiltonian

$$\bar{H}_D = \bar{H}_c + \sum_a \lambda^a(t) \bar{\Phi}_a \quad (4.1)$$

and we get the Hamilton equations (2.6) on T^*Q with $\bar{H}$ replaced by $\bar{H}_D$.

The arbitrary Dirac multipliers $\lambda^a(t)$, enforcing the restriction from T^*Q to $\bar{\gamma}_1$, are the phase-space analogue of the non-projectable velocity functions $g^a(q, \dot{q})$. This can be checked a posteriori by means of the first half of the final Hamilton equations: from them can be extracted the natural T^*Q -oriented functional form of the g^a such that $g^a(q, \dot{q}) \stackrel{0}{=} \lambda^a$. At the Lagrangian level, any other functional form of the g^a would be allowed, reflecting itself in a redefinition of the functional form of the primary constraints $\bar{\Phi}_a$. The only restriction, apart from covariance requirements, is that the new $\bar{\Phi}'_a$ be functionally independent, similarly to the $\bar{\Phi}_a$, and should determine the same submanifold $\bar{\gamma}_1$.

With the Dirac Hamiltonian $\bar{H}_D$ we can get the Dirac evolution vector field on $R \times T^*Q$, see Eqs. (2.11), $\bar{\Gamma}_D = (\partial/\partial t) + \bar{X}_{\bar{H}_D} = (\partial/\partial t) + \bar{X}_{\bar{H}_c} + \sum_a \lambda^a(t) \bar{X}_a$ and verify that it satisfies Eq. (2.10) if we define the following contact 2-form

$$\tilde{\bar{\omega}}_D = \bar{\omega} - d\bar{H}_D \wedge dt \quad (4.2)$$

One has

$$\begin{cases} i_{\bar{X}_a} \bar{\omega} = -d\bar{\Phi}_a \\ i_{\bar{X}_a} \tilde{\bar{\omega}}_D = -d\bar{\Phi}_a + \bar{\pi}_D(\bar{\Phi}_a) dt \end{cases} \quad \begin{aligned} \bar{\pi}_D(\bar{\Phi}_a) &= \sum_b \lambda^b \{ \bar{\Phi}_a, \bar{\Phi}_b \} + \{ \bar{\Phi}_a, \bar{H}_c \} \stackrel{0}{=} \\ &\stackrel{0}{=} \dot{\bar{\Phi}}_a \end{aligned} \quad (4.3)$$

We have the following situation: the vector fields $\bar{X}_{a'} \in \ker \bar{\omega}_1$ are tangent to $\bar{\gamma}_1$ while the remaining $\bar{X}_a$ are not. Therefore, the vector field $\bar{\Gamma}_D$ is not in general tangent to $R \times \bar{\gamma}_1$ and cannot generate a consistent set of first-order Hamilton equations on $R \times \bar{\gamma}_1$. We must add the requirement that $\bar{\Gamma}_D - (\partial/\partial t) \in \ker \bar{\omega}_1$, i.e. that $\bar{X}_{\bar{H}_c} + \sum_a \lambda^a \bar{X}_a$ be tangent to $\bar{\gamma}_1$. From Eqs. (4.3) this implies

$$\dot{\bar{\Phi}}_a \stackrel{0}{=} \bar{\pi}_D(\bar{\Phi}_a) = \{ \bar{\Phi}_a, \bar{H}_c \} + \sum_b \lambda^b \{ \bar{\Phi}_a, \bar{\Phi}_b \} \stackrel{0}{=} \big|_{\bar{\gamma}_1} 0 \quad (4.4)$$

where $=|_{\bar{\gamma}_1}$ means restricted to $\bar{\gamma}_1$. This is the starting point for the Dirac-Bergmann algorithm [1-4, 13, 46]. The solution of Eqs. (4.4) may either determine some of the Dirac multipliers λ^a or introduce secondary constraints $\bar{\Phi}_{a_2}^{(2)} \approx 0$ (with the obvious notation $\bar{\Phi}_a = \bar{\Phi}_{a_1}^{(1)}$) determining a restriction of $\bar{\gamma}_1$ to a closed submanifold $\bar{\gamma}_2$. Then all the analysis has to be repeated, until a final closed submanifold $\bar{\gamma}_F$ is determined. $\bar{\gamma}_F$ is determined by a sequence $\bar{\Phi}_{a_1}^{(1)} = \bar{\Phi}_a, \bar{\Phi}_{a_2}^{(2)}, \dots$ of constraints such that their associated vector fields $\bar{X}_{a_n}^{(n)} = \{ \cdot, \bar{\Phi}_{a_n}^{(n)} \}$ on T^*Q can be divided into two classes [10, 13]: those tangent to $\bar{\gamma}_F$ and those not tangent. In the Dirac terminology, to the former correspond first-class constraints, while to the latter correspond the second-class ones (which are an even number when all the variables are bosonic). The final Dirac Hamiltonian

$$\bar{H}_D^F = \bar{H}_C^F + \sum_A \lambda^A(t) \bar{\Phi}_A \quad (4.5)$$

will contain a final canonical Hamiltonian $\bar{H}_C^F$, whose associated vector field $\bar{X}_{\bar{H}_C^F}$ is tangent to $\bar{\gamma}_F$ (so that $\bar{H}_C^F$ has a vanishing Poisson bracket with all the constraints), and all the primary first-class constraints $\bar{\Phi}_A$ have arbitrary multipliers $\lambda^A(t)$. Therefore, $\bar{\Gamma}_D^F = (\partial/\partial t) + \bar{X}_{\bar{H}_D^F}$ will be tangent to $R \times \bar{\gamma}_F$ and will determine a consistent set of first-order Hamilton equations when restricted to $\bar{\gamma}_F$ [13, 46]. $\bar{\omega}_F$, the restriction of $\bar{\omega}$ to $\bar{\gamma}_F$, will have a kernel spanned by the vector fields $\bar{X}_A$ and $\bar{X}_a$, where the $\bar{X}_a$ are the vector fields associated with all the secondary, tertiary, etc., constraints which are first class (we assumed that all of them are generators of gauge transformations for our system). The final degenerate contact form $\bar{\bar{\omega}}_D^F = \bar{\omega}_F - d\bar{H}_D^F \Delta dt$ will have a kernel composed of $\ker \bar{\omega}_F$ plus the vector field $\bar{\Gamma}_C^F = (\partial/\partial t) + \bar{X}_{\bar{H}_C^F}$. Therefore, $\bar{\Gamma}_F = \bar{\Gamma}_C^F + \sum_A \lambda^A(t) \bar{X}_A$ will describe an evolution which is a mixture of a deterministic part and of gauge transformations. As all the vector fields $\bar{X}_a$ are assumed to generate gauge transformations, they must be contained in $\bar{X}_{\bar{H}_C^F}$:

$$\bar{X}_{\bar{H}_C^F} = \bar{X}_d + \sum_a \bar{c}^a(q,p) \bar{X}_a \quad (4.6)$$

where $\bar{X}_d$ describes the deterministic part of the evolution, i.e. $\bar{\Gamma}_d = (\partial/\partial t) + \bar{X}_d$ is a second-order vector field [40, 41, 61]. In the general case [34] there may exist higher-order first-class constraints not generating gauge transformations, so that they are not contained in $\bar{X}_{\bar{H}_C^F}$. Dirac proposed to generalize $\bar{\Gamma}_F$ to $\bar{\Gamma}_E = \bar{\Gamma}_D + \sum_a \mu^a(t) \bar{X}_a + \sum_A \lambda^A(t) \bar{X}_A$. In so doing, we are extending the genuine gauge transformations of our system to generalized gauge transformations.

The observables of our system, i.e. the gauge-invariant quantities, are those functions on $\bar{\gamma}_F$ which satisfy $\bar{X}_A \bar{f}_0 = \bar{X}_a \bar{f}_0 = 0$. They are equivalence classes of functions on T^*Q which have weakly vanishing Poisson brackets with all the first- and second-class constraints [65].

It can be shown [46, 53, 66] that the presymplectic manifold $\bar{\gamma}_F$ is foliated by the action of the vector fields $\bar{X}_A, \bar{X}_a$ and the leaves of this foliation are the so-called gauge orbits. By definition all the points in a gauge orbit are connected by gauge transformations. The reduced phase-space $\bar{\gamma}_R$ is the quotient of $\bar{\gamma}_F$ with respect to this foliation. A copy of it can be obtained from T^*Q by defining first the Dirac brackets with respect to all the second-class constraints, by introducing global gauge-fixing constraints for all the first-class constraints, and then by defining the final Dirac brackets.

When $\bar{\gamma}_R$ exists as a manifold, when the $\bar{X}_A, \bar{X}_a$ define a Lie algebra $\mathfrak{g}$ and when many mathematical hypotheses are satisfied [46, 53, 54, 66] $\bar{\gamma}_F$ is a bundle associated to a principal fibre bundle with basis $\bar{\gamma}_R$ and with a structural group G whose Lie algebra is $\mathfrak{g}$. The fibres are the gauge orbits and the ∞ -dimensional group of the gauge transformations realizes the changes of the local sections. If, moreover, $\bar{\gamma}_R = T^*\delta$, δ is the reduced configuration space of the original configuration space Q .

This description of the T^*Q formulation of singular systems is only qualitative and is oversimplified. At each stage of the analysis, many mathematical requirements have to be introduced, which will be reflected in restrictions on the original Lagrangian $L(q, \dot{q})$. Moreover, in the generic case Eqs. (4.4) will not have a unique solution, and each solution will require a separate discussion leading to a set of disjoint final closed submanifolds $\bar{\gamma}_{FK}$. This means that the analysis of the original Euler–Lagrange equations associated with $L(q, \dot{q})$ must imply the existence of an equal number of disjoint classes of solutions.

While all the pieces of the previous discussion are spread in the physical and mathematical literature (including many topological aspects ignored here), their Lagrangian and TQ counterparts have been much less investigated. The starting points should be: to find the Lagrangian analogues of the Dirac–Bergmann algorithm and to understand TQ. Actually, TQ is carrying more information than T^*Q because, owing to the degenerate nature of the Legendre transformation in the singular case, all TQ is mapped only in $\bar{\gamma}_I$, the primary constraint submanifold of T^*Q . This means that TQ is intrinsically a presymplectic manifold which is not isotropically embedded [15] in a larger unphysical symplectic manifold, such as $\bar{\gamma}_I$ in T^*Q . Moreover, its presymplectic 2-form ω_L has a much larger kernel [46] than $\bar{\omega}_I$, whose extra vector fields are connected with local dynamical invariances of the original Euler–Lagrange equations, which are independent of the local invariances of $L(q, \dot{q})$, as will be shown in the next section.

Let us end this section by noting that in the favourable case in which $\bar{\gamma}_R = T^*\delta$ one would expect that a final submanifold $\bar{\gamma}_F$ of TQ can be reduced to $T\delta$ and that, also considering $T^*(TQ)$ as in the regular case, the situation in Fig. 1 should be realized with all the arrows being well-defined projections [$\bar{q}_I$ and $\bar{q}_F$ are the primary and final constraint submanifolds in $T^*(TQ)$].

When δ exists, it is possible to define a projection from Q to δ , the configuration space of gauge orbits [81]. Even when δ does not exist globally, the connection between $\bar{\gamma}_R$ and $\bar{\gamma}_R$ is given by a mapping τ such that $\tau_0 \pi = \bar{\pi}_0 FL$. We are not aware of any detailed study of these problems.

5. SINGULAR LAGRANGIANS

Let us define a class of singular systems which is simple enough to be analysed without superfluous complications, but which contains as many interesting cases as possible. Let $L(q, \dot{q})$ be such that in the phase space we get the following situation. There are i) k_1 primary constraints $\bar{\Phi}_\alpha$, $\alpha = 1, \dots, k_1$, which do not generate secondary constraints and which at the end will be first class; ii) k_2 primary constraints $\bar{\Phi}_\beta$, $\beta = k_1 + 1, \dots, k_1 + k_2$, each one of which will generate a secondary constraint $\bar{\Phi}'_\beta$, and at the end both $\bar{\Phi}_\beta$ and $\bar{\Phi}'_\beta$ will be first class; iii) $2k_3$ primary constraints $\bar{\Phi}_r$, $r = k_1 + k_2 + 1, \dots, k_1 + k_2 + 2k_3$, which do not generate secondary constraints and which at the end will constitute pairs of second-class constraints; iv) k_4 primary constraints $\bar{\Phi}_s$, $s = k_1 + k_2 + 2k_3 + 1, \dots, k_1 + k_2 + 2k_3 + k_4$, each one of which will generate a secondary constraint $\bar{\Phi}'_s$, so that at the end each pair $\bar{\Phi}_s, \bar{\Phi}'_s$ is second class. No tertiary constraints are generated, so that the total number of constraints is $k_1 + 2k_2 + 2k_3 + 2k_4$ and $k_1 + 2k_2$ are first class: the dimension of the reduced phase space will be $2[n - (k_1 + 2k_2) - k_3 - k_4]$. The Dirac Hamiltonian is

$$\bar{H}_0 = \bar{H}_c + \sum_\alpha \lambda^\alpha \bar{\Phi}_\alpha + \sum_\beta \lambda^\beta \bar{\Phi}_\beta + \sum_r \lambda^r \bar{\Phi}_r + \sum_s \lambda^s \bar{\Phi}_s \quad (5.1)$$

and $\bar{\gamma}_1$ is determined by $\bar{\Phi}_\alpha \approx 0$, $\bar{\Phi}_\beta \approx 0$, $\bar{\Phi}_r \approx 0$, $\bar{\Phi}_s \approx 0$.

Equations (4.4) have the following form

$$\left\{ \begin{array}{l} \dot{\bar{\Phi}}_\alpha \equiv \{\bar{\Phi}_\alpha, \bar{H}_0\} \approx 0 \\ \dot{\bar{\Phi}}_\beta \equiv \{\bar{\Phi}_\beta, \bar{H}_0\} \approx 0 \\ \dot{\bar{\Phi}}_r \equiv \{\bar{\Phi}_r, \bar{H}_0\} \approx 0 \\ \dot{\bar{\Phi}}_s \equiv \{\bar{\Phi}_s, \bar{H}_0\} \approx 0 \end{array} \right. \quad (5.2)$$

The α equations are void, the β ones have the solutions $\bar{\Phi}'_\beta \approx 0$, the r ones determine the λ^r , the s ones have the solutions $\bar{\Phi}'_s \approx 0$. For the sake of simplicity we shall assume the following conditions to be satisfied (in the generic case $\lambda^r \neq 0$ and $=$ has to be replaced by $\approx$):

$$\left\{ \begin{array}{l} \lambda^r = 0 \quad \{\bar{\Phi}_\alpha, \bar{H}_c\} = \{\bar{\Phi}_\alpha, \bar{\Phi}_A\} = 0 \quad A = \alpha, \beta, r, s \\ \{\bar{\Phi}_\beta, \bar{H}_c\} = \bar{\Phi}'_\beta \quad \{\bar{\Phi}_\beta, \bar{\Phi}_A\} = 0 \quad A = \alpha, \beta, r, s \quad \{\bar{\Phi}'_\beta, \bar{\Phi}'_{\beta'}\} = 0 \\ \{\bar{\Phi}_r, \bar{H}_c\} = \{\bar{\Phi}_r, \bar{\Phi}_A\} = 0 \quad A = \alpha, \beta, s \quad \{\bar{\Phi}_r, \bar{\Phi}_{r'}\} \neq 0 \\ \{\bar{\Phi}_s, \bar{H}_c\} = \bar{\Phi}'_s \quad \{\bar{\Phi}_s, \bar{\Phi}_A\} = 0 \quad A = \alpha, \beta, r, s \quad \{\bar{\Phi}'_s, \bar{\Phi}'_{s'}\} = 0 \end{array} \right. \quad (5.3)$$

$\bar{\gamma}_1$ is restricted to $\bar{\gamma}_2$ by $\bar{\Phi}'_\beta \approx 0$, $\bar{\Phi}'_s \approx 0$ and $\bar{H}_0$ becomes

$$\bar{H}'_0 = \bar{H}_c + \sum_\alpha \lambda^\alpha \bar{\Phi}_\alpha + \sum_\beta \lambda^\beta \bar{\Phi}_\beta + \sum_s \lambda^s \bar{\Phi}_s \quad (5.4)$$

The new equations (4.4) are

$$\begin{cases} \dot{\bar{\Phi}}'_\beta \equiv \{\bar{\Phi}'_\beta, \bar{H}_D\} \approx_{\bar{\gamma}_2} 0 \\ \dot{\bar{\Phi}}'_s \equiv \{\bar{\Phi}'_s, \bar{H}_D\} \approx_{\bar{\gamma}_2} 0 \end{cases} \quad (5.5)$$

The β equations are void, while the s equations determine λ^s . Again let us assume

$$\begin{cases} \lambda^s = 0 & \{\bar{\Phi}'_\beta, \bar{H}_c\} = \{\bar{\Phi}'_\beta, \bar{\Phi}_A\} = 0 \quad A = \alpha, \beta, r, s \\ \{\bar{\Phi}'_s, \bar{H}_c\} = \{\bar{\Phi}'_s, \bar{\Phi}_A\} = 0 \quad A = \alpha, \beta, r & \{\bar{\Phi}'_s, \bar{\Phi}_{s'}\} \neq 0 \text{ if } s \neq s' \end{cases} \quad (5.6)$$

The hypothesis that $\bar{\Phi}'_\beta$ is first class implies also that $\{\bar{\Phi}'_\beta, \bar{\Phi}_r\} \approx_{\bar{\gamma}_2} 0$, but again we assume

$$\{\bar{\Phi}'_\beta, \bar{\Phi}_r\} = 0 \quad (5.7)$$

As the $\bar{\Phi}'_\beta$ are assumed to be generators of gauge transformations, the equations $\{\bar{\Phi}'_\beta, \bar{H}_c\} = \bar{\Phi}'_\beta$, $\{\bar{\Phi}'_\beta, \bar{H}_c\} = 0$ must imply [see also Eqs. (4.6)]:

$$\begin{aligned} \bar{H}_c &= \bar{H}_d + \sum_\beta \bar{c}^\beta(q, p) \bar{\Phi}'_\beta \\ \begin{cases} \{\bar{H}_d, \bar{\Phi}'_\beta\} = \{\bar{H}_d, \bar{\Phi}'_s\} = \{\bar{H}_d, \bar{\Phi}_A\} = 0 & A = \alpha, \beta, r \\ \{\bar{H}_d, \bar{\Phi}'_s\} = \bar{\Phi}'_s \approx 0 & \{\bar{\Phi}'_\beta, \bar{c}^{\beta'}\} = \delta^{\beta\beta'} \\ \{\bar{c}^\beta, \bar{\Phi}'_{\beta'}\} = \{\bar{c}^\beta, \bar{\Phi}'_s\} = \{\bar{c}^\beta, \bar{\Phi}_A\} = 0 & A = \alpha, r, s \end{cases} \end{aligned} \quad (5.8)$$

Therefore, $\bar{\gamma}_F = \bar{\gamma}_2$ and the final Dirac Hamiltonian is

$$\bar{H}_D^F = \bar{H}_d + \sum_\beta \bar{c}^\beta \bar{\Phi}'_\beta + \sum_\alpha \lambda^\alpha \bar{\Phi}_\alpha + \sum_\beta \lambda^\beta \bar{\Phi}_\beta \quad (5.9)$$

Its associated Hamilton equations imply: i) $g^A(q, \dot{q}) \stackrel{0}{=} \lambda^A$, $A = \alpha, \beta$; ii) $g^A(q, \dot{q}) \stackrel{0}{=} 0$, $A = r, s$; iii) $\dot{\bar{c}}^\beta \stackrel{0}{=} \{\bar{c}^\beta, \bar{H}_D^F\} = \lambda^\beta$ so that the functions $\bar{c}^\beta$ in Eqs. (5.8) are arbitrary. Their arbitrariness is induced by the velocity functions g^β .

Let us remark that the primary constraints $\bar{\Phi}_A(q, p) \approx 0$, $A = \alpha, \beta, r, s$, are a consequence of the non-invertibility of $k_1 + k_2 + 2k_3 + k_4$ of the equations $p_i(q, \dot{q}) = \partial L / \partial \dot{q}^i$ in $\dot{q}^i$. Therefore we have

$$\bar{\Phi}_A(q, p) \Big|_{p = \partial L / \partial \dot{q}} = \bar{\Phi}(q, p(q, \dot{q})) \equiv 0 \quad (5.10)$$

That is, the primary constraints vanish identically at the Lagrangian level. Two consequences of Eqs. (5.10) are

$$\begin{cases} 0 \equiv \frac{\partial}{\partial \dot{q}^i} \bar{\Phi}_A(q, p(q, \dot{q})) = A_{ij} \frac{\partial \bar{\Phi}_A}{\partial p_j} \\ 0 \equiv \frac{\partial}{\partial q^i} \bar{\Phi}_A(q, p(q, \dot{q})) = \frac{\partial}{\partial q^i} \bigg|_p \bar{\Phi}_A + \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} \frac{\partial \bar{\Phi}_A}{\partial p_j} \end{cases} \quad (5.11)$$

The first one tells us that $\partial \bar{\Phi}_A / \partial p_j$ are the null eigenvector (assumed linearly independent) of the Hessian matrix A. The second one gives the following result ($A, B = \alpha, \beta, r, s$):

$$\begin{aligned} \{\bar{\Phi}_A, \bar{\Phi}_B\} &= \frac{\partial \bar{\Phi}_A}{\partial q^i} \frac{\partial \bar{\Phi}_B}{\partial p_i} - \frac{\partial \bar{\Phi}_A}{\partial p_i} \frac{\partial \bar{\Phi}_B}{\partial q^i} = \\ &= - \frac{\partial \bar{\Phi}_A}{\partial p_i} R_{ij} \frac{\partial \bar{\Phi}_B}{\partial p_j} \stackrel{\text{def}}{=} - \bar{R}_{AB}(q, p) \end{aligned} \quad (5.12)$$

That is, the antisymmetric matrix R of Eqs. (2.14) becomes a function projectable to T^*Q when it is saturated with the null eigenvalues of A. The antisymmetric matrix $\bar{R}$ has rank $2k_3$ and the simplifying ansatz (5.3) leaves only the minor $(\bar{R}_{rr})$ non vanishing.

Let us now revert to the Lagrangian $L(q, \dot{q})$, whose Euler-Lagrange equations are given by Eqs. (2.2). For the class of systems previously defined, the Hessian matrix A will have $k_1 + k_2 + 2k_3 + k_4$ null eigenvalues with associated null eigenvectors ${}_{\alpha}\xi_0^i(q, \dot{q})$, ${}_{\beta}\xi_0^i(q, \dot{q})$, ${}_{r}\xi_0^i(q, \dot{q})$, ${}_{s}\xi_0^i(q, \dot{q})$ and $n - (k_1 + k_2 + 2k_3 + k_4)$ non-null eigenvalues $\mu_e(q, \dot{q})$ with eigenvectors $\chi_e^i(q, \dot{q})$. We shall assume that all the eigenvectors are orthonormal and that the functional form of the primary constraints $\bar{\Phi}_A$ has been chosen in such a form to have

$${}_{A\alpha} \xi_0^i(q, \dot{q}) = \frac{\partial \bar{\Phi}_A(q, p)}{\partial p_i} \bigg|_{p=\partial L / \partial \dot{q}} \quad A = \alpha, \beta, r, s \quad (5.13)$$

In general this orthonormalization is only possible locally and amounts to an abelianization of the original 1st-class constraints. Therefore the following discussion is essentially local, but it avoids complications in the presentation of the results. The matrix A has the form $A_{ij} = \sum_e \mu_e \chi_{ei} \chi_{ej}$ and we can define a quasi-inverse [67] $B^{ij} = \sum_e \mu_e^{-1} \chi_e^i \chi_e^j$ (the nearest analogue to $B = A^{-1}$ of Section 2) such that

$$B^{ik} A_{kj} = A_{jk} B^{ki} = \delta_j^i - \sum_A \frac{\xi_0^i}{A} \frac{\xi_0^k}{A} \delta_{kj} \quad (5.14)$$

B is simultaneously a left and a right inverse of A in the terminology of Ref. [67]. $\det A = 0$ implies that the Euler-Lagrange equations (2.2) are a system of differential equations with different orders 2, 1, 0 in the generic case. They are invariant under the transformations $\dot{q}^i \rightarrow \dot{q}^i + \sum_A a^A(t) {}_{A}\xi_0^i$, q^i and $\dot{q}^i$ fixed, with $a^A(t)$ arbitrary function. These local dynamical symmetries of the Euler-Lagrange equations are independent of the local Noether invariances of the action which we are going to study. By using the matrix B we can rewrite Eqs. (2.2) in the following form:

$$\left\{ \begin{array}{l} -B^{ij}L_j = \ddot{q}^i - \Lambda^i - \sum_A a^A(t) \xi_{A0}^i \stackrel{!}{=} 0 ; \Lambda^i = B^{ij}\alpha_j \\ \xi_{\alpha 0}^i L_i = \xi_{\alpha 0}^i \alpha_i \stackrel{!}{=} \chi_{\alpha} \stackrel{!}{=} 0 \\ \xi_{\beta 0}^i L_i = \xi_{\beta 0}^i \alpha_i \stackrel{!}{=} \chi_{\beta} \stackrel{!}{=} 0 \\ \xi_{\tau 0}^i L_i = \xi_{\tau 0}^i \alpha_i \stackrel{!}{=} \chi_{\tau} \stackrel{!}{=} 0 \\ \xi_{s 0}^i L_i = \xi_{s 0}^i \alpha_i \stackrel{!}{=} \chi_s \stackrel{!}{=} 0 \end{array} \right. \quad \left\{ \begin{array}{l} \chi_{ei}(\ddot{q}^i - \Lambda^i) \stackrel{!}{=} 0 \\ \xi_{A0i} \ddot{q}^i \stackrel{!}{=} a^A(t) \end{array} \right. \quad (5.15)$$

$\chi_{ei}(\ddot{q}^i - \Lambda^i) \stackrel{!}{=} 0$ are the only genuine second-order differential equations. Those χ_A which do not vanish identically are either first-order differential equations, when $\chi_A = \chi_A(q, \dot{q})$, or holonomic constraints $\chi_A(q)$. However, from the point of view of T^*Q the χ_A fall in two unequivalent classes according to whether they are projectable on T^*Q or not. On this point there is a lot of confusion in the literature and no clear resolution of this problem, as has been said in the Introduction. In what follows we shall show that one has: i) $\chi_{\alpha} \equiv 0$; ii) $\chi_{\beta}(q, \dot{q}) = \bar{\Phi}'_{\beta}(q, p)$; iii) $\chi_s(q, \dot{q}) = \bar{\Phi}'_s(q, p)$; iv) the χ_{τ} are not projectable on T^*Q . All the holonomic constraints are of the type χ_{β} : in $L(q, \dot{q})$ there is some variable which is a linear or non-linear Lagrange multiplier not determined by Eqs. (2.2), whose variation gives $\chi_{\beta}(q) \stackrel{!}{=} 0$ and whose associated momentum vanishes (so that it is one of the primary constraints $\bar{\Phi}_{\beta}$). The case of a holonomic constraint whose Lagrange multiplier is determined by Eqs. (2.2) requires in T^*Q a tertiary and a quaternary constraint (and is not contained in our class of systems) so that, at the end, there are two pairs of second-class constraints: one pair is the holonomic constraint and its conjugated variable, the other is made of the Lagrange multiplier and its momentum (as an example consider $L = \frac{1}{2} \dot{x}^2 + xy$).

The content of the equations $\chi_{\tau} \stackrel{!}{=} 0$ is recovered in T^*Q from the first half of the final Hamilton equations on $\bar{\gamma}_F, \dot{q}^i \stackrel{!}{=} \{q^i, \bar{H}_B^F\}$ as a consequence of $\lambda^{\tau} = 0$, see Eq. (5.3).

This state of affairs induces the introduction of the following terminology: the non-identically vanishing χ 's, which are projectable on T^*Q , are secondary Lagrangian constraints, because they will become the secondary Hamiltonian constraints; the non-projectable χ 's (the χ_{τ}) are genuine first-order differential equations of motion. The primary Lagrangian constraints would be the $\bar{\Phi}_A[q, p(q, \dot{q})]$, but they vanish identically, see Eq. (5.10). Let us remark that further genuine first-order equations and/or higher-order Lagrangian constraints may be generated by suitable combinations of the L_i and their time derivatives.

When the system admits only first-class constraints, it is known that the Lagrangian is (quasi-) invariant under the gauge transformations generated by them. This is the content of the second Noether theorem [48-51] according to which, with each null eigenvalue $\mu_A = 0$ of the Hessian matrix, is associated a set of local variations $\delta_A q^i$, depending upon an arbitrary function $\epsilon^A(t)$ and its derivatives up to a certain maximum order J_A , such that the Lagrangian is (quasi-) invariant under them and this invariance produces $J_A + 1$ identities involving L and its derivatives. Instead, in the first Noether theorem we have (quasi-) invariance of L under a set of global transformations δq^i and only one identity is produced, implying the existence of a constant of the motion.

However, for a generic singular Lagrangian there will also be null eigenvalues of the Hessian matrix associated with primary second-class constraints. It turns out [47] that the second Noether theorem can be generalized to include this case in the following way. With each null eigenvalue $\mu_A = 0$ of the Hessian matrix is associated a set of local variations $\delta_A q^i$, depending upon an arbitrary function $\epsilon^\lambda(t)$ and its derivatives up to a certain maximum order J_A , such that the Lagrangian is, in general, only weakly quasi-invariant under them and this invariance produces $J_A + 1$ identities. Weak quasi-invariance means that the variation $\delta_A L$ of L is a total time derivative (or zero) plus a term linear in $\epsilon^\lambda(t)$ and whose coefficient vanishes by using all the genuine first-order equations of motion which can be extracted from the Euler-Lagrange equations and their time derivatives.

The concept of weak quasi-invariance is an adaptation to the second Noether theorem of one of the extensions of the first Noether theorem [68]. Let us illustrate the theorem for the class of singular Lagrangians under consideration. The following local variations:

$$\left\{ \begin{array}{l} \delta_\alpha q^i = \epsilon^\alpha(t) \xi_{\alpha 0}^i \\ \delta_\beta q^i = \epsilon^\beta(t) \xi_{\beta 1}^i + \dot{\epsilon}^\beta(t) \xi_{\beta 0}^i \\ \delta_r q^i = \epsilon^r(t) \xi_{r 0}^i \\ \delta_s q^i = \epsilon^s(t) \xi_{s 1}^i + \dot{\epsilon}^s(t) \xi_{s 0}^i \end{array} \right. \quad \begin{array}{l} \delta_A \dot{q}^i = \frac{d}{dt} \delta_A q^i \\ A = \alpha, \beta, r, s \end{array} \quad (5.16)$$

are associated with the null eigenvalues $\mu_\alpha = \mu_\beta = \mu_r = \mu_s = 0$ of the Hessian matrix. We have anticipated with the notation the fact that the coefficient of the highest derivative of $\epsilon^\lambda(t)$ is the null eigenvector $\xi_{\lambda 0}^i$. Let us remark that, in general, the value of J_A has to be determined case by case or, with an a posteriori method, by using the Dirac-Bergmann algorithm in T^*Q to see how many secondary constraints are implied by a primary constraint and by checking how many of them are effective in generating the $\delta_A q^i$.

Given a variation δq^i , we get the following formal expression for the variation δL :

$$\delta L = \frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i = \delta q^i L_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right) \quad (5.17)$$

Therefore the (weak quasi-) invariance associated with the variations (5.16) implies

$$\left\{ \begin{array}{l} \delta_\alpha L = \frac{d}{dt} (\epsilon^\alpha F_0) \equiv \delta_\alpha q^i L_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta_\alpha q^i \right) \\ \delta_\beta L = \frac{d}{dt} (\epsilon^\beta F_1 + \dot{\epsilon}^\beta F_0) \equiv \delta_\beta q^i L_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta_\beta q^i \right) \\ \delta_r L = \frac{d}{dt} (\epsilon^r F_0) + \epsilon^r D_r \equiv \delta_r q^i L_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta_r q^i \right) \\ \delta_s L = \frac{d}{dt} (\epsilon^s F_1 + \dot{\epsilon}^s F_0) + \epsilon^s D_s \equiv \delta_s q^i L_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta_s q^i \right) \end{array} \right. \quad (5.18)$$

with $D_t(q, \dot{q}) \stackrel{0}{=} 0$, $D_s(q, \dot{q}) \stackrel{0}{=} 0$. Some or all of the ${}_A F_A(q, \dot{q})$ may vanish. By introducing the quantities

$$\left\{ \begin{aligned} G_\alpha &= \frac{\partial L}{\partial \dot{q}^i} \delta_\alpha q^i - \varepsilon^\alpha F_0 = \varepsilon^\alpha G_0 \\ G_\beta &= \frac{\partial L}{\partial \dot{q}^i} \delta_\beta q^i - \varepsilon^\beta F_1 - \dot{\varepsilon}^\beta F_0 = \varepsilon^\beta G_1 + \dot{\varepsilon}^\beta G_0 \\ G_r &= \frac{\partial L}{\partial \dot{q}^i} \delta_r q^i - \varepsilon^r F_0 = \varepsilon^r G_0 \\ G_s &= \frac{\partial L}{\partial \dot{q}^i} \delta_s q^i - \varepsilon^s F_1 - \dot{\varepsilon}^s F_0 = \varepsilon^s G_1 + \dot{\varepsilon}^s G_0 \end{aligned} \right. \quad (5.19)$$

we obtain the following Noether identities:

$$\left\{ \begin{aligned} \dot{G}_\alpha &\equiv -\delta_\alpha q^i L_i = -\varepsilon^\alpha \xi_0^i L_i \stackrel{0}{=} 0 \\ \dot{G}_\beta &\equiv -\delta_\beta q^i L_i = -(\varepsilon^\beta \xi_1^i L_i + \dot{\varepsilon}^\beta \xi_0^i L_i) \stackrel{0}{=} 0 \\ \dot{G}_r - \varepsilon^r D_r &\equiv -\delta_r q^i L_i = -\varepsilon^r \xi_0^i L_i \stackrel{0}{=} 0 \\ \dot{G}_s - \varepsilon^s D_s &\equiv -\delta_s q^i L_i = -(\varepsilon^s \xi_1^i L_i + \dot{\varepsilon}^s \xi_0^i L_i) \stackrel{0}{=} 0 \end{aligned} \right. \quad (5.20)$$

As ε^α , $\dot{\varepsilon}^\alpha$ are independent arbitrary functions, the identities (5.20) may be rewritten in the following form

$$\left\{ \begin{aligned} \alpha G_0 &\equiv 0 \\ 0 &\equiv \alpha \dot{G}_0 \equiv -\alpha \xi_0^i L_i = -\chi_\alpha \stackrel{0}{=} 0 \end{aligned} \right. \quad (5.21a)$$

$$\left\{ \begin{aligned} \beta G_0 &\equiv 0 \\ 0 &\equiv \beta \dot{G}_0 \equiv -\beta G_1 - \beta \xi_0^i L_i \stackrel{0}{=} -\beta G_1 \Rightarrow \beta G_1 \equiv -\beta \xi_0^i L_i = -\chi_\beta \stackrel{0}{=} 0 \\ \beta G_1 &\equiv -\beta \xi_1^i L_i \stackrel{0}{=} 0 \end{aligned} \right. \quad (5.21b)$$

$$\begin{cases} {}_r\dot{G}_0 \equiv 0 \\ 0 \equiv {}_r\dot{G}_0 \equiv D_r - \xi_0^i L_i \stackrel{0}{=} D_r \Rightarrow D_r \equiv \xi_0^i L_i = \chi_r \stackrel{0}{=} 0 \end{cases} \quad (5.21c)$$

$$\begin{cases} {}_s\dot{G}_0 \equiv 0 \\ 0 \equiv {}_s\dot{G}_0 \equiv -{}_sG_1 - \xi_0^i L_i \stackrel{0}{=} -{}_sG_1 \Rightarrow {}_sG_1 \equiv -\xi_0^i L_i = -\chi_s \stackrel{0}{=} 0 \\ {}_s\dot{G}_1 \equiv D_s - \xi_1^i L_i \stackrel{0}{=} D_s \Downarrow \Rightarrow D_s \equiv \xi_1^i L_i - \frac{d}{dt}(\xi_0^i L_i) \stackrel{0}{=} 0 \end{cases} \quad (5.21d)$$

From Eqs. (5.10) we see that we can identify ${}_AG_0(q, \dot{q})$ with $\bar{\Phi}_A(q, p(q, \dot{q}))$, thus fixing the functional forms of the primary constraints associated with the chosen orthonormal eigenvectors ${}_A\xi_0^i$. As Eqs. (5.21) are identities involving functions of $q^i, \dot{q}^i, \ddot{q}^i$, the coefficients of the $\ddot{q}^i$'s have to vanish identically. Then from Eqs. (2.2) and ${}_A\dot{G}_0 \equiv 0$ we get

$$\begin{cases} A_{ij} {}_A\xi_0^j \equiv 0 \\ \chi_\alpha \equiv 0 \\ {}_\beta G_1 \equiv -\chi_\beta \stackrel{0}{=} 0 \\ D_r \equiv \chi_r \stackrel{0}{=} 0 \\ {}_sG_1 \equiv -\chi_s \stackrel{0}{=} 0 \end{cases} \quad A = \alpha, \beta, r, s \quad (5.22)$$

Therefore the ${}_A\xi_0^i$ are the null eigenvectors of A ; χ_α vanishes identically; $D_r \stackrel{0}{=} 0$ by using the genuine first-order equations of motion $\chi_r \stackrel{0}{=} 0$; ${}_\beta G_1$ and ${}_sG_1$ are equal but opposite in sign to the Lagrangian constraints $\chi_\beta \stackrel{0}{=} 0, \chi_s \stackrel{0}{=} 0$.

Analogously, from ${}_\beta\dot{G}_1 \equiv -{}_ \beta\xi_1^i L_i, {}_s\dot{G}_1 \equiv D_s - {}_s\xi_1^i L_i$, we get

$$\begin{cases} A_{ij} {}_\beta\xi_1^j \equiv \frac{\partial {}_\beta G_1}{\partial \dot{q}^i} \equiv -\frac{\partial \chi_\beta}{\partial \dot{q}^i} \\ A_{ij} {}_s\xi_1^j \equiv \frac{\partial {}_s G_1}{\partial \dot{q}^i} \equiv -\frac{\partial \chi_s}{\partial \dot{q}^i} \end{cases} \quad \begin{cases} \dot{q}^i \frac{\partial {}_\beta G_1}{\partial q^i} \equiv -\xi_1^i \alpha_i \equiv -\dot{q}^i \frac{\partial \chi_\beta}{\partial q^i} \\ D_s \equiv \dot{q}^i \frac{\partial {}_s G_1}{\partial q^i} + \xi_1^i \alpha_i \equiv -\dot{q}^i \frac{\partial \chi_s}{\partial q^i} + \xi_1^i \alpha_i \stackrel{0}{=} 0 \end{cases} \quad (5.23)$$

By comparing (5.21d) and (5.23) we see that $D_s \stackrel{0}{=} 0$ owing to a combination of the Euler-Lagrange equations and their first time derivative, which is independent of $\ddot{q}^i$: this is a non-projectable higher-order genuine first-order equation of motion independent of Eqs. (5.15). For this class of systems there are no tertiary Lagrangian constraints. Equations (5.16), (5.19), and (5.23) imply

$$A_{ij} \delta_A q^j \equiv \frac{\partial G_A}{\partial \dot{q}^i} \quad (5.24)$$

From Eqs. (5.21a,b) we also have

$$\begin{cases} \alpha \xi_0^i L_i \equiv 0 \\ \beta \xi_1^i L_i - \frac{d}{dt} (\beta \xi_0^i L_i) \equiv 0 \end{cases} \quad (5.25)$$

These equations state that the null eigenvalues $\mu_\alpha = \mu_\beta = 0$ imply that the Euler-Lagrange equations and their time derivatives are not independent. The relations (5.25) are called contracted Bianchi identities.

Let us remark that the simplicity of these results relies on the choice of the most relevant orthonormal basis for the null eigenvectors in the vector space associated with the null eigenvalues $\mu_A = 0$ of the Hessian matrix. For every other choice the statement of the second Noether theorem has to be further generalized but the content of the Noether identities remains unchanged. Moreover, if in Eqs. (5.16) the $\dot{\epsilon}^\beta, \dot{\epsilon}^\gamma$ are replaced by arbitrary functions η^β, η^γ , then the concept of weak quasi-invariance has to be generalized in the sense that besides the D_r, D_s terms, also ${}_\beta G_1, {}_s G_1$ terms will appear in δL . As ${}_\beta G_1 \stackrel{0}{=} 0, {}_s G_1 \stackrel{0}{=} 0$, owing to $\ddot{q}$ -independent consequences of the Euler-Lagrange equations, nothing changes except that the discussion becomes more involved. As a consequence of the decoupling of the Noether transformations, when an orthonormal basis is used, Eqs. (5.23) imply: ${}_A \xi_1^i = B^{ij}(\partial_A G_1 / \partial \dot{q}^j)$.

The replacement of $\dot{\epsilon}^A$ by η^A is equivalent to the Dirac extended Hamiltonian, with generalized gauge transformations for those values of A which are associated with first-class constraints.

It is now evident that the form (5.21) of the Noether identities reproduces at the Lagrangian level the Dirac-Bergmann algorithm in T^*Q . ${}_A G_0$ are the primary constraints $\bar{\Phi}_A$; the secondary Lagrangian constraints ${}_\beta G_1 \equiv -\chi_\beta, {}_s G_1 \equiv -\chi_s$ are the secondary constraints $\bar{\Phi}'_\beta, \bar{\Phi}'_s$; $D_r = \chi_r \stackrel{0}{=} 0$ and $D_s \stackrel{0}{=} 0$ determine the otherwise arbitrary velocity functions $g^r(q, \dot{q}), g^s(q, \dot{q})$ associated with $\mu_r = \mu_s = 0$ and replace the determination of λ^r and λ^s in T^*Q . As Eqs. (5.1), (5.13) imply $g^A \stackrel{0}{=} \lambda^A \stackrel{0}{=} {}_A \xi_0^i (\dot{q}^i - \{q^i, \bar{H}_D\}) = {}_A \xi_0^i (\dot{q}^i - (\partial \bar{H}_C) / \partial p_i)$, the content of the genuine (non-projectable) first-order equations of motion $D_r = \chi_r \stackrel{0}{=} 0, D_s = {}_s \xi_1^i L_i - (d/dt)({}_s \xi_0^i L_i) \equiv -(\partial \chi_s / \partial q^i) - {}_s \xi_1^i \alpha_i \stackrel{0}{=} 0$ is recovered by means of the equations $\dot{q}^i \stackrel{0}{=} \{q^i, \bar{H}_D\}$. Let us remark that the arbitrary functions $\epsilon^A(t)$, implying undetermined velocities, are completely unrelated to the arbitrary functions $a^A(t)$ of Eqs. (5.15), implying arbitrary accelerations.

D_r and D_s are those functional forms of the velocity functions g^r, g^s such that the rectangular matrix $({}_A \xi_0^i \partial D_A / \partial \dot{q}^i)$, $A = \alpha, \beta, r, s, a = r, s$, has the minor $({}_b \xi_0^i \partial D_a / \partial \dot{q}^i)$, $a, b = r, s$, non-singular, and carry the same information of the choice $g^r \stackrel{0}{=} \lambda^r \stackrel{0}{=} 0, g^s \stackrel{0}{=} \lambda^s \stackrel{0}{=} 0$, being functions of them. Since D_r, D_s are linear combinations of the Euler-Lagrange equations L_i and of their first-time derivative, we have also $\dot{D}_r \stackrel{0}{=} 0, \dot{D}_s \stackrel{0}{=} 0$. These equations determine $a^r(t), a^s(t)$ as functions of $q^i, \dot{q}^i, a^\alpha, a^\beta$, due to the non-singular nature of the previously defined minor and by using Eqs. (5.15). This allows to find the final form of the genuine second-order Euler-Lagrange equations contained in Eqs. (5.15).

To understand better the connection between the Noether identities (5.18) or (5.20) and the Dirac-Bergmann algorithm we must consider the Legendre transformation. As the Lagrangian is

singular, this transformation also becomes singular. For instance, the following vector fields are projected onto the null vector field on T^*Q :

$$Z_A = \xi_0^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i} \rightarrow \xi_0^i(q, \dot{q}) A_{ij}(q, \dot{q}) \frac{\partial}{\partial p_j} = 0 \quad (5.26)$$

because the ξ_0^i are the null eigenvectors of the Hessian matrix. The meaning of Eqs. (5.26) will be elucidated in the next section.

The velocity functions $g^\lambda(q, \dot{q})$ are the intrinsically non-projectable functions, which arise because the equations $p_i = \partial L / \partial \dot{q}^i$ cannot be inverted to get all the velocities in terms of q and p . This explains why only the primary constraints submanifold $\bar{\gamma}_1$ of T^*Q is relevant for the phase-space description of the singular systems. As in the Dirac-Bergmann approach one starts by considering $\bar{\gamma}_1$ embedded in T^*Q , to be able to use its symplectic structure, one can introduce a generalized Legendre transformation in the following sense. Given a function $f(q, \dot{q})$, let us rewrite it as $\bar{f}[q, p(q, \dot{q}), g^\lambda(q, \dot{q})]$. Then its projection to a function over $R \times T^*Q$ (the extended phase space) will be $\bar{f}[\bar{q}, p, \lambda^\lambda(t)]$. This prescription, which coincides with the usual one for projectable functions, is well defined owing to our convention about the functional form of $g^\lambda(q, \dot{q})$.

By using the Hamilton equations $\dot{p}_i - \{p_i, \bar{H}_D\} \approx -L_i \stackrel{0}{=} 0$ when $\dot{q}^i - \{q^i, \bar{H}_D\} \stackrel{0}{=} 0$, Eqs. (5.20) could be rewritten in the following form:

$$\frac{d\bar{G}_A}{dt} \stackrel{0}{\approx} \epsilon^A(t) \bar{D}_A + \delta_{Aq^i} (\dot{p}_i - \{p_i, \bar{H}_D\}) \quad A = \alpha, \beta, \gamma, \delta \quad (5.27)$$

with $\bar{G}_A, \delta_{Aq^i}$ functions of $q, p, \lambda^\beta(t), \epsilon^\lambda(t)$ and $\bar{D}_A$ functions of $q, p, \lambda^\beta(t)$. Here ' $\stackrel{0}{\approx}$ ' means modulo $\bar{\Phi}_A \approx 0$ and $\dot{q}^i - \{q^i, \bar{H}_D\} \stackrel{0}{=} 0$. As we have

$$\frac{d\bar{G}_A}{dt} = \frac{\partial \bar{G}_A}{\partial t} \Big|_{\lambda^\beta} + \sum_B \dot{\lambda}^B(t) \frac{\partial \bar{G}_A}{\partial \lambda^B} + \dot{q}^i \frac{\partial \bar{G}_A}{\partial q^i} + \dot{p}_i \frac{\partial \bar{G}_A}{\partial p_i} \quad (5.28)$$

where in the first term of the right-hand side of the equation $\partial/\partial t$ acts on $\epsilon^\lambda(t)$, the identities (5.27) imply

$$\begin{cases} \frac{\partial \bar{G}_A}{\partial \lambda^B} \equiv 0 & \Rightarrow \bar{G}_A = \bar{G}_A(q, p, \epsilon^\lambda(t)) \\ \delta_{Aq^i} \equiv \frac{\partial \bar{G}_A}{\partial p_i} = \{q^i, \bar{G}_A\} \end{cases} \quad (5.29)$$

In Eqs. (5.29) we have used ' $\equiv$ ' and not ' $\stackrel{0}{\approx}$ ', because the lacking terms in $\bar{\Phi}_A$ and $\dot{q}^i - \{q^i, \bar{H}_D\}$ cannot introduce dependences upon $\lambda^\lambda(t)$ and $\dot{p}_i$. By remembering Eqs. (5.16), (5.19) we can rewrite Eqs. (5.29) in the following form:

$$\left\{ \begin{array}{l} \bar{G}_A = \sum_{j=0}^{J_A} \varepsilon^{A(j)}(t) {}_A \bar{G}_{J_A-j}(q, p) \\ \bar{\delta}_A q^i = \sum_{j=0}^{J_A} \varepsilon^{A(j)}(t) {}_A \bar{\xi}_{J_A-j}^i(q, p) = \sum_{j=0}^{J_A} \varepsilon^{A(j)}(t) \{q^i, {}_A \bar{G}_{J_A-j}\} \end{array} \right. \left\{ \begin{array}{l} \varepsilon^{A(j)}(t) = \frac{d^j \varepsilon^A(t)}{dt^j} \\ J_A = J_T = 0, J_B = J_S = 1 \end{array} \right. \quad (5.30)$$

Equations (5.30) show that ${}_A G_{J_A-j}(q, \dot{q})$, ${}_A \xi_{J_A-j}^i(q, \dot{q})$ are projectable functions, that the constraints $\bar{G}_A$ are the phase-space generators of the local Noether transformations $\delta_A q^i$ of Eqs. (5.16) and that Eqs. (5.13) hold. Moreover, by writing $F_A = \sum_{j=0}^{J_A} \varepsilon^{A(j)}(t) {}_A F_{J_A-j}$ and by using Eqs. (5.19), (5.30), we see that the functions ${}_A F_{J_A-j}(q, \dot{q})$ are projectable to the T^*Q functions ${}_A \bar{F}_{J_A-j}(q, p)$.

By using the results of the Appendix about the phase-space formulation of the first Noether theorem, we can now formulate the phase-space version of the second Noether theorem. With each null eigenvalue of the Hessian matrix is associated a set of local transformations

$$\left\{ \begin{array}{l} \bar{\delta}_A q^i(q, p, \varepsilon^A) = \sum_{j=0}^{J_A} \varepsilon^{A(j)}(t) {}_A \bar{\xi}_{J_A-j}^i(q, p) = \delta_A q^i(q, \dot{q}, \varepsilon^A) \\ \bar{\delta}_A p_i(q, p, \varepsilon^A) = \sum_{j=0}^{J_A} \varepsilon^{A(j)}(t) {}_A \bar{\eta}_{J_A-j, i}(q, p) \end{array} \right. \quad (5.31)$$

under which the phase-space Lagrangian $\bar{L}_D = p_i \dot{q}^i - \bar{H}_D$ is weakly quasi-invariant:

$$\begin{aligned} \delta_A \bar{L}_D &= \bar{\delta}_A p_i \dot{q}^i + p_i \frac{d}{dt} \bar{\delta}_A q^i - \frac{\partial \bar{H}_D}{\partial q^i} \bar{\delta}_A q^i - \frac{\partial \bar{H}_D}{\partial p_i} \bar{\delta}_A p_i = \\ &= \bar{\delta}_A p_i \bar{L}_q^i - \bar{\delta}_A q^i \bar{L}_{p_i} + \frac{d}{dt} (p_i \bar{\delta}_A q^i) \equiv \\ &\equiv \frac{d}{dt} \bar{F}_A + \varepsilon^A(t) \bar{D}_A + \varepsilon^{A(J_A+1)}(t) \bar{\Phi}_A \end{aligned} \quad (5.32)$$

Here $\bar{F}_A(q, p, \varepsilon^A) = F_A(q, \dot{q}, \varepsilon^A)$, $\bar{D}_A(q, p, \lambda^B) \stackrel{0}{=} D_A[q, p(q, \dot{q}), g^B(q, \dot{q})]$ and

$$\left\{ \begin{array}{l} \bar{L}_q^i = \dot{q}^i - \{q^i, \bar{H}_D\} \stackrel{0}{=} 0 \\ \bar{L}_{p_i} = \dot{p}_i - \{p_i, \bar{H}_D\} \stackrel{0}{=} 0 \end{array} \right. \quad (5.33)$$

are the Hamilton equations with respect to $\bar{H}_D$. The presence of the term in $\bar{\Phi}_A$ is induced by the term $p_i (d/dt) \delta_A q^i$, which contains

$$\varepsilon^{A(J_A+1)}(t) p_i \bar{\xi}_A^i = \varepsilon^{A(J_A+1)}(t) (\bar{\Phi}_A + \bar{F}_0) \quad (5.34)$$

From Eqs. (5.32) we get the following exact form of Eqs. (5.27):

$$\frac{d\bar{G}_A}{dt} \equiv \varepsilon^A(t) \bar{D}_A + \varepsilon^{A(J_A+1)}(t) \bar{\Phi}_A + \bar{\delta}_A q^i \bar{L}_{pi} - \bar{\delta}_A p_i \bar{L}_q^i \quad (5.35)$$

implying again Eqs. (5.29), (5.30), and moreover

$$\left\{ \begin{array}{l} \bar{\delta}_A p_i \equiv \{p_i, \bar{G}_A\} \\ \frac{\partial \bar{G}_A}{\partial t} + \frac{\partial \bar{G}_A}{\partial q^i} \dot{q}^i + \frac{\partial \bar{G}_A}{\partial p_i} \dot{p}_i = \frac{\partial \bar{G}_A}{\partial t} + \{\bar{G}_A, \bar{H}_0\} + \bar{\delta}_A q^i \bar{L}_{pi} - \bar{\delta}_A p_i \bar{L}_q^i \equiv \\ \equiv \varepsilon^A(t) \bar{D}_A + \varepsilon^{A(J_A+1)}(t) \bar{\Phi}_A + \bar{\delta}_A q^i \bar{L}_{pi} - \bar{\delta}_A p_i \bar{L}_q^i \end{array} \right. \quad (5.36)$$

where we used $\{\bar{f}, \bar{g}\} = \{\bar{f}, p_k\} \{q^k, \bar{g}\} - \{\bar{f}, q^k\} \{p_k, \bar{g}\}$. The last line of Eqs. (5.36) may be written in the following form:

$$\bar{\Gamma}_0(\bar{G}_A) = \frac{\partial \bar{G}_A}{\partial t} + \{\bar{G}_A, \bar{H}_0\} \equiv \varepsilon^A(t) \bar{D}_A + \varepsilon^{A(J_A+1)}(t) \bar{\Phi}_A \quad (5.37)$$

These are the Noether identities in T^*Q , whose explicit form is

$$\left\{ \begin{array}{l} {}_A \bar{G}_0 \equiv \bar{\Phi}_A \approx 0 \\ \frac{d{}_A \bar{G}_0}{dt} \equiv -{}_A \bar{G}_1 + \{q^i, {}_A \bar{G}_0\} \bar{L}_{pi} - \{p_i, {}_A \bar{G}_0\} \bar{L}_q^i \equiv -{}_A \bar{G}_1 \\ \vdots \\ \frac{d{}_A \bar{G}_{J_A}}{dt} \equiv \bar{D}_A + \{q^i, {}_A \bar{G}_{J_A}\} \bar{L}_{pi} - \{p_i, {}_A \bar{G}_{J_A}\} \bar{L}_q^i \equiv \bar{D}_A \end{array} \right. \quad (5.38)$$

The Dirac-Bergmann algorithm becomes evident when we rewrite Eqs. (5.38) in the form

$$\left\{ \begin{array}{l} {}_A \bar{G}_0 \equiv \bar{\Phi}_A \approx 0 \\ \bar{\Gamma}_0({}_A \bar{G}_0) = \{{}_A \bar{G}_0, \bar{H}_c\} + \sum_B \lambda^B(t) \{{}_A \bar{G}_0, \bar{\Phi}_B\} \equiv -{}_A \bar{G}_1 \approx 0 \\ \vdots \\ \bar{\Gamma}_0({}_A \bar{G}_{J_A}) = \{{}_A \bar{G}_{J_A}, \bar{H}_c\} + \sum_B \lambda^B(t) \{{}_A \bar{G}_{J_A}, \bar{\Phi}_B\} \equiv \bar{D}_A \approx 0 \end{array} \right. \quad (5.39)$$

by using Eqs. (5.33).

As $J_\alpha = J_r = 0$, $J_\beta = J_s = 1$, Eqs. (5.39) have the same content as Eqs. (5.2) to (5.9), since $\bar{D}_\alpha = \bar{D}_\beta = 0$; $\bar{D}_r \approx 0$, $\bar{D}_s \approx 0$ amount to the determination of λ^r, λ^s ; ${}_\beta \bar{G}_1 = \bar{\Phi}'_\beta$, ${}_s \bar{G}_1 = \bar{\Phi}'_s$.

We shall finish this section with a characterization of the local Noether transformations, which are gauge transformations, i.e. are generated by first-class constraints, such as $\delta_\alpha q^i$, $\delta_\beta q^i$. If we evaluate the variation of the Euler-Lagrange equations (2.2), under the transformations $\delta_A q^i$, $A = \alpha, \beta, r, s$, we get [56]

$$\begin{aligned} \delta_A L_i &= L_i|_{q+\delta_A q} - L_i|_q = \left(\frac{\partial^2 L}{\partial q^i \partial q^j} - \frac{d}{dt} \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \right) \delta_A q^j - \\ &- R_{ij} \delta_A \dot{q}^j - \frac{d}{dt} (A_{ij} \delta_A \dot{q}^j) = J_i(\delta_A q^k) \end{aligned} \quad (5.40)$$

Here $J_i(\delta q)$ are the Jacobi equations [57], which can be obtained from the second variation of the action $S = \int dt L$ and which vanish, when restricted to the extremals, solutions of $L_i \stackrel{0}{=} 0$, if $\delta q^i|_{L_i \stackrel{0}{=}} 0$ are deviations between two neighbouring extremals. After some manipulations Eqs. (5.40) become

$$\begin{aligned} \delta_A L_i &= J_i(\delta_A q^k) = \left(\frac{\partial L_i}{\partial q^j} + \dot{R}_{ij} \right) \delta_A q^j - \frac{d}{dt} (R_{ij} \delta_A q^j + A_{ij} \delta_A \dot{q}^j) = \\ &= \frac{\partial L_j}{\partial q^i} \delta_A q^j - \frac{d}{dt} (R_{ij} \delta_A q^j + A_{ij} \delta_A \dot{q}^j) = \\ &= \frac{\partial}{\partial q^i} (\delta_A q^j L_j + G_A) - \frac{\partial \delta_A q^j}{\partial q^i} L_j - \frac{d}{dt} \left(\frac{\partial G_A}{\partial q^i} + R_{ij} \delta_A q^j + A_{ij} \delta_A \dot{q}^j \right) \end{aligned} \quad (5.41)$$

where in the last line we have added and subtracted $\partial G_A / \partial q^i$ and used $[(\partial / \partial q^i), (d/dt)] = 0$.

By using the Noether identities (5.20) we get

$$\begin{aligned} \delta_A L_i &= J_i(\delta_A q^k) \equiv \varepsilon^A(t) \frac{\partial D_A}{\partial q^i} - \frac{\partial \delta_A q^j}{\partial q^i} L_j - \\ &- \frac{d}{dt} \left(\frac{\partial G_A}{\partial q^i} + R_{ij} \delta_A q^j + A_{ij} \delta_A \dot{q}^j \right) \end{aligned} \quad (5.42)$$

When $A_{ij} \neq 0$, by using Eqs. (5.15) $\delta_A \dot{q}^i$ may be written as

$$\begin{aligned} \delta_A \dot{q}^i &= \left(\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k} + \ddot{q}^k \frac{\partial}{\partial \dot{q}^k} \right) \delta_A q^i = \\ &= \Gamma(\delta_A q^i) - \frac{\partial \delta_A q^i}{\partial \dot{q}^k} B^{kh} L_h \end{aligned} \quad (5.43)$$

where Γ is the evolution vector field associated with the second-order Euler-Lagrange equations (5.15)

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k} + \Lambda^k \frac{\partial}{\partial \dot{q}^k} + \sum_A \alpha^A(t) Z_A \quad (5.44)$$

with Z_A given by Eqs. (5.26). Equations (5.42) become

$$\begin{aligned}
\delta_A L_i &= J_i(\delta_A q^k) = \varepsilon^A(t) \frac{\partial D_A}{\partial q^i} - \frac{\partial \delta_A q^j}{\partial q^i} L_j - \\
&\quad - \frac{d}{dt} \left(\frac{\partial G_A}{\partial q^i} + R_{ij} \delta_A q^j + A_{ij} P(\delta_A q^j) - A_{ij} \frac{\partial \delta_A q^j}{\partial \dot{q}^k} B^{kh} L_h \right) \stackrel{0}{=} \\
&\stackrel{0}{=} \varepsilon^A(t) \frac{\partial D_A}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial G_A}{\partial q^i} + R_{ij} \delta_A q^j + A_{ij} P(\delta_A q^j) \right)
\end{aligned} \tag{5.45}$$

Since a gauge transformation, restricted by $L_i \stackrel{0}{=} 0$, is a mapping between neighbouring extremals, it must satisfy $J_i(\delta_A q^k) \stackrel{0}{=} 0$. This implies that for a gauge transformation $\delta_\alpha q^i$, $\delta_\beta q^i$ the following equations must be satisfied (remember that $D_\alpha = D_\beta \equiv 0$):

$$\begin{cases} \frac{\partial D_A}{\partial q^i} = 0 \\ \frac{\partial G_A}{\partial q^i} + R_{ij} \delta_A q^j + A_{ij} P(\delta_A q^j) = 0 \end{cases} \tag{5.46}$$

Either both or only one of them are not satisfied by $\delta_r q^i$, $\delta_s q^i$, D_r , D_s .

Finally, let us notice that

$$\begin{aligned}
\delta_A \frac{\partial L}{\partial \dot{q}^i} &= \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \delta_A q^j + A_{ij} \delta_A \dot{q}^j \stackrel{0}{=} \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \delta_A q^j + A_{ij} P(\delta_A q^j) = \\
&= \frac{\partial G_A}{\partial q^i} + R_{ij} \delta_A q^j + A_{ij} P(\delta_A q^j) - \frac{\partial G_A}{\partial q^i} + \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} \delta_A q^j = \\
&= - \frac{\partial \bar{G}_A(q, P(q, \dot{q}))}{\partial q^i} + \frac{\partial G_A}{\partial q^i} + R_{ij} \delta_A q^j + A_{ij} P(\delta_A q^j) = \\
&= \{P_i, \bar{G}_A\}|_{P=\partial L/\partial \dot{q}} + \frac{\partial G_A}{\partial q^i} + R_{ij} \delta_A q^j + A_{ij} P(\delta_A q^j) = \\
&= \overline{\delta_A P_i}|_{P=\partial L/\partial \dot{q}} + \frac{\partial G_A}{\partial q^i} + R_{ij} \delta_A q^j + A_{ij} P(\delta_A q^j)
\end{aligned} \tag{5.47}$$

where we have used Eqs. (5.36). Therefore, $\delta_A \partial L / \partial \dot{q}^i = \delta_A p_i|_{P=\partial L/\partial \dot{q}}$ for the gauge transformations and also for those transformations $\delta_r q^i$, $\delta_s q^i$ for which the second set of Eqs. (5.46) are satisfied, but not the first set. In both these cases for a projectable function $f(q, \dot{q}) = \bar{f}(q, p)$ we get $\delta_A \bar{f} \stackrel{0}{=} \{\bar{f}, \bar{G}_A\}$

and, when $L_i \stackrel{0}{=} 0$, the vector field $Y_A = \delta_A q^i \partial / \partial q^i + \delta_A \dot{q}^i (\partial / \partial \dot{q}^i)$ is sent by the Legendre transformation into the vector field $\bar{X}_A = \{\cdot, \bar{G}_A\}$ on T^*Q :

$$\begin{aligned} Y_A &\rightarrow \frac{\partial \bar{G}_A}{\partial p_i} \frac{\partial}{\partial q^i} \Big|_p + \left(\frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} \frac{\partial \bar{G}_A}{\partial p_i} + A_{ji} \delta_A \dot{q}^i \right) \frac{\partial}{\partial p_j} \stackrel{0}{=} \\ &\stackrel{0}{=} \{\cdot, \bar{G}_A\} + \left(\frac{\partial \bar{G}_A}{\partial q^i} + R_{ji} \delta_A q^i + A_{ji} P(\delta_A q^i) \right) \frac{\partial}{\partial p_j} \end{aligned} \quad (5.48)$$

When the second set of Eqs. (5.46) does not hold (as may happen for Y_r, Y_s), the corresponding vector fields are not projectable [see also Eq. (5.47)], but $\delta_A q^i = \bar{\delta}_A q^i = \{q^i, \bar{G}_A\}$.

6. MORE ON THE GAUGE TRANSFORMATIONS

In the Lagrangian approach we have seen that L is weakly quasi-invariant under the local Noether transformations (5.16):

$$\delta_A q^i = \sum_j \varepsilon^{A(j)}(t) \xi_{j, T_A - j}^i(q, \dot{q}) \quad J_\alpha = J_\tau = 0, \quad J_\beta = J_s = 1 \quad (6.1)$$

When some of the $\xi_{j, T_A - j}^i$ depend on the $\dot{q}^i$, as in the models with reparametrization and/or supersymmetry invariance, $\delta_A \dot{q}^i = (d/dt)\delta_A q^i$ depend on $\dot{q}^i$ and so on. When $\delta_A q^i$ is velocity-dependent, so that it is not a transformation of the configuration space Q , the theory should be defined in the infinite jet bundle [69] over the extended configuration space $R \times Q$. This space has local coordinates $\{t, q^i, \overset{(1)}{q}^i, \overset{(2)}{q}^i, \dots\}$ and each of its points can be identified with the equivalence class of all the curves $\{t, c^i(t)\}$ in $R \times Q$, which at the time t pass through the point $\{q^i\}$ of Q and have there a point of tangency of infinite order: $d^k c_1^i(t)/dt^k = d^k c_2^i(t)/dt^k$ for every k at the given t . This implies $(d\overset{(k)}{q}^i - \overset{(k+1)}{q}^i dt)_c = 0$, where the suffix c means restriction to the equivalence class $\{t, c^i(t)\}$ for every $t \in R$, i.e. to the infinite jet $j_t^\infty(c)$ associated with the curve c . Velocity-dependent transformations appear also in the study of the first Noether theorem [70], see for instance the Runge-Lenz vectorial constant of the motion in the Kepler problem, and in the study of the invariance transformations (dynamical symmetries) of differential equations such as the Euler-Lagrange equations. In this last case, as shown in Ref. [71], there are only two kinds of invariance transformations when the number of degrees of freedom is higher than one: i) the point transformations of $R \times Q$ [$\delta t = \delta t(t, q)$, $\delta_0 q^i = \delta_0 q^i(t, q)$, $\delta q^i = \delta_0 q^i + \dot{q}^i \delta t$] extended to the higher derivatives; ii) the Lie-Bäcklund transformations (or tangent transformations of infinite order preserving the tangency of infinite order of two curves). These latter have δt and/or δq^i depending upon $\overset{(1)}{q}^i$ and possibly upon $\overset{(k)}{q}^i$, $k > 1$. For instance, the non-point canonical transformations of T^*Q become Lie-Bäcklund transformations with $\delta t = 0$, when rephrased in the second-order formalism of the Euler-Lagrange equations. Also the local dynamical symmetries of these equations in the singular case, described before Eqs. (5.15), are Lie-Bäcklund transformations.

In the second-order Lagrangian formalism, with the Noether transformations (6.1) one associates the following vector fields:

$$\begin{aligned} Y_A &= \delta_A q^i \frac{\partial}{\partial q^i} + \delta_A \dot{q}^i \frac{\partial}{\partial \dot{q}^i} = \\ &= \delta_A q^i \frac{\partial}{\partial q^i} + \left[\Gamma(\delta_A q^i) - \frac{\partial \delta_A q^i}{\partial \dot{q}^k} B^{kh} L_h \right] \frac{\partial}{\partial \dot{q}^i} \end{aligned} \quad (6.2)$$

where Eqs. (5.43) and (5.44) have been used. Y_A should be thought of as a restriction of a vector field over the infinite jet bundle, when acting on functions only of $q^i, \dot{q}^i$ as the Lagrangian. Only when the genuine second-order Euler-Lagrange equations are satisfied does Y_A not depend upon the accelerations. However, as shown by Eqs. (5.44), Y_A carries the information of the local dynamical symmetries of the Euler-Lagrange equations, because it depends upon the arbitrary functions $a^\lambda(t)$, describing the undetermined accelerations connected with those Euler-Lagrange equations which are only first or zeroth order. Equations (5.44) also show that the existence of the local dynamical symmetries is associated with the vector fields Z_A of Eqs. (5.26), which belong to the kernel of the Legendre transformation to T^*Q .

To avoid the use of the infinite jet bundle it is convenient to pass to the first-order formalism of TQ [or $R \times TQ = J^1(R, Q)$, the first jet bundle]. Here the genuine second-order Euler-Lagrange equations are replaced by pairs of first-order differential equations, as shown in Section 2 for the regular case. But with an Euler-Lagrange equation which is already of first-order (or of

zeroth-order), we cannot associate such a pair. In this case both the first and the second half of Eqs. (2.18) will contain arbitrary functions: the first half will contain the analogue of the arbitrary velocity functions $g^\lambda(q, \dot{q})$ [or of the Dirac multipliers $\lambda^\lambda(t)$ of T^*Q]; the second half will contain the arbitrary functions $a^\lambda(t)$ and the vector fields $\tilde{Z}_A$ will play the role of generators of extra local invariances [the remnant in TQ of the local dynamical symmetries of the original Euler–Lagrange equations], whose effect will be fully displayed in $T^*(TQ)$.

Before describing the TQ approach, a final point has to be stressed about the second-order Lagrangian description in the singular case. As we have seen, the local Noether transformations (6.1) fall into two classes: the gauge transformations associated with first-class constraints and the others associated with second-class ones. While the former, when restricted to the extremals, become Jacobi fields satisfying Eqs. (5.45) and describe the arbitrariness of the solutions of the Euler–Lagrange equations, the latter have the associated arbitrariness removed at that stage, when their velocity functions $g(q, \dot{q})$ (or their Dirac multipliers) are determined, so that the corresponding degrees of freedom are uniquely determined by first- and/or zeroth-order equations of motion. The final state of affairs is that, given a set of Cauchy data compatible with the first- and zeroth-order equations, the genuine second-order Euler–Lagrange equations will determine their evolution modulo the arbitrary functions associated with the gauge transformations. To get well-defined extremals we have to fix ad hoc the arbitrary functions in such a way that given two neighbouring trajectories stemming from the same Cauchy data there must exist an infinitesimal gauge transformation connecting them. This means that the vector fields (6.2) associated with gauge transformations must form an involutive distribution, so that the Frobenius theorem [53] will ensure its local integrability. The local existence of the extremals will be assured if, furthermore, the commutators of such vector fields with the second-order vector field, describing the deterministic part of the evolution, close on the vector fields of the gauge transformations [1a]. Finally, to assure perturbative global integrability every multiple commutator of gauge transformations must give gauge transformations. Therefore, we must assume that our singular Lagrangian is such that all these conditions are satisfied, i.e. that it allows the existence of the so-called gauge algebra [58].

To make these statements more precise let us introduce the vector fields associated with the commutator of two Noether transformations (6.1)

$$\left\{ \begin{aligned} Y_{[A,B]} &= \delta_{[A,B]} q^i \frac{\partial}{\partial q^i} + \frac{d\delta_{[A,B]} q^i}{dt} \frac{\partial}{\partial \dot{q}^i} \\ \delta_{[A,B]} q^i &= Y_A(\delta_B q^i) - Y_B(\delta_A q^i) = \\ &= \delta_A q^\kappa \frac{\partial \delta_B q^i}{\partial q^\kappa} - \delta_B q^\kappa \frac{\partial \delta_A q^i}{\partial q^\kappa} + \delta_A \dot{q}^\kappa \frac{\partial \delta_B q^i}{\partial \dot{q}^\kappa} - \delta_B \dot{q}^\kappa \frac{\partial \delta_A q^i}{\partial \dot{q}^\kappa} \end{aligned} \right. \quad (6.3)$$

Equations (5.17) and (5.18) stating the weak quasi-invariance of L under $\delta_A q^i$ become

$$\delta_A L = Y_A L = \mathcal{L}_{Y_A} L \equiv \frac{dF_A}{dt} + \varepsilon^A D_A \quad (6.4)$$

where $\mathcal{L}_Y$ is the Lie derivative with respect to the vector field Y . As the second member of Eqs. (6.4) may depend on $\dot{q}^i$, to evaluate the action of $Y_{[A,B]}$ on L we must, for consistency, consider Y_A and $Y_{[A,B]}$ as Lie–Bäcklund vector fields, i.e.

$$Y_A = \sum_K^\infty \frac{d^K \delta_A q^i}{dt^K} \frac{\partial}{\partial \dot{q}^i} \Rightarrow [Y_A, \frac{d}{dt}] = 0 \quad (6.5)$$

and analogously for $Y_{[A,B]}$. Then we get

$$\begin{aligned} \delta_{[A,B]} L &= \mathcal{L}_{Y_{[A,B]}} L = \delta_{[A,B]} q^i L_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta_{[A,B]} q^i \right) = \\ &= \frac{\partial L}{\partial q^i} [Y_A (\delta_B q^i) - Y_B (\delta_A q^i)] + \frac{\partial L}{\partial \dot{q}^i} \frac{d}{dt} [Y_A (\delta_B q^i) - Y_B (\delta_A q^i)] = \\ &= Y_A (\delta_B L) - Y_B (\delta_A L) - \\ &\quad - [Y_A \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta_B \dot{q}^i - Y_B \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta_A \dot{q}^i + Y_A \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta_B \ddot{q}^i - Y_B \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta_A \ddot{q}^i] = \\ &\equiv Y_A \left(\frac{dF_B}{dt} + \varepsilon^B D_B \right) - Y_B \left(\frac{dF_A}{dt} + \varepsilon^A D_A \right) + \frac{d}{dt} [Y_B \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta_A q^i - Y_A \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta_B q^i] + \\ &\quad + [Y_B \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{d}{dt} Y_B \left(\frac{\partial L}{\partial \dot{q}^i} \right)] \delta_A q^i - [Y_A \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{d}{dt} Y_A \left(\frac{\partial L}{\partial \dot{q}^i} \right)] \delta_B q^i = \quad (6.6) \\ &= \frac{d}{dt} [Y_A (F_B) - Y_B (F_A) + Y_B \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta_A q^i - Y_A \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta_B q^i] + \\ &\quad + \varepsilon^B Y_A (D_B) - \varepsilon^A Y_B (D_A) + Y_B (L_i) \delta_A \dot{q}^i - Y_A (L_i) \delta_B \dot{q}^i = \\ &\Rightarrow \frac{d}{dt} F_{[A,B]} + \varepsilon^B Y_A (D_B) - \varepsilon^A Y_B (D_A) + \\ &\quad + J_i (\delta_B q^i) \delta_A \dot{q}^i - J_i (\delta_A q^i) \delta_B \dot{q}^i \end{aligned}$$

where we have used Eqs. (6.4), (6.5), (5.40), and we have defined

$$\begin{aligned} F_{[A,B]} &= Y_A (F_B) - Y_B (F_A) + Y_B \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta_A q^i - Y_A \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta_B q^i = \\ &= Y_B (G_A) - Y_A (G_B) + \delta_{[A,B]} q^i \frac{\partial L}{\partial \dot{q}^i} \end{aligned} \quad (6.7)$$

Equations (6.6) show that $Y_{[A,B]}$ generates a generalized weak quasi-invariance only if $\delta_A q^i, \delta_B q^i$ are gauge transformations: $D_A = D_B = 0, J_i(\delta_A q) \stackrel{0}{=} J_i(\delta_B q) \stackrel{0}{=} 0$. Then by defining $G_{[A,B]} = Y_B(G_A) - Y_A(G_B)$, the Noether identities implied by Eqs. (6.6) are

$$\frac{d}{dt} G_{[A,B]} \equiv -\delta_{[A,B]} q^i L_i + \delta_A q^i J_i(\delta_B q^K) - \delta_B q^i J_i(\delta_A q^K) \quad (6.8)$$

As noted at the end of the previous section, the vector fields of the gauge transformations are projectable to T^*Q when $L_i \stackrel{0}{=} 0$ and this allows one to check that the phase-space generator of $\delta_{[A,B]} q^i$ is $\{\bar{G}_A, \bar{G}_B\}$.

To check definitely that $\delta_{[A,B]} q^i$ is a gauge transformation, let us study the variation of L under $\delta_{[D,[A,B]]} q^i$ with $\delta_D q^i$ being a gauge transformation:

$$\begin{aligned} \delta_{[D,[A,B]]} L &= \delta_{[D,[A,B]]} q^i L_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta_{[D,[A,B]]} q^i \right) \equiv \\ &\equiv Y_D \left(\frac{dF_{[A,B]}}{dt} + J_i(\delta_B q^K) \delta_A q^i - J_i(\delta_A q^K) \delta_B q^i \right) + Y_{[A,B]} \left(\frac{dF_D}{dt} \right) + \\ &+ \frac{d}{dt} \left[Y_{[A,B]} \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta_D q^i - Y_D \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta_{[A,B]} q^i \right] + \\ &+ J_i(\delta_{[A,B]} q^K) \delta_D q^i - J_i(\delta_D q^K) \delta_{[A,B]} q^i = \\ &= \frac{d}{dt} \left[Y_{[A,B]}(G_D) - Y_D(Y_A(G_B) - Y_B(G_A)) + \frac{\partial L}{\partial \dot{q}^i} \delta_{[D,[A,B]]} q^i \right] + \\ &+ J_i(\delta_{[A,B]} q^K) \delta_D q^i - J_i(\delta_D q^K) \delta_{[A,B]} q^i + Y_D [J_i(\delta_B q^K) \delta_A q^i - J_i(\delta_A q^K) \delta_B q^i] \end{aligned} \quad (6.9)$$

As $\delta_A q^i, \delta_B q^i, \delta_D q^i$ are gauge transformations, for which Eqs. (5.46) hold, Eqs. (5.45) imply

$$\begin{aligned} Y_D [J_i(\delta_A q^K)] &= Y_D \left[-\frac{\partial \delta_A q^j}{\partial q^i} L_j + \frac{d}{dt} \left(A_{ij} \frac{\partial \delta_A q^j}{\partial \dot{q}^k} B^{kh} L_h \right) \right] = \\ &= -Y_D \left(\frac{\partial \delta_A q^j}{\partial q^i} \right) L_j - \frac{\partial \delta_A q^j}{\partial q^i} J_i(\delta_D q^K) + \\ &+ \frac{d}{dt} \left[Y_D (A_{ij} \frac{\partial \delta_A q^j}{\partial \dot{q}^k} B^{kh}) L_h + A_{ij} \frac{\partial \delta_A q^j}{\partial \dot{q}^k} B^{kh} J_h(\delta_D q^K) \right] \stackrel{0}{=} 0 \end{aligned} \quad (6.10)$$

Therefore $\delta_{[D,[A,B]]} q^i$ will generate a generalized weak quasi-invariance only if

$$J_i(\delta_{[A,B]} q^K) \stackrel{0}{=} 0 \quad (6.11)$$

that is if $\delta_{[A,B]} q^i$ is a Jacobi field. From Eqs. (6.3) the gauge transformation $\delta_{[A,B]} q^i$ will have the following expression (see also Ref. [25]):

$$\begin{aligned}
\delta_{[A,B]} q^i &= [Y_A, Y_B] q^i = \\
&= \sum_j^{J_A+1} \sum_k^{J_B+1} \varepsilon^{A(j)} \varepsilon^{B(k)} \left[A_{J_A-j}^h \frac{\partial \xi_{J_B-k}^i}{\partial q^h} - B_{J_B-k}^h \frac{\partial \xi_{J_A-j}^i}{\partial q^h} + \right. \\
&\quad \left. + (A_{J_A-j+1}^h + A_{J_A-j}^h) \frac{\partial \xi_{J_B-k}^i}{\partial \dot{q}^h} - (B_{J_B-k+1}^h + B_{J_B-k}^h) \frac{\partial \xi_{J_A-j}^i}{\partial \dot{q}^h} \right] = \\
&= \sum_j^{J_A+1} \sum_k^{J_B+1} \varepsilon^{A(j)} \varepsilon^{B(k)} \left[\sum_c \sum_h^{J_c} \mathcal{C}_{(A|J_A-j)(B|J_B-k)}^{(c|J_c-h)}(q, \dot{q}) \xi_{J_c-h}^i + \right. \\
&\quad \left. + \mathcal{D}_{(A|J_A-j)(B|J_B-k)}^{i i'}(q, \dot{q}) L_{i'} \right] = \\
&= \sum_c \sum_h^{J_c} \mathcal{S}_h(t|q, \dot{q}) \xi_{J_c-h}^i(q, \dot{q}) + \mathcal{P}_{AB}^{i i'}(t|q, \dot{q}) L_{i'}
\end{aligned} \tag{6.12}$$

where c spans all the gauge transformations, $A_{J_A+1}^i = B_{J_B+1}^i = 0$, $\mathcal{C}$ and $\mathcal{D}$ vanish if one of their lower indices is equal to -1 , and with the ϱ 's defined as follows:

$$\left\{ \begin{aligned} \mathcal{S}_h(t|q, \dot{q}) &= \sum_j^{J_A+1} \sum_k^{J_B+1} \varepsilon^{A(j)}(t) \varepsilon^{B(k)}(t) \mathcal{C}_{(A|J_A-j)(B|J_B-k)}^{(c|J_c-h)}(q, \dot{q}) \\ \mathcal{P}_{AB}^{i i'}(t|q, \dot{q}) &= \sum_j^{J_A+1} \sum_k^{J_B+1} \varepsilon^{A(j)}(t) \varepsilon^{B(k)}(t) \mathcal{D}_{(A|J_A-j)(B|J_B-k)}^{i i'}(q, \dot{q}) \end{aligned} \right. \tag{6.13}$$

By using Eqs. (5.43), (5.44), the functions $\mathcal{C}$ and $\mathcal{D}$ are defined by the following equations:

$$\left\{ \begin{aligned} \sum_c \sum_h^{J_c} \mathcal{C}_{(A|J_A-j)(B|J_B-k)}^{(c|J_c-h)} \xi_{J_c-h}^i &= A_{J_A-j}^h \frac{\partial \xi_{J_B-k}^i}{\partial q^h} - B_{J_B-k}^h \frac{\partial \xi_{J_A-j}^i}{\partial q^h} + \\ &\quad + (A_{J_A-j+1}^h + A_{J_A-j}^h) \frac{\partial \xi_{J_B-k}^i}{\partial \dot{q}^h} - (B_{J_B-k+1}^h + B_{J_B-k}^h) \frac{\partial \xi_{J_A-j}^i}{\partial \dot{q}^h} \\ \mathcal{D}_{(A|J_A-j)(B|J_B-k)}^{i i'} &= \left(\frac{\partial B_{J_B-k}^h}{\partial \dot{q}^r} \frac{\partial \xi_{J_A-j}^i}{\partial \dot{q}^h} - \frac{\partial A_{J_A-j}^h}{\partial \dot{q}^r} \frac{\partial \xi_{J_B-k}^i}{\partial \dot{q}^h} \right) B^{rs} \end{aligned} \right. \tag{6.14}$$

Therefore the gauge transformations satisfy what is called an open algebra [59], which closes only by using the genuine second-order Euler-Lagrange equations. This algebra is not open only when the gauge transformations are velocity-independent: the functions $\mathcal{C}$ depend upon the arbitrary functions $a^\Lambda(t)$ owing to the presence of the vector fields Z_Λ in Γ . However, this dependence disappears when we project to T^*Q . If $\hat{\mathcal{C}}$ is the part of $\mathcal{C}$ independent of the $a^\Lambda(t)$'s, then $\hat{\mathcal{C}}$ is projectable to T^*Q , $\hat{\mathcal{C}}(q, \dot{q}) = \bar{\mathcal{C}}(q, p)$, and the phase-space counterpart of the first set of Eqs. (6.14) is

$$\{ \bar{G}_{A-j}, \bar{G}_{B-k} \} = \sum_c \sum_h \bar{\mathcal{C}}_{(A|j_A-j)(B|j_B-k)}^{(c|h)} \bar{G}_{c-h} \quad (6.15)$$

where the $\bar{G}_c$'s are all the first-class constraints of the model. When the $\bar{\mathcal{C}}$ are constants we can speak of a Lie algebra of the gauge transformations. By studying the higher commutators of the gauge transformations one would find the Lagrangian analogue of the higher structure functions introduced in Refs. [24, 26] as a preliminary to the phase-space BRS approach.

The check that the commutator of the gauge transformations with the deterministic part of the evolution operator [the $\bar{H}_d$ of Eqs. (5.8) in T^*Q] is again a gauge transformation, is implicitly contained in the Noether identities (5.21a) (5.21b) [or (5.39) in T^*Q].

Therefore, if L is such that Eqs. (6.14) hold, the integrability conditions for the local existence of the extremals implied by the Frobenius theorem are satisfied. In T^*Q all the previous conditions are called the Dirac test, whose non-triviality is connected with the assumption that all the secondary, tertiary, etc., first-class constraints are generators of gauge transformations. This hypothesis may be rephrased by saying that L must be such that the $\bar{H}_c$ of Eq. (5.8) contains all such constraints.

7. SINGULAR SYSTEMS: THE TQ APPROACH

As we said in the previous section, the TQ approach allows the avoidance of the infinite jet bundle retaining all the information of the Lagrangian approach, as well as the local dynamical symmetries of the Euler-Lagrange equation, which are lost in T^*Q owing to Eq. (5.26). We shall follow the scheme of Section 2. See also Refs. [35, 40, 43].

Let us go back to $\tilde{L}_v$ of Eqs. (2.13) and to the variational principle of Eqs. (2.15). Since the inverse of the Hessian matrix $\tilde{A}$ does not exist, the first-order differential equations (2.17) implied by the variational principle:

$$\begin{cases} \tilde{L}_{q^i} = -(\tilde{A}_{ij} \dot{v}^j + \tilde{R}_{ij}(\dot{q}^j - v^j) - \tilde{\alpha}_i) \stackrel{0}{=} 0 \\ \tilde{L}_{v^i} = \tilde{A}_{ij}(\dot{q}^j - v^j) \stackrel{0}{=} 0 \end{cases} \quad (7.1)$$

cannot be put in the normal form of Eqs. (2.18).

For the sake of simplicity we shall restrict the analysis to the class of models defined in Section 5. By using the null eigenvectors ${}_A \tilde{\xi}_0^i$, $A = \alpha, \beta, r, s$, of $\tilde{A}$ we get from Eqs. (7.1):

$$\begin{cases} {}_A \tilde{\xi}_0^i \tilde{L}_{v^i} = 0 \\ {}_A \tilde{\xi}_0^i \tilde{L}_{q^i} = -[{}_A \tilde{\xi}_0^i \tilde{R}_{ij}(\dot{q}^j - v^j) - \tilde{\chi}_A] \stackrel{0}{=} 0 \end{cases} \quad (7.2)$$

where $\tilde{\chi}_A$ was defined in Eqs. (5.15).

The solution of the second set of Eqs. (7.1) is

$$\begin{cases} \dot{q}^i \stackrel{0}{=} v^i + \sum_A \tilde{K}^A(t, q, v) {}_A \tilde{\xi}_0^i = \sum_e (\tilde{x}_{eK} v^K) \tilde{x}_e^i + \sum_A \tilde{\lambda}^A(t) {}_A \tilde{\xi}_0^i \\ \tilde{K}^A(t, q, v) = \tilde{\lambda}^A(t) - {}_A \tilde{\xi}_{0K} v^K \end{cases} \quad (7.3)$$

where $\tilde{\lambda}^A(t)$ are the analogues of the Dirac multipliers of T^*Q , as ${}_A \tilde{\xi}_0^i(q, v) = {}_A \tilde{\xi}_0^i(q, p) = \{q^i, \tilde{\Phi}_A\}$ and as $\tilde{L}_{v^i}$ is the TQ analogue of the Hamilton equations $\dot{q}^i \stackrel{0}{=} \{q^i, \tilde{H}_D\} = \{q^i, \tilde{H}_c\} + \sum_A \tilde{\lambda}^A(t) \{q^i, \tilde{\Phi}_A\}$. By remembering the definition of the matrix $\tilde{\mathcal{R}}_{AB}$ in Eq. (5.12), the second set of Eqs. (7.2) may be rewritten as

$$\sum_B \tilde{\mathcal{R}}_{AB} (\tilde{\lambda}^B - {}_B \tilde{\xi}_{0K} v^K) = \tilde{\chi}_A - {}_A \tilde{\xi}_0^i \tilde{L}_{q^i} \stackrel{0}{=} \tilde{\chi}_A \quad (7.4)$$

For the class of models of Section 5 the $2k_3$ -rank matrix $\tilde{\mathcal{R}}$ has only the minor $(\tilde{\mathcal{R}}_{rr})$ non vanishing. Therefore, Eqs. (7.4) become

$$\left\{ \begin{array}{l} \tilde{\chi}_\alpha = \tilde{\xi}_0^i \tilde{L}_{qi} = 0 \\ \tilde{\chi}_\beta = \tilde{\Phi}'_\beta(q, v) = \tilde{\xi}_0^i \tilde{L}_{qi} \stackrel{0}{=} 0 \\ \tilde{\chi}_s = \tilde{\Phi}'_s(q, v) = \tilde{\xi}_0^i \tilde{L}_{qi} \stackrel{0}{=} 0 \\ \tilde{\chi}_r = \tilde{D}_r = \sum_{r'} \tilde{\Omega}_{rr'} (\tilde{\lambda}^{r'} - \tilde{\xi}_{0K}^{r'} v^K) + \tilde{\xi}_0^i \tilde{L}_{qi} \stackrel{0}{=} \sum_{r'} \tilde{\Omega}_{rr'} (\tilde{\lambda}^{r'} - \tilde{\xi}_{0K}^{r'} v^K) \end{array} \right. \quad (7.5)$$

where we used the fact that $\chi_\alpha \equiv 0$, see Eqs. (5.22). We see that $\tilde{\chi}_\beta \stackrel{0}{=} 0, \tilde{\chi}_s \stackrel{0}{=} 0$ are the TQ expression of the secondary Lagrangian constraints χ_β, χ_s and that there are no TQ primary constraints since $\tilde{\Phi}_A(q, v) = \Phi_A(q, p(q, \dot{q})) \equiv 0$. The last line in Eqs. (7.5) determines the Dirac multipliers $\tilde{\lambda}^r$

$$\tilde{\lambda}^r \stackrel{0}{=} \tilde{\xi}_{0K}^{r'} v^K + \sum_{r'} (\tilde{\Omega}^{-1})_{rr'} \tilde{\chi}_{r'} \quad (7.6)$$

so that Eqs. (7.3) become

$$\begin{aligned} \dot{q}^i &\stackrel{0}{=} v^i + \sum_{r,r'} (\tilde{\Omega}^{-1})_{rr'} \tilde{\chi}_{r'} \tilde{\xi}_0^i + \sum_{A \neq r} \tilde{K}^A \tilde{\xi}_0^i = \\ &= \sum_e (\tilde{\chi}_{eK} v^K) \tilde{\xi}_e^i + \sum_r \left[\tilde{\xi}_{0K}^{r'} v^K + \sum_{r'} (\tilde{\Omega}^{-1})_{rr'} \tilde{\chi}_{r'} \right] \tilde{\xi}_0^i + \\ &\quad + \sum_{A \neq r} \tilde{\lambda}^A \tilde{\xi}_0^i \end{aligned} \quad (7.7)$$

Before going on with the discussion, let us remark that Eqs. (7.3) do not imply $\dot{q}^i \stackrel{0}{=} v^i$ even if some $\tilde{\lambda}$'s are determined like in Eqs. (7.6). At the end of the discussion this will imply that the evolution vector field $\tilde{\Gamma}_F$, analogue to the $\bar{\Gamma}_F$ of Section 4, on the final submanifold $\tilde{\gamma}_F \subset TQ$ defined by the Lagrangian constraints like $\tilde{\chi}_\beta, \tilde{\chi}_s$, is not a second-order vector field unless the arbitrary functions $\tilde{\lambda}^A(t)$ are not chosen such that $\tilde{K}^A = \tilde{\lambda}^A - \tilde{\xi}_{0K}^A v^K = 0$. This restriction will be necessary to recover the Euler-Lagrange equations $\ddot{q}^i \stackrel{0}{=} \Lambda^i(q, \dot{q}) + \sum_A a^A(t) \tilde{\xi}_0^i(q, \dot{q})$ of Eqs. (5.15) from the TQ approach. As Eqs. (7.1) will leave $\tilde{\lambda}^\alpha, \tilde{\lambda}^\beta$ arbitrary, the restriction $\tilde{K}^\alpha = \tilde{K}^\beta = 0$ will become a restriction on the gauge-fixing constraints for the first-class constraints of $T^*(TQ)$ (see next Section). Instead $\tilde{\lambda}^r$ is determined by Eqs. (7.6), in analogy with $\lambda^r = 0$ in T^*Q , and the condition $\tilde{K}^r = 0$ implies $\tilde{\chi}_r(q, v) = \tilde{D}_r(q, v) = 0$ as $\tilde{\Omega}_{rr'} \neq 0$. It turns out that the Lagrangian $\tilde{L}_v$ on TQ does not describe the genuine first-order equations of motion $D_A(q, \dot{q}) \stackrel{0}{=} 0$, $A = r, s$ associated to the various families of second class constraints. One has to add $\tilde{L}_v$ the holonomic (in TQ) constraints $\tilde{D}_A(q, v) = 0$, $A = r, s$, to recover the original Euler-Lagrange [Eqs. (5.15)] from the TQ approach, when $\tilde{K}^\alpha = \tilde{K}^\beta = 0$. Therefore, for singular systems with genuine first-order equations of motion, i.e. with second-class constraints, the right Lagrangian on TQ is $\tilde{L}'_v = \tilde{L}_v + q_1^r \tilde{D}_r + q_2^s \tilde{D}_s$, where $q_1^r(t), q_2^s(t)$ are Lagrange multipliers. This is consistent, because $D_A(q, \dot{q}) \stackrel{0}{=} 0$, $A = r, s$, are already first-order

equations, while the transition from the Lagrangian to the TQ formalism amounts to replace every second-order equation with a pair of first-order equations. While in T^*Q $D_A(q, \dot{q}) \stackrel{0}{=} 0$, $A = r, s$, is a consequence of the Hamilton equations $\dot{q}^i \stackrel{0}{=} \{q^i, \bar{H}_D\}$, in TQ only equations like (7.6) are obtained in the singular case, and the evolution vector field $\tilde{\Gamma}_F$ is of second-order only with the restrictions $\tilde{K}^A = 0$, $\tilde{D}_r = \tilde{D}_s = 0$. While for first-class constraints ($D \equiv 0$) $\tilde{K}^\alpha = \tilde{K}^\beta = 0$ is only a restriction on the arbitrary functions $\tilde{\lambda}^\alpha, \tilde{\lambda}^\beta$, for second-class constraints we must restrict $\tilde{\gamma}_F$ to a submanifold $\tilde{\gamma}'_F \subset \tilde{\gamma}_F$, defined by $\tilde{D}_r = \tilde{D}_s = 0$. While a theorem of the second paper in Ref. [40] assured the existence of a second-order vector field on some submanifold of TQ, we have now obtained a precise specification of this submanifold, but we have also shown that the extra conditions $\tilde{K}^A = 0$ are needed to reproduce Eqs. (5.15).

Coming back to our discussion, we get from Eq. (7.7)

$$\tilde{\xi}_{0i}^i (\dot{q}^i - v^i) \stackrel{0}{=} \sum_r (\tilde{\theta}^{-1})_{rr} \tilde{D}_{r1} = 0 \quad \text{WHEN} \quad \tilde{D}_r = 0 \quad (7.8)$$

By using the quasi-inverse $\tilde{B}$ of $\tilde{A}$, see Eqs. (5.14) and (7.7), the solution of the first set of Eqs. (7.1) may be written in the following form

$$\begin{aligned} \dot{v}^i &\stackrel{0}{=} \tilde{B}^{ij} [\tilde{\alpha}_j - \tilde{R}_{jk} (\dot{q}^k - v^k)] + \sum_A a^A(t) \tilde{\xi}_{A0}^i \stackrel{0}{=} \\ &\stackrel{0}{=} \tilde{\Lambda}^i - \tilde{B}^{ij} \tilde{R}_{jk} \left[\sum_{r1} \tilde{\xi}_{r0}^k (\tilde{\theta}^{-1})_{rr} \tilde{D}_{r1} + \sum_{A \neq r} \tilde{K}^A \tilde{\xi}_{A0}^k \right] + \sum_A a^A \tilde{\xi}_{A0}^i \end{aligned} \quad (7.9)$$

where $\tilde{\Lambda}^i = \tilde{B}^{ij} \tilde{\alpha}_j$. The arbitrary functions $a^A(t)$, see Eqs. (5.15), are the TQ version of the local dynamical symmetries of the Euler-Lagrange equations.

To have a consistent evolution, the Lagrangian constraints $\tilde{\chi}_\beta, \tilde{\chi}_s$ must be preserved in time:

$$\frac{d \tilde{\chi}_\beta}{dt} \stackrel{0}{=} 0 \quad \frac{d \tilde{\chi}_s}{dt} \stackrel{0}{=} 0 \quad (7.10)$$

These equations will determine tertiary Lagrangian constraints and/or $\tilde{\lambda}'$'s. For the class of models of Section 5 $\tilde{\chi}_\beta \stackrel{0}{=} 0$ is void, while $\tilde{\chi}_s \stackrel{0}{=} 0$ will determine $\tilde{\lambda}'^s$ (in analogy with $\lambda^s = 0$). According to the previous discussion $\tilde{\lambda}'^s$ will have the following form

$$\tilde{\lambda}'^s \stackrel{0}{=} \tilde{\xi}_{0k}^s v^k + \sum_{s'} \tilde{T}_{ss'} \tilde{D}_{s'} \quad \det(\tilde{T}_{ss'}) \neq 0 \quad (7.11)$$

so that $\tilde{K}^s = 0$, and $\tilde{\xi}_{0i}^i (\dot{q}^i - v^i) \stackrel{0}{=} 0$ will be obtained by imposing $\tilde{D}_s = 0$, i.e. the TQ version of the first-order genuine equations of motion $D_s(q, \dot{q}) = -\partial \chi_s / \partial q^i - \xi_{0i}^s \stackrel{0}{=} 0$ of Eqs. (5.23).

For this class of models, the TQ algorithm ends at this point: we have consistent evolution only on the submanifold $\tilde{\gamma}'_F \subset TQ$ determined by the Lagrangian constraints $\tilde{\chi}_\beta = \tilde{\chi}_s = 0$. If we consider

the contact form $\tilde{\omega}_L$ on $R \times TQ$ of Eq. (2.21), the solution of the equations $i_{\tilde{\gamma}} \tilde{\omega}_L = 0$, $i_{\tilde{\gamma}} dt = 1$ is no more Eq. (2.25). Instead, we get the following $\tilde{\Gamma}_F$ on $R \times \tilde{\gamma}_F$:

$$\tilde{\Gamma}_F = \tilde{\Gamma}_c + \sum_{A=\alpha, \beta} \tilde{\lambda}^A(t) \tilde{X}_A + \sum_A a^A(t) \tilde{Z}_A \quad (7.12)$$

where

$$\left\{ \begin{aligned} \tilde{\Gamma}_c &= \frac{\partial}{\partial t} + \left[\dot{q}^i - \sum_{A=\alpha, \beta} (\tilde{\xi}_{\alpha\beta}^{ik} v^k) \tilde{\xi}_{\alpha\beta}^i \right] \frac{\partial}{\partial q^i} + \\ &+ \left\{ \tilde{\lambda}^i - \tilde{B}^{ij} \tilde{R}_{jh} \left[\sum_{r,r'} \tilde{\xi}_{r,r'}^h (\tilde{Q}^{-1})_{rr'} \tilde{D}_{r'} + \sum_{s,s'} \tilde{\xi}_{s,s'}^h \tilde{T}_{ss'} \tilde{D}_{s'} - \right. \right. \\ &\quad \left. \left. - \sum_{A=\alpha, \beta} (\tilde{\xi}_{\alpha\beta}^{ik} v^k) \tilde{\xi}_{\alpha\beta}^h \right] \right\} \frac{\partial}{\partial v^i} \\ \tilde{X}_A &= \tilde{\xi}_{\alpha\beta}^i \frac{\partial}{\partial q^i} - \tilde{B}^{ij} \tilde{R}_{jk} \tilde{\xi}_{\alpha\beta}^k \frac{\partial}{\partial v^i} \quad A = \alpha, \beta \\ \tilde{Z}_A &= \tilde{\xi}_{\alpha\beta}^i \frac{\partial}{\partial v^i} \end{aligned} \right. \quad (7.13)$$

Since $\tilde{\lambda}^{\alpha, \beta}$ and a^A are arbitrary functions, we have on $R \times \tilde{\gamma}_F$: $i_{\tilde{\Gamma}_c} \tilde{\omega}_L = i_{\tilde{X}_A} \tilde{\omega}_L = i_{\tilde{Z}_A} \tilde{\omega}_L = 0$, i.e. all the vector fields (7.13) lie in $\ker \omega_L$, with the $\tilde{Z}_A$ describing its vertical part [46] (along the fibres of TQ). Owing to the assumption that all the first-class constraints are generators of gauge transformations, see Eqs. (4.6) and (5.8), it is possible to split $\tilde{\Gamma}_c$ in the following way:

$$\tilde{\Gamma}_c = \tilde{\Gamma}_d + \sum_{\beta} \tilde{C}^{\beta}(q, v) \tilde{X}'_{\beta} \quad (7.14)$$

with $i_{\tilde{X}'_{\beta}} \tilde{\omega}_L = 0$. Therefore on $R \times \tilde{\gamma}_F$, $\tilde{\omega}_L$ is a degenerate contact form, that is the ω_L of Eqs. (2.20) defines on $\tilde{\gamma}_F$ only a presymplectic structure, as $\ker \omega_L = \{\tilde{X}_{\alpha}, \tilde{X}_{\beta}, \tilde{X}'_{\beta}, \tilde{Z}_A\}$. The foliation with gauge orbits of $\tilde{\gamma}_F$ contains more information on the analogous case of $\tilde{\gamma} \subset T^*Q$, owing to the presence of the extra gauge vector fields $\tilde{Z}_A$, which belong to the kernel of the Legendre transformation, Eqs. (5.26). Instead we have $\tilde{X}_{\alpha} \rightarrow \bar{X}_{\alpha} = \{\cdot, \bar{\Phi}_{\alpha}\}$, $\tilde{X}_{\beta} \rightarrow \bar{X}_{\beta} = \{\cdot, \bar{\Phi}_{\beta}\}$, $\tilde{X}'_{\beta} \rightarrow \bar{X}'_{\beta} = \{\cdot, \bar{\Phi}'_{\beta}\}$.

As already said, a second-order vector field $\tilde{\Gamma}_F$ exists if we restrict $\tilde{\gamma}_F$ to the submanifold $\tilde{\gamma}'_F \subset \tilde{\gamma}_F$ defined by the equations $\tilde{D}_r = \tilde{D}_s = 0$ and the arbitrary functions $\tilde{\lambda}^{\alpha}$, $\tilde{\lambda}^{\beta}$ to ${}_{\alpha} \tilde{\xi}_{\alpha\beta}^{ik} v^k$, ${}_{\beta} \tilde{\xi}_{\alpha\beta}^{ik} v^k$ respectively. Then Eq. (7.12) becomes

$$\tilde{\Gamma}'_F = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \tilde{\Lambda}^i(q, v) \frac{\partial}{\partial v^i} + \sum_A a^A(t) \tilde{Z}_A \quad (7.15)$$

But $\tilde{\Gamma}'_F$ will be tangent to $R \times \tilde{\gamma}'_F$ only if $\tilde{D}_r = \tilde{\Gamma}'_F(\tilde{D}_r) \stackrel{0}{=} 0$, $\tilde{D}_s = \tilde{\Gamma}'_F(\tilde{D}_s) \stackrel{0}{=} 0$. These equations determine $a^r(t)$, $a^s(t)$ in the same way as $\dot{D}_r \stackrel{0}{=} 0$, $\dot{D}_s \stackrel{0}{=} 0$ determine them at the Lagrangian level. Now, $\tilde{\Gamma}'_F$ reproduce the genuine second-order Euler-Lagrange equations contained in Eqs. (5.15).

8. SINGULAR SYSTEMS: THE $T^*(TQ)$ APPROACH

Let us now consider the $T^*(TQ)$ description of the class of singular systems defined in Section 5. Already for regular systems this description implied second-class constraints and Eqs. (2.33), showing the weak quasi-invariance of $\tilde{L}_v$ under the Noether transformations generated by them, give the associated Noether identities following the scheme of Section 5. For our class of singular systems the new feature of $T^*(TQ)$ is that some of the constraints (2.26) become first class. Indeed Eqs. (2.29) give now the result $\det \tilde{C} = (\det \tilde{A})^2 = 0$.

As said in the previous section, we must use the modified Lagrangian $\tilde{L}'_v = \tilde{L}_v + q_1^r(t)\tilde{D}_r(q,v) + q_2^s(t)\tilde{D}_s(q,v)$ with the Lagrange multipliers $q_1^r(t), q_2^s(t)$ to implement the Lagrangian genuine first-order equations of motion in TQ as holonomic constraints. In $T^*(TQ \oplus R^{2k_3+k_4})$, besides the constraints (2.26) there are also the constraints $\tilde{\pi}_{1r} = \partial \tilde{L}'_v / \partial \dot{q}_1^r = 0, \tilde{\pi}_{2s} = \partial \tilde{L}'_v / \partial \dot{q}_2^s = 0$. The Hamiltonian (2.30) is replaced by $\tilde{H}'_D = \tilde{H}_D - q_1^r \tilde{D}_r - q_2^s \tilde{D}_s + \mu_1^r \tilde{\pi}_{1r} + \mu_2^s \tilde{\pi}_{2s}$, where μ_1^r, μ_2^s are new Dirac multipliers. Instead of Eqs. (2.31) we get

$$\left\{ \begin{array}{l} \dot{\tilde{\chi}}_i \equiv \{ \tilde{\chi}_i, \tilde{H}'_D \}' = - \left(\tilde{A}_{ij} \lambda_1^j + \tilde{R}_{ij} \lambda_1^j + \frac{\partial \tilde{E}}{\partial q_i^1} - q_1^r \frac{\partial \tilde{D}_r}{\partial q_i^1} - q_2^s \frac{\partial \tilde{D}_s}{\partial q_i^1} \right) \approx 0 \\ \dot{\tilde{\pi}}_i \equiv \{ \tilde{\pi}_i, \tilde{H}'_D \}' = \tilde{A}_{ij} (\lambda_1^j - v^j) - q_1^r \frac{\partial \tilde{D}_r}{\partial v^i} - q_2^s \frac{\partial \tilde{D}_s}{\partial v^i} \approx 0 \\ \dot{\tilde{\pi}}_{1r} \equiv \{ \tilde{\pi}_{1r}, \tilde{H}'_D \}' = \tilde{D}_r \approx 0 \\ \dot{\tilde{\pi}}_{2s} \equiv \{ \tilde{\pi}_{2s}, \tilde{H}'_D \}' = \tilde{D}_s \approx 0 \end{array} \right. \quad (8.1)$$

From $\tilde{\chi}_0^i \dot{\tilde{\pi}}_i = - (q_1^r \tilde{Z}_A \tilde{D}_r + q_2^s \tilde{Z}_A \tilde{D}_s) \approx 0$, remembering that the matrix $(\tilde{Z}_a \tilde{D}_b)$, $a, b = r, s$, is non singular (see Section 5), we get the secondary constraints $\tilde{q}_1^r \approx 0, \tilde{q}_2^s \approx 0$. As $\dot{\tilde{q}}_1^r = \dot{\tilde{q}}_2^s = 0$ imply $\mu_1^r = \mu_2^s = 0$, we can forget from now on the pairs of second-class constraints $\tilde{q}_1^r, \tilde{\pi}_{1r}, \tilde{q}_2^s, \tilde{\pi}_{2s}$ and $\tilde{H}'_D = \tilde{H}_D$ of Eq. (2.30). Then, as in Eqs. (7.3) the solutions of the equations $\dot{\tilde{\pi}}_i \approx 0$ are

$$\left\{ \begin{array}{l} \lambda_1^i \approx v^i + \sum_A \tilde{K}_1^A(t, q, v) \tilde{\xi}_0^i = \sum_e (\tilde{\chi}_{eK} v^K) \tilde{\chi}_e^i + \sum_A \tilde{\lambda}_1^A(t) \tilde{\xi}_0^i \\ \tilde{K}_1^A(t, q, v) = \tilde{\lambda}_1^A(t) - \tilde{\xi}_0^K v^K \end{array} \right. \quad (8.2)$$

The $\tilde{\lambda}_1^A$'s are restricted by the equations

$$\begin{aligned} \tilde{\xi}_0^i \dot{\tilde{\chi}}_i &\equiv - \tilde{\xi}_0^i \left(\tilde{R}_{ij} \lambda_1^j + \frac{\partial \tilde{E}}{\partial q_i^1} \right) \approx - \tilde{\xi}_0^i \left[\tilde{R}_{ij} (v^j + \sum_B \tilde{K}_1^B \tilde{\xi}_0^j) + \frac{\partial \tilde{E}}{\partial q_i^1} \right] = \\ &= - \left(\sum_B \tilde{\rho}_{AB} \tilde{K}_1^B - \tilde{\chi}_A \right) \approx 0 \end{aligned} \quad (8.3)$$

where we have used Eqs. (2.12), (2.14), (2.3) and (5.15). But these are Eqs. (7.4), whose solutions for our class of models are

$$\left\{ \begin{array}{l} \tilde{\lambda}_1^r = \tilde{\xi}_{0K}^r v^K + \sum_{r'} (\tilde{\alpha}^{-1})_{rr'} \tilde{D}_{r'} \approx \tilde{\xi}_{0K}^r v^K \\ \tilde{\chi}_\beta \approx 0 \\ \tilde{\chi}_s \approx 0 \\ \lambda_i^i \approx \sum_e (\tilde{\chi}_{eK} v^K) \tilde{\chi}_e^i + \sum_r (\tilde{\xi}_{0K} v^K) \tilde{\xi}_{r0}^i + \sum_{A \neq r} \tilde{\lambda}_1^A \tilde{\xi}_{A0}^i = v^i + \sum_{A \neq r} \tilde{K}_1^A \tilde{\xi}_{A0}^i \end{array} \right. \quad (8.4)$$

Therefore we recover the secondary constraints $\tilde{\chi}_\beta, \tilde{\chi}_s$. The other equations to be solved are

$$\dot{\tilde{\chi}}_i \approx - \left(\tilde{A}_{ij} \lambda_2^j + \tilde{R}_{ij} \sum_{A \neq r} \tilde{K}_1^A \tilde{\xi}_{A0}^j - \tilde{\alpha}_i \right) \approx 0 \quad (8.5)$$

In analogy with Eqs. (7.9), their solution is

$$\lambda_2^i \approx \tilde{\lambda}^i - \tilde{B}^{ij} \tilde{R}_{jk} \sum_{A \neq r} \tilde{K}_1^A \tilde{\xi}_{A0}^k + \sum_A \tilde{\lambda}_2^A(t) \tilde{\xi}_{A0}^i \quad (8.6)$$

After having substituted λ_1^i, λ_2^i given by Eqs. (8.4), (8.6) in $\tilde{H}_D$ of Eqs. (2.30), we must evaluate $\dot{\tilde{\chi}}_\beta \approx 0, \dot{\tilde{\chi}}_s \approx 0$. For our class of singular models $\dot{\tilde{\chi}}_\beta \approx 0$ are void, while $\dot{\tilde{\chi}}_s \approx 0$ have the solution $\tilde{\lambda}_1^s = 0$. Therefore, the final Dirac Hamiltonian in $T^*(TQ)$ is

$$\begin{aligned} \tilde{H}_D^F = & \tilde{E} + \left[v^i - \sum_{A \neq r} (\tilde{\xi}_{0K}^i v^K) \tilde{\xi}_{A0}^i \right] \tilde{\chi}_i + \\ & + \left[\tilde{\lambda}^i + \tilde{B}^{ij} \tilde{R}_{jk} \sum_{A \neq r} (\tilde{\xi}_{0K}^i v^K) \tilde{\xi}_{A0}^k \right] \tilde{\pi}_i + \\ & + \sum_{A \neq r} \tilde{\lambda}^A(t) \left(\tilde{\xi}_{A0}^i \tilde{\chi}_i - \tilde{B}^{ij} \tilde{R}_{jk} \tilde{\xi}_{A0}^k \tilde{\pi}_i \right) + \\ & + \sum_A a^A(t) \tilde{\xi}_{A0}^i \tilde{\pi}_i \end{aligned} \quad (8.7)$$

in agreement with Eqs. (7.12), (7.13) if we redefine $\tilde{\lambda}_1^A = \tilde{\lambda}^A, \tilde{\lambda}_2^A = a^A$. The assumption (4.6), (5.8) means here that $\tilde{E}$ must contain the $\tilde{\chi}_\beta$'s linearly.

Finally, we have to impose $\tilde{D}_r = \{\tilde{D}_r, \tilde{H}_D\}' \stackrel{0}{=} 0, \tilde{D}_s = \{\tilde{D}_s, \tilde{H}_D\}' \stackrel{0}{=} 0$ to get the determination of $a^r(t), a^s(t)$ as in Sections 5 and 7.

The primary and secondary constraints $\tilde{\chi}_i, \tilde{\pi}_i, \tilde{\chi}_\beta, \tilde{\chi}_s$ can therefore be decomposed into the following two groups (if for the sake of simplicity $\tilde{\mathcal{R}}_{t\alpha} = \tilde{\mathcal{R}}_{t\beta} = 0$):

$$\begin{array}{cc}
 \text{1st-CLASS} & \text{2nd-CLASS}
 \end{array}
 \quad (8.8)$$

$$\left\{ \begin{array}{l}
 \tilde{\chi}_0^i \tilde{\chi}_i \approx 0 \\
 \tilde{\chi}_\beta^i \tilde{\chi}_i \approx 0 \\
 \tilde{\chi}_\beta \approx 0 \\
 \tilde{\chi}_0^i \tilde{\pi}_i \quad A = \alpha, \beta
 \end{array} \right.
 \quad
 \left\{ \begin{array}{l}
 \tilde{\chi}_e^i \tilde{\chi}_i \approx 0 \\
 \tilde{\chi}_e^i \tilde{\pi}_i \approx 0 \\
 \tilde{\chi}_0^i \tilde{\chi}_i \approx 0 \\
 \tilde{\chi}_0^i \tilde{\pi}_i \approx 0 \\
 \tilde{\mathcal{D}}_r \approx 0
 \end{array} \right.
 \quad
 \left\{ \begin{array}{l}
 \tilde{\chi}_s^i \tilde{\chi}_i \approx 0 \\
 \tilde{\chi}_s \approx 0 \\
 \tilde{\chi}_0^i \tilde{\pi}_i \approx 0 \\
 \tilde{\mathcal{D}}_s \approx 0
 \end{array} \right.$$

The constraints $\tilde{\chi}_0^i \tilde{\chi}_i, A = \alpha, \beta, r, s$, are the $T^*(TQ)$ counterpart of the primary constraints Φ_A of T^*Q , while $\tilde{\chi}_\beta, \tilde{\chi}_s$ are the counterpart of the secondary constraints Φ'_β, Φ'_s . Owing to the velocity dependence the $T^*(TQ)$ constraints have different functional forms from the T^*Q ones but carry the same information. The division into first- and second-class constraints is based on the fact that their non-vanishing Poisson brackets, see Eqs. (2.28), are

$$\left\{ \begin{array}{l}
 \{\tilde{\chi}_e^i \tilde{\chi}_i, \tilde{\chi}_e^j \tilde{\chi}_j\}' \approx -\tilde{\chi}_e^i \tilde{R}_{ij} \tilde{\chi}_e^j = -\tilde{\mathcal{R}}_{ee'} \\
 \{\tilde{\chi}_e^i \tilde{\chi}_i, \tilde{\chi}_e^j \tilde{\pi}_j\}' \approx -\tilde{\chi}_e^i \tilde{A}_{ij} \tilde{\chi}_e^j = -\tilde{\mu}_e \delta_{ee'} \\
 \{\tilde{\chi}_e^i \tilde{\chi}_i, \tilde{\chi}_0^j \tilde{\chi}_j\}' \approx -\tilde{\chi}_e^i \tilde{R}_{ij} \tilde{\chi}_0^j = -\tilde{\mathcal{R}}_{er} \\
 \{\tilde{\chi}_0^i \tilde{\chi}_i, \tilde{\chi}_0^j \tilde{\chi}_j\}' \approx -\tilde{\mathcal{R}}_{rr'} \\
 \{\tilde{\chi}_s^i \tilde{\chi}_i, \tilde{\chi}_s^j \tilde{\chi}_j\}' = \tilde{T}_{ss'} \neq 0
 \end{array} \right.
 \quad
 \left\{ \begin{array}{l}
 \{\tilde{\chi}_0^i \tilde{\pi}_i, \tilde{\mathcal{D}}_r\}' = -\delta_{rr'} \\
 \{\tilde{\chi}_0^i \tilde{\pi}_i, \tilde{\mathcal{D}}_s\}' = -\delta_{ss'}
 \end{array} \right.
 \quad (8.9)$$

Here we have assumed the same simplifying hypothesis as Eqs. (5.3) and moreover that $\tilde{\mathcal{D}}_r, \tilde{\mathcal{D}}_s$ have vanishing Poisson brackets with all the constraints except the ones shown in Eqs. (8.9). To further simplify the matter let us assume that L is such that also $(\tilde{\mathcal{R}}_{er}) = 0$, so that the three blocks of second-class constraints in Eqs. (8.8) are decoupled. In this simplified situation, if we denote by $\tilde{\mathcal{C}}'$ the matrix associated with Eqs. (8.9), we get

$$\det \tilde{\mathcal{C}}' = (\pi_e \tilde{\mu}_e)^2 \det(-\tilde{\mathcal{R}}_{rr'}) (\det \tilde{T})^2 \quad (8.10)$$

The only remnants of the second-class constraints of the regular case are $\tilde{\chi}_e^i \tilde{\chi}_i, \tilde{\chi}_e^i \tilde{\pi}_i$, i.e. those constraints associated with the non-null eigenvectors of $\tilde{A}$. The Dirac brackets with respect to these constraints would have the form (2.38) by using the quasi-inverse $\tilde{B}$ of $\tilde{A}$. However, now the constraints $\tilde{\chi}_e^i \tilde{\chi}_i, \tilde{\chi}_e^i \tilde{\pi}_i$ allow only the elimination of a subset of the variables $v^i, \tilde{\pi}_i$, so that these Dirac brackets are not connected with the T^*Q Poisson brackets. However, they can be used to define an analogue of Eqs. (2.22), when restricted to functions only of q^i, v^i . It turns out that these TQ brackets admit neutral elements [3], having zero bracket with every TQ function in accordance with the presymplectic structure ω_L of TQ in the singular case.

The most relevant feature of the $T^*(TQ)$ approach is the appearance of the extra first-class constraints ${}_A \tilde{\xi}_0^i \tilde{\pi}_i$, $A = \alpha, \beta$, connected with the vector fields $\tilde{Z}_A$ of Eqs. (7.13):

$$\tilde{Z}_A = {}_A \tilde{\xi}_0^i \frac{\partial}{\partial v^i} \approx \{ \cdot, {}_A \tilde{\xi}_0^i \tilde{\pi}_i \} \quad (8.11)$$

Their associated gauge freedom, as already stated, is connected with the gauge freedom of the accelerations in the Lagrangian formalism. This last freedom, which is completely lost in the T^*Q approach, reappears in the intrinsically first-order formalism of TQ just by means of the $\tilde{Z}_A$, see Eqs. (7.12) and (8.7). Instead the gauge freedom associated to $\tilde{Z}_r, \tilde{Z}_s$ is removed, because ${}_r \tilde{\xi}_0^i \tilde{\pi}_i, \tilde{D}_r, {}_s \tilde{\xi}_0^i \tilde{\pi}_i, \tilde{D}_s$ are pairs of second-class constraints. In the Lagrangian formulation the genuine first-order equations $D_r(q, \dot{q}) \stackrel{0}{=} 0, D_s(q, \dot{q}) \stackrel{0}{=} 0$ determine the originally arbitrary velocity functions $g^r(q, \dot{q}), g^s(q, \dot{q})$; then $\dot{g}^r, \dot{g}^s$ determine the components ${}_r \xi_{0i} \dot{q}^i, {}_s \xi_{0i} \dot{q}^i$ of the accelerations, destroying the gauge freedom associated with Z_r, Z_s . As we have seen in TQ only $\tilde{\lambda}^r, \tilde{\lambda}^s$ are determined by Eqs. (7.6), (7.11), but the addition of the holonomic constraints $\tilde{D}_r = \tilde{D}_s = 0$ again removes the gauge freedom of $\tilde{Z}_r, \tilde{Z}_s$. With regard to the true gauge transformations, their gauge freedom can be removed at the Lagrangian level by imposing gauge-fixing conditions $\zeta_\alpha(q, \dot{q}) = \zeta_\beta(q, \dot{q}) = \zeta'_\beta(q, \dot{q}) = 0$. In this way $g^\alpha(q, \dot{q}), g^\beta(q, \dot{q})$ and the induced arbitrary $\bar{c}^\beta(q, p)$, associated with $\chi'_\beta = \bar{\Phi}'_\beta$, see Eqs. (5.8), are determined. Then $\dot{g}^\alpha, \dot{g}^\beta$ determine ${}_a \xi_{0i} \dot{q}^i, {}_\beta \xi_{0i} \dot{q}^i$, breaking the gauge freedom generated by Z_α, Z_β . Instead, in TQ the conditions $\tilde{f}_\alpha(q, v) = \tilde{f}_\beta(q, v) = \tilde{f}'_\beta(q, v) = 0$, implying $\tilde{\lambda}^A = {}_A \tilde{\xi}_{0k} v^k$, $A = \alpha, \beta$, in order to reproduce the genuine second-order Eqs. (5.15), fix the gauge freedom associated with $\tilde{X}_\alpha, \tilde{X}_\beta, \tilde{X}'_\beta$, but not the ones associated with $\tilde{Z}_\alpha, \tilde{Z}_\beta$. Therefore, extra independent gauge-fixing conditions are needed. The extra conditions are not required in the T^*Q approach, where, instead of the 'covariant' gauge fixings, the Dirac gauge-fixing constraints $\bar{f}_\alpha(q, p) \approx 0, \bar{f}_\beta(q, p) \approx 0, \bar{f}'_\beta(q, p) \approx 0$ are the only ones needed. In $T^*(TQ)$ besides $\tilde{f}_\alpha, \tilde{f}_\beta, \tilde{f}'_\beta$ extra conditions $\tilde{\eta}_{A=\alpha, \beta}$ are needed to fix the gauge freedom of $\tilde{Z}_{A=\alpha, \beta}$. This means that the Faddeev-Popov measure of the path integral in $T^*(TQ)$ will be more complicated than the corresponding one in T^*Q . But this is the price to pay for unification of the TQ and T^*Q description of the singular system, because in this way, as shown in Fig. 1 (see Section 4), we will get a simultaneous description of the reduced spaces $\tilde{\gamma}_R, \tilde{\gamma}_R$ even when the reduced configuration space δ does not exist globally.

As we shall treat the BRST approach in a future paper, let us add here some remarks concerning it. Since the $T^*(TQ)$ description is based on a phase space, the natural approach is the BFM one. To this end we need the structure functions associated with the first-class constraints (8.8) and with the deterministic part of the canonical Hamiltonian contained in $\tilde{H}_D^E$ of Eq. (8.7)

$$\begin{aligned} \tilde{H}_C &= \tilde{E} + \left[v^i - \sum_{A=\alpha, \beta} ({}_A \tilde{\xi}_{0k} v^k) {}_A \tilde{\xi}_0^i \right] \tilde{\chi}_i + \\ &+ \left[\tilde{\lambda}^i + \tilde{B}^{ij} \tilde{R}_{jk} \sum_{A=\alpha, \beta} ({}_A \tilde{\xi}_{0h} v^h) {}_A \tilde{\xi}_0^k \right] \tilde{\pi}_i = \\ &= \tilde{H}_d + \sum_{\beta} \tilde{C}^\beta(q, v, \tilde{p}_i, \tilde{\pi}) \tilde{\chi}_\beta \end{aligned} \quad (8.12)$$

While the structure functions of the constraints ${}_a \tilde{\xi}_0^i \tilde{\chi}_i, {}_\beta \tilde{\xi}_0^i \tilde{\chi}_i, \tilde{\chi}_\beta$ are related to the corresponding ones of T^*Q , see Eqs. (6.15), taking into account the change in the functional form of the constraints in

going from T^*Q to $T^*(TQ)$ and their extra velocity dependence, there will be the new structure functions of the ${}_A\tilde{\xi}_0^i \tilde{\pi}_i$, $A = \alpha, \beta$, among themselves and with the other first-class constraints. Also the structure functions of the first-class constraints with respect to $\tilde{H}_d$ will be different from those in T^*Q .

In any case, there will be extra ghosts associated with the new constraints ${}_A\tilde{\xi}_0^i \tilde{\pi}_i$. At a local level it should be possible to eliminate them by going to T^*Q and then to the standard Lagrangian BRST fixed action. But it is not evident that they are irrelevant from a global point of view, being connected with the local dynamical symmetries of the Euler-Lagrange equations and not of the action.

Let us finish this section by noting that the constraints (8.8) define a final submanifold $\bar{q}_F$ of $T^*(TQ)$. $\bar{q}_F$ is foliated with gauge orbits by the canonical gauge transformations generated by the first-class constraints ${}_A\tilde{\xi}_0^i \tilde{\chi}_i$, ${}_B\tilde{\xi}_0^i \tilde{\chi}_i$, $\tilde{\chi}_\beta$, ${}_A\tilde{\xi}_0^i \tilde{\pi}_i$, $A = \alpha, \beta$. For our class of singular models $\dim \bar{\gamma}_R = 2[n - (k_1 + 2k_2 + k_3 + k_4)]$. As $\text{rank } \tilde{A} = n - (k_1 + k_2 + 2k_3 + k_4)$, in $T^*(TQ)$, of dimension $4n$, we have $k_1 + 2k_2 + (k_1 + k_2) = 3k_2 + 2k_1$ first-class constraints and $2[n - (k_1 + k_2 + 2k_3 + k_4)] + 2k_3 + 2k_4 + 4k_3 + 2k_4 = 2[n - (k_1 + k_2) + k_3 + k_4]$ second-class ones. Therefore we get $\dim \bar{q}_F = 2(n - k_3 - k_4) - k_2$, and going to the quotient with respect to the foliation of the gauge transformations we recover $\bar{\gamma}_R$ as the reduced phase space. An isomorphic copy of $\bar{\gamma}_R$ can be obtained by adding the restricted class of gauge-fixing constraints: $\tilde{\xi}_\alpha \approx 0$, $\tilde{\xi}_\beta \approx 0$, $\tilde{\xi}'_\beta \approx 0$, $\tilde{\eta}_A \approx 0$, $A = \alpha, \beta$, implying $\tilde{\lambda}^A = {}_A\tilde{\xi}_{0k} v^k$. If now we revert to the original form of the constraints $\tilde{\chi}_i \approx 0$, $\tilde{\pi}_i \approx 0$, we can use the fact that they are explicitly solved in the momenta $\tilde{p}_i$, $\tilde{\pi}_i$ to replace the constraints $\tilde{\xi}_\alpha$, $\tilde{\xi}_\beta$, $\tilde{\xi}'_\beta$, $\tilde{\eta}_A$, $A = \alpha, \beta$, with equivalent constraints independent of the momenta. This will determine a section of $\bar{q}_F$ that is isomorphic to the reduced space $\bar{\gamma}_R$, which was obtained from TQ . As in Section 2 for the regular case, it is the peculiar form of the constraints $\tilde{\chi}_i$, $\tilde{\pi}_i$, which allows a unified treatment of the TQ and T^*Q approaches by means of $T^*(TQ)$ also in the singular case.

Let us add a remark on the Dirac gauge-fixing procedure in phase space when there are secondary first-class constraints such as $\tilde{\Phi}'_\beta$ in T^*Q or $\tilde{\chi}'_\beta$ in $T^*(TQ)$. When we add a gauge-fixing constraint $\tilde{\xi}_\alpha \approx 0$ to a primary first-class constraint without secondaries such as $\tilde{\Phi}_\alpha$, its preservation in time $\dot{\tilde{\xi}}_\alpha \stackrel{0}{=} \{\tilde{\xi}_\alpha, \bar{H}_B^F\} \approx 0$ will determine the Dirac multiplier λ^α and then the velocity function $g^\alpha(q, \dot{q})$. When there are secondary first-class constraints such as $\tilde{\Phi}'_\beta$ generating gauge transformations, they are present in $\bar{H}_c$ with coefficients $\bar{c}^\beta(q, p)$, see Eqs. (4.6) and (5.8), such that $\dot{\bar{c}}^\beta \stackrel{0}{=} \lambda^\beta$. If we use $\bar{H}_B^F$, the gauge-fixing $\tilde{\xi}_\beta \approx 0$ to $\tilde{\Phi}_\beta$ will determine λ^β from $\dot{\tilde{\xi}}_\beta \stackrel{0}{=} 0$ and this will fix $\dot{\bar{c}}^\beta$, i.e. the time derivative of the arbitrary function $\bar{c}^\beta$. Then the gauge-fixing $\tilde{\xi}'_\beta \approx 0$ to $\tilde{\Phi}'_\beta$ cannot be arbitrary, as the equation $\dot{\tilde{\xi}}'_\beta \stackrel{0}{=} 0$ must be consistent with the fixed $\dot{\bar{c}}^\beta$ and only determine the integration constant to fix $\bar{c}^\beta$. A second possibility is to modify $\bar{H}_B^F$ to $\bar{H}_D'^F$ by replacing $\lambda^\beta(t)$ with $\dot{u}^\beta(t)$ and $\bar{c}^\beta(q, p)$ with $u^\beta(t)$, so that the term $u^\beta \tilde{\Phi}'_\beta + \dot{u}^\beta \tilde{\Phi}_\beta$ of $\bar{H}_D'^F$ will reproduce $\delta_\beta q^i$ of Eqs. (5.16): as before, $\tilde{\xi}'_\beta \approx 0$ is not completely arbitrary. The final possibility is to use the extended Dirac Hamiltonian $\bar{H}_E^F$, in which $\bar{c}^\beta(q, p)$ is replaced by $\lambda'^\beta(t)$ and now there is no restriction on $\tilde{\xi}'_\beta$, i.e. on the choice of the section of $\bar{\gamma}_F$ isomorphic to $\bar{\gamma}_R$. See also Ref. [36, 72] for a discussion of these problems.

9. CANONICAL QUANTIZATION OF FIRST-CLASS CONSTRAINTS: THE MULTITEMPORAL APPROACH

In Section 3 we described the canonical quantization of $T^*(TQ)$ for regular systems, which have only second-class constraints in the $T^*(TQ)$ description. For singular systems there are, in general, first- and second-class constraints both in T^*Q and $T^*(TQ)$. Therefore we start with the canonical commutation relation (3.1) or (3.6) and treat the second-class constraints as described in Section 3. Concerning the first-class constraints one possibility is to add gauge-fixing constraints and to treat the resulting set of second-class constraints in the previous way. In this way, the differences between T^*Q and $T^*(TQ)$ would disappear, at least locally, because this 'non-covariant' quantization would amount to quantizing $\bar{\gamma}_R$ in both cases.

The other possibility is to try to make a 'covariant' quantization of the first-class constraints without introducing gauge fixings. Let us consider a given set of first-class constraints $\bar{\Phi}_a \approx 0$, $a = 1, \dots, N$, and a given Hamiltonian $\bar{H}$ on a certain $2n$ -dimensional phase space Γ [which could be either T^*Q or $T^*(TQ)$], so that the Dirac Hamiltonian is $\bar{H}_D = \bar{H} + \sum_a \lambda^a(t) \bar{\Phi}_a$.

The simplest case is when we have an Abelian algebra

$$\{\bar{\Phi}_a, \bar{\Phi}_b\} = \{\bar{\Phi}_a, \bar{H}\} = 0 \quad (9.1)$$

The Hamilton equations imply the following equations for a generic function

$$\frac{d\bar{A}}{dt} \equiv \{\bar{A}, \bar{H}\} + \sum_a \lambda^a(t) \{\bar{A}, \bar{\Phi}_a\} \quad (9.2)$$

These equations are formulated in T^*Q , and even if their initial data are restricted to the submanifold $\bar{\gamma}$ defined by the constraints, in which case their solutions will lie on $\bar{\gamma}$, the gauge freedom associated with the gauge orbits in $\bar{\gamma}$ is left. Therefore it is convenient to consider Eqs. (9.2) as giving the off-shell description. The problem with these equations is that to solve them we have to prefix the gauge freedom by choosing a well-defined set of multipliers $\lambda^a(t)$: this means that only one trajectory will evolve from each initial point on a gauge orbit, instead of the family of trajectories connected by the canonical gauge transformations. As each such family is parametrized by the λ^a 's and we assume that all the gauge orbits are isomorphic for the sake of simplicity, we can get an equivalent description by introducing N parameters τ^a (modulo a shift in their origin) by the equations

$$d\tau^a = \lambda^a(t) dt \quad (9.3)$$

Then to parametrize the evolution in such a way as to take into account which member of a given family is the trajectory under examination, we redefine the coordinates $q^i(t)$, $p_i(t)$ as $q^i(t, \tau^a)$, $p_i(t, \tau^a)$ by assuming the standard Poisson brackets at equal values of t and of all the τ^a 's. In this way the Hamilton equations (9.2) with Hamiltonian $\bar{H}_D$ are replaced by the following set of coupled $N + 1$ Hamilton equations:

$$\begin{cases} \frac{\partial}{\partial t} \bar{A}(q(t, \tau^a), p(t, \tau^a)) = \{\bar{A}, \bar{H}\} \\ \frac{\partial}{\partial \tau^a} \bar{A}(q(t, \tau^a), p(t, \tau^a)) = \{\bar{A}, \bar{\Phi}_a\} \end{cases} \quad (9.4)$$

where $\bar{H}$ is the Hamiltonian for t , and $\bar{\Phi}_a$ the one for τ^a . The integrability conditions of Eqs. (9.4) are just Eqs. (9.1). This approach has been developed in the study of relativistic particle mechanics [60, 73] and called 'the multitemporal approach' because there the τ^a are connected with the individual proper-time coordinates of the various particles. See Ref. [74] for its application to Newton mechanics reformulated with first-class constraints and the clarification of the mentioned connection.

However, the second set of Eqs. (9.4) has a group theoretical interpretation. The vector fields $\bar{X}_a = \{\cdot, \bar{\Phi}_a\}$ on phase space satisfy an Abelian Lie algebra and, when restricted to $\bar{\gamma}$, are tangent to it. When this algebra can be integrated to a symplectic action [54] of an Abelian (local) Lie group G on the given phase space, the restriction of this action to $\bar{\gamma}$ will allow the construction of the gauge orbits, leaves of the gauge foliation, so that each gauge orbit is a homogeneous space for G . (If the action of G on $\bar{\gamma}_F$ is proper, regular, and free [54, 66], then $\bar{\gamma}_R = \bar{\gamma}_F/G$ is a manifold.) The equations $\partial \bar{A} / \partial \tau^a = \bar{X}_a \bar{A}$, with the initial conditions $\bar{A}_0 = \bar{A}(q_0, p_0)$, where q_0, p_0 are the coordinates of a point on a given orbit, are nothing else than the Lie equations for the realization of the Lie group on the given orbit as a transformation group [3]. Their integration defines the finite action of the group $(q_0, p_0) \xrightarrow{g} [q^i(g, q_0, p_0), p_i(g, q_0, p_0)]$ consistently with the composition law of the group and allows the orbit to be reconstructed starting from (q_0, p_0) , at least locally. Therefore, the parameters τ^a are coordinates for the group manifold of G . The connection of Eqs. (9.4) with the Lagrangian generalized Lie equations of Ref. [25] is now under study: the starting point should be to reformulate Eqs. (9.4) by using the vector fields (6.2).

To define the on-shell description one has to find a $2(n - N)$ -dimensional symplectic basis of observables, i.e. of gauge invariant functions $\bar{U}(q, p)$ satisfying

$$\{\bar{U}, \bar{\Phi}_a\} = 0 \quad a = 1, \dots, N \quad (9.5)$$

Each such symplectic basis $\{\bar{U}_\alpha\}$ gives a realization of the reduced phase space $\bar{\gamma}_R$ as a section of $\bar{\gamma}_F \subset \Gamma$; this can be seen by completing it to a local basis of Γ near $\bar{\gamma}_F$ by adding the $\bar{\Phi}_a$ as new momenta and by choosing a set of gauge variables $\bar{\zeta}_a$ such that $\{\bar{\zeta}_a, \bar{\Phi}_b\} = \delta_{ab}$, $\{\bar{\zeta}_a, \bar{U}_\alpha\} = 0$ [42]. This is the reason why we introduced strong observables instead of the weak ones satisfying only $\{\bar{U}, \bar{\Phi}_a\} \approx 0$. Each set of gauge-fixing constraints $\bar{\zeta}_a = \text{const}$ will identify a section of $\bar{\gamma}_F \subset \Gamma$ isomorphic to $\bar{\gamma}_R$. From Eqs. (9.5), (9.4) we see that the observables are τ^a independent (constant on the gauge orbits) and satisfy $d\bar{U}/dt = \{\bar{U}, H\}$.

When we have secondary first-class constraints, so that $\bar{H}_D = \bar{H}_d + \Sigma_\alpha \lambda^\alpha \bar{\Phi}_\alpha + \Sigma_\beta (\bar{c}^\beta \bar{\Phi}'_\beta + \lambda^\beta \bar{\Phi}_\beta)$, we can use the extended Dirac Hamiltonian $\bar{H}_E = \bar{H}_d + \Sigma_\alpha \lambda^\alpha \bar{\Phi}_\alpha + \Sigma_\beta (\lambda'^\beta \bar{\Phi}'_\beta + \lambda^\beta \bar{\Phi}_\beta)$, as explained at the end of Section 8, to apply the multitemporal approach. There will be new parameters τ'^β defined by Eq. (9.3), $\dot{\tau}'^\beta(t) = \lambda'^\beta(t)$. To recover the description with the modified $\bar{H}_D = \bar{H}_d + \Sigma_\alpha \lambda^\alpha \bar{\Phi}_\alpha + \Sigma_\beta (\lambda'^\beta \bar{\Phi}'_\beta + \lambda^\beta \bar{\Phi}_\beta)$, with λ'^β equal to the u^β of Section 8, we have only to parametrize the submanifold $(\tau^\beta, \tau'^\beta)$ of the group manifold in the form $[\tau^\beta(t^\beta), \tau'^\beta(t'^\beta)]$ and to restrict ourselves to those one-parameter subgroups for which $\dot{\tau}^\beta(t) = \dot{\lambda}^\beta(t) = \dot{\tau}'^\beta(t)$.

The canonical quantization of the multitemporal approach is done in two steps [60]: an off-shell one and an on-shell one. Let us assume that at the quantum level we have :

$$[\hat{\Phi}_a, \hat{\Phi}_b] = [\hat{\Phi}_a, \hat{H}] = 0 \quad (9.6)$$

i.e. that there are no Schwinger terms due to ordering problems (a central extension could be tolerated, with the only effect of creating a projective representation instead of a vector one). The

off-shell description is obtained by replacing the Eqs. (9.4) with the following set of coupled Schrödinger equations

$$\begin{cases} i \frac{\partial}{\partial t} \psi(q; t, \tau^a) \stackrel{e}{=} \hat{H} \psi(q; t, \tau^a) \\ i \frac{\partial}{\partial \tau^a} \psi(q; t, \tau^a) \stackrel{e}{=} \hat{\Phi}_a \psi(q; t, \tau^a) \quad a = 1, \dots, N \end{cases} \quad (9.7)$$

whose integrability conditions are Eqs. (9.6) (so that they are broken in the presence of anomalies). To define the Hilbert space we have to introduce an off-shell scalar product compatible with Eqs. (9.7)

$$(\psi_1, \psi_2) = \int d^n q \psi_1^* K \psi_2, \quad \frac{\partial}{\partial t} (\psi_1, \psi_2) \stackrel{e}{=} \frac{\partial}{\partial \tau^a} (\psi_1, \psi_2) \stackrel{e}{=} 0, \quad a = 1, \dots, N \quad (9.8)$$

where K is a suitable kernel. Assuming that $\hat{H}$, $\hat{\Phi}_a$ are self-adjoint with respect to this scalar product with, in general, a continuous spectrum, we see that the finite classical canonical gauge transformations $\exp(\tau^a \bar{X}_a)$ are replaced by the unitary gauge transformations $\exp(i\tau^a \hat{\Phi}_a)$. At the classical level the gauge transformations generated the foliation of $\bar{\gamma}$, so that the reduced phase space $\bar{\gamma}_R$ was obtained by going to the quotient with respect to it. Let us introduce an equivalence relation in the space of the solutions of the Eqs. (9.7) in the following way: two solutions ψ_1, ψ_2 are equivalent, $\psi_1 \approx \psi_2$, if they are connected by a unitary gauge transformation. By taking the quotient of the previous space with respect to the equivalence relation, we get the quantum equivalent of the reduced phase space $\bar{\gamma}_R$. The wave functions $\tilde{\psi}$ in this space are equivalence classes of the solutions of Eqs. (9.7). A realization of them will satisfy the following equations:

$$\begin{cases} i \frac{\partial}{\partial t} \tilde{\psi} \stackrel{e}{=} \hat{H} \tilde{\psi} \\ \hat{\Phi}_a \tilde{\psi} \stackrel{e}{=} 0 \quad a = 1, \dots, N \end{cases} \quad (9.9)$$

Therefore, the $\tilde{\psi}$ are τ^a independent and the second set of Eqs. (9.9) is the quantum transcription of the definition of the classical observables, Eq. (9.5).

If we choose to realize the $\tilde{\psi}$ by using a member ψ of their associated equivalence classes, then ψ would not be normalizable in the scalar product (9.8) [76]. Therefore, we have to define a new Hilbert space for the wave functions $\tilde{\psi}(q, t)$ with an on-shell scalar product. When the $\hat{\Phi}_a$ are resolved in N momenta, $\hat{\Phi}_a = \hat{p}_a - \hat{f}(q, \hat{p}_\alpha)$, $a = 1, \dots, N$, $\alpha = N+1, \dots, n$, this scalar product will have the form

$$\langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle = \int d^{n-N} q \tilde{\psi}_1^* \hat{K} \tilde{\psi}_2 \quad (9.10)$$

where the measure runs over dq^α , $\alpha = N + 1, \dots, n$. The compatibility with Eqs. (9.9) forces the kernel $\tilde{K}$ (in general non local) to be such that we have

$$\frac{\partial}{\partial t} \langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle \stackrel{!}{=} \frac{\partial}{\partial q^a} \langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle \stackrel{!}{=} 0 \quad a=1, \dots, N \quad (9.11)$$

When the constraints are not resolved in the momenta, the situation is more complex, but the measure of Eqs. (9-10) will again be $(n-N)$ -dimensional. See Ref. [75] for the construction of such scalar products in the case of two scalar relativistic particles with an action-at-a-distance manifestly covariant interaction described by two first-class constraints. In this case one gets two coupled integrable Klein-Gordon equations, of integro-differential type due to the form of the mutual interaction, and four Poincaré-invariant non-local on-shell scalar products have been found, generalizing the two possible scalar products of the single Klein-Gordon equation. See also Ref. [76] for a different approach to this problem.

In the previous method of quantization no gauge-fixing constraint has been introduced. As a consequence, there is a quantum counterpart of the classical ambiguity in the choice of the section in $\bar{\gamma}$ isomorphic to $\bar{\gamma}_R$ after the definition of a symplectic basis $\{\bar{U}_\alpha\}$ of observables (the ambiguity of the constants in the gauge-fixing conditions $\bar{\gamma}_a = \text{const}$). It appears in the fact that the wave functions $\tilde{\psi}$ are functions $\tilde{\psi}(q^\alpha; t; q^a)$, $\alpha = N + 1, \dots, n$, if we consider the case of $\hat{\Phi}_a$ resolved in the momenta for the sake of simplicity. While the q^α are physical degrees of freedom, the q^a are gauge variables (and the elementary solutions for $\tilde{\psi}$ depend upon them only through a phase, see Ref. [75]). Therefore $\tilde{\psi}$ depends upon the $N + 1$ parameters t, q^a . Each line in the parametric space of the q^a 's is in one-to-one correspondence with a classical section in $\bar{\gamma}$ isomorphic to $\bar{\gamma}_R$. See Refs. [60, 74] for the definition of the multitemporal evolution operator connected with the existence of this parametric space.

Let us now consider the case in which Eqs. (9.1) are replaced by a non-Abelian Lie algebra with structure constants C_{ab}^c and C_{a0}^c .

$$\begin{cases} \{\bar{\Phi}_a, \bar{\Phi}_b\} = C_{ab}^c \bar{\Phi}_c \\ \{\bar{\Phi}_a, \bar{H}\} = C_{a0}^c \bar{\Phi}_c \end{cases} \quad (9.12)$$

In this case the definition (9.3) of the parameters τ^a gives rise to non-integrable multitemporal Hamilton equations. Equations (9.4) must now be replaced by the following equations:

$$\begin{cases} \frac{\partial}{\partial t} \bar{A} \stackrel{!}{=} D(t, \tau^c) \{\bar{A}, \bar{H}\} + D^b(t, \tau^c) \{\bar{A}, \bar{\Phi}_b\} \\ \frac{\partial}{\partial \tau^a} \bar{A} \stackrel{!}{=} D_a(t, \tau^c) \{\bar{A}, \bar{H}\} + D_a^b(t, \tau^c) \{\bar{A}, \bar{\Phi}_b\} \end{cases} \quad (9.13)$$

whose integrability conditions, after having used the Jacobi identity and Eqs. (9.72), are

$$\frac{\partial D_\alpha^\gamma}{\partial \tau^\beta} - \frac{\partial D_\beta^\gamma}{\partial \tau^\alpha} + C_{\gamma\sigma}^\delta D_\alpha^\sigma D_\beta^\sigma = 0 \quad (9.14)$$

Here we have replaced the index a by $\alpha = (0, a)$, so that $\tau^0 = t$, $D_0^0 = D$, $D_0^a = D^a$, and so on. Equations (9.14) are the Maurer–Cartan equations, implying that D_α^β must be a realization of the adjoint representation of the Lie algebra as the τ^α are coordinates of the group manifold. When $C_{a0}^b = 0$, we are left with the Lie algebra of the $\bar{\Phi}_a$ and $D = 1$, $D^a = D_a = 0$, $\partial D_a^b / \partial t = 0$: then Eqs. (9.3) are replaced by $D_b^a(\tau^c) d\tau^b = \lambda^a(\tau) d\tau$. These quantities become the analogue of the BRS ghosts (when interpreted as Maurer–Cartan forms [77]) on $\bar{\gamma}_1 \subset T^*Q$, not on T^*Q to which the BFM approach applies [24–27]. In this case the second set of Eqs. (9.13) are again the Lie equations for the realisation of the Lie group on the gauge orbits as a transformation group. In principle, the D_α^β are known from the theory of Lie algebras (see, for instance, Ref. [3]), but in practice their expressions are so complicated as to make Eqs. (9.13) difficult to study.

In the generic case Eqs. (9.12) still hold, but with structure functions $\bar{C}_{\alpha\beta}^\gamma(q, p)$. For this case Batalin has introduced the notion of quasi-group [78] and Eqs. (9.13) turn out to be integrable only when restricted to the constraint submanifold $\bar{\gamma}_F$, where Eqs. (9.14) hold with $\bar{C}_{\alpha\beta}^\gamma$ replaced by $\bar{C}_{\alpha\beta}^{\gamma_F}$, so that also the $D_\alpha^\beta(t, \tau^c)$ have to be replaced by $\bar{D}_\alpha^\beta(t, \tau^c, q, p)$ in general.

In the non-Abelian case and in the case of structure functions there is the possibility of getting a simpler but local description by means of a process of Abelianization. The final constraint submanifold $\bar{\gamma}_F$ is described by the independent constraints $\bar{\Phi}_a \approx 0$ satisfying Eqs. (9.12). We can, however, choose other functional forms $\bar{\Phi}'_a$, as long as they are functionally independent, to define $\bar{\gamma}_F$. The only modification is an alteration of the original C_{ab}^c (or $\bar{C}_{ab}^c$): in general, even if we started from a Lie algebra, the modified $\bar{\Phi}'_a$ will satisfy only a quasi-algebra with structure functions $\bar{C}'^c$. In particular, we can look for a functional form such that $\bar{C}'^c = 0$. This is always possible, for instance by solving the original constraints in a subset ω_a of the variables q^i, p_i (ω_a will denote the complementary set): $\bar{\Phi}'_a = \omega_a - \bar{f}_a(\omega_a) = \bar{F}_a^b \bar{\Phi}_b \approx 0$. On one hand, the Poisson brackets $\{\bar{\Phi}'_a, \bar{\Phi}'_b\}$ cannot depend on the coordinates ω_a ; on the other hand, they must vanish using the $\bar{\Phi}'_a$, which depend on the ω_a . Therefore $\{\bar{\Phi}'_a, \bar{\Phi}'_b\} = 0$ must be true. However, to find the $\bar{\Phi}'_a$ we have to use the implicit function theorem and this shows that in general the equations $\bar{\Phi}'_a \approx 0$ will not describe the whole $\bar{\gamma}_F$ but only one of its submanifolds, on whose boundary the functions $\bar{F}_a^b$ will have singularities. Therefore the Abelianization procedure is only a local process and we have to find enough Abelian forms $\bar{\Phi}'_a^{(k)}$ to cover the whole $\bar{\gamma}_F$. In the generic case $\bar{\gamma}_F$ may turn out to be the union of disjoint submanifolds $\bar{\gamma}^{(r)}$ with each $\bar{\gamma}^{(r)}$ defined by a family of Abelian constraints $\bar{\Phi}'_a^{(k_r)}$, $k_r = 1, \dots, m_r$. Each set $\bar{\Phi}'_a^{(k_r)}$ will identify a chart of an atlas on $\bar{\gamma}^{(r)}$ with well-defined transition functions. Moreover, there can be lower dimensional submanifolds of $\bar{\gamma}_F$, which require a special treatment and are a sign of the existence of non-isomorphic gauge orbits (see Ref. [79] for such an analysis in the case of the Nambu string). Very little is known about these problems as in the existing literature [26] one is satisfied to quote the general theorems, based on the theory of function groups, for the existence of the Abelian representation [80].

If we restrict ourselves to a local chart of $\bar{\gamma}_F$ with local Abelian constraints $\bar{\Phi}'_a$, the next step is to find a new Hamiltonian $\bar{H}' = \bar{H} + \bar{T}^a \bar{\Phi}'_a$ such that $\{\bar{H}', \bar{\Phi}'_a\} = 0$. Then, we can develop the classical theory as in the Abelian case with the Hamilton Eqs. (9.4) holding in the local chart and by defining local observables $\{\bar{U}, \bar{\Phi}'_a\} = 0$ (global ones may not exist). The transition functions between two overlapping charts will be given by suitable canonical transformations in the original phase space, restricted to the overlap of the two charts.

In principle, the quantum theory should follow the same off- and on-shell steps of the Abelian case when the algebra (9.12) is reproduced at the quantum level with the quantum constraints at the right of the quantum structure functions $\hat{\bar{C}}'$ s. As by the redefinition $\bar{H} \rightarrow \bar{H}' = \bar{H} + \bar{T}^a \bar{\Phi}'_a$ we can

get $C_{a0}^b = 0$ (or $\bar{C}_{a0}^b = 0$) in Eqs. (9.12), Eqs. (9.13) can be simplified to have $D = 1$, $D^a = D_a = 0$. Therefore, the off-shell Schrödinger equations are

$$\begin{cases} i \frac{\partial}{\partial t} \psi = \hat{H} \psi \\ i \frac{\partial}{\partial a} \psi = D_a^b(z^c) \hat{\Phi}_b \psi \end{cases} \quad (9.15)$$

and the on-shell ones are still given by Eqs. (9.9). In the absence of anomalies (only central extensions are allowed) these equations are integrable and the discussion goes on as in the Abelian case. However, it is not clear how problems such as the existence of non-isomorphic gauge orbits will influence the quantum description. On the other hand, we could make local quantizations of the Abelian constraints in their local charts (as in the old non-covariant quantization of the string), but then it is not clear which are the ‘quantum transition functions’ connecting the local Hilbert spaces associated with the various charts. The problem should probably be reformulated in the language of geometric quantization [17, 53]. Another problem in this approach to quantization, in the case of structure functions, is the formulation of what could be called the ‘quantum Frobenius theorem’ for gauge transformations, to ensure their global integrability also at the quantum level. At the classical level this information is hidden in the higher structure functions mentioned in Section 6.

The modern attitude of treating these quantization problems is to resort to the BRS quantization. There, one restricts the original infinite-dimensional gauge group, whose rigid part has the algebra (9.12), to some subgroup for which the Frobenius theorem implementation is hidden either in the requirement $\delta^2 = 0$ for the operator δ describing the BRS transformations in the Lagrangian approach or in $\{\bar{\Omega}, \bar{\Omega}\} = 0$, where $\bar{\Omega}$ is the BRS charge carrying the information on the higher structure functions, in the BFV phase-space approach. However, it is not clear how problems such as the existence of non-isomorphic gauge orbits are formulated in the BRS approach.

10. PATH INTEGRAL ON $T^*(TQ)$ FOR SINGULAR SYSTEMS

In this section we shall study the modification of the path integral (3.12) on $T^*(TQ)$ when the system is singular. We shall restrict ourselves to the class of singular systems defined in Section 5 with the further simplifications introduced in Section 8 after Eqs. (8.9). To define the Faddeev-Popov measure, we shall introduce gauge-fixing constraints $\tilde{\xi}_\alpha \approx 0$, $\tilde{\xi}_\beta \approx 0$, $\tilde{\xi}'_\beta \approx 0$, $\tilde{\eta}_{A=\alpha,\beta} \approx 0$ to transform the first-class constraints of Eqs. (8.8) into second-class constraints with the restriction of giving $\tilde{\chi}^A = A_{\xi_0 k} v^k$, $A = \alpha, \beta$.

For the sake of simplicity we shall assume that each gauge-fixing constraint has vanishing Poisson bracket with all the constraints with a different kind of lower index and that all the gauge-fixing constraints have vanishing Poisson brackets among themselves (see Ref. [64] about relaxing this last assumption). Finally, as the original constraints $\tilde{\chi}_i, \tilde{\pi}_i$ are explicitly solved in the momenta, there is always the possibility of choosing all the gauge fixings as functions $\tilde{\xi}_\alpha, \tilde{\xi}_\beta, \tilde{\xi}'_\beta, \tilde{\eta}_{A=\alpha,\beta}$ only of the coordinates q^i, v^i .

Therefore Eq. (3.12) is now replaced by the following expression:

$$\begin{aligned}
 K(q, q_0, t-t_0) = & \int_{q_0 t_0}^{q t} dq d\tilde{p} dv d\tilde{\pi} e^{-\frac{i}{\hbar} \int_{t_0}^t dt (\dot{q}^i \tilde{p}_i + \dot{v}^i \tilde{\pi}_i - \tilde{E})} \\
 & \prod_e |\tilde{\mu}_e| \delta(\tilde{x}_e^i \tilde{\chi}_i) \delta(\tilde{x}_e^i \tilde{\pi}_i) \prod_r \sqrt{|\det(-\tilde{g}_{rr})|} \delta(\tilde{\xi}_0^i \tilde{\chi}_i) \delta(\tilde{\xi}_0^i \tilde{\pi}_i) \delta(\tilde{p}_r) \\
 & \prod_s |\det \tilde{\tau}| \delta(\tilde{\xi}_0^i \tilde{\chi}_i) \delta(\tilde{\chi}_s) \delta(\tilde{\xi}_0^i \tilde{\pi}_i) \delta(\tilde{p}_s) \prod_\alpha \det | \{ \tilde{\xi}_0^i \tilde{\chi}_i, \tilde{\xi}'_\alpha \} | \\
 & \delta(\tilde{\xi}_0^i \tilde{\chi}_i) \delta(\tilde{\xi}'_\alpha) \prod_\beta \delta(\tilde{\xi}_0^i \tilde{\chi}_i) \delta(\tilde{\xi}'_\beta) \delta(\tilde{\chi}_\beta) \delta(\tilde{\xi}'_\beta) \\
 & \det \begin{vmatrix} \{ \tilde{\xi}_0^i \tilde{\chi}_i, \tilde{\xi}'_{\beta_3} \} & \{ \tilde{\xi}_0^i \tilde{\chi}_i, \tilde{\xi}'_{\beta_4} \} \\ \{ \tilde{\chi}_{\beta_2}, \tilde{\xi}'_{\beta_3} \} & \{ \tilde{\chi}_{\beta_2}, \tilde{\xi}'_{\beta_4} \} \end{vmatrix} \prod_{A=\alpha,\beta} \delta(\tilde{\xi}_0^i \tilde{\pi}_i) \delta(\tilde{\eta}_A) \det | \{ \tilde{\xi}_0^i \tilde{\pi}_i, \tilde{\eta}_A \} |
 \end{aligned} \tag{10.1}$$

where we used Eqs. (8.7) to find the action and Eqs. (8.9), (8.10) for the measure.

As $A_{\xi_0}^i, \tilde{x}_e^i$ are an orthonormal basis of vectors, if we denote them as e_a^i , we have $(\det |e_a^i|)^2 = \det |e_A \cdot e_B| = 1$. As a consequence we get

$$\begin{cases} \prod_e \delta(\tilde{x}_e^i \tilde{\chi}_i) \prod_A \delta(\tilde{\xi}_0^i \tilde{\chi}_i) = \prod_i \delta(\tilde{\chi}_i) \\ \prod_e \delta(\tilde{x}_e^i \tilde{\pi}_i) \prod_A \delta(\tilde{\xi}_0^i \tilde{\pi}_i) = \prod_i \delta(\tilde{\pi}_i) \end{cases} \tag{10.2}$$

Therefore, when the integrations over $\tilde{p}_i$ can be done before those on v^i , we get the following expression for the TQ path integral:

$$K(q, q_0, t-t_0) = \int_{q_0 t_0}^{q t} \mathcal{D}q \mathcal{D}v e^{-\frac{i}{\hbar} \int_{t_0}^t dt \tilde{L}_v} \prod_e |\tilde{\mu}_e| \sqrt{\det(-\tilde{\mathcal{R}}_{rr'})} \prod_r \delta(\tilde{D}_r) \prod_s |\det \tilde{T}| \delta(\tilde{\chi}_s) \delta(\tilde{D}_s) \prod_\alpha \delta(\tilde{\xi}_\alpha) \det |\{ \tilde{\xi}_0^i \tilde{\chi}_i, \tilde{\xi}_\alpha' \}|$$

(10.3)

$$\prod_\beta \delta(\tilde{\chi}_\beta) \delta(\tilde{\xi}_\beta) \delta(\tilde{\xi}_\beta') \det \begin{vmatrix} \{ \tilde{\xi}_0^i \tilde{\chi}_i, \tilde{\xi}_{\beta_3} \} & \{ \tilde{\xi}_0^i \tilde{\chi}_i, \tilde{\xi}_{\beta_4}' \} \\ \{ \tilde{\chi}_{\beta_2}, \tilde{\xi}_{\beta_3} \} & \{ \tilde{\chi}_{\beta_2}, \tilde{\xi}_{\beta_4}' \} \end{vmatrix} \prod_{A=\alpha, \beta} \delta(\tilde{\eta}_A) \det |\{ \tilde{\xi}_0^i \tilde{\eta}_i, \tilde{\eta}_A \}|$$

where all the functions $\tilde{f}$ inside the measure have to be evaluated at $\tilde{p}_i = \partial \tilde{L} / \partial v^i$, $\tilde{\pi}_i = 0$. Equation (10.3) has to be compared with Eq. (3.14).

To get the T^*Q expression of the path integral is more difficult, because rank $\tilde{A} = n - (k_1 + k_2 + 2k_3 + k_4)$ does not allow all the v^i to be obtained from $\tilde{\chi}_i = \tilde{p}_i - \partial \tilde{L} / \partial v^i = 0$ in terms of q^i , $\tilde{p}_i$. Only the subset $\tilde{\xi}_0^i \tilde{\chi}_i$ of the second-class constraints can be inverted to get rank $\tilde{A}$ velocities in terms of q^i , $\tilde{p}_i$. Moreover, all the $\tilde{\pi}_i$ -independent constraints $\tilde{\xi}_0^i \tilde{\chi}_i$, $\tilde{\chi}_\beta$, $\tilde{\chi}_s$, notwithstanding the fact that they contain the same information on the T^*Q constraints $\tilde{\Phi}_A$, $\tilde{\Phi}_\beta$, $\tilde{\Phi}_s$, are velocity-dependent. And also the gauge-fixing constraints $\tilde{\xi}_\alpha$, $\tilde{\xi}_\beta$, $\tilde{\xi}_\beta'$, counterparts of the T^*Q ones $\tilde{\xi}_\alpha$, $\tilde{\xi}_\beta$, $\tilde{\xi}_\beta'$, even if taken $\tilde{\pi}_i$ -independent are velocity-dependent. This shows that the $T^*(TQ)$ path integral is much more general than the T^*Q path integral:

$$\int_{q_0 t_0}^{q t} \mathcal{D}q \mathcal{D}p e^{-\frac{i}{\hbar} \int_{t_0}^t dt (\dot{q}^i p_i - \tilde{H}_A)} \prod_r \delta(\tilde{\Phi}_r) \sqrt{\det(-\tilde{\mathcal{R}}_{rr'})} \prod_s \delta(\tilde{\Phi}_s) \delta(\tilde{\Phi}_s') \det \tilde{T} \prod_\alpha \delta(\tilde{\Phi}_\alpha) \delta(\tilde{\xi}_\alpha) \det |\{ \tilde{\Phi}_\alpha, \tilde{\xi}_\alpha' \}|$$

$$\prod_\beta \delta(\tilde{\Phi}_\beta) \delta(\tilde{\Phi}_\beta') \delta(\tilde{\xi}_\beta) \delta(\tilde{\xi}_\beta') \det \begin{vmatrix} \{ \tilde{\Phi}_{\beta_1}, \tilde{\xi}_{\beta_3} \} & \{ \tilde{\Phi}_{\beta_1}, \tilde{\xi}_{\beta_4}' \} \\ \{ \tilde{\Phi}_{\beta_2}, \tilde{\xi}_{\beta_3} \} & \{ \tilde{\Phi}_{\beta_2}, \tilde{\xi}_{\beta_4}' \} \end{vmatrix}$$

(10.4)

After doing the integration over $\tilde{\pi}_i$, Eqs. (10.1) become

$$K(q, q_0, t-t_0) = \int_{q_0 t_0}^{q t} \mathcal{D}q \mathcal{D}\tilde{p} \mathcal{D}v e^{-\frac{i}{\hbar} \int_{t_0}^t dt (\dot{q}^i \tilde{p}_i - \tilde{E})} \prod_{A=\alpha, \beta} \delta(\tilde{\xi}_0^i \tilde{\chi}_i) \delta(\tilde{\eta}_A) \prod_e \delta(\tilde{\mu}_e \tilde{\chi}_i) \prod_r \delta(\tilde{\xi}_0^i \tilde{\chi}_i) \delta(\tilde{D}_r) \prod_s \delta(\tilde{\xi}_0^i \tilde{\chi}_i) \delta(\tilde{\chi}_s) \delta(\tilde{D}_s) \prod_\alpha \delta(\tilde{\xi}_\alpha) \prod_\beta \delta(\tilde{\chi}_\beta) \delta(\tilde{\xi}_\beta) \delta(\tilde{\xi}_\beta') \prod_e |\tilde{\mu}_e| \sqrt{\det(-\tilde{\mathcal{R}}_{rr'})} \det \tilde{T} \det |\{ \tilde{\xi}_0^i \tilde{\chi}_i, \tilde{\xi}_\alpha' \}|$$

$$\det \left| -\tilde{\xi}_0^i \frac{\partial \tilde{\eta}_A}{\partial v^i} \right|_{A=\alpha, \beta} \det \begin{vmatrix} \{ \tilde{\xi}_0^i \tilde{\chi}_i, \tilde{\xi}_{\beta_3} \} & \{ \tilde{\xi}_0^i \tilde{\chi}_i, \tilde{\xi}_{\beta_4}' \} \\ \{ \tilde{\chi}_{\beta_2}, \tilde{\xi}_{\beta_3} \} & \{ \tilde{\chi}_{\beta_2}, \tilde{\xi}_{\beta_4}' \} \end{vmatrix}$$

(10.5)

If we develop v^i on the orthonormal basis of the eigenvectors of $\tilde{A}$, $v^i = \Sigma_A \tilde{v}^A \tilde{\xi}_0^i + \Sigma_e \tilde{v}^e \tilde{\chi}_e^i$, we get $\tilde{E}(q, v) = \tilde{E}(q, \tilde{v}^e)$, because $\partial \tilde{E} / \partial \tilde{v}^A = \tilde{\xi}_0^i \partial \tilde{E} / \partial v^i = \tilde{\xi}_0^i \tilde{A}_{ij} v^j = 0$ (remember that in general this can only be done locally). Therefore, the constraints $\tilde{\chi}_e^i \tilde{\chi}_i$ can be solved, at least locally, in $\tilde{v}^e$

$$\begin{cases} \tilde{\chi}_e^i \tilde{\chi}_i = \sum_{e'} \hat{F}_{ee'} [\tilde{v}^{e'} - \hat{g}^{e'}(q, \hat{p}, \tilde{v}^A)] \\ \prod_e \delta(\tilde{\chi}_e^i \tilde{\chi}_i) = \prod_e \frac{\delta(\tilde{v}^e - \hat{g}^e)}{|\det(\hat{F}_{ee'})|} \end{cases} \quad (10.6)$$

As the Jacobian from v^i to $\tilde{v}^e$, $\tilde{v}^A$ is one, the $\tilde{v}^e$ can be integrated and, as $\tilde{E}$ is $\tilde{v}^A$ independent, $\tilde{E}(q, \tilde{f}^e) = \bar{H}_d(q, \hat{p})$. Equations (10.5) become, disregarding the locality of the inversion,

$$K(q, q_0, t-t_0) = \int_{q_0 t_0}^{q t} \mathcal{D}q \mathcal{D}\tilde{p} e^{-\frac{i}{\hbar} \int_{t_0}^t dt (\dot{q}^i \hat{p}_i - \bar{H}_d)} \\ \cdot \int \frac{\prod_A \mathcal{D}\tilde{v}^A}{|\det(\hat{F}_{ee'})|} \prod_{A=d||\tilde{p}} \delta(\hat{\eta}_A) \delta(\hat{\xi}_0^i \tilde{\chi}_i) \dots \quad (10.7)$$

where the dots mean all the other terms in Eq. (10.5) and the measure is evaluated at $\tilde{v}^e = \tilde{f}^e$.

Now, at least locally, we have

$$\begin{cases} \hat{\xi}_0^i \tilde{\chi}_i |_{\tilde{v}^e = \tilde{f}^e} = \sum_B \hat{F}_{AB} \bar{\Phi}_B(q, \hat{p}) \\ \tilde{\chi}_\beta |_{\tilde{v}^e = \tilde{f}^e} = \sum_{\beta'} \hat{F}_{\beta\beta'} \bar{\Phi}'_{\beta'}(q, \hat{p}) \\ \tilde{\chi}_s |_{\tilde{v}^e = \tilde{f}^e} = \sum_{s'} \hat{F}_{ss'} \bar{\Phi}'_{s'}(q, \hat{p}) \end{cases} \quad (10.8)$$

If the gauge-fixings $\tilde{\xi}_\alpha$, $\tilde{\xi}_\beta$, $\tilde{\xi}'_\beta$, are chosen in such a way that their restriction to $\tilde{v}^e = \tilde{f}^e$ coincides with $\bar{\xi}_\alpha$, $\bar{\xi}_\beta$, $\bar{\xi}'_\beta$, Eq. (10.7) becomes

$$K(q, q_0, t-t_0) = \int_{q_0 t_0}^{q t} \mathcal{D}q \mathcal{D}\tilde{p} e^{-\frac{i}{\hbar} \int_{t_0}^t dt (\dot{q}^i \hat{p}_i - \bar{H}_d)} \\ \prod_A \delta(\bar{\Phi}_A) \prod_\alpha \delta(\bar{\xi}_\alpha) \prod_\beta \delta(\bar{\Phi}'_\beta) \delta(\bar{\xi}_\beta) \delta(\bar{\xi}'_\beta) \prod_s \delta(\bar{\Phi}'_s) \cup \quad (10.9)$$

with

$$U = \int \prod_A \mathcal{D}\tilde{v}^A \prod_{A=\alpha,\beta} \delta(\hat{\tilde{\eta}}_A) \prod_r \delta(\tilde{D}_r) \prod_s \delta(\tilde{D}_s) \frac{\pi_e |\tilde{\mu}_e| \sqrt{\det(-\tilde{Q}_{rr})} \det \tilde{T}}{|\det(\tilde{F}_{ee}) \det(\tilde{F}_{AB}) \det(\tilde{F}_{\beta\rho}) \det(\tilde{F}_{ss})|} \quad (10.10)$$

$$\det \left| \left\{ \tilde{\xi}_0^i \hat{\tilde{\chi}}_i, \tilde{\xi}_{\alpha 1}^i \right\}' \right| \det \left| -\tilde{\xi}_0^i \frac{\partial \hat{\tilde{\eta}}_{\alpha 1}^i}{\partial v^i} \right|_{A=\alpha,\beta} \det \left| \left\{ \tilde{\xi}_{\beta 1}^i \hat{\tilde{\chi}}_i, \tilde{\xi}_{\beta 3}^i \right\}' \right| \left\{ \tilde{\xi}_0^i \hat{\tilde{\chi}}_i, \tilde{\xi}_{\beta 4}^i \right\}' \right|$$

$$\left| \left\{ \tilde{\chi}_{\beta 2}, \tilde{\xi}_{\beta 3} \right\}' \right| \left\{ \tilde{\chi}_{\beta 2}, \tilde{\xi}_{\beta 4} \right\}' \right|$$

If gauge-fixing constraints $\tilde{\eta}_{A=\alpha,\beta} \approx 0$ exist such that the associated U reproduce the determinant terms in the measure of Eq. (10.4), then Eqs. (10.4) and (10.9) may agree. However, in general, too many local operations are required, so that globally the $T^*(TQ)$ path integral is more general than the T^*Q one.

To recover a measure for the path integral on the configuration space, one should start from the TQ path integral (10.3) and evaluate the integrations on the velocities.

To apply the BRS approach one should either apply the standard BFV method to the $T^*(TQ)$ path integral or to find a BRS gauge-fixing for the Lagrangian $\tilde{L}_v$ to be used inside a TQ path integral. In both cases the measures should contain the terms corresponding to the second-class constraints. As there are the extra gauge freedoms associated with the first-class constraints $\tilde{\xi}_0^i \tilde{\pi}_i$, $A = \alpha, \beta$, extra ghosts and antighosts would be required and it is not yet clear how to make contact with the standard BRS [for $L(q, \dot{q})$] and BFV [for T^*Q] approaches, which do not contain these extra variables; however a bridge must exist, at least perturbatively, as in the transition from $T^*(TQ)$ to T^*Q .

11. CONCLUSIONS

We have obtained a unification of the Lagrangian, velocity- and phase-space descriptions of constrained systems by means of the phase space over the velocity space. It seems that this possibility was precedently unnoticed. For instance, Ref. [38] uses the Whitney sum of TQ and T^*Q (i.e. a space with independent coordinates (q^i, v^i, p_i)), instead of $T^*(TQ)$, while in Ref. [53], exercise 5.3 L, $T^*(TQ)$ is introduced but only used to identify the relevant Lagrangian submanifolds, having in mind the geometric quantization approach. Along the way we had to clarify some intriguing points, such as the difference between Lagrangian constraints and genuine first-order equations of motion in TQ , the connection between the extra gauge vector fields Z_A on TQ and $T^*(TQ)$, and the local dynamical symmetries of the Euler-Lagrange equations, the identification of the Noether gauge transformations as Jacobi fields, the necessity for the Lagrangian to generate a gauge algebra to ensure the global integrability of its extremals, the identification of the general Noether identities and their connection with the Dirac-Bergmann algorithm.

We made the analysis for a sufficiently general class of models, which incorporates most of the interesting examples. If we relax the simplifying hypothesis (5.2) to (5.8), nothing new happens except that the discussion becomes more involved. For more general singular systems the chains of Noether identities (and therefore also the chains of higher order constraints deriving from the primary ones in T^*Q) could have the last identity (the last constraint) non trivial and new phenomena like ramifications of chains could occur. This means that the D term in Eqs. (5.21) may be the product of various functions and we have a bifurcation of the dynamics according to which of these functions is vanishing. In other words, the Euler-Lagrange equations admit different classes of solutions, and to each class will correspond a different pattern of first- and second-class constraints.

Besides these problems we postpone to future papers also the case of a Hessian matrix of variable rank. This requires an analysis of its non-null eigenvalues and the definition of new kinds of constraints, besides the first- and second-class ones, to resolve the ‘pathological’ examples and understand when the Dirac conjecture fails. However, it is not clear whether there may be physically relevant examples of this type.

Concerning quantization, we have introduced the path integral on $T^*(TQ)$ and shown how to get the measure of it on TQ . In the regular case also that on T^*Q is easily recovered, while in the singular case the $T^*(TQ)$ description is more general than the T^*Q one, which in general can be obtained only locally. This is due to the extra gauge freedom present in TQ . It is still unclear whether this new path integral will be useful for practical calculation, but conceptually it allows the unification of the physics of the reduced velocity- and phase-spaces, $\tilde{\gamma}_R$ and $\bar{\gamma}_R$, also at the quantum level and this could be relevant at the global level.

We have also delineated how to make the canonical quantization of first-class constraints without introducing gauge fixings with its two off- and on-shell steps and associated scalar products. We pointed out some of the non-trivial problems involved: When there are non-isomorphic gauge orbits, must we quantize them as independent systems?; Is the reduced phase space $\bar{\gamma}_R$ a manifold when this happens? (in field theory the Gribov ambiguity will complicate the situation even more; see Ref. [81] for the space of orbits of Yang-Mills theory); Are the mathematical structures of geometric quantization needed to glue the various local Hilbert spaces that arise with the Abelianization process equivalent to the covariant canonical quantization followed by the reduction to the physical states, and are we losing something of the non-Abelian theory by using it? The last questions have some aspects in common with the problem of the equivalence of the BFM approach (where the independence from the gauge-fixing function is based on formal changes of variables inside the path integral) with all the possible non-covariant quantizations based on Dirac brackets: how to build the explicit isomorphisms among all the associated Hilbert spaces even when the Gribov problem is

absent and the Dirac brackets may be globally defined? A related problem, which is under investigation, is the connection between the presymplectic structure of the final constraint submanifold $\bar{\gamma}_F$ and the Dirac brackets on one hand, the BFV approach [26,27] and the Dewitt-Vilkowsky action [82] on the other hand.

Another question under study is how and when the BFV method can be obtained starting from the BRST approach. This is why we have not touched the BRST gauge-fixing of $\tilde{L}_v$ and the BFV treatment of $T^*(TQ)$ in this paper, referring the reader to the standard literature.

The formal extension of many of the results of this paper to field theory should not be difficult and will be done elsewhere. However, at a more rigorous level all the problems of the infinite dimensional Hamiltonian systems [13, 18, 83] have to be faced and the canonical quantization will be possible only after a better understanding of the unsolved problems quoted in the Introduction.

Finally, a clear reformulation of the topological problems and of the classical background of the anomalies (when it exists) would be needed starting from the finite-dimensional case for the sake of simplicity to delineate which should be the complete set of specifications (manifold structure, properties of the Lagrangian, ...), making a system well defined and consistent.

Acknowledgement

I should like to thank K. Sundermeyer for helpful discussions about the Noether identities. I also acknowledge the Theory Division for giving me the opportunity of realizing this work in a stimulating atmosphere.

APPENDIX: THE FIRST NOETHER THEOREM IN PHASE SPACE

For a regular system with Lagrangian L , the phase-space action has been given in Eqs. (2.7) and the Hamilton Eqs. (2.6) are

$$\begin{cases} \bar{L}_q^i = \dot{q}^i - \{q^i, \bar{H}\} \stackrel{e}{=} 0 \\ \bar{L}_{p_i} = \dot{p}_i - \{p_i, \bar{H}\} \stackrel{e}{=} 0 \end{cases} \quad (A1)$$

Let us assume that $\bar{L}$ is quasi-invariant under global transformations $\overline{\delta q^i}, \overline{\delta p_i}$:

$$\begin{aligned} \delta \bar{L} &= \overline{\delta p_i} \dot{q}^i + p_i \frac{d}{dt} \overline{\delta q^i} - \delta \bar{H} = \\ &= \overline{\delta p_i} \bar{L}_q^i - \overline{\delta q^i} \bar{L}_{p_i} + \frac{d}{dt} (p_i \overline{\delta q^i}) \stackrel{e}{=} \frac{d}{dt} \bar{F}(q, p) \end{aligned} \quad (A2)$$

By introducing $\bar{G} = p_i \overline{\delta q^i} - \bar{F}$, we get the following Noether identity

$$\frac{d}{dt} \bar{G} \stackrel{e}{=} \overline{\delta q^i} \bar{L}_{p_i} - \overline{\delta p_i} \bar{L}_q^i \stackrel{e}{=} 0 \quad (A3)$$

implying that $\bar{G}$ is a constant of the motion. Equations (A3) imply

$$\begin{cases} \overline{\delta q^i} \stackrel{e}{=} \frac{\partial \bar{G}}{\partial p_i} = \{q^i, \bar{G}\} \\ \overline{\delta p_i} \stackrel{e}{=} -\frac{\partial \bar{G}}{\partial q^i} = \{p_i, \bar{G}\} \\ \delta \bar{H} = \frac{\partial \bar{H}}{\partial q^i} \overline{\delta q^i} + \frac{\partial \bar{H}}{\partial p_i} \overline{\delta p_i} \stackrel{e}{=} -\{\bar{G}, \bar{H}\} \stackrel{e}{=} \frac{\partial \bar{G}}{\partial t} = 0 \end{cases} \quad (A4)$$

and show that $\bar{G}$ is the generator of a canonical transformation, that $\{\bar{G}, \bar{H}\} = 0$ in accordance with $\bar{G} \stackrel{e}{=} 0$, and that the fact that the generator is a constant of the motion implies a special feature of the canonical transformation, i.e. that the functional form of the Hamiltonian does not change, $\delta_0 \bar{H} = \bar{H}'(q, p) - \bar{H}(q, p) = 0$.

This is the phase-space formulation of the first Noether theorem, whose Lagrangian form is that if L is quasi-invariant under the global transformations $\delta q^i(q, \dot{q}) = \overline{\delta q^i}(q, p)$, i.e. $\delta L = (d/dt) F(q, \dot{q})$ with $F(q, \dot{q}) = \bar{F}(q, p)$, t. we get the Noether identity

$$\frac{d}{dt} G(q, \dot{q}) \stackrel{e}{=} -\delta q^i L_i \stackrel{e}{=} 0 \quad G(q, \dot{q}) = \bar{G}(q, p) \quad (A5)$$

REFERENCES

- [1a] P.A.M. Dirac, *Can. J. Math.* **2**, 129 (1950), *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Monographs Series (Yeshiva University, New York, 1964).
- [1b] J.L. Anderson and P.G. Bergmann, *Phys. Rev.* **83**, 1018 (1951).
P.G. Bergmann and J. Golberg, *Phys. Rev.* **98**, 531 (1955).
- [2] A.J. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian Systems*, *Contributi del Centro Linceo Interdisciplinare di Scienze Matematiche, Fisiche e loro Applicazioni*, No 22, *Accademia Nazionale dei Lincei*, Roma, 1975.
- [3] E.C.G. Sudarshan and N. Mukunda, *Classical Mechanics: a Modern Perspective* (Wiley, New York, 1974).
- [4] K. Sundermeyer, *Constrained Dynamics with Applications to Yang-Mills Theory, General Relativity, Classical Spin, Dual String Model*, *Lecture Notes in Physics* (Springer, Berlin, 1982), vol. 169.
- [5] *Constraint's Theory and Relativistic Dynamics*, Firenze Workshop 1986, eds. G. Longhi and L. Lusanna (World Scientific, Singapore, 1987).
- [6] A. Ashtekar, *New Perspectives in Canonical Gravity* (Bibliopolis, Naples, 1988).
- [7] I.T. Todorov, Sissa report, Trieste, 1980 (unpublished); *in Quantum Theory, Groups, Fields and Particles*, ed. A.O. Barut (Reidel, Dordrecht, 1983).
- [8] *Relativistic Action-at-a-Distance: Classical and Quantum Aspects*, ed. J.A. Llosa, *Lecture Notes in Physics* (Springer, Berlin, 1982), vol. 162.
- [9] L. Lusanna *in Proc. VII Seminar on Problems of High Energy Physics and Quantum Field Theory*, Protvino, USSR, 1984, vol. 1, p. 123; *Nuovo Cimento* **65B**, 135 (1981).
- [10] A. Lichnerowicz, *C.R. Acad. Sci. Paris*, **A280**, 523 (1975).
- [11] J. Sniatycki, *in Proc. Journées relativistes*, Caen, 1970; *in Proc. Biennial Seminar Math. Cong.* **2**, 125 (1972); *Ann. Inst. H. Poincaré* **20**, 365 (1974).
- [12] W. Tulczyiew, *Symposia Math.* **14**, 247 (1974).
- [13] M.J. Gotay, *Presymplectic manifolds, geometric constraint theory and the Dirac-Bergmann theory of constraints*, Dissertation, Univ. of Maryland, Phys. Dept. preprint 80-063, 1979 (unpublished).
M.J. Gotay, J.M. Nester and G. Hinds, *J. Math. Phys.* **19**, 2388 (1978).
- [14] J. Sniatycki, *in Lecture Notes in Mathematics* 1037, *Proc. Clausthal Workshop*, 1981, ed. A. Dold and B. Eckmann (Springer, Berlin, 1983); *J. Geom. Phys.* **2**, 1 (1985).
- [15] M.J. Gotay and J. Sniatycki, *Commun. Math. Phys.* **82**, 377 (1981).
- [16] M.J. Gotay, *J. Math. Phys.* **27**, 2051 (1986).
N. Woodhouse, *Geometric Quantization* (Clarendon, Oxford, 1980).
- [17] A. Ashtekar and M. Stillerman, *J. Math. Phys.* **27**, 1319 (1986).
- [18] C.A. Hurst, *in Recent Developments in Mathematical Physics*, *Proc. XXVI Schladming*, 1987, eds. H. Mitter and L. Pittner (Springer, Berlin, 1987).
- [19] C.J. Isham, *in Relativity, Groups and Topology II*, *Proc. XL Les Houches*, 1983, eds. B.S. De Witt and R. Stora (North Holland, Amsterdam, 1984).
- [20] R.P. Feynman, *Acta Phys. Pol.* **24**, 697 (1963).
L.D. Faddeev and V.N. Popov, *Phys. Lett.* **25B**, 30 (1967).
B.S. De Witt, *Phys. Rev.* **162**, 1195 and 1239 (1967).
- [21] A.A. Slavnov, *Theor. Math. Phys.* **10**, 153 (1972).

- [22] C. Becchi, A. Rouet and R. Stora, Phys. Lett. **52B**, 344 (1974); Commun. Math. Phys. **42**, 127 (1975); Ann. Phys. (N.Y.) **98**, 287 (1976); CNRS Centre de Physique Théorique, Marseille, Colloquium on Recent Progress in Lagrangian Field Theory, June 1974.
- [23] I.V. Tyutin, FIAN **39** (1975) (unpublished).
- [24] E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. **55B**, 224 (1975); preprint CERN/TH. 2332 (1977) (unpublished).
I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **69B**, 309 (1977).
- [25] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **102B**, 27 (1981); Nucl. Phys., **B234**, 106 (1984).
- [26] M. Henneaux, Phys. Rep. **126**, 1 (1985); Classical Foundations of BRS Symmetry (Bibliopolis, Napoli, 1988).
- [27] B. Konstant and S. Sternberg, Ann. Phys. (N.Y.) **176**, 49 (1987).
M. Henneaux and C. Teitelboim, Commun. Math. Phys. **115**, 213 (1988).
- [28] I.A. Batalin and E.S. Fradkin, Phys. Lett. **128B**, 303 (1983); Sov. J. Nucl. Phys. **39**, 145 (1984); J. Math. Phys. **25**, 2426 (1984); Phys. Lett. **180B**, 157 (1986); Riv. Nuovo Cimento **9**, 1 (1986); Nucl. Phys. **B279**, 514 (1987).
L.D. Faddeev and S.L. Shatashvili, Phys. Lett. **167B**, 225 (1986).
- [29] R. Stora, Differential Algebras in Field Theory, Talk at the Second Summer Meeting on Quantum Mechanics of Fundamental Systems, Santiago, Chile, 1987.
- [30] R.O. Jackiw, in Relativity, Groups and Topology II, Proc. XL Les Houches 1983, eds. B.S. De Witt and R. Stora (North Holland, Amsterdam, 1984).
- [31] L. Faddeev, in Recent Developments in Mathematical Physics, Proc. XXVI Schlading, 1987, eds. H. Mitter and L. Pittner (Springer, Berlin, 1987).
- [32] L. Alvarez-Gaumé, in Fundamental Problems of Gauge Theories, Proc. 6th E. Majorana School, Erice, 1985, eds. G. Velo and A.S. Wightman (Plenum, New York, 1986), ASI B141.
- [33] R. Cawley, Phys. Rev. Lett. **42**, 413 (1979); Phys. Rev. **D21**, 2988 (1980).
A. Frenkel, Phys. Rev. **D21**, 2986 (1980).
G.R. Allcock, Phil. Trans. R. Soc. London **A279**, 487 (1975); Kinam Rev. Fis. **2**, 335 (1980).
R. Sugano and T. Kimura, Prog. Theor. Phys. **69**, 1241 (1983).
R. Di Stefano, Phys. Rev. **D27**, 1752 (1983).
- [34] M.J. Gotay and J.M. Nester, in Group Theoretical Methods in Physics, Austin, 1978. Lecture Notes in Physics **94** (Springer, Berlin, 1979).
M.J. Gotay, J. Phys. **A16**, L141 (1983).
D.M. Appleby, J. Phys. **A15**, 1191 (1982).
A. Cabo, J. Phys. **A19**, 629 (1986).
- [35] R. Sugano and H. Kamo, Prog. Theor. Phys. **67**, 1966 (1982).
R. Sugano, Prog. Theor. Phys. **68**, 1377 (1982).
- [36] R. Sugano and T. Kimura, Prog. Theor. Phys. **69**, 252 (1983); **69**, 1256 (1983); Nuovo Cimento **87A**, 57 (1985).
R. Sugano, Prog. Theor. Phys. **70**, 36 (1983).
R. Sugano, T. Sohikawa and T. Kimura, Prog. Theor. Phys. **73**, 1025 (1985).
R. Sugano, Y. Saito and T. Kimura, Prog. Theor. Phys. **76**, 283 (1986).
- [37] M.J. Gotay and J.M. Nester, J. Phys. **A17**, 3063 (1984).
- [38] R. Skinner and R. Rusk, J. Math. Phys. **24**, 2589 and 2595 (1983).
- [39] K. Kamimura, Nuovo Cimento **69B**, 33 (1982).
- [40] M.J. Gotay and J.M. Nester, Ann. Inst. H. Poincaré **A30**, 129 (1979); **A32**, 1 (1980).
- [41] R. Skinner, J. Math. Phys. **24**, 2581 (1983).
- [42] S. Shanmugadhasan, J. Math. Phys. **14**, 677 (1973).

- [43] R.L. Schafir, J. Phys. **A15**, L331 (1982).
- [44] C. Batlle, J. Gomis, J.M. Pons and N. Roman, J. Math. Phys. **27**, 2963 (1986).
- [45] N. Mukunda, Ann. Phys. (NY) **99**, 408 (1976); Phys. Scr. **21**, 783 (1980).
- [46] G. Marmo, N. Mukunda and J. Samuel, Riv. Nuovo Cimento **6**, 1 (1983).
- [47] L. Lusanna, Nuovo Cimento **52B**, 141 (1979).
- [48] E. Noether, Nachr. Ges. Wiss. Göttingen, Math. Phys. Klasse, **H.2**, 235 (1918); English Translation in Transport Theory and Statistical Physics **1**, 183 (1971).
- [49] J.D. Logan, Invariant Variational Principles (Academic, New York, 1977).
B.F. Plybon, J. Math. Phys. **12**, 57 (1971).
- [50] N.P. Konopleva and V.N. Popov, Gauge Fields (Harwood, New York, 1981).
- [51] B.M. Barbashov and V.V. Nesterenko, Fortschr. Phys. **31**, 535 (1983).
- [52] P.G. Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1953).
P.G. Bergmann, I. Goldberg, A. Janis and E. Newman, Phys. Rev. **103**, 807 (1956).
P.G. Bergmann and A.J. Janis, Phys. Rev. **111**, 1191 (1958).
J.G. Fletcher, Rev. Mod. Phys. **32**, 65 (1960).
V.I. Ogiyevetsky and I.V. Polubarinov, Nucl. Phys. **76**, 677 (1966).
W.R. Davis and J.W. York Jr., Nuovo Cimento **65B**, 1 (1970).
R. Sugano, Prog. Theor. Phys. **70**, 36 (1983).
A.M. Chervyakov and V.V. Nesterenko, Theor. Math. Phys. **64**, 82 (1985).
L. Castellani, Ann. Phys. (NY) **143**, 357 (1982).
A. Barducci, R. Giachetti, J. Gomis and E. Sorace, J. Phys. **A17**, 3277 (1984).
J. Gomis, K. Kamimura and J.M. Pons, Prog. Theor. Phys. **71**, 1051 (1984); Europhys. Lett. **2**, 187 (1986).
I. Gràcia and J.M. Pons, Gauge generators, Dirac's conjecture and degrees of freedom for constrained systems, Barcelona Univ. preprint UB-ECM-PF 1/88.
- [53] R. Abraham and J.E. Marsden, Foundations of Mechanics (Benjamin, Mass., 1979), 2nd edn.
- [54] G. Marmo, E.J. Saletan, A. Simoni and B. Vitale, Dynamical Systems (Wiley, Chichester, 1985).
- [55] I.A. Batalin and E.S. Fradkin, Phys. Lett. **122B**, 157 (1983); Lett. Nuovo Cimento **38**, 393 (1983); Phys. Rev. **D28**, 2567 (1983).
- [56] Y. Dothan, Nuovo Cimento **11A**, 499 (1972).
- [57] R. Courant and D. Hilbert, Methods of Mathematical Physics (Interscience, New York, 1953), vol. 1.
B.S. De Witt, in Relativity, Groups and Topology, Proc. XIII Les Houches, 1963, eds. C. De Witt and B.S. De Witt (Gordon and Breach, London, 1964).
L.S. Schulman, Techniques and Applications of Path Integrations (Wiley, New York, 1981).
- [58] I.A. Batalin and G.A. Vilkovisky, J. Math. Phys. **26**, 172 (1985).
- [59] D.Z. Freedman and P. van Nieuwenhuizen, Phys. Rev. **D14**, 912 (1976).
- [60] L. Lusanna, in Proc. IV M. Grossmann Meeting, Roma, 1985, ed. R. Ruffini (Elsevier, Amsterdam, 1986).
- [61] G. Morandi, C. Ferrario, G. Lo Vecchio, G. Marmo and C. Rubano, The inverse problem in the calculus of variations and the geometry of the tangent bundle, Dept. of Physics, Naples University, 1987, preprint.
- [62] R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integral (McGraw Hill, New York, 1965).
E.S. Abers and B.W. Lee, Phys. Rep. **9**, 1 (1973).

- L. Faddeev and A. Slavnov, Gauge fields: Introduction to Quantum Theory (Benjamin/Cummings, London, 1980).
- B.S. De Witt, Rev. Mod. Phys. **29**, 377 (1957).
- [63] L.S. Schulman, Techniques and Applications of Path Integration (Wiley, New York, 1981).
- J. Gervais and A. Jevicki, Nucl. Phys. **B10**, 93 (1980).
- A.Y. Shiekh, Canonical transformations in quantum mechanics, ICTP preprint IC/87/39, Trieste, 1987.
- H. Fukutaka and T. Kashiwa, Canonical transformations in the path integral and the operator formalism, Kyushu Univ. preprint 87-HE-5, 1987.
- J.R. Klauder, in Recent developments in mathematical physics, Proc. XXVI Schlading, 1987, eds. H. Mitter and L. Pittner (Springer, Berlin, 1987).
- V. De Alfaro, S. Fubini, G. Furlan and M. Roncadelli, Phys. Lett. **B200**, 323 (1987); Nucl. Phys. **B296**, 402 (1988).
- [64] P. Senjanovic, Ann. Phys. (NY) **100**, 227 (1976).
- T. Maskawa and H. Nakajima, Prog. Theor. Phys. **56**, 1295 (1976).
- V.V. Nesterenko, Theor. Math. Phys. **69**, 1038 (1986).
- P. Ditsas, Ann. Phys. (NY) **167**, 36 (1986).
- W. Garczynski, in Fields and geometry, Proc. XXII Karpacz School, 1986, ed. A. Jadczyk (World Scientific, Singapore, 1986).
- [65] P.G. Bergmann, Rev. Mod. Phys. **33**, 510 (1961).
- [66] A. Trautman, Differential Geometry for Physicists (Bibliopolis, Naples, 1984).
- [67] P.G. Bergmann, R. Penfield, R. Schiller and H. Zatzkis, Phys. Rev. **80**, 81 (1950).
- [68] E. Candotti, C. Palmieri and B. Vitale, Nuovo Cimento **70A**, 233 (1970); **7A**, 271 (1972).
- C. Palmieri and B. Vitale, Nuovo Cimento **66A**, 299 (1970).
- [69] V. Aldaya and J.A. De Azcarraga, J. Math. Phys. **19**, 1869 and 1876 (1978); Riv. Nuovo Cimento **3**, 1 (1980).
- F.A.E. Pirani, D.C. Robinson and W.F. Shadwick, Local Jet Bundle Formulation of Bäcklund Transformations (Reidel, Dordrecht, 1979).
- [70] W. Sarlet and F. Cantrijn, SIAM Rev. **23**, 467 (1981).
- [71] N.H. Ibragimov, Sov. Math. Dokl **17**, 1242 (1976).
- R.L. Anderson and N.H. Ibragimov, Lie-Bäcklund Transformations in Applications (SIAM, Philadelphia, 1979).
- F. Guil Guerrero and L. Martinez Alonso, J. Phys. **A13**, 689 (1980).
- [72] A. Burnel, Phys. Rev. **D26**, 442 (1982).
- [73] Ph. Droz-Vincent, Lett. Nuovo Cimento **1**, 839 (1969); Phys. Scr. **2**, 129 (1970); Nuovo Cimento **12B**, 1 (1972); Ann. Inst. H. Poincaré **20A**, 269 (1974); **27A**, 407 (1977); **32A**, 377 (1980); Rep. Math. Phys. **8**, 79 (1975).
- [74] G. Longhi, L. Lusanna and J.M. Pons, On the multitime formulation of the non-relativistic classical particle dynamics, Firenze Univ. preprint, 1987 (to appear in J. Math. Phys.); in Constraint's Theory and Relativistic Dynamics, Firenze Workshop, 1986, eds. G. Longhi and L. Lusanna (World Scientific, Singapore, 1987).
- G. Longhi, in Actas de los Encuentros Relativistas, St. Ander, 1984 (Universidad de Valencia, 1984).
- [75] G. Longhi and L. Lusanna, Phys. Rev. **D34**, 3707 (1986).
- [76] R. Marnelius, Acta Phys. Pol. **B13**, 669 (1982).
- [77] L. Bonora and P. Cotta Ramusino, Commun. Math. Phys. **87**, 589 (1983).
- [78] I.A. Batalin, J. Math. Phys. **22**, 1837 (1981).

- [79] G. Colomo, G. Longhi and L. Lusanna, in preparation.
- [80] J.A. Schouten and M. van der Kulk, *Pfaff's Problem and its Generalizations* (Clarendon, Oxford, 1949).
L.P. Eisenhart, *Continuous Groups of Transformations* (Princeton Univ. Press, Princeton, 1933).
- [81] P.K. Mitter and C. M. Viallet, *Commun. Math. Phys.* **79**, 457 (1981).
O. Babelon and C.M. Viallet, *Commun. Math. Phys.* **81**, 515 (1981).
C.M. Viallet, *in Fields and Geometry*, Proc. XXII Karpacz, 1986, ed. A. Jadczyk (World Scientific, Singapore 1986).
- [82] G.A. Vilkovisky, *Nucl. Phys.* **B234**, 125 (1984); *in Quantum Theory of Gravity*, Ed. S. Christensen (Hilger, Bristol, 1984).
B.S. De Witt, *in Quantum Field Theory and Quantum Statistics*, eds. A. Batalin, C.J. Isham and G.A. Vilkovisky (Hilger, Bristol, 1987), vol. 1.
- [83] R. Schimdt, *Infinite Dimensional Hamiltonian Systems* (Bibliopolis, Naples, 1987).

ADDITIONAL NOTES

The general theory of singular Lagrangians and Hamiltonian constraints without the limitations of constant rank of the Hessian matrix and without the restriction of no tertiary constraints has been developed in the meanwhile in Ref. [A.1]. The pathological examples quoted in the Introduction are studied and fully resolved. The development of the theory requires a careful analysis of the D-terms (and of their Hamiltonian counterparts) and the introduction of new concepts: 3rd and 4th class constraints, proliferations of constraints, ramification and joinings of chains. The Noether identities for classical field theory are also given. The Lagrangian gauge-fixing problem is analysed in the finite-dimensional case and then some aspects of the classical and quantum BRS theory are elucidated in Ref. [A.2].

A topic, which is not treated either in the present paper or in the additional references, is the Hamilton-Jacobi theory for singular systems: it can be found in Ref. [A.3].

ADDITIONAL REFERENCES

- [A.1] L. Lusanna, The second Noether Theorem as the basis of the theory of singular Lagrangians and Hamiltonian constraints, Firenze University preprint 1989 (submitted to Lecture Notes in Physics).
- [A.2] L. Lusanna, On the BRS's, Firenze University preprint 1989.
- [A.3] P.G. Bergmann, Phys. Rev. **144**, 1078 (1966).
A. Komar, Phys. Rev. **D18**, 1881 (1978).
D. Dominici, G. Longhi, J. Gomis and J.M. Pons, J. Math. Phys. **25**, 2439 (1984).

Figure caption

Fig. 1 The relevant spaces and their relationships for a singular system.

Fig. 1