CERN-TH.5446/89

Characters of the $N=4$ superconformal algebra with two central extensions

J. L. Petersen¹ and A. Taormina²

CERN, CH-1211 Geneva 23, Switzerland.

ABSTRACT

We derive formulas for the characters of the $N=4$ algebra $\mathcal{A}_\gamma$ which contains two $\widehat{SU}(2)$ Kac-Moody subalgebras. We present results for unitary, highest weight, massive representations in the Ramond and Neveu-Schwarz sectors. Massive characters are given for the $\tilde{\mathcal{A}}_\gamma$ algebra, which is closely related to $\mathcal{A}_\gamma$, with no dimension $\frac{1}{2}$ operators but rather composite operators in the OPE's. Comparisons with previously known results in relevant limits are performed.

¹ On leave of absence from The Niels Bohr Institute, University of Copenhagen, Denmark. Address after 1st September 1989: The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark.

² Address after 1st October 1989: The Enrico Fermi Institute, University of Chicago, 5630 South Ellis Ave, Chicago IL 60637, USA.

CERN-TH.5446/89

June 1989

1 Introduction.

N -extended superconformal algebras (SCA's) have received a great deal of attention over the past few years, in particular because they appear in the study of 2-dimensional (super)-conformal theories, relevant for statistical mechanics and string theory. The conventional $N = 0, 1, 2, 3, 4$ SCA's, i.e. the Lie superalgebras with a single dimension 2 operator (the energy-momentum tensor), N dimension $3/2$ supersymmetry operators which square to the energy-momentum tensor, and no operator of dimension smaller than or equal to zero, can be viewed as subalgebras of a new, equally conventional $N=4$ SCA, $\mathcal{A}_\gamma$ [1,2,3,4], parametrized by the central charge, c , of the Virasoro algebra and one additional parameter $\gamma \geq 0$, or, equivalently, by the levels k^+ and k^- of the two commuting Kac-Moody (KM) algebras embedded in it, $SU(2)^+$ and $SU(2)^-$.

Some interesting realizations of $\mathcal{A}_\gamma$ on quaternionic spaces [2] and its fairly rich representation theory [4] suggest that it might be of interest to string compactification. In any case, the next step in the analysis of these questions consists in deriving character formulas for the unitary representations of $\mathcal{A}_\gamma$.

In this paper, we present characters for the (generic) massive, or zero-Witten index, representations. (For these notations, see [4,5,6]). The characters for the massless representations will be considered elsewhere [7].

In section 2, we introduce some notations and describe the zero-mode levels of the representations in the Neveu-Schwarz (NS) and Ramond (R) sectors, and give results following from 'spectral flow' arguments [8].

In section 3, we describe the pattern of singular vectors in R-Verma modules, using a generalization of known techniques for $SU(2)$ characters [5,9]. As a result, we may write down massive characters in the R sector, and using spectral flow, in the NS sector. Some interesting factorization properties for which we have so far no explanation emerge.

In section 4, we consider the algebra $\tilde{\mathcal{A}}_\gamma$ [3,4] introduced by Knizhnik [10], Goddard and Schwimmer [11]. It is simpler than $\mathcal{A}_\gamma$ in that no dimension $\frac{1}{2}$ generators appear, but the algebra only closes on composite operators. However, from our results we immediately obtain character formulas for massive unitary representations of $\tilde{\mathcal{A}}_\gamma$, and we compare with results in [12] in a special case.

In section 5, we investigate the limit $k^- \rightarrow \infty$ (or $k^+ \rightarrow \infty$) where the $N = 4$ algebra of [13] arises so that again we may compare with known results [5].

In section 6, we make some concluding remarks.

2 Notations and formal results.

The generators of $\mathcal{A}_\gamma$ are the Virasoro operator $L(z)$, the four supercharges of dimension $\frac{3}{2}$, $G_\pm(z)$ and $\bar{G}_\pm(z)$, the six $SU(2)^\pm$ operators of dimension 1, $T^{\pm\pm}(z)$, $T^{\pm 3}(z)$, one dimension 1 current $U(z)$ and four dimension $\frac{1}{2}$ operators $Q_\pm(z)$, $\bar{Q}_\pm(z)$. Their explicit commutation relations are reproduced in the Appendix.

In this paper, we only consider unitary, highest weight representations. They are characterized by a zero-mode multiplet $\{|\Omega_i\rangle\}$ of states satisfying

$$\begin{aligned} L_0|\Omega_i\rangle &= T_n^\pm|\Omega_i\rangle = (G_a)_\pm|\Omega_i\rangle = (Q_a)_\pm|\Omega_i\rangle = U_n|\Omega_i\rangle = 0, \\ L_0|\Omega_i\rangle &= h|\Omega_i\rangle, \quad iU_0|\Omega_i\rangle = u|\Omega_i\rangle, \end{aligned} \quad (2.1)$$

with $n > 0$ integer, $r > 0$ integer in R and half-integer in NS, and $i = \pm, 3$; $a = \pm, \pm K$.

In the NS sector, the zero-mode multiplet has a unique highest weight state (hws) $|\Omega, hws\rangle$ with highest values ℓ^+, ℓ^- of the third component of isospin for $SU(2)^+$ and $SU(2)^-$, satisfying

$$T_0^{\pm\pm}|\Omega, hws\rangle = 0; \quad T_0^{\pm 3}|\Omega, hws\rangle = \ell^\pm|\Omega, hws\rangle; \quad T_0^{-3}|\Omega, hws\rangle = \ell^-|\Omega, hws\rangle. \quad (2.2)$$

The R sector is more complicated. In the generic massive case which we consider in this paper, the representation of the zero-mode operators

$$T_0^\pm, (G_a)_0, (Q_a)_0, \quad a = \pm, \pm K; \quad i = \pm, 3 \quad (2.3)$$

can be reduced into sixteen multiplets of $(SU(2)^+, SU(2)^-)$, in each of which there is a state $|\Omega_a\rangle$ satisfying

$$T_0^{\pm\pm}|\Omega_a\rangle = 0; \quad T_0^{\pm 3}|\Omega_a\rangle = \ell_a^\pm|\Omega_a\rangle \quad (2.4)$$

$a=1, \dots, 16$. We choose to let (ℓ^+, ℓ^-) denote the two maximal values of ℓ_a^+ and ℓ_a^- that occur in the list. There is no unique 'natural' highest weight state, however. Instead, the states $|\Omega_+\rangle$ and $|\Omega_-\rangle$ with respectively ℓ_+^+ and ℓ_-^- maximal satisfy

$$\begin{aligned} T_0^{\pm 3}|\Omega_+\rangle &= \ell^+|\Omega_+\rangle, \quad T_0^{-3}|\Omega_+\rangle = (\ell^- - 1)|\Omega_+\rangle, \\ (G_+)_0|\Omega_+\rangle &= (G_+K)_0|\Omega_+\rangle = 0 \\ (Q_+)_0|\Omega_+\rangle &= (Q_+K)_0|\Omega_+\rangle = 0, \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} T_0^{\pm 3}|\Omega_-\rangle &= (\ell^+ - 1)|\Omega_-\rangle, \quad T_0^{-3}|\Omega_-\rangle = \ell^-|\Omega_-\rangle, \\ (G_+)_0|\Omega_-\rangle &= (G_-K)_0|\Omega_-\rangle = 0 \\ (Q_+)_0|\Omega_-\rangle &= (Q_-K)_0|\Omega_-\rangle = 0. \end{aligned} \quad (2.6)$$

The quantum number pattern of the sixteen $SU(2)^+ \otimes SU(2)^-$ multiplets is shown in Figure 1.

The states $|\Omega_+>$ and $|\Omega_->$ are related by

$$|\Omega_-> \propto (G_-K)_0 (Q_-K)_0 |\Omega_+> ; |\Omega_+> \propto (G_+K)_0 (Q_+K)_0 |\Omega_-> \quad (2.7)$$

up to an overall normalization. The vacuum structure (or multiplet structure) consists then of a total of

$$16(2\ell^+ - 1)(2\ell^- - 1) \quad (2.8)$$

states, half of which are bosonic and half of which are fermionic.

All of this discussion pertains to the massive case for which one has [4]

$$kh_R > u^2 + (\ell^+ + \ell^- - 1)^2 + \frac{k^+ k^-}{4} \quad (2.9)$$

in the R sector, and

$$kh_{NS} > u^2 + (\ell^+ - \ell^-)^2 + (k^- \ell^+ + k^+ \ell^-) \quad (2.10)$$

$$0 \leq \ell^\pm < \frac{k^\pm}{2} - \frac{1}{2}$$

in the NS sector. For massless representations, the inequalities involving h_R and h_{NS} are replaced by equalities, and the norm of $|\Omega_+>$ or $|\Omega_->$ correspondingly vanishes in the R sector, as can be seen from the algebra.

The NS and R sectors are related by 'spectral flow' [8]. For $\mathcal{A}_r$ there are essentially two interesting ways of relating the sectors:

Flow in the $SU(2)^+$ quantum number :

$$L_{0,R} = L_{0,NS} - T_{0,NS}^{+3} + \frac{k^+}{4} ;$$

$$T_{0,R}^{+3} = T_{0,NS}^{+3} - \frac{k^+}{2} ;$$

$$T_{0,R}^{-3} = T_{0,NS}^{-3} . \quad (2.11)$$

The R highest weight state $|\Omega_+>_R$ is obtained from the NS hws by

$$|\Omega_+>_R = (T_{-1}^{+3})^{k^+ - 2\ell^+} |\Omega, hws>_{NS}, \quad (2.12)$$

with quantum numbers

$$h_R = h_{NS} - \ell_{NS}^+ + \frac{k^+}{4} ; \ell_R^+ = \frac{k^+}{2} - \ell_{NS}^+ ; \ell_R^- = \ell_{NS}^- . \quad (2.13)$$

Flow in the $SU(2)^-$ quantum number :

$$L_{0,R} = L_{0,NS} - T_{0,NS}^{-3} + \frac{k^-}{4} ;$$

$$T_{0,R}^{-3} = T_{0,NS}^{-3} - \frac{k^-}{2} ;$$

$$T_{0,R}^{+3} = T_{0,NS}^{+3} . \quad (2.14)$$

The R highest weight state $|\Omega_->_R$ is obtained from the NS hws by

$$|\Omega_->_R = (T_{-1}^{+3})^{k^- - 2\ell^-} |\Omega, hws>_{NS}, \quad (2.15)$$

with quantum numbers

$$h_R = h_{NS} - \ell_{NS}^- + \frac{k^-}{4} ; \ell_R^- = \frac{k^-}{2} - \ell_{NS}^- ; \ell_R^+ = \ell_{NS}^+ . \quad (2.16)$$

In either sector, we may define the characters quite generally by the trace over the irreducible part of the Verma module :

$$Ch^{\mathcal{A}_r, I}(k^+, k^-, h, \ell^+, \ell^-, u ; q, z_+, z_-, \chi, y) \equiv \text{Tr}(q^{L_0 - \frac{3}{24}} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}} \chi^{U_0} y^F), \quad (2.17)$$

where $I = NS$ or R and F is the 'Fermi-number' operator, anticommuting with $G_\pm$ and $Q_\pm$. In this paper, however, we shall suppress dependences on u, χ and y by putting $\chi = y = 1$. It is a trivial matter to reintroduce those variables.

Using the spectral flows (2.11) and (2.14), we get

$$Ch^{\mathcal{A}_r, NS}(k^+, k^-, h, \ell^+, \ell^- ; q, z_+, z_-)$$

$$= \frac{k^+}{q^{\frac{1}{4}}} z_+^{k^+} Ch^{\mathcal{A}_r, R}(k^+, k^-, h - \ell^+ + \frac{k^+}{4}, \frac{k^+}{2} - \ell^+ + 1, q, q^{\frac{1}{2}} z_+, z_-)$$

$$= \frac{k^-}{q^{\frac{1}{4}}} z_-^{k^-} Ch^{\mathcal{A}_r, R}(k^+, k^-, h - \ell^- + \frac{k^-}{4}, \ell^+ + 1, \frac{k^-}{2} - \ell^- ; q, z_+, q^{\frac{1}{2}} z_-) . \quad (2.18)$$

In the next section, we shall calculate $Ch^{\mathcal{A}_r, R}$ directly and determine $Ch^{\mathcal{A}_r, NS}$ by spectral flow. It will be a consistency check that the two expressions (2.18) agree.

It will turn out to be very convenient to express our results in terms of characters of the 'conventional' $N=4$ SCA [13], given by [5] as

$$Ch^{NS}(k, h, \ell ; q, z_+, z_-) = q^{h - \frac{1}{4}} F^{NS}(q, z_+, z_-)$$

$$* \prod_{n=1}^{\infty} (1 - q^n)^{-2} (1 - z_+^2 q^n)^{-1} (1 - z_-^2 q^{n-1})^{-1}$$

$$* \sum_{m=-\infty}^{m=+\infty} q^{(k+1)m^2 + (2\ell+1)m} \{ z_+^{2[m(k+1)+\ell]} - z_+^{-2[m(k+1)+\ell+1]} \} \quad (2.19)$$

and

$$\begin{aligned}
Ch^R(k, h, \ell; q, z_+, z_-) &= q^{h-\frac{k}{4}} F^R(q, z_+, z_-) (1 + z_+^{-1} z_-) (1 + z_+^{-1} z_-^{-1}) \\
&\quad * \prod_{n=1}^{\infty} (1 - q^n)^{-2} (1 - z_+^2 q^n)^{-1} (1 - z_+^{-2} q^{n-1})^{-1} \\
&\quad * \sum_{m=-\infty}^{m=\infty} q^{(k+1)m^2 + 2\ell m} \{ z_+^{2[m(k+1)+\ell]} - z_+^{-2[m(k+1)+\ell]} \}.
\end{aligned} \tag{2.20}$$

Here, we have introduced the fermionic non-zero mode partition functions

$$F^{\ell}(q, z_+, z_-) = \prod_{n \in \mathbb{N}_{\ell}} (1 + z_+ z_- q^n) (1 + z_+^{-1} z_-^{-1} q^n) (1 + z_+^{-1} z_- q^n) (1 + z_+ z_-^{-1} q^n), \tag{2.21}$$

where $\mathbb{N}_S = \{\text{positive half-integers}\}$, $\mathbb{N}_R = \{\text{positive integers}\}$. Compared to [5], which corresponds to $z_- = 1, z_+ = z$, we have introduced the trivial generalization that the four G 's ($(G^2, -G^1)$ and $(\tilde{G}^1, \tilde{G}^2)$ in [5] correspond to $\sqrt{2}(G_+, -G_-)$ and $\sqrt{2}(G_{+\kappa}, G_-)$ here) transform under an additional *global* $S\widehat{U}(2)^-$ in addition to the $S\widehat{U}(2)$ KM algebra usually considered. The transformation properties are taken to correspond to those of our G_a 's in the Appendix. This accounts for the z_- -dependence in (2.19) and (2.20). These equations are related by the spectral flow :

$$Ch^{NS}(k, h, \ell; q, z_+, z_-) = q^{\frac{k}{4}} z_+^k Ch^R(k, h - \ell + \frac{k}{4}, \frac{k}{2} - \ell; q, z_+, z_-). \tag{2.22}$$

3 The pattern of singular vectors in the Verma module.

We consider the *Ramond module* of $\mathcal{A}_\gamma$ built on the zero-mode multiplet characterized by (ℓ^+, ℓ^-) defined as in the previous section. For the most part, we think of $|\Omega_+\rangle$ as our highest weight vector, in which case all operators $(G_+)_0, (G_{+\kappa})_0, (Q_+)_0, (Q_{+\kappa})_0$ may be eliminated from descendants. It is then a trivial matter to write down the character for the full reducible module. It is given by

$$Ch_{\text{full},+}^{\mathcal{A}_\gamma,R}(k^+, k^-, h, \ell^+, \ell^-; q, z_+, z_-) = q^{h-\frac{k}{4}} z_+^{2\ell^+} z_-^{2\ell^-} Ch_{0,+}^{\mathcal{A}_\gamma,R}(k^+, k^-, q, z_+, z_-) \tag{3.1}$$

where

$$\begin{aligned}
Ch_{0,+}^{\mathcal{A}_\gamma,R}(k^+, k^-, q, z_+, z_-) &= (F^R(q, z_+, z_-))^2 (1 + z_+^{-1} z_-^{-1})^2 (1 + z_+^{-1} z_-)^2 \\
&\quad * \prod_{n=1}^{\infty} (1 - q^n)^{-4} (1 - z_+^2 q^n)^{-1} (1 - z_+^{-2} q^{n-1})^{-1} (1 - z_-^2 q^n)^{-1} (1 - z_-^{-2} q^{n-1})^{-1}.
\end{aligned} \tag{3.2}$$

This expression is symmetric in $+$ and $-$, see further below. From this trace we should subtract the contributions of modules built on singular vectors, add the ones from doubly

subtracted ones, etc. [5,9] in order to get the characters of the irreducible part.

The pattern of singular vectors for $S\widehat{U}(2)$, the $N = 4$ SCA of [13] considered by [5] and the pattern for $\mathcal{A}_\gamma$ are rather similar, the main complication in our case being the presence of the two commuting $S\widehat{U}(2)^+$ and $S\widehat{U}(2)^-$.

For us, a *singular vector* is simply one obeying all the conditions of $|\Omega_+\rangle$ or $|\Omega_-\rangle$, respectively, (2.1) and (2.5) or (2.6). For the precise definition of singular vectors for $S\widehat{U}(2)$ and $N = 4$ SCA, we refer to [5,9], but the situation is quite analogous.

For the case of $S\widehat{U}(2)$, it can be proven that if $|\chi\rangle = |H; L\rangle$ is a singular vector with conformal dimension H and isospin L , then the vectors

$$\begin{aligned}
\alpha_L |\chi\rangle &= (T_+^+)^{k-2L+1} |\chi\rangle = |H + k - 2L + 1; k - L + 1\rangle \\
\beta_L |\chi\rangle &= (T_0^-)^{2L+1} |\chi\rangle = |H; -L - 1\rangle
\end{aligned} \tag{3.3}$$

are singular as well. Starting from a hws, we then have (in a slightly symbolic notation) the following pattern of singular vectors

$$\begin{aligned}
|A_n\rangle &= (\beta\alpha)^n |\Omega\rangle, & n = 1, 2, \dots \\
|B_n\rangle &= (\alpha\beta)^n |\Omega\rangle, & n = 0, 1, \dots \\
|A'_n\rangle &= (\alpha\beta)^{n-1} \alpha |\Omega\rangle, & n = 1, 2, \dots \\
|B'_n\rangle &= (\beta\alpha)^n \beta |\Omega\rangle, & n = 0, 1, 2, \dots
\end{aligned} \tag{3.4}$$

One easily forms the characters for the full module of each of these, and the final character is obtained by subtracting modules built on states with an odd number of α, β -operators, and adding those with an even number of α, β -operators.

The case of the $N = 4$ SCA considered in [5] is entirely analogous with the exception that the formula for β has to be modified. Indeed, the appropriate form turns out to be

$$\begin{aligned}
\beta_L |H; L\rangle &= (T_0^-)^{2L-1} \{ 2L(G_-)_0 (G_{-\kappa})_0 - (H - \frac{k}{4}) T_0^- \} |H; L\rangle \\
&= |H; -L\rangle
\end{aligned} \tag{3.5}$$

In our case, we find singular vectors generated by operators $\alpha_{L\pm}^{\pm}, \beta_{L\pm}^{\pm}$ pertaining to the two $S\widehat{U}(2)^{\pm}$ KM algebras.

In complete analogy with the $S\widehat{U}(2)$ case, it turns out to be very easy to verify that, if $|\chi, H, L^+, L^-, +\rangle$ is singular of the $|\Omega_+\rangle$ type, then the following vectors are singular :

$$\begin{aligned}
\alpha_{L^+}^+ |\chi; H; L^+, L^-, +\rangle &= (T_{-1}^{++})^{k^+ - 2L^+ + 1} |\chi\rangle \\
&= |H + k^+ - 2L^+ + 1; k^+ - L^+ + 1; L^-, +\rangle
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\beta_{L^-}^-|\chi; H; L^+; L^-; +\rangle &= (T_0^{--})^{2L^-+1}|\chi\rangle \\ &= |H; L^+; -L^- - 1; +\rangle.\end{aligned}\quad (3.7)$$

Similarly, α^- and β^+ are obtained by acting on singular vectors of the $|\Omega_+\rangle$ type by interchanging $SU(2)^\pm$. We shall choose to do our calculations based on $|\Omega_+\rangle$, so we get from (2.7)

$$\begin{aligned}\alpha_{L^-}^-|\chi\rangle &= (G_{+\kappa})_0(Q_{+\kappa})_0(T_{-1}^{--})^{k^- - 2L^- - 1}(Q_{-\kappa})_0(G_{-\kappa})_0|\chi\rangle \\ &\equiv |H + k^- - 2L^- - 1; L^+; k^- - L^- - 1; +\rangle \\ \beta_{L^+}^+|\chi\rangle &= (G_{+\kappa})_0(Q_{+\kappa})_0(T_0^{+-})^{2L^+ - 1}(Q_{-\kappa})_0(G_{-\kappa})_0|\chi\rangle \\ &\equiv |H; -L^+ + 1; L^-; +\rangle.\end{aligned}\quad (3.8)$$

Here we may employ (2.5) or (2.6) and the algebra to eliminate $(G_{+\kappa})_0$ and $(Q_{+\kappa})_0$. The results however are very complicated looking states, somehow generalizing (3.5), and the above forms are easier to work with.

Just as it was conjectured for the ordinary $N = 4$ SCA that all singular vectors arose by acting with α, β , [5], we make the analogous conjecture here for $\mathcal{A}_\gamma$.

Now, since we have both $\alpha^\pm$ and $\beta^\pm$ at our disposal, the pattern appears at first very complicated and intractable. However, the situation is rescued by the following crucial observations (notice that $\alpha_{L^+}^+, \beta_{L^+}^+$ do not change the $SU(2)^-$ quantum numbers and vice-versa). If $|\chi\rangle$ is a singular vector of the $|\Omega_+\rangle$ type,

$$\begin{aligned}\alpha_{L^+}^+ \beta_{L^-}^- |\chi\rangle &= \beta_{L^-}^- \alpha_{L^+}^+ |\chi\rangle \\ \alpha_{L^-}^- \beta_{L^+}^+ |\chi\rangle &\propto \beta_{L^+}^+ \alpha_{L^-}^- |\chi\rangle \\ \alpha_{L^+}^+ \alpha_{L^-}^- |\chi\rangle &= \alpha_{L^-}^- \alpha_{L^+}^+ |\chi\rangle \\ \beta_{L^+}^+ \beta_{L^-}^- |\chi\rangle &= \beta_{L^-}^- \beta_{L^+}^+ |\chi\rangle\end{aligned}\quad (3.9)$$

The first two relations are rather easily established, whereas we have verified the last two by explicit, lengthy calculations.

Let us denote by V_O the full character (3.1) of the module built on $O|\Omega_+\rangle$. Then a careful analysis of the structure of the singular vectors that arise leads to the following subtraction scheme in the construction of the massive character for a unitary representation with highest weight state $|\Omega_+\rangle$:

$$\begin{aligned}V_1 &= V_{\alpha^+} - V_{\beta^+} - V_{\alpha^-} - V_{\beta^-} + V_{\alpha^+\alpha^-} + V_{\alpha^+\beta^-} + V_{\beta^+\alpha^-} + V_{\beta^+\beta^-} \\ &+ V_{\alpha^-\beta^-} + V_{\beta^-\alpha^+} + V_{\beta^-\beta^+} + V_{\beta^+\alpha^+} - V_{\alpha^+\beta^+} - V_{\alpha^-\alpha^-} - \dots\end{aligned}\quad (3.10)$$

Thus again : minus signs (plus signs) for modules involving an odd (even) number of $\alpha^\pm, \beta^\pm$ operators.

The quantum numbers (H, L^+, L^-) of the relevant states may be given as (with $n = 1, 2, \dots$ for the A -type states, and $n = 0, 1, \dots$ for the B -type states)

$$\begin{aligned}|A_n^+\rangle &= (\beta^+ \alpha^+)^n |\Omega_+\rangle = |h + n^2 k^+ - (2\ell^+ - 1)n; \ell^+ - nk^+; \ell^- - 1\rangle \\ |B_n^+\rangle &= (\alpha^+ \beta^+)^n |\Omega_+\rangle = |h + n^2 k^+ + (2\ell^+ - 1)n; \ell^+ + nk^+; \ell^- - 1\rangle \\ |A_n^+\rangle &= (\alpha^+ \beta^+)^{n-1} \alpha^+ |\Omega_+\rangle = |h + n^2 k^+ - (2\ell^+ - 1)n; -\ell^+ + nk^+ + 1; \ell^- - 1\rangle \\ |B_n^+\rangle &= (\beta^+ \alpha^+)^n \beta^+ |\Omega_+\rangle = |h + n^2 k^+ + (2\ell^+ - 1)n; -\ell^+ - nk^+ + 1; \ell^- - 1\rangle \\ |A_n^-\rangle &= (\beta^- \alpha^-)^n |\Omega_+\rangle = |h + n^2 k^- - (2\ell^- - 1)n; \ell^-; \ell^- - nk^- - 1\rangle \\ |B_n^-\rangle &= (\alpha^- \beta^-)^n |\Omega_+\rangle = |h + n^2 k^- + (2\ell^- - 1)n; \ell^-; \ell^- + nk^- - 1\rangle \\ |A_n^-\rangle &= (\alpha^- \beta^-)^{n-1} \alpha^- |\Omega_+\rangle = |h + n^2 k^- - (2\ell^- - 1)n; \ell^-; -\ell^- + nk^- \rangle \\ |B_n^-\rangle &= (\beta^- \alpha^-)^n \beta^- |\Omega_+\rangle = |h + n^2 k^- + (2\ell^- - 1)n; \ell^-; -\ell^- - nk^- \rangle.\end{aligned}\quad (3.11)$$

Due to the commutation rules (3.9) and the explicit form of the character for the full reducible module built on a singular vector of the $|\Omega_+\rangle$ type (3.1), the massive character in the R sector (3.10) factorizes into

$$\begin{aligned}Ch^{A,R}(k^+, k^-, h, \ell^+, \ell^-; q, z_+, z_-) &= Ch_{0,+}^{A,R}(k^+, k^-; q, z_+, z_-) q^{h - \frac{1}{2}\ell^+} \\ &\quad * \sum_{m=-\infty}^{\infty} q^{m^2 k^+ + (2\ell^+ - 1)m} \{z_+^{2(mk^+ + \ell^+)} - z_+^{-2(mk^+ + \ell^+ - 1)}\} \\ &\quad * \sum_{n=-\infty}^{\infty} q^{n^2 k^- + (2\ell^- - 1)n} \{z_-^{2(nk^- + \ell^-)} - z_-^{-2(nk^- + \ell^- - 1)}\} z_-^{-2}.\end{aligned}\quad (3.12)$$

We notice that by (3.1), this expression is symmetric in $(+, -)$.

By explicit comparison with the formula (2.20), we find the following surprising result

$$\begin{aligned}Ch^{A,R}(k^+, k^-, h, \ell^+, \ell^-; q, z_+, z_-) &= q^{h - \frac{1}{2}\ell^+} \\ &\quad * q^{-h_1 + \frac{\ell^+ - 1}{2}} Ch^R(k^+ - 1, h_1, \ell^+ - \frac{1}{2}; q, z_+, z_-) \\ &\quad * q^{-h_2 + \frac{\ell^- - 1}{2}} Ch^R(k^- - 1, h_2, \ell^- - \frac{1}{2}; q, z_-, z_+)\end{aligned}\quad (3.13)$$

(h_1, h_2 are arbitrary dimensions compatible with the massiveness). We do not know any interpretation for this factorization, except in the limit $k^- \rightarrow \infty, k^+$ fixed (or vice-versa), see section 5, but its origin is in (3.9).

We may now use the spectral flow rules, (2.18), to obtain

$$\begin{aligned}
Ch^{A_1, NS}(k^+, k^-, h, \ell^+, \ell^-; q, z_+, z_-) &= q^{h-\frac{h}{24}} \\
* \quad q^{-h_1+\frac{k^+-1}{4}} Ch^{NS}(k^+-1, h_1, \ell^+, q, z_+, z_-) \\
* \quad q^{-h_2+\frac{k^+-1}{4}} Ch^{NS}(k^+-1, h_2, \ell^-, q, z_-, z_+).
\end{aligned} \tag{3.14}$$

These factorized forms would seem to suggest that modular properties of our new characters may be tractable to the same extent as the ones for the ordinary $N=4$ SCA [5,6]. For this, however, we still need to derive the massless characters [7].

4 Characters of the $\tilde{\mathcal{A}}_\gamma$ algebra

As pointed out in [11], the $\tilde{\mathcal{A}}_\gamma$ algebra is a direct sum

$$\mathcal{A}_\gamma = \tilde{\mathcal{A}}_\gamma \oplus \mathcal{A}_{QV} \tag{4.1}$$

where $\mathcal{A}_{QV}$ is the algebra of the four free fermions Q_a and the generator $U(z)$ of the $U(1)$ algebra. $\tilde{\mathcal{A}}_\gamma$ only contains generators $\tilde{L}, \tilde{T}^\pm, \tilde{G}_a$ commuting with those and corresponding to a Virasoro algebra of central charge $\tilde{c} = c - 3$, two $SU(2)^\pm$ KM algebras of levels $\tilde{k}^\pm = k^\pm - 1$ and four supercharges $\tilde{G}_a$ (see [11,4] for details). Therefore, in some sense, it is more natural to consider characters for $\tilde{\mathcal{A}}_\gamma$. However, the commutation relations of $\tilde{\mathcal{A}}_\gamma$ only close on composite operators forming an algebra of the kind considered by Knizhnik [10]. In particular, the anticommutator of two supercharges is, in real notations, (see [1,4,3] for details) :

$$\begin{aligned}
\{\tilde{G}_{a,m}, \tilde{G}_{b,n}\} &= 2\tilde{L}_{m+n} \delta_{ab} + \frac{4}{k} (m-n) [\tilde{k}^- i \alpha_{ab}^{+i} \tilde{T}^{+i} + \tilde{k}^+ i \alpha_{ab}^{-i} \tilde{T}^{-i}]_{m+n} \\
&+ \frac{2\tilde{k}^+ \tilde{k}^-}{k} \delta_{ab} \delta_{m+n,0} (m^2 - \frac{1}{4}) - \frac{2}{k} \delta_{ab} [(\tilde{T}^{+i})^2_{m+n} + (\tilde{T}^{-i})^2_{m+n}] \\
&- \frac{16}{k} (\alpha^{+i} \alpha^{-j})_{ab} (\tilde{T}^{+i} \tilde{T}^{-j})_{m+n}.
\end{aligned} \tag{4.2}$$

Therefore, not even $\{\tilde{G}_{a,m}, \tilde{G}_{a,m}\} = 0$ is satisfied, and the analysis of singular vectors in $\tilde{\mathcal{A}}_\gamma$ would not seem to follow the simple pattern indicated above.

However, it follows from (4.1) that the characters for $\mathcal{A}_\gamma$ factorize into ones for $\tilde{\mathcal{A}}_\gamma$ and for $\mathcal{A}_{QV}$, and the last ones are rather trivial.

Indeed, in the NS sector, we simply find for the $\mathcal{A}_{QV}$ character

$$Ch^{A_{QV}, NS}(u; q, z_+, z_-) = q^{u^2/k-1/8} F^{NS}(q, z_+, z_-) * \prod_{n=1}^{\infty} (1-q^n)^{-1}, \tag{4.3}$$

while in the R sector, the $\mathcal{A}_{QV}$ character is

$$Ch^{A_{QV}, R}(u; q, z_+, z_-) = q^{u^2/k+1/4-1/8} F^R(q, z_+, z_-) \prod_{n=1}^{\infty} (1-q^n)^{-1} (1+z_+^{-1} z_-^{-1}) (1+z_+^{-1} z_-^{-1}) z_+. \tag{4.4}$$

Here, we used that the $\{Q_a, U\}$ system provides a central charge of 3, and that U gives a contribution $h_U = u^2/k$. In the R case, we further used that the fermionic Fock vacuum has (ℓ^+, ℓ^-) quantum numbers $(\frac{1}{2}, 0)$ for $|\Omega_+ \rangle$ and $(0, \frac{1}{2})$ for $|\Omega_- \rangle$. The equation (4.4) suggests a derivation based on $|\Omega_+ \rangle$, but the result is symmetric. The four free Ramond fermions contribute $\frac{4}{16} = \frac{1}{4}$ to h .

In these formulas, the trace is built using the operators L_0 and $T_0^{\pm 3}$ constructed from U and the four Q_a 's [11,4]. These in fact are the ones that have been subtracted from the original L_0 and $T_0^{\pm 3}$ to form $\tilde{L}_0$ and $\tilde{T}_0^{\pm 3}$.

Finally we notice that in the NS sector, we simply have $\tilde{\ell}^\pm = \ell^\pm$, whereas in the R sector, $\tilde{\ell}^\pm = \ell^\pm - \frac{1}{2}$, the $\frac{1}{2}$ coming from fermionic zero-modes.

We then get for the $\tilde{\mathcal{A}}_\gamma$ characters:

$$\begin{aligned}
Ch^{\mathcal{A}_\gamma, I}(\tilde{k}^+, \tilde{k}^-, \tilde{h}, \tilde{\ell}^+, \tilde{\ell}^-; q, z_+, z_-) &= q^{\tilde{k}^+-\tilde{\ell}^2/24} [F^I(q, z_+, z_-)]^{-1} f_I(z_+, z_-) \prod_{n=1}^{\infty} (1-q^n) \\
* \quad q^{-h_1+\tilde{k}^+/4} Ch^I(\tilde{k}^+, h_1, \tilde{\ell}^+; q, z_+, z_-) & q^{-h_2+\tilde{k}^-/4} Ch^I(\tilde{k}^-, h_2, \tilde{\ell}^-; q, z_-, z_+),
\end{aligned} \tag{4.5}$$

where $I = NS$ or R , and where the second, third and fourth factors are the contribution from $Ch^{\mathcal{A}_{QV}, I}$ with

$$f_{NS}(z_+, z_-) = 1; \quad f_R(z_+, z_-) = (1+z_+^{-1} z_-^{-1})^{-1} (z_+ + z_-)^{-1}. \tag{4.6}$$

These factors of $Ch^{\mathcal{A}_{QV}, I}$ may be removed from either one of the two character factors. We give the form (4.5) to emphasize the symmetry. The central charge $\tilde{c}$ is

$$\tilde{c} = c - 3 = \frac{6\tilde{k}^+ \tilde{k}^-}{k} - 3 = \frac{6\tilde{k}^+ \tilde{k}^- + 3(\tilde{k}^+ + \tilde{k}^-)}{\tilde{k}^+ + \tilde{k}^- + 2}. \tag{4.7}$$

Our characters (4.5) agree, in the special case where $\tilde{k}^+ = \tilde{k}^- = 1$ and $z_\pm \rightarrow 1$, with the expression given in [12]. To see this, it is convenient to rewrite them as [5]

$$\begin{aligned}
Ch^{\mathcal{A}_\gamma, I}(\tilde{k}^+, \tilde{k}^-, \tilde{h}, \tilde{\ell}^+, \tilde{\ell}^-; q, z_+, z_-) &= q^{h-\tilde{c}/24} q^{-i(\tilde{\ell}^++1/2)/(\tilde{k}^++1)-i(\tilde{\ell}^-+1/2)/(\tilde{k}^-+1)+1/4} \\
* \quad F^I(q, z_+, z_-) f_I^{-1}(z_+, z_-) &\prod_{n=1}^{\infty} (1-q^n)^{-1} \chi_{\tilde{k}^+-1}^{\tilde{\ell}^+}(q, z_+) \chi_{\tilde{k}^--1}^{\tilde{\ell}^-}(q, z_-),
\end{aligned} \tag{4.8}$$

where $\tilde{\ell}_{NS}^\pm = \tilde{\ell}^\pm$ and $\tilde{\ell}_R^\pm = \tilde{\ell}^\pm - \frac{1}{2}$ and where $\chi_k^\ell(q, z)$ is the level k $SU(2)$ character for isospin ℓ [9]:

$$\begin{aligned} \chi_k^\ell(q, z) &= q^{(\ell+\frac{1}{2})^2/(k+2)-1/8} \prod_{n=1}^{\infty} (1-q^n)^{-1} (1-z^2 q^n)^{-1} (1-z^{-2} q^{n-1})^{-1} \\ &\quad * \sum_{m=-\infty}^{m=+\infty} q^{(k+2)m^2+(2\ell+1)m} \left\{ z^{2((k+2)m+\ell)} - z^{-2((k+2)m+\ell+1)} \right\}. \end{aligned} \quad (4.9)$$

For $\tilde{k}^\pm = 1$, the only possible isospin values are $\tilde{\ell}_{NS}^\pm = \tilde{\ell}_R^\pm = 0$. Note that in the limit $z_\pm \rightarrow 1$,

$$\lim_{z_+ \rightarrow 1} \chi_0^0(q, z_+) = \lim_{z_- \rightarrow 1} \chi_0^0(q, z_-) = 1, \quad (4.10)$$

and

$$Ch^{\mathcal{A}, I}(1, 1, \tilde{h}, 0, 0; q, 1, 1) = q^{\tilde{h}-1/8} f_I^{-1}(1, 1) \prod_{n \in N_I} (1+q^n)^4 \prod_{n=1}^{\infty} (1-q^n)^{-1}. \quad (4.11)$$

With $f_{NS}^{-1}(1, 1) = 1$ and $f_R^{-1}(1, 1) = 4$, this agrees with [12].

5 The limit $k^- \rightarrow \infty$, k^+ fixed.

In this limit, the ordinary extended $N = 4$ SCA [5, 13, 14] emerges [1, 2], as is immediately seen from the commutation relations in the Appendix. The interpretation in terms of the behaviour of the characters, however, is obscured by the fact that the remaining operators in $\mathcal{A}_\gamma$ do not decouple. To get a clearer picture, it is best to consider $\mathcal{A}_\gamma$ in this limit.

We see that for $k^- \rightarrow \infty$, k^+ fixed, the following rescalings should be performed

$$\begin{aligned} \tilde{\ell}_n^{-i} &= \frac{1}{\sqrt{k}} \tilde{T}_n^{-i}, \quad n \neq 0 \\ u_n &= \frac{1}{\sqrt{k}} U_n, \quad n \neq 0 \\ (\psi_a)_n &= -\frac{1}{\sqrt{k}} (Q_a)_n \quad \text{for all } n \in \mathbb{Z}. \end{aligned} \quad (5.1)$$

Then $\tilde{\mathcal{A}}_\gamma$ reduces as follows for $k^- \rightarrow \infty$: the operators $\tilde{\ell}_n^{-i}$ ($n \neq 0$) become oscillators of three free scalar fields (without zero-modes) and decouple completely from the rest of $\tilde{\mathcal{A}}_\gamma$, except that they have isospin 1 under the global (unscaled!) $\tilde{T}_0^{-i}$'s. They build a Virasoro generator L' with $c' = 3$. All the composite operators in (4.2) decouple with the exception of L' , and we see that the set

$$\{L_{N=4} \equiv \tilde{L} - L', \tilde{G}_\alpha, \tilde{T}^{++i}\} \quad (5.2)$$

precisely form an $N = 4$ SCA with one central extension $\tilde{k}^+$ and with

$$c_{N=4} = \tilde{c} - c' = c - 6 = 6\tilde{k}^+ \quad (5.3)$$

as is required. In addition, the $\tilde{G}_\alpha$ operators transform as always under the global $SU(2)^-$.

It is now an easy matter with the above formulas to consider the limit $k^- \rightarrow \infty$ (in (2.19),(2.20)), this limit is well behaved for $z_\pm = \exp i\theta_\pm$, $\theta_\pm$ real) with the result

$$Ch^{\mathcal{A}, I}(\tilde{k}^+, \tilde{k}^- \rightarrow \infty, \tilde{h}, \tilde{\ell}^+, \tilde{\ell}^-; q, z_+, z_-) = \chi_\ell(q, z_+, z_-) = \chi_\ell^I(k^+, \tilde{h}, \tilde{\ell}^+; q, z_+, z_-) \quad (5.4)$$

with $I = NS, R$. Here,

$$\chi_\ell(q, z_+, z_-) = q^{-\frac{1}{2}\ell} \prod_{n=1}^{\infty} (1 - z_-^2 q^n)^{-1} (1 - z_-^{-2} q^n)^{-1} (1 - q^n)^{-1} \quad (5.5)$$

is the contribution from the three free scalar fields $\tilde{\ell}_n^{-i}$ ($n \neq 0$) and

$$\chi_{NS}^{NS}(z_-) = z_-^{-2\ell^-} + z_-^{-2\ell^-+2} + \dots + z_-^{2\ell^-} \quad (5.6)$$

$$\chi_{SR}^R(z_-) = z_-^{-(2\ell^- - 1)} + \dots + z_-^{2\ell^- - 1} \quad (5.7)$$

are the classical characters of the relevant $SU(2)^-$ multiplet. (In the Ramond case, the $SU(2)^-$ isospin of the analog of $|\Omega_+\rangle$ has $\tilde{\ell}^- - \frac{1}{2}$ due to the fermionic zero-mode subtraction. If we consider the limit $k^+ \rightarrow \infty$, k^- fixed, the symmetry is restored).

Then (5.4) provides a check that our characters in the limit $k^- \rightarrow \infty$ behave in a way consistent with the algebra and known results.

6 Conclusions.

In this paper, we have given the characters of the algebras $\mathcal{A}_\gamma$ and $\tilde{\mathcal{A}}_\gamma$ for irreducible, unitary, highest weight, massive representations.

Our results are found in (3.13), (3.14), (4.5), (4.8) (with (2.19),(2.20),(2.21),(4.9)).

We have found that despite the appearance of two $\widetilde{SU}(2)$ KM algebras which render the pattern of singular vectors inherently complicated, the situation remains tractable due to the commutativity properties summarized in (3.9). These properties give rise to the rather surprising factorized form of the character obtained. This form in turn promises to make modular properties manageable. However, an analysis of the characters of massless representations will have to be carried out before definite statements can be made [7].

We have performed consistency checks by examining the cases $\tilde{k}^+ = \tilde{k}^- = 1$ (see [12]), and the limit $k^- \rightarrow \infty$, k^+ fixed (or vice-versa).

Acknowledgments

We thank M. Günaydin for useful discussions on the Ramond zero-mode multiplet, and we are grateful to K. Schoutens and A. Van Proeyen for illuminating remarks.

Appendix

For convenience, we present explicitly the commutation relations of the $\mathcal{A}_r$ algebra [1,2] in the complex notation:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}\delta_{m+n,0}m(m^2-1), \quad (A.1)$$

$$[L_m, \phi_n] = [(d_\phi - 1)m - n]\phi_{m+n}$$

where d_ϕ is the conformal dimension of ϕ . Thus, $d_Q = \frac{1}{2}$, $d_{T^{\pm i}} = d_U = 1$, $d_G = \frac{3}{2}$. We have moreover

$$c = 6\frac{k^+k^-}{k}, \quad \gamma = \frac{k^-}{k}, \quad 1 - \gamma = \frac{k^+}{k}, \quad k = k^+ + k^-. \quad (A.2)$$

The other commutation relations are

$$\begin{aligned} \{G_{+,m}, G_{-,n}\} &= L_{m+n} + \frac{c}{6}(m^2 - \frac{1}{4})\delta_{m+n,0} - (n-m)\left[\frac{k^-}{k}T_{m+n}^{+3} + \frac{k^+}{k}T_{m+n}^{-3}\right], \\ \{G_{+,m}, G_{-,K,n}\} &= L_{m+n} + \frac{c}{6}(m^2 - \frac{1}{4})\delta_{m+n,0} - (n-m)\left[\frac{k^-}{k}T_{m+n}^{+3} - \frac{k^+}{k}T_{m+n}^{-3}\right], \\ \{G_{\pm,m}, G_{+,K,n}\} &= \pm \frac{k^\mp}{k}(n-m)T_{m+n}^{\pm\pm}, \quad \{G_{\pm,m}, G_{-,K,n}\} = \pm \frac{k^\pm}{k}(n-m)T_{m+n}^{\mp\pm}, \end{aligned} \quad (A.3)$$

$$[T_m^{\pm 3}, T_n^{\pm 3}] = 2T_{m+n}^{\pm 3} + mk^\pm \delta_{m+n,0}, \quad [T_m^{\pm 3}, T_n^{\pm 3}] = \frac{m}{2}k^\pm \delta_{m+n,0},$$

$$[T_m^{\pm 3}, T_n^{\pm 1}] = T_{m+n}^{\pm 1}, \quad [T_m^{\pm 3}, T_n^{\pm 1}] = -T_{m+n}^{\pm 1}.$$

Let $\tau_i^{\pm\beta}(i = +, -, 3; \alpha, \beta = 1, 2)$ be the standard doublet representation of $SU(2)$:

$$\tau^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \quad (A.4)$$

Then the $Q_{a,n}$'s transform under the $T_m^{\pm i}$ according to the form

$$[T_m^{\pm i}, (Q_1, Q_2)_n] = (Q_1, Q_2)_{m+n} \tau^i \quad (A.5)$$

where

$$(Q_1, Q_2) = (Q_{+,K}, Q_-) \quad \text{or} \quad (Q_+, -Q_{-,K}) \quad \text{for } SU(2)^+ \quad (A.6)$$

and

$$(Q_1, Q_2) = (Q_{-,K}, Q_-) \quad \text{or} \quad (Q_+, -Q_{+,K}) \quad \text{for } SU(2)^- \quad (A.7)$$

For the G 's, the structure of the commutation relations is

$$[T_m^{\pm i}, (G_1, G_2)_n] = ((G_1, G_2)_{m+n} \mp 2\frac{k^\pm}{k}m(Q_1, Q_2)_{m+n}) \tau^i, \quad (A.8)$$

with (G_1, G_2) given similarly as in (A.6) and (A.7). Finally we have,

$$\begin{aligned} \{Q_{\pm,m}, G_{\mp,n}\} &= \mp \frac{1}{2}(T_{m+n}^{+3} - T_{m+n}^{-3}) + \frac{1}{2}U_{m+n}, \\ \{Q_{\pm,K,m}, G_{\mp,K,n}\} &= \mp \frac{1}{2}(T_{m+n}^{+3} + T_{m+n}^{-3}) + \frac{1}{2}U_{m+n}, \\ \{Q_{\pm,m}, G_{+,K,n}\} &= \frac{1}{2}T_{m+n}^{\pm\pm}, \quad \{Q_{\pm,m}, G_{-,K,n}\} = -\frac{1}{2}T_{m+n}^{\mp\pm}, \\ \{Q_{+,K,m}, G_{\pm,n}\} &= -\frac{1}{2}T_{m+n}^{\pm\pm}, \quad \{Q_{-,K,m}, G_{\pm,n}\} = \frac{1}{2}T_{m+n}^{\mp\pm}, \\ [U_m, G_{a,n}] &= mQ_{a,m+n}, \quad [U_m, U_n] = -m\frac{k}{2}\delta_{m+n,0} \\ \{Q_{+,m}, Q_{-,n}\} &= \{Q_{+,K,m}, Q_{-,K,n}\} = -\frac{k}{4}\delta_{m+n,0} \end{aligned} \quad (A.9)$$

The commutators not explicitly indicated are all zero. The hermiticity properties are as follows :

$$\begin{aligned} L_n^\dagger &= L_{-n}, & U_n^\dagger &= -U_{-n} \\ G_{+,n}^\dagger &= G_{-, -n}, & Q_{+,n}^\dagger &= -Q_{-, -n} \\ G_{+,K,n}^\dagger &= G_{-,K, -n}, & Q_{+,K,n}^\dagger &= -Q_{-,K, -n} \\ (T_n^{\pm 3})^\dagger &= T_{-n}^{\pm 3}, & (T_n^{\pm 1})^\dagger &= T_{-n}^{\pm 1}. \end{aligned} \quad (A.10)$$

References

- [1] A. Sevrin, W. Troost and A. Van Proeyen, Phys. Lett. **208B** (1988) 447.
- [2] A. Sevrin, Ph. Spindel, W. Troost and A. Van Proeyen, Nucl. Phys. **B308** (1988) 662; Nucl. Phys. **B311** (1988) 465.
- [3] A. Van Proeyen, Realizations of $N=4$ superconformal algebras, KUL-TF-89/12, to be published in Classical and Quantum Gravity.
- [4] M. Günaydin, J.L. Petersen, A. Taormina and A. Van Proeyen, Nucl. Phys. **B322** (1989) 402.
- [5] T. Eguchi and A. Taormina, Phys. Lett **196B** (1987) 75; **200B** (1988) 315 ; **210B** (1988) 125.

- [6] T. Eguchi, H. Ooguri, A. Taormina and S.-K. Yang, Nucl. Phys. **B315** (1989) 193.
- [7] J.L. Petersen and A. Taormina, work in progress.
- [8] F. Defever, S. Schrans and K. Thielemans, Phys. Lett. **212B** (1988) 467.
- [9] V. Kac, Infinite dimensional Lie algebras (Cambridge U. P., Cambridge, 1975).
- [10] V.G. Knizhnik, Theor. Math. Phys., **66** (1986) 68.
- [11] P. Goddard and A. Schwimmer, Phys. Lett. **214B** (1988) 209.
- [12] K. Schoutens, Nucl. Phys. **B314** (1989) 519.
- [13] M. Ademollo et al., Phys. Lett. **62B** (1976) 105; Nucl. Phys. **B111** (1976) 77.
- [14] M. Yu, Nucl. Phys. **B294** (1987) 890.

Figure Caption

Figure 1: The sixteen $SU(2)^+ \otimes SU(2)^-$ hws of the multiplets appearing at the ground state level in the R sector are labeled by their quantum numbers (ℓ_a^+, ℓ_a^-) , and (n) for $n = 1, 2, 4$ gives their multiplicity. In particular, $|\Omega_+ \rangle$ and $|\Omega_- \rangle$ have quantum numbers $(\ell^+, \ell^- - 1)$ and $(\ell^+ - 1, \ell^-)$ respectively, and they are unique. The dots and crosses represent bosonic and fermionic states respectively, or vice-versa.