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ALGEBRAIC CHARACTERIZATION OF ERGODICITY

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ABSTRACT

In this paper we continue the study of the hidden symmetries revealed by the path-integral formulation of *classical* mechanics. We find that, besides the *BRS* and anti-*BRS* symmetry recently discovered, this formulation of classical mechanics presents a genuine *supersymmetry* associated to the energy conservation of the system. We prove that any dynamical system with this supersymmetry unbroken is ergodic. Moreover the supersymmetric invariant ground state turns out to be a Gibbs (and KMS) state. KAM and integrable systems have this supersymmetry spontaneously broken but they present many more hidden graded symmetries besides the above-mentioned supersymmetry. These are symmetries associated to each constant of motion (or to each invariant sub-space of non-zero measure) and each of them makes a $S(2)$ superalgebra. For integrable systems this superalgebra is maximal and is $S(2)^n$ where n is the number of degrees of freedom of the system.

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1. INTRODUCTION

Recently we proposed a path-integral formulation of classical Hamiltonian dynamics [1] which may be considered the path-integral counterpart of the operatorial approach to classical mechanics due to Koopman and von Neumann [2]. Our classical path-integral measure gives weight "one" to the classical paths and weight "zero" to all others. It turns out that this measure can be rewritten as the exponential of an action $\tilde{S}$, which, however, does not only depend on the original phase-space coordinates ϕ^a , but also on anticommuting ghosts c^a and $\bar{c}_a$ and an auxiliary field λ_a . The formalism of ref. [1] does not only describe the evolution of scalar probability densities on phase-space, but also of p-form valued fields. They have their origin in the fact that the ghosts c^a provide a basis for the cotangent space. Furthermore, the Hamiltonian $\tilde{\mathcal{H}}$ associated with $\tilde{S}$ can be identified with the Lie-derivative along the Hamiltonian flow. The action $\tilde{S}$ was found to be invariant under a *BRS* and an anti-*BRS* transformation, which mix the bosonic degrees of freedom with the ghosts. Together with a *Sp*(2)-symmetry of the purely fermionic part of the action, they form an *ISp*(2) symmetry algebra. All its generators have natural differential geometric interpretations. In particular, the *BRS* operator can be put in correspondence with the exterior derivative on phase-space. For further details we have to refer to ref. [1].

The purpose of the present paper is to further study the symmetries of the *classical* path-integral. In particular in Section (2), we shall show that the action $\tilde{S}$ is invariant under a set of hidden graded symmetries, which are related to the constants of motion of the original Hamiltonian $H(\phi^a)$. Thus any conservative system has at least one of these graded symmetries coming from the conservation of energy. This particular graded symmetry is a genuine *supersymmetry* in the sense that it closes on the Hamiltonian $\tilde{\mathcal{H}}$. Contrary to the *ISp*(2)-invariance that is related only to the geometry of the phase-space, these new graded symmetries are sensitive to the form of the Hamiltonian of the system. So their knowledge can be used to extract information on the dynamics of the models under study. In Section (3) we shall show that *any system* with the above-mentioned *supersymmetry* unbroken is ergodic [3] and vice-versa. So KAM and integrable systems [4], which are non-ergodic, have that supersymmetry always spontaneously broken. In Section (4) we will also prove that the supersymmetric invariant ground states of the systems are automatically Gibbs states and that the KMS condition [5] can be derived as a sort of Ward-identity of this supersymmetry. It has been a longstanding problem to characterize Gibbs states and many ideas have been evoked [5]: stability, locality, KMS-condition, etc. It might be that the supersymmetry found here embodies all these features and is really at the heart of the problem. Work is in progress in this direction [6].

2. HIDDEN SUPERSYMMETRY AND OTHER GRADED SYMMETRIES

Let us now briefly recall the operatorial and path-integral approach to classical mechanics^{*}. We consider a Hamiltonian system defined by a $2n$ -dimensional phase-space $\mathcal{M}_{2n}$ (with local coordinates $\phi^a, a = 1 \cdots 2n$) and a time-independent Hamiltonian $H(\phi^a)$. Then Hamilton's equations read[†]

$$\dot{\phi}^a(t) = \omega^{ab} \partial_b H(\phi(t)) \quad (2.1)$$

where ω^{ab} is the symplectic matrix [4]. It gives rise to the Poisson bracket:

$$\{A_1, A_2\} = \partial_a A_1 \omega^{ab} \partial_b A_2 \quad (2.2)$$

The time evolution of some observable $A = A(\phi^a(t))$ is given by

$$\frac{d}{dt} A = \{A, H\} \quad (2.3)$$

We shall also consider probability distributions $\varrho = \varrho(\phi^a, t)$. Their dynamics is governed by the Liouville equation:

$$\frac{\partial}{\partial t} \varrho = -\{\varrho, H\} = -\hat{L} \varrho \quad (2.4)$$

where

$$\hat{L} = \omega^{ab} \partial_b H \partial_a \quad (2.5)$$

is the Liouville operator or "Liouvillian". Equation (2.4) is formally solved by

$$\varrho(\phi, t) = e^{\hat{L}t} \varrho(\phi, 0) \quad (2.6)$$

Many features of a dynamical system are encoded in the spectrum of $\hat{L}$. The one we are most interested in here is that of ergodicity. It is characterized in the following way:

^{*} In the following we assume the reader familiar with ref. [1].

[†] In this paper we assume a topological trivial situation where ω^{ab} and its inverse ω_{ab} have their ϕ^a -independent form everywhere.

a system is ergodic if, at fixed energy, the eigenstate ϱ_o of $\hat{L}$ with eigenvalue zero is non-degenerate[3] . This means that, for a given energy, the equation

$$\hat{L}\varrho_o = 0 \quad (2.7)$$

has only one solution. Because of eq. (2.4), this solution is time-independent. Generally speaking, the solutions of (2.7) are functions of the constants of motion $I_i(\phi^a)$, $i = 1, 2, \dots$.*

$$\varrho_o = F(I_i) \quad (2.8)$$

where

$$\{H, I_i\} = 0 \quad (2.9)$$

If the system is completely integrable, for instance, it has n constants of motion (including the energy). If, on the other side, the system is ergodic, the only constant of motion is the energy $E = H(\phi^a)$, so that $\varrho_o = \varrho_o(E)$. For a fixed energy this is a non-degenerate function: it is just a constant[3] on the energy hyper-surface. If there exists a further constant of motion $I(\phi^a)$, we may set $\varrho_o = F(E, I(\phi))$, which is in general not constant for fixed E . Hence any F of the form above, normalizable on the energy hyper-surface, gives rise to a solution of eq. (2.7) and therefore the system is not ergodic. The reader might be a bit puzzled by the abstract definition of ergodicity[3] we have used. It is easy to realize anyhow that this abstract definition corresponds to the intuitive understanding of ergodicity, in the sense that a non-ergodic system will "fill densely"[†] subsections of the energy manifold and it will not be able to "fill densely" the whole energy manifold. In fact let us choose the $\varrho_o = F(E, I(\phi))$ mentioned above: if we start moving with a particle at constant energy E , this particle will also have well-defined values of all the other constants of motion (because of the defined values of its initial conditions), in particular of $I(\phi)$. Now imagine that this value is I_0 , then $I(\phi) = I_0$ will describe a particular sub-surface on the energy hyper-surface, and except for the one-dimensional systems, this sub-surface will not coincide with the whole energy-surface. This is the trivial reason why systems with the above $\varrho_o(E, I(\phi))$ are not ergodic.

* This is easy to prove by inserting $\varrho(I)$ in $\hat{L}$ given by (2.5) and using eq.(9) supplemented by the definition of Poisson brackets (2.2).

† We use this wording in a loose sense, and refer the reader to[3] for a more technical notation. All through this paper we will try to be as pedagogical and intuitive as possible so as to be able to address the subject to particle-physicists.

Now we turn to the classical path-integral introduced in ref. [1]

$$K(\phi_f, c_f, t_f | \phi_i, c_i, t_i) = \int \mathcal{D}''\phi \mathcal{D}\lambda \mathcal{D}''c \mathcal{D}\bar{c} \exp i \int_{t_i}^{t_f} dt \tilde{\mathcal{L}} \quad (2.10)$$

with the Lagrangian

$$\tilde{\mathcal{L}} = \lambda_a \dot{\phi}^a + i\bar{c}_a \dot{c}^a - \tilde{\mathcal{H}} \quad (2.11)$$

and the "super-Hamiltonian"

$$\tilde{\mathcal{H}} = \lambda_a \omega^{ab} \partial_b H + i\bar{c}_a \omega^{ac} \partial_c \partial_b H c^b \quad (2.12)$$

The functional integral (2.10) is over the ϕ^a and c^a -paths with fixed end-points $\phi_{i,f}^a$ and $c_{i,f}^a$, respectively. The variables λ_a and $\bar{c}_a$ are the momenta conjugate to ϕ^a and c^a . In (2.10) we integrate these momenta over all possible values at the end-points. The integral (2.10) can be done in closed form. The result is:

$$K(\phi_f^a, c_f^a, t_f | \phi_i^a, c_i^a, t_i) = \delta^{(2n)}(\phi_f^a - \phi_{cl}^a(t_f, \phi_i)) \delta^{(2n)}(c_f^a - c_{cl}^a(t_f, c_i, [\phi])) \quad (2.13)$$

Here ϕ_{cl}^a and c_{cl}^a denote solutions of Hamilton's equation of motion (2.1) and of the ghost equation of motion

$$[\partial_t \delta_b^a - \omega^{ac} \partial_c \partial_b H(\phi)] c^b = 0 \quad (2.14)$$

with initial values ϕ_i^a and c_i^a respectively at $t = t_i$. We recall that (2.14) is the Jacobi equation associated to eq. (2.1)[‡]. The kernel K can be used to propagate generalized density functions $\tilde{\varrho}$ which depend on the ghosts:

$$\tilde{\varrho}(\phi^a, c^a, t) = \sum_{p=0}^{2n} \frac{1}{p!} \varrho_{a_1 \dots a_p}^{(p)}(\phi^a, t) c^{a_1} \dots c^{a_p} \quad (2.15)$$

These functions are solutions of the diffusion equation

$$i\partial_t \tilde{\varrho}(\phi^a, c^a, t) = \tilde{\mathcal{H}} \tilde{\varrho}(\phi^a, c^a, t) \quad (2.16)$$

where a "Schrödinger-like" representation [1] of $\tilde{\mathcal{H}}$ has been used[°]. Since $\tilde{\mathcal{H}}$ reduces to $-i\tilde{\mathcal{L}}$, in the $p = 0$ -sector, eq. (2.16) is a generalization of the Liouville equation (2.4).

‡ For further details, in particular about the role of the ghosts and the physical relevance of the Jacobi-fields, we refer the reader to ref. [1].

° This means that we replace $\lambda_a \rightarrow -i\partial_a$ and $\bar{c}_a \rightarrow \frac{\partial}{\partial c^a}$ in eq. (2.12).

For a given $\tilde{\varrho}$ we can compute classical averages according to

$$\langle A \rangle_t = \int d^{2n}\phi d^{2n}c A(\phi^a, c^a) \tilde{\varrho}(\phi^a, c^a, t) \quad (2.17)$$

where A is any observable depending on ϕ^a and the ghosts. Furthermore, time-dependent correlation functions can be obtained by taking derivatives of the generating functional

$$\begin{aligned} Z_{\tilde{\varrho}}[J, \Lambda, \bar{\eta}, \eta] &= \\ &= \int \mathcal{D}\phi \mathcal{D}\lambda \mathcal{D}c \mathcal{D}\bar{c} \tilde{\varrho}(\phi(0), c(0)) \exp i \int_0^\infty dt \{ \tilde{\mathcal{L}} + \mathcal{L}_s \} \end{aligned} \quad (2.18)$$

with

$$\mathcal{L}_s \equiv J_a(t)\phi^a(t) + \Lambda^a(t)\lambda_a(t) + \bar{\eta}_a(t)c^a(t) + \bar{c}_a(t)\eta^a(t) \quad (2.19)$$

Here $\tilde{\varrho}$ denotes some initial density distribution specified at $t = 0$.

Next we discuss the symmetries of $\tilde{S} = \int dt \tilde{\mathcal{L}}$. In [1] we found that $\tilde{S}$ is invariant under the transformations generated by the $ISp(2)$ charges:

$$\begin{aligned} Q &= ic^a \lambda_a, \\ \bar{Q} &= i\bar{c}_a \omega^{ab} \lambda_b \\ Q_g &= c^a \bar{c}_a \\ K &= \frac{1}{2} \omega_{ab} c^a c^b, \\ \bar{K} &= \frac{1}{2} \omega^{ab} \bar{c}_a \bar{c}_b \end{aligned} \quad (2.20)$$

We also remind the reader of the basic (graded) commutation relations on which the "operatorial" formalism of our theory is based^{*}:

$$\begin{aligned} [\phi^a, \lambda_b] &= i\delta_b^a \\ [\bar{c}_b, c^a] &= \delta_b^a \end{aligned} \quad (2.21)$$

As an example of the $ISp(2)$ transformations, we only write down

• By "operatorial" here we mean the *classical* operator formalism that can be associated to our *classical* path-integral: for details see ref. [1].

the (anti-)BRS-transformation of ϕ^a here:

$$\begin{aligned}\delta\phi^a &\equiv [\epsilon Q + \bar{\epsilon}\bar{Q}, \phi^a] \\ &= \epsilon c^a + \bar{\epsilon}\bar{c}_b\omega^{ba}\end{aligned}\tag{2.22}$$

The generators (2.20) are not the only generators commuting with $\tilde{\mathcal{H}}$. Let us consider the quantities

$$\begin{aligned}N &= c^a\partial_a H \\ \bar{N} &= \bar{c}_a\omega^{ab}\partial_b H\end{aligned}\tag{2.23}$$

Using (2.21) it is easy to verify that they are conserved:

$$[\tilde{\mathcal{H}}, N] = [\tilde{\mathcal{H}}, \bar{N}] = 0\tag{2.24}$$

Similarly, for all conserved quantities I_i fulfilling (2.9), we can define

$$\begin{aligned}N_i &= c^a\partial_a I_i \\ \bar{N}_i &= \bar{c}_a\omega^{ab}\partial_b I_i\end{aligned}\tag{2.25}$$

and, using (2.9), we again find that these new charges are conserved:

$$[\tilde{\mathcal{H}}, N_i] = [\tilde{\mathcal{H}}, \bar{N}_i] = 0\tag{2.26}$$

If we furthermore assume that the I_i 's are in involution, *i.e.*, that

$$\{I_i, I_j\} = 0\tag{2.27}$$

then the N_i 's and $\bar{N}_i$'s commute[∇] among themselves:

$$[N_i, N_j] = [\bar{N}_i, N_j] = [\bar{N}_i, \bar{N}_j] = 0\tag{2.28}$$

Moreover, eq. (2.9) implies that

$$[N, N_i] = [\bar{N}, N_i] = [\bar{N}, \bar{N}_i] = [\bar{N}, \bar{N}_i] = 0\tag{2.29}$$

The charges (2.23) and (2.25) have a simple physical interpretation. Consider a classical trajectory $\phi^a(t)$ and apply the (anti-)BRS transformation (2.22) to it. Since c^a and $\bar{c}_a$

[∇] Note that the parentheses in (2.27) are the standard Poisson brackets, while the one in eq.(28) are the classical commutators defined in ref.[1]. The relations that we will derive using commutators also hold in the enlarged Poisson brackets structure associated to $\tilde{\mathcal{H}}$ (see ref.[1] for details).

are Jacobi- fields, the new trajectory $\phi^a(t) + \delta\phi^a(t)$ is again a classical solution. However, in general, the old and new trajectories will have different energies. It is easy to see that the energy difference is essentially given by N and $\bar{N}$:

$$\begin{aligned}\delta E &\equiv H(\phi^a + \delta\phi^a) - H(\phi^a) \\ &= \delta\phi^a \partial_a H \\ &= \epsilon c^a \partial_a H + \bar{\epsilon} \bar{c}_b \omega^{ba} \partial_a H \\ &\equiv \epsilon N + \bar{\epsilon} \bar{N}\end{aligned}\tag{2.30}$$

Since the energy is conserved for both the old and the new trajectories, their energy difference must be conserved too. This explains why the charges N and $\bar{N}$ are conserved. The same argument also applies to N_i and $\bar{N}_i$ with regard to the conservation of I_i .

Next let us look at the algebra of the following linear combinations of our symmetry generators:

$$\begin{aligned}Q_H &\equiv Q - \beta N \quad , \quad Q_i \equiv Q - N_i \\ \bar{Q}_H &\equiv \bar{Q} + \beta \bar{N} \quad , \quad \bar{Q}_i \equiv \bar{Q} + N_i\end{aligned}\tag{2.31}$$

with β an arbitrary parameter. In the differential operator representation of ref. [1], they can be written as[†]

$$\begin{aligned}Q_H &= e^{\beta H} Q e^{-\beta H} \quad , \quad Q_i = e^{I_i} Q e^{-I_i} \\ \bar{Q}_H &= e^{-\beta H} \bar{Q} e^{\beta H} \quad , \quad \bar{Q}_i = e^{-I_i} \bar{Q} e^{I_i}\end{aligned}\tag{2.32}$$

Using (2.21) one can easily verify that

$$[Q_H, Q_H] = [\bar{Q}_H, \bar{Q}_H] = [Q_i, Q_j] = [\bar{Q}_i, \bar{Q}_j] = 0\tag{2.33}$$

$$[Q_H, Q_i] = [\bar{Q}_H, \bar{Q}_i] = 0\tag{2.34}$$

$$[Q_H, \bar{Q}_H] = 2i\beta\tilde{\mathcal{H}} \equiv 2\beta l_{(dH)}^\sharp\tag{2.35}$$

$$[Q_i, \bar{Q}_j] = 2(l_{(dI_i)}^\sharp + l_{(dI_j)}^\sharp) = [Q_j, \bar{Q}_i]\tag{2.36}$$

$$[Q_H, \bar{Q}_i] = (\beta l_{(dH)}^\sharp + l_{(dI_i)}^\sharp) = [Q_i, \bar{Q}_H]\tag{2.37}$$

$$[l_{(dH)}^\sharp, l_{(dI_i)}^\sharp] = [l_{(dI_i)}^\sharp, l_{(dI_j)}^\sharp] = 0\tag{2.38}$$

Here we have used the notation $(dH)^\sharp \equiv \omega^{ab} \partial_b H \partial_a$ and $(dI_i)^\sharp \equiv \omega^{ab} \partial_b I_i \partial_a$ for the Hamiltonian vector fields generated by the gradients of H and I_i , respectively. Furthermore

[†] See Witten[7] and [8] for a very similar construction.

l_v denotes the Lie derivative along some vector field v [1], [4]. The most interesting of these relations is eq. (2.35). It shows that the anticommutator of Q_H and $\bar{Q}_H$ is the super-Hamiltonian and thus these operators, unlike Q and $\bar{Q}$, are genuine supersymmetry generators. Let us also recall from [1] that the action of $\tilde{\mathcal{H}}$ (expressed as a differential operator) on functions of the form (2.15) is the same as the action of the Lie-derivative $l_{(dH)^\sharp}$ on the components $\varrho_{a_1 \dots a_p}^{(p)}$. Quite generally, considering the ghosts c^a as a basis in the cotangent space to the phase-space, the Lie derivative along any vector field $v = v^a \partial_a$ can be written as

$$l_v = v^a \partial_a + c^b \partial_b v^a \frac{\partial}{\partial c^a} \quad (2.39)$$

In particular, $(dI_i)^\sharp$ generates a Hamiltonian flow with "Hamiltonian" $\tilde{\mathcal{H}}_i \equiv -i l_{(dI_i)^\sharp}$. Interpreted in this way, eqs. (2.35) and (2.36) tell us that the new operators (2.31) generate a kind of "square-root" of a (symplectic) diffeomorphism. This becomes even more obvious by defining the following operators:

$$\begin{aligned} \Omega_H &\equiv \frac{1}{\sqrt{2}}(Q_H + \bar{Q}_H) \\ \Omega_i &\equiv \frac{1}{\sqrt{2}}(Q_i + \bar{Q}_i) \end{aligned} \quad (2.40)$$

In fact

$$\Omega_H^2 = \beta l_{(dH)^\sharp}, \quad \Omega_i^2 = l_{(dI_i)^\sharp} \quad (2.41)$$

Because all the $Q_i, \bar{Q}_i$ and $l_{(dI_i)^\sharp}$, defined above, commute with the Hamiltonian $\tilde{\mathcal{H}}$, as it is easy to check

$$[Q_i, \tilde{\mathcal{H}}] = [\bar{Q}_i, \tilde{\mathcal{H}}] = [l_{(dI_i)^\sharp}, \tilde{\mathcal{H}}] = 0 \quad (2.42)$$

we have that each constant of motion I_i of the original Hamiltonian $H(\phi)$ gives rise to a graded symmetry of the "super-Hamiltonian" $\tilde{\mathcal{H}}$. The three generators of this graded symmetry $Q_i, \bar{Q}_i$, and $l_{(dI_i)^\sharp}$ make an $S(2)$ superalgebra in the classification of De Crombrugghe and Rittenberg [9]. The fact that the I_i 's have vanishing Poisson brackets among themselves (they are in involution [4]) and with $H(\phi)$ implies that different Q_i commute (the same for different $\bar{Q}_i$ and different $l_{(dI_i)^\sharp}$), see eqs. (2.33), (2.36) and (2.38). So for integrable systems[†] the symmetry of $\tilde{\mathcal{H}}$ is maximal and is $S(2)^n$. For systems that are not integrable, but have some $m < n$ constants of motion in involution, the symmetry

[†] They have n -constants of motion I_i in involution.

is $S(2)^m$. If (as it is in most of the cases) they are not in involution, then each of them makes an $S(2)$ symmetry but the overall symmetry is not the direct sum of $S(2)$'s. For ergodic systems, *i.e.*, those that have only $H(\phi)$ as constant of motion, the only symmetry is the supersymmetry of eq. (2.35).

3. ERGODIC SYSTEMS

Up to now we have only shown that the action $\tilde{S}$ is invariant under the supersymmetry whose generators are the first of eq. (2.40) and under the graded symmetries whose generators are the second of eq. (2.40) together with $l_{(dI_i)}$, but we have not asked the question if these symmetries can be spontaneously broken. The reader might wonder how a symmetry can be spontaneously broken in systems with a finite number of degrees of freedom and with no-quantum fluctuations. The answer to the second question is easy: here we do not have quantum fluctuations but we are doing a sort of "thermodynamic formalism" of classical mechanics on the line of ref. [10]. Hence fluctuations enter through the initial distribution $\varrho(\phi, c)$ and the "interactions" among the classical paths indicated by $\tilde{\mathcal{L}}$. For the first question, of how can we break a symmetry in a system with a finite number of degrees of freedom (or finite volume), we should remember that the usual proof (see for instance pag.15-17 of ref. [11]) require an infinite volume because the partition function $Z = \sum_n e^{-\beta E_n}$ is positive and can develop a zero (sign of spontaneous symmetry breaking) only in the infinite volume limit. This is not any more the case for a supersymmetric system where the partition function, being of the form $Z = \sum_n (-1)^F e^{\beta E_n}$ [12] is not anymore positive, so it can develop a zero even in a finite volume. In fact, there are examples of supersymmetric quantum mechanical models [12] with broken supersymmetry. The next question to ask is: how does the ground state transform? Is it invariant like the action? Let us examine here only the supersymmetry (2.35) and not the other graded-symmetries. For simplicity, let us assume that we are only interested in such observables A which do not depend on the ghosts: $A = A(\phi)$. Then in order to obtain a non-zero expectation value (see eq. (2.17)), the density $\tilde{\varrho}$ has to contain $2n$ ghosts:

$$\tilde{\varrho}(\phi^a, c^a, t) = \varrho(\phi^a, t) c^1 \dots c^{2n} \quad (3.1)$$

It is easy to verify that if $\tilde{\varrho}$ solves eq. (2.16), then $\varrho(\phi^a, t)$ obeys the standard Liouville equation (2.4)*. Moreover the classical average has the standard form (without ghosts)

* $l_{(dH)} c^1 \dots c^{2n}$ vanishes because $c^1 \dots c^{2n}$ is proportional to K^n , where K is one of the $ISp(2)$ generators. Interpreting $c^1 \dots c^{2n}$ as the volume form on phase space, this is exactly the Liouville theorem.

in this case:

$$\langle A \rangle_t = \int d^{2n} \phi A(\phi) \varrho(\phi, t) \quad (3.2)$$

Since $\tilde{\varrho}$ of eq. (3.1) contains a maximum number of ghosts it is trivially annihilated by Q_H :

$$Q_H \tilde{\varrho} = c^a (\partial_a - \beta \partial_a H) \tilde{\varrho}(\phi, c) = 0 \quad (3.3)$$

Invariance under $\bar{Q}_H$ requires

$$\bar{Q}_H \tilde{\varrho} = \frac{\partial}{\partial c^a} \omega^{ab} (\partial_a + \beta \partial_b H) \tilde{\varrho}(\phi, c) = 0 \quad (3.4)$$

This equation can be satisfied only if

$$\varrho(\phi^a) = k e^{-\beta H(\phi^a)} \quad (3.5)$$

where k is a constant. This result tells us that the supersymmetric invariant state has to be a Gibbs state [5], [11], which means a state that depends only on the Hamiltonian. We easily see that by properly choosing k we can obtain the canonical, the microcanonical or the grandcanonical distributions. Moreover the parameter β naturally plays the role of the $\frac{1}{KT}$ with T the temperature and K the Boltzmann constant. We feel that this result is a real striking surprise: it tells us that the canonical or better the Gibbs distribution is selected among all other possible states by the requirement of being the one that is supersymmetric invariant under the *universal* supersymmetry present for any dynamical system. It has been a challenge [5] for many years to find the physical reasons behind the Gibbs states, and many concepts like stability, locality, ergodicity, KMS conditions, have been explored. It might be that this supersymmetry really embodies all these features: work is in progress on this issue [6].

Besides being supersymmetric invariant, it is easy to see that the state (3.4) is also a "vacuum"[‡] of the super-Hamiltonian, $\tilde{\mathcal{H}} \tilde{\varrho}_o = 0$, and that correspondingly the state $\varrho(\phi)_o$ of eq. (3.5) is annihilated by the Liouvillian: $\hat{L} \varrho_o = 0$. Similarly we can ask which ϱ_o is invariant under the fermionic symmetry of $Q_i, \bar{Q}_i$. In complete analogy with what

[‡] We will indicate with the subscript "o" the fact that it is a state at zero eigenvalue of $\tilde{\mathcal{H}}$.

we did before, we find that the state is[‡]:

$$\varrho_o' = e^{-I_i(\phi^a)} \quad (3.6)$$

which again is a "vacuum" of $\tilde{\mathcal{H}}$ or $\hat{L}$ because it is built out of constants of motion. The fact that unbroken $(Q_H, \bar{Q}_H)$ -supersymmetry requires a ground state of the form $ke^{-\beta H}$ can be qualitatively understood as follows. Let us consider a system with no further constant of motion beyond H , at a fixed energy E , and let us pick a specific classical trajectory $\phi^a(t)$ with $H(\phi^a(t)) = E$. We mentioned already that, by applying a generic (anti-) BRS transformation to $\phi(t)$ we change its energy by the amount given in (2.30). However, we can "consistently truncate" the full phase-space path-integral to an integral over paths in one energy hyper-surface by just inserting a δ -function $\delta^{2n}(H(\phi) - E)$. This spoils the invariance of the theory under general BRS transformations, because the new trajectory, with energy $E + \delta E$, does not appear anymore in the same path-integral. As a way out, one has to impose restrictions on the ghosts c^a and $\bar{c}_a$, namely one has to require that they do not have any components orthogonal to the energy hypersurface. Referring back to eq. (2.23) we see that this is the same as requiring that N and $\bar{N}$ have to vanish. This can be implemented by appropriate δ -functions in the path-integral. The theory restricted in this way is again (anti-) BRS invariant, and the transformation law for ϕ^a is again given by (2.22). Furthermore, since ϕ^a commute with N and $\bar{N}$, it coincides with the transformation law of the $(Q_H, \bar{Q}_H)$ -supersymmetry. Now we have a simple physical picture of what ergodicity and supersymmetry mean in our setting. At least locally, we can use the (truncated) transformations (2.22) to deform any physical path into any other one with the same energy. Clearly physics can be invariant under these transformations (or the symmetry be unbroken) only if the ground state $\varrho_o(\phi)$ does not distinguish between different trajectories or phase-space points, respectively, of a given energy. This qualitatively explains why an unbroken supersymmetry requires the density function to be of the Gibbs form $\varrho_o = F(H)$.

Let us now proceed and go on to the proof, promised in the Introduction, that *ergodic systems are the only ones for which the supersymmetry (2.35) is unbroken, while for KAM and integrable systems it is spontaneously broken*. The proof goes as follows: let us assume that our system has another constant of motion $I(\phi)$ besides the energy $H(\phi)$ [◊]. Then, besides $\varrho_o(H)$, any normalizable function of $I(\phi)$, $\varrho_o'(I)$, would be an eigenfunction of $\hat{L}$

‡ We indicate with a prime "′" the fact that it is a function of I .

◊ So the system is not ergodic.

with zero eigenvalue and the corresponding $\tilde{\varrho}'_0(\phi, c) = \varrho'_0 c^1 \dots c^{2n}$ would be a "vacuum" of the super-Hamiltonian: $\tilde{\mathcal{H}}\tilde{\varrho}'_0 = 0$. Having both $\tilde{\varrho}_0(H)$ and $\tilde{\varrho}'_0(I)$ as ground states, not only ergodicity is spoiled but also supersymmetry. In fact, $\tilde{\varrho}'_0(I)$ is not supersymmetric invariant, i.e.,

$$\bar{Q}_H \tilde{\varrho}'_0 \neq 0 \quad (3.7)$$

just because the solutions of the equation $\bar{Q}_H \tilde{\varrho} = 0$ are only states of the form $\tilde{\varrho} = k e^{-\beta H} c^1 \dots c^{2n}$ and $\tilde{\varrho}'_0(I)$ is not of this form[•]. So for systems with more than one constant of motion, the supersymmetry is broken. In fact the ground state could be any normalizable function of them[∇]

$$\tilde{\varrho}''_0 = F[\tilde{\varrho}_0, \tilde{\varrho}'_0] = \tilde{F}[H, I, c] \quad (3.8)$$

and this is not supersymmetric invariant! Therefore integrable systems or systems with few integrals of motion besides the energy, have this supersymmetry spontaneously broken. The system in fact has no "stability" reasons to choose the supersymmetric vacuum instead of the most general state (3.8): both of them are ground states of the "effective" theory because both of them satisfy the "operatorial" condition: $\tilde{\mathcal{H}}\tilde{\varrho} = 0$. In general it might be that we are not able to find analytic constants of motion, but still the system is non-ergodic presenting some invariant[†] surfaces, of measure different from zero, on the energy

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- Saying "not supersymmetric invariant" means that there is no constant value of β , for which the state can be put in the form $k e^{-\beta H}$. If, for example, we find that ϱ_0 is of the form $\varrho_0 = k_1 e^{-\beta_1 H} + k_2 e^{-\beta_2 H}$, this is still supersymmetric, because it is annihilated by a particular $\bar{Q}_H$ with a β given by $\beta = \frac{\ln[k_1 e^{-\beta_1 H} + k_2 e^{-\beta_2 H}]}{H}$. As H is a constant on the energy surface, we can put the state in the form $e^{-\beta H}$ with a constant β . So, to be able to find a constant β such that a state ϱ_0 could still be put in the form $k e^{-\beta H}$, the only condition is that ϱ_0 has to be a function of H only (Gibbs state condition). In fact then we would have

$$\varrho_0 = F(H) = e^{-\beta H}$$

with $\beta = -\frac{\ln F(H)}{H}$ and this β is constant on the energy manifold. This would not be so if the ϱ_0 had been of the form $\varrho_0 = F(H, I)$. The associated β would be $\beta = -\frac{\ln F(H, I)}{H}$. Now, as I is not constant on surfaces of constant H , β itself is not a constant. From this analysis we understand that the freedom we have for β in the definition of $\bar{Q}_H$ is very powerful, since that freedom has allowed us to rescue situations like the one above (the sum of the two exponentials), that would otherwise not be judged supersymmetric. Note also that β is basically the concept of temperature: this concept is usually defined as an equivalence class. Two systems belong to the same (temperature) class if, once they are put in touch, the overall Hamiltonian is just the sum of the two without any interaction term. This concept of equivalence class might be related to other equivalence classes that appear naturally when we have nilpotent operators as Q_H and Q_i . We hope to come back to this issue in future publications.

[∇] This is due to the fact that in $\hat{L}$ we have only first-order derivatives.

[†] Invariant under the Hamiltonian flow.

manifold : KAM [4] systems are of this kind. In this case the ϱ'_o associated to this surface could be built in the following way:

$$\varrho'_o = \begin{cases} 1, & \text{on the invariant surface} \\ 0, & \text{elsewhere} \end{cases} \quad (3.9)$$

This ϱ'_o is an acceptable state[†], because it has norm different from zero

$$\int d^{2n} \phi \varrho'_o = V$$

where V is the volume of the invariant surface in phase-space. This V is different from zero because KAM systems are characterized by invariant surfaces of non-zero measure. So ϱ'_o is of finite norm and moreover it is at a zero eigenvalue of $\hat{L}$ because it is built out of *invariant* surfaces. Anyhow the associated $\tilde{\varrho}$ is not supersymmetric invariant because it is not of the form $ke^{-\beta H}$. So also for KAM systems the supersymmetry is spontaneously broken!. Let us also note that, like in field theory, tuning the value of the superpotential H it is possible to go from a situation of broken supersymmetry (KAM) to one of unbroken supersymmetry (ergodic). This is what happens, for example, for the Toda system in which, increasing the value of the energy above some limit, the few left over invariant surfaces get destroyed and the system goes from a KAM to an ergodic or chaotic regime. The "maximum" of ergodicity breaking would be for an integrable system: there we have at least n functions as "vacua" of $\tilde{\mathcal{H}}$. They are those built out of the action variables I_i and annihilated by $\tilde{Q}_i$:

$$\tilde{\varrho}_o^{[i]} = e^{-I_i} c^1 \dots c^{2n}$$

and they are not supersymmetric invariant. Of course, out of these functions we can build many more ground states, so integrable systems have a real "plethora" of non-supersymmetric invariant ground states.

The reader might have realized by now some very strange features of our supersymmetry breaking: differently than in usual supersymmetry [12] we have a supersymmetry breaking without having a lifting of the ground-states off the zero value of $\tilde{\mathcal{H}}$, in fact the various non-supersymmetric $\tilde{\varrho}_0$ that we have built above, are all annihilated by $\tilde{\mathcal{H}}$:

[†] The only problem is that this state is not analytical in ϕ , and this is a typical feature of KAM [4] systems.

$\tilde{\mathcal{H}}\tilde{\varrho}'_o = 0$ This is due to the *non-positive definite* character of our scalar product in (2.10) so that we can have a zero-norm state as $||\tilde{Q}_H\tilde{\varrho}'_o||^2$ without the state being zero. This can be understood from ref. [1] and from the form of $\tilde{Q}_H$ derived from (2.31) and (2.20). The non-positive-definite character is imputable to the presence of the matrix ω^{ab} in $\tilde{Q}_H$: more details of this phenomenon will be provided in a forthcoming publication [13]. The reader at this point, alerted by the non-positive character of our scalar product, might wonder why we called the state at zero eigenvalue the ground states or "vacua" of $\tilde{\mathcal{H}}$. Being a non-positive definite operator $\tilde{\mathcal{H}}$, its "true ground state" might be at some very negative eigenvalue λ , $\lambda \ll 0^*$, and what we have proved on the zero-eigenvalue state might imply nothing at all on the supersymmetry breaking, just because the zero-eigenvalue states are not ground states. We will show now how to rescue the result that we have presented before: namely that for a non-ergodic system the supersymmetry is spontaneously broken. Let us suppose that the real vacuum of $\tilde{\mathcal{H}}$ is some state $\tilde{\varrho}_{(\lambda)}$ at some very negative eigenvalue λ , and that this state is supersymmetric invariant

$$\tilde{Q}_H\tilde{\varrho}_{(\lambda)} = 0$$

If the system is non-ergodic, following what we did before, we can build out of the constants of motion I_i some state $\tilde{\varrho}'_o$ that is not supersymmetric invariant and is at zero eigenvalue of $\tilde{\mathcal{H}}$. Let us now construct out of $\tilde{\varrho}_{(\lambda)}$ and ϱ'_o , the following new state

$$\tilde{\varrho}'_{\lambda o} = k' \varrho'_o \tilde{\varrho}_{(\lambda)} \quad (3.10)$$

where k' is a normalization constant to guarantee that the new state is normalized. The state (3.10) is at eigenvalue λ and so degenerate with the ground state $\tilde{\varrho}_{(\lambda)}$ but it is not supersymmetric invariant because of the presence of the ϱ'_o . So again, even if the true ground state is at a negative eigenvalue, supersymmetry is broken!!

The next desirable thing to do would be to construct an index perhaps similar to the the Witten index [12] $\sum_n (-1)^F$, as a tool to check if our supersymmetry is broken or not. This is not easy in our case because our path-integral for classical mechanics is by no means similar to the supersymmetric quantum mechanics of Witten. Somehow we have to find a manner to count the states at zero eigenvalue of $\tilde{\mathcal{H}}$. The desirable object would be something like the sum of the dimensions of the zero cohomology classes of the operators

* Or it may even be that $\tilde{\mathcal{H}}$ is not bounded below, but still the following argument will work.

Q_H, Q_i :

$$\Delta = \sum_i \# \left(\frac{\text{Ker } Q_i}{\text{Im } Q_i} \right)_o + \# \left(\frac{\text{Ker } Q_H}{\text{Im } Q_H} \right)_o \quad (3.11)$$

If this number were 1, then the system would be ergodic, because it means that only one zero-cohomology class is non empty and this can only be the one of Q_H . If it were larger than 1, it would indicate that there are some other constants of motion besides the energy,^{*} if it were $(n+1)$ the system would be integrable. The reader will immediately realize the power of being able to calculate this Δ . Unfortunately, a priori if we do not know how to integrate the system, we also do not know the I_i and Q_i and so we do not know how to build the first terms in (3.11). It would be nice if the Δ could be proven to be the dimension of the zero cohomology class of a unique operator Ω built out of the Q_H .

$$\Delta = \# \left(\frac{\text{Ker } \Omega[Q_H]}{\text{Im } \Omega[Q_H]} \right)_o \quad (3.12)$$

If that were possible, then given Q_H , most probably Δ could be calculated as an expectation value, under the classical path-integral of (2.10), of some polynomial in $\phi, c, \bar{c}, \lambda$, on the same line as the Witten index[12], or the other topological invariants of topological field theory[14], [15]. The reader might ask why we restrict ourselves to a ϱ_o with no-ghosts and so to zero-cohomological classes in (3.11). If we had considered all the higher classes, then Δ would had a chance to be identifiable with the Euler characteristic of some manifold. The reason we limited ourselves to the zero cohomology is that the higher form sectors are more related to the chaotic behaviour of the system[16] than to the ergodic/integrable/KAM character. We will come back on these issues in ref.[13]

* We do not know if we could associate an intermediate value of Δ to KAM systems, because the KAM systems, as we explained above, have non-analytic ϱ_o .

4. KMS CONDITION

As we said already in the introduction, the Gibbs nature of equilibrium states has been studied very extensively in the literature[5] . In particular, a lot of work has gone into the study of the *deep* physical reasons for the Gibbs states on one side and, on the other, in replacing the Gibbs-state condition by a more algebraic characterization of them which could be used for infinite systems. This algebraic characterization is known as KMS conditions[5] , [11] . In this section we will derive it from the supersymmetry that we have seen as being responsible for the Gibbs nature of the states. To this end we consider the time-evolution[1] of two observables $A_1(\phi^a)$ and $A_2(\phi^a)$ which are independent of the ghosts:

$$A_{1,2}(\phi^a, t) = e^{i\tilde{\mathcal{H}}t} A_{1,2}(\phi^a, 0) e^{-i\tilde{\mathcal{H}}t} \quad (4.1)$$

The condition $\bar{Q}_H \tilde{\varrho} = 0$ implies that the following expression is zero:[†]

$$\int d^{2n}\phi d^{2n}c A_1(\phi^a, 0) Q_H A_2(\phi^a, t) \bar{Q}_H \tilde{\varrho}(\phi, c) = 0 \quad (4.2)$$

The above expression can be written explicitly as

$$\int d^{2n}\phi d^{2n}c A_1(\partial_a + \beta \partial_a H) A_2(t) \cdot \frac{\partial}{\partial c^e} \omega^{ef} (\partial_f - \beta \partial_f H) \tilde{\varrho}(\phi, c) = 0 \quad (4.3)$$

Doing an integration by parts in $\frac{\partial}{\partial c^e}$ and in ∂_f and remembering the form of the Poisson brackets (2.2), the expression (4.3) reduces to

$$\langle \{A_1(0), H\} A_2(t) \rangle_{\tilde{\varrho}} = \frac{1}{\beta} \langle \{A_1(0), A_2(t)\} \rangle_{\tilde{\varrho}} \quad (4.4)$$

where $\langle \cdot \rangle_{\tilde{\varrho}}$ is defined in (3.2). The LHS of (4.4) is nothing else than $\langle \dot{A}_1(0) A_2(t) \rangle$ and so (4.4) can be re-written as

$$\langle \dot{A}_1(0) A_2(t) \rangle_{\tilde{\varrho}} = \frac{1}{\beta} \langle \{A_1(0), A_2(t)\} \rangle_{\tilde{\varrho}} \quad (4.5)$$

This equation is known as the classical KMS condition[17] . It can be obtained[11] from its, better known, quantum mechanical counterpart[5]

$$\langle A_1(t) A_2(0) \rangle = \langle A_2(0) A_1(t + i\beta\hbar) \rangle \quad (4.6)$$

in the limit $\hbar \rightarrow 0$, but it can also be derived by purely classical means[17] . It is not a surprise, however, that we are able to derive eq. (4.5) from the supersymmetric invariance

[†] The reader may wonder why we use the following expression: any other would be zero on $\bar{Q}_H \tilde{\varrho} = 0$. The reason is that this particular expression will lead to the KMS condition.

of $\tilde{\varrho}$, since it is known that the classical KMS-condition uniquely fixes ϱ to be of the form $\varrho = ke^{-\beta H}$ like supersymmetry does. Similarly we could derive the classical fluctuation-dissipation theorem [11], [18] from the supersymmetric-invariance of $\tilde{\varrho}$ since this theorem crucially depends on ϱ being of the form (3.5).*

The derivation of the KMS condition presented here indicates that the KMS is like a Ward identity for this supersymmetry. From the eqs. (4.2), (4.4) and (4.5) it is clear that, in deriving the KMS condition from the supersymmetry, we did not use the explicit form of the ground state which usually is not known for complicated (infinite-dimensional) systems. Our derivation of (4.5) only embodies the *algebraic* character of the supersymmetry.

5. CONCLUSIONS

A lot of efforts have gone [5], [17] into trying to understand the physical reasons behind the KMS conditions. Now an equally strong effort should go into uncovering the physical reasons of the supersymmetry that generates the KMS condition as a Ward identity. We think that it should be easier and more illuminating to understand the physics behind a full symmetry rather than the one behind just a single Ward identity. The supersymmetry might really throw a new light on the issue of stability invoked [5] to understand the KMS condition. We also think that the graded symmetries discovered for integrable systems will help in unveiling the geometric "secrets" still behind the issue of integrability.

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This work is dedicated to the memory of Pepu Salamen.

* For details see ref. [17], [18]. In this context it is also interesting to note that our auxiliary field λ_a is the same as the field $\hat{\psi}_a$ contained in the third of ref. [18]. Therefore we can identify it with the response field describing the linear response of the system to external perturbations.

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