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EXTENDED HIGHER SPIN
SUPERALGEBRAS AND THEIR
MASSLESS REPRESENTATIONS

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Abstract

Three two-parameter sequences of infinite dimensional extended higher spin superalgebras are constructed, which give rise to consistent equations of motion of interacting gauge fields of all spins in four dimensions. In the Yang-Mills sector of spin-1 gauge fields, these higher spin superalgebras reduce to $u(n) \oplus u(m)$, $o(n) \oplus o(m)$ and $usp(n) \oplus usp(m)$ with arbitrary integer parameters $n \geq 0$ and $m \geq 0$ (n and m are assumed to be even for symplectic algebras). Massless unitary representations of the proposed higher spin superalgebras are analyzed. It is shown that all these superalgebras obey the admissibility condition which requires them to possess massless unitary representations with just the same spectra of spins as it follows from the structure of the related higher spin gauge fields.

1. Introduction.

During the last two years it became evident that a broad class of new nontrivial gauge theories of massless higher spin fields exists. This conclusion is based mainly on the results of refs./1,2/. Namely, in ref./1/, a class of actions was proposed which describe consistent interactions of $d=4$ higher spin gauge fields in the lowest (cubic) order in interactions. These actions are explicitly general coordinate invariant and contain the Einstein action with the cosmological term in the spin-2 sector. Gauge theories of higher spins were shown to be unusual in that respect that higher spin interactions contain both positive and negative powers of the parameter which is proportional to the value of the cosmological constant /1/. This property prevents one from taking the flat limit in the phase with unbroken higher spin gauge symmetries. (It is worth mentioning that the fact of non-analyticity in the cosmological constant has nothing to do with physical regimes in which higher spin gauge symmetries are expected to be broken spontaneously.) In fact, it is the nonanalyticity of the higher spin gauge interactions in the cosmological constant which served as the main obstacle that prevented for a long time from constructing a consistent interaction of massless higher spin fields with gravity*.

* Such attempts were undertaken e.g. in refs./3/. However, because of using expansion procedures over the flat background, these attempts were not successful. Note also that some non-gravitational cubic interactions of massless higher spins in the flat space were constructed in refs./4/. However we believe that, in nontrivial cases, it will be necessary to introduce the gravitational interaction of massless fields and to add some interactions containing negative powers of the cosmological constant in order to extend cubic higher spin interactions to the highest orders.

In ref./2/, the formulation of consistent equations of motion of massless fields of all spins was proposed in terms of some free differential algebra. One of the advantages of this approach is due to using the expansion procedure in powers of curvatures which is much more efficient than the expansion procedure used in ref. /1/. Consequently, the results obtained in ref./2/ exceed considerably those of ref./1/. In particular, this manifests itself in the fact that, when constructing consistent equations of motion along the lines of ref./2/, one needs to use some specific sets of fields, the higher spin supermultiplets. In this respect, the situation with higher spin theories is analogous to that with supergravity-type theories /5/ which are known to admit consistent formulations only when one starts from the appropriate sets of fields corresponding to one or another supergravitational supermultiplet.

From the viewpoint of the approach of ref./2/, higher spin supermultiplets are described by the sets of gauge 1-forms and Weyl 0-forms corresponding to one or another of infinite dimensional higher spin superalgebras proposed in refs./6-8/. The higher spin superalgebras used in refs./1,2/ can be thought of as higher spin counterparts of the usual superalgebras of anti-de Sitter (AdS) supersymmetry $osp(N,4)$, which can be used for building supergravitational models along the lines of the MacDowell-Mansouri approach /9/. The essentially new point discovered in ref./2/ consists of the fact that only some part of higher spin superalgebras proposed in refs./6-8/ gives rise to consistent equations of motion beyond the cubic order. For the simplest case of higher spin superalgebras with trivial internal symmetries, exactly the same list of admissible higher spin su-

peralgebras was obtained in ref./10/ from the very different viewpoint. Namely, it was shown in ref./10/ that only those higher spin superalgebras lead to consistent equations of motion in ref./2/ which possess massless unitary representations with just the same spectra of spins as predicted by the structure of the gauge fields originating from these superalgebras. If such unitary representations exist, then, following to the terminology of ref./10/, we say that corresponding higher spin superalgebras obey the admissibility condition.

In the present paper, we construct three two-parameter sequences of infinite dimensional $d=4$ extended higher spin superalgebras, denoted as $shs_u(n;m|4)$, $shs_o(n;m|4)$ and $shs_{usp}(n;m|4)$ (shs means super-higher-spin), which give rise to consistent equations of motion for interacting massless fields of all spins. Their Yang-Mills subalgebras, which give rise to spin-1 massless fields, reduce to $u(n) \oplus u(m)$, $o(n) \oplus o(m)$, and $usp(n) \oplus usp(m)$ respectively (n and m are assumed to be even for the symplectic case). It seems very likely that three sequences of extended higher spin superalgebras mentioned above exhaust all extended higher spin superalgebras which can lead to consistent dynamics of massless higher spin fields in $d=4$.

This conclusion is supported by the analysis of massless unitary representations of the proposed higher spin superalgebras. In accordance with the results by Plato and Fronsdal /11/, these representations reduce to tensor products of pairs of Dirac's singletons /12/. It is shown that all higher spin superalgebras, proposed in the paper, obey the admissibility condition. Conversely, all sensible types of tensor products of pairs of singletons are shown to correspond to one of the proposed higher spin

superalgebras. Specifically, our results agree with the results by Bergshoeff, Salam, Sezgin and Tani $\ddot{g}$ obtained in ref./13/, where the admissibility condition was verified for one of the extended higher spin superalgebras of ref./7/ in the context of the analysis of excitations of the supermembrane.

The rest of the paper is organized as follows. In sect.2, we recall the main facts about higher spin superalgebras and their automorphisms which enable one to extract various types of extended higher spin superalgebras out the superalgebras of ref./2/. In sect.3, we construct all types of extended higher spin superalgebras as those subalgebras of the superalgebras $shs_u(n;n|4)$ used in ref./2/, which correspond to consistent truncations of equations of motion of massless higher spins. In sect.4, massless unitary representations of the proposed extended higher spin superalgebras are constructed in terms of tensor products of Dirac's singletons. Finally, in sect.5, some perspectives and open questions are discussed briefly.

2. Higher Spin Superalgebras and Their Automorphisms

In ref./2/, the dynamics of massless higher spin fields was described in terms of the higher spin Lie superalgebra $shs^{\mathbb{Z}}(1)$ of ref./8/ originating from the associative algebra A with the generating elements q_{α} , $r_{\dot{\beta}}$ and K obeying the relations

$$[q_{\alpha}, q_{\beta}] = 2i\epsilon_{\alpha\beta}, \quad [r_{\dot{\alpha}}, r_{\dot{\beta}}] = 2i\epsilon_{\dot{\alpha}\dot{\beta}}, \quad [q_{\alpha}, r_{\dot{\beta}}] = 0; \quad (2.1a)$$

$$\{K, q_{\alpha}\} = 0, \quad \{K, r_{\dot{\beta}}\} = 0, \quad K^2 = 1. \quad (2.1b)$$

Here the labels $\alpha, \beta \dots$ and $\dot{\alpha}, \dot{\beta} \dots$ take on values 1, 2 and serve as indices of two-component spinors. K possesses the pro-

properties of the Klein operator.

Let f be the automorphism of boson-fermion parity defined by the relation

$$f(\mathcal{P}(q, z, K)) = \mathcal{P}(-q, -z, K) \quad (2.2)$$

for any $\mathcal{P} \in A$. Then the full higher spin superalgebra $\text{shs}^f(1)$ proposed in ref./8/ can be defined in the standard fashion

$$[\mathcal{P}_1, \mathcal{P}_2] = \mathcal{P}_1 \mathcal{P}_2 - (-1)^{\pi_1 \pi_2} \mathcal{P}_2 \mathcal{P}_1 \quad (2.3)$$

for polynomials $\mathcal{P}_{1,2}$ possessing definite boson-fermion parity, $f(\mathcal{P}_i) = (-1)^{\pi_i} \mathcal{P}_i$.

When investigating irreducible representations of higher spin superalgebras, it is more convenient /10/ to use the following set of generating elements:

$$\begin{aligned} a &= \frac{1}{2}(q_1 + iz_2), & a^+ &= \frac{1}{2}(z_2 - iq_2), \\ b &= \frac{1}{2}(z_1 + iq_2), & b^+ &= \frac{1}{2}(q_1 - iz_2), \end{aligned} \quad (2.4)$$

which obey the Heisenberg commutation relations

$$[a, a^+] = [b, b^+] = 1, \quad [a^{(+)}, b^{(+)}] = 0; \quad (2.5a)$$

$$\{K, a^{(+)}\} = \{K, b^{(+)}\} = 0, \quad K^2 = 1. \quad (2.5b)$$

In fact, speaking on $\text{shs}^f(1)$, we assume the real form of the originally complex higher spin superalgebra, which is extracted by the hermiticity conditions

$$(a)^\dagger = a^+, \quad (b)^\dagger = b^+, \quad (K)^\dagger = K \quad (2.6)$$

(equivalently, $(q_\alpha)^\dagger = r_\alpha$; for more details about extracting real forms of complex Lie superalgebras with the aid of hermiticity

conditions (involutions) see ref./7/).

The higher spin superalgebra $\text{shs}^f(1)$ of ref./8/ corresponds to the simplest case with the Abelian Yang-Mills subalgebra in the spin-1 sector. It was shown in ref./2/ that there exists a simple way for constructing extended higher spin superalgebras by allowing all physical quantities to take on their values in some (originally complex) associative algebra B . From the algebraic viewpoint, this means that we start from the associative algebra $A \otimes B$ in place of A . In practice, we confine ourselves to the case when B is identified with one of the matrix algebras $\text{Mat}_n(\mathbb{C})$, and we shall use the designation $A^n = A \otimes \text{Mat}_n(\mathbb{C})$. According to the approach of ref./2/, this implies that a dynamics of gauge higher spin fields is described in terms of the gauge potentials* (1-forms)

$$\omega_\kappa^\ell(q, \tau, K|x) = \sum_{u, v=0}^{\infty} \sum_{A=0}^{\frac{1}{2}} \frac{1}{2^i u! v!} (K)^A \omega_{\kappa}^{A \ell}{}_{\alpha_1 \alpha_2 \dots \alpha_u, \dot{\beta}_1 \dot{\beta}_2 \dots \dot{\beta}_v} (x) \cdot \quad (2.7)$$

$$q^{\alpha_1} q^{\alpha_2} \dots q^{\alpha_u} \tau^{\dot{\beta}_1} \tau^{\dot{\beta}_2} \dots \tau^{\dot{\beta}_v}$$

* Let us note that in ref./2/ broader superalgebras of higher spin and auxiliary fields were used which contained $\text{shs}^f(1)$ as a subalgebra. These superalgebras possess /8/ two Klein-type operators, R and Q , for dotted and undotted spinorial indices respectively. The higher spin superalgebras of the type of $\text{shs}^f(1)$ arise as subalgebras formed by elements containing Q and R via the combination $K = QR$. Correspondingly, the gauge fields $\omega_i^{A j} \dots$ of the present paper coincide with the fields $\omega_i^{AA j} \dots$ of ref./2/. As for the fields $\omega_i^{AB j}$ with $A+B=1$, which also emerged in ref./2/, these were shown in ref./14/ to be auxiliary fields possessing no dynamical degrees of freedom.

where k and ℓ are matrix indices taking on their values from 1 to n , and x denotes space-time coordinates x_0, x_1, x_2, x_3 . According to ref./15/, the fields (2.7) with some fixed indices A , k and ℓ and all u and v such that $u+v=2(s-1)$ describe one massless spin $s \geq 1$. This implies that the set of fields (2.7) describes infinite chain of spins $1 \leq s < \infty$ with the multiplicity $2n^2$ for every spin (lower spin fields $0 \leq s < 1$, which are described by appropriate Weyl 0-forms /2/, also emerge with this multiplicity).

The gauge fields (2.7) are assumed to obey the hermiticity conditions

$$(\omega_{\mu}^{A \ell}(q, z))^{\dagger} = -\omega_{\ell}^{A k}((-1)^A q, (-1)^A z) \quad (2.8)$$

at the condition $(q_{\alpha})^{\dagger} = r_{\alpha}$ (cf. eq.(2.6)). In the sector of spin-1 fields (these are the fields ω which do not depend on q_{α} and r_{α}), the conditions (2.8) correspond to the Yang-Mills gauge group $U(n) \otimes U(n)$. Let us denote the corresponding higher spin superalgebra as $\text{shs}_{\mu}(n; n|4)$.

Here 4 stands for four spinorial generating elements q_{α} and r_{β} , i.e., we reserve the notation $\text{shs}_{\mu}(n; m|2k)$ for the case when "spinorial" indices are allowed to take on $2k$ real values. This generalization to arbitrary k is analogous to the generalization $\text{osp}(N, 2k)$ of $\text{osp}(N, 4)$, while the meaning of the parameters n and m at $n \neq m$ will be clarified in Sect.3.

There exists a simple method for extracting subalgebras with the aid of automorphisms: given system of automorphisms τ_v of an arbitrary algebra $\mathcal{D}$, the set of elements $d \in \mathcal{D}$ obeying the conditions $\tau_v(d) = d$ for all $v = 1, 2, \dots$ forms some subalgebra of $\mathcal{D}$. If τ_v are automorphisms of $\text{shs}_{\mu}(n; n|4)$, which admit extensions

to the corresponding free differential algebras used in ref./2/ (these free differential algebras can, roughly speaking, be thought of as deformations of $\text{shs}_u(n;n|4)$), then one can build in this way consistent truncations of the higher spin equations of motion suggested in ref./2/.

The class of automorphisms which can be used for constructing higher spin subalgebras of $\text{shs}_u(n;n|4)$ leading to consistent equations of motion of massless fields, is restricted considerably by the requirement that these automorphisms should reduce to the trivial (identity) automorphism when restricted to the AdS subalgebra $\text{sp}_{\text{dS}}(4) \subset \text{shs}_u(n;n|4)$ which gives rise to the gravitational gauge fields. (The subalgebra $\text{sp}_{\text{dS}}(4)$ is spanned by various quadratic polynomials constructed from q_μ and r_μ (equivalently, from a , a^+ , b and b^+) and belongs to the center of B , i.e. elements of $\text{sp}_{\text{dS}}(4)$ are proportional to the unit $n \times n$ matrix $\hat{O}_\mu^{\mu\ell}$.) This restriction originates from the requirement that the gravitational field (spin-2 singlet) should survive in all truncations of higher spin equations of motion. Let us emphasize that the latter requirement seems to be inevitable because it concerns both the dynamical and background components of the gravitational fields. Actually, even working with free fields in the flat space one needs to use the flat metric tensor in order to construct lagrangians and equations of motion.

Since the effect of introducing the associative algebra B consists /2/ of replacing ordinary product law by the associative product in B (matrix product for $B = \text{Mat}_n(\mathbb{C})$), all automorphisms of B serve as automorphisms of the higher spin free differential algebra of ref./2/ as well. In the subsequent analysis of higher spin subalgebras, we shall use automorphisms μ_U of $B = \text{Mat}_n(\mathbb{C})$

which correspond to similarity transformations with various non-degenerate matrices $U_i^k \in GL(n)$,

$$\mu_U(\delta_i^j) = U_i^k \delta_k^l U^{-1}{}^j{}_l, \quad \delta \in \text{Mat}_n(\mathbb{C}). \quad (2.9)$$

Since higher spin gauge fields are assumed to obey the hermiticity conditions (2.8), it suffices in practice to consider automorphisms μ_U with unitary matrices U ($U^\dagger = U^{-1}$) which respect the property (2.8) (this implies that μ_U with unitary U serve as automorphisms of the real higher spin superalgebra $\text{shs}_u(n; n|2k)$). Let us also mention that μ_U leaves invariant the unit element of $\text{Mat}_n(\mathbb{C})$ since $\mu_U(\delta_i^j) = \delta_i^j$. This simple fact ensures that μ_U (2.9) acts trivially on the gravitational subalgebra $\text{sp}_{\text{dS}}(4)$ defined previously.

Similarly, all automorphisms of A can trivially be extended to $A^n = A \otimes \text{Mat}_n(\mathbb{C})$. Among these automorphisms, the automorphism of boson-fermion parity f , defined in eq.(2.2), will be of most importance for us. The automorphism f acts in the free differential algebra of ref./2/ as well since the boson-fermion parity is well-defined in any relativistic system. On the other hand, the automorphism $\mathcal{Z}$ of $\text{shs}_u(n, n|2k)$ that changes sign of K (i.e. $\mathcal{Z}(\mathcal{P}_i^j(q, r, K)) = \mathcal{P}_i^j(q, r, -K)$), cannot be extended /2/ to some automorphism of the free differential algebra of ref./2/ when the fermion sector is nontrivial because this free differential algebra contains explicitly the Klein operator K . As shown in refs./2, 10/ for the $n=1$ case, one can use $\mathcal{Z}$ for building consistent truncations only in the pure bosonic theory in which case the operators $\Pi_\pm = \frac{1}{2}(1 \pm K)$ behave as projectors belonging to the center of the algebra. In what follows, we disregard the automorphism $\mathcal{Z}$ since, firstly, these projectors and their fur-

ther generalizations used in the next section can easily be constructed without using $\mathcal{R}$ and, secondly, it was shown in ref./10/ for the $n=1$ case that all nontrivial superalgebras, which can be extracted with the aid of $\mathcal{R}$, do not obey the admissibility condition.

Another important possibility for constructing automorphisms of higher spin superalgebras is based on antiautomorphisms of the original associative algebra A^n . Namely, let σ be some anti-automorphism of A^n (i.e., $\sigma(a \cdot b) = \sigma(b)\sigma(a)$; $\sigma(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1 \sigma(a_1) + \lambda_2 \sigma(a_2)$). Then the mapping

$$\beta(P) = -i \pi(P) \sigma(P) \quad (2.10)$$

forms an automorphism of the Lie superalgebra constructed from the associative algebra A^n according to eq.(2.3) ($\pi(P) = 0(1)$) for even (odd) elements of the superalgebra; for more detail about antiautomorphisms and their relationship with automorphisms of Lie superalgebras see ref./7/). In order to define the action of σ on $A^n = A \otimes \text{Mat}_n(\mathbb{C})$, it suffices to describe how it acts separately on A and on $\text{Mat}_n(\mathbb{C})$. The action of the antiautomorphism σ on A was defined in refs./7,8,2/ by the relations

$$\sigma(q_\alpha) = i q_\alpha, \quad \sigma(z_\beta) = i z_\beta, \quad \sigma(K) = K. \quad (2.11)$$

Note that it was in fact verified in refs./7,8/ that this choice for the restriction of σ to A is physically unique up to automorphisms of A mentioned previously. In ref./2/, the action of σ on $\text{Mat}_n(\mathbb{C})$ was identified with the usual matrix transposition that enabled one to construct consistent higher spin truncations with the spin-1 gauge groups $SO(n) \otimes SO(n)$. This choice is a particular case of the following quite general definition of anti-

automorphisms via bilinear forms.

Let $M = \mathbb{C}^n$ be the representation space of the associative algebra $B = \text{Mat}_n(\mathbb{C})$, and $Q(x, y)$ be some non-degenerate bilinear form on M . Then the antiautomorphism σ^Q of the algebra B can be defined as follows:

$$Q(x, \sigma^Q(b)y) = Q(\sigma^Q(b)x, y) \quad (2.12)$$

$$\forall x, y \in M, b \in B.$$

If

$$Q(x, y) = \eta_Q^{ij} x_i y_j, \quad (2.13)$$

then

$$\sigma^Q(b)_{\kappa}{}^{\ell} = \eta_Q^{\ell p} b_p{}^q \eta_Q^{-1} q_{\kappa}. \quad (2.14)$$

Moreover, it can be shown that any antiautomorphism of $\text{Mat}_n(\mathbb{C})$ is of the form (2.14) with some non-degenerate metrics $\eta_Q^{\kappa\ell}$.

If σ^{Q_1} and σ^{Q_2} are two antiautomorphisms of $\text{Mat}_n(\mathbb{C})$ originating from the bilinear forms Q_1 and Q_2 , then $\sigma^{Q_1} \circ \sigma^{Q_2} = \mu_{U_{1,2}}$, where $\mu_{U_{1,2}}$ is the automorphism (2.9) with the matrix $U_{1,2}$ of the form

$$(U_{1,2})_{\kappa}{}^{\ell} = \eta_1^{-1} \eta_2^{\ell p} \quad (2.15)$$

In practice, this means that, when extracting subalgebras of Lie superalgebras by the condition $\tau_{\nu}(a) = a$ with some set of automorphisms τ_{ν} , it suffices to use only one automorphism, say τ_1 , generated by some antiautomorphism of the original associative algebra while all other conditions $\tau_{\nu}(a) = a$ with $\nu \neq 1$ can equivalently be replaced by the conditions which do not contain automorphisms based on the antiautomorphisms of the associative algebra. Note also that antiautomorphisms of $\text{Mat}_n(\mathbb{C})$, which are of the form $\mu_{\nu} \circ \sigma$, can be defined by the relation (2.14)

with the metrics η' of the form

$$\eta'_{\mu\nu} = \eta^{\rho\sigma} U^{-1}_{\rho\mu} U_{\sigma\nu} \quad (2.16)$$

where $\eta^{\rho\sigma}$ is the "metric tensor" that gives rise to G . For an arbitrary automorphism μ , μ^*G is some antiautomorphism of $\text{Mat}_n(\mathbb{C})$. Using the fact mentioned previously that any antiautomorphism of $\text{Mat}_n(\mathbb{C})$ is of the form (2.14), one easily proves that any automorphism μ of $\text{Mat}_n(\mathbb{C})$ is of the form (2.9).

We confine ourselves to consideration of the antiautomorphisms originating from either symmetric forms Q_S or antisymmetric forms Q_A . In the complex case, it is always possible to choose such bases that

$$Q_S(x, y) = \sum_{i=1}^n x_i y_i \quad (2.17)$$

and

$$Q_A(x, y) = \sum_{i=1}^{n/2} (x_i y_{i+n/2} - x_{i+n/2} y_i) \quad (2.18)$$

(n is assumed to be even for the case of the antisymmetric form Q_A). Evidently, the antiautomorphism (2.14), originating from the form Q_S (2.17), coincides with ordinary matrix transposition, and it is this choice for G which was discussed in ref./2/. Generally, the antiautomorphisms of $A^n = A \otimes \text{Mat}_n(\mathbb{C})$, which are defined by eq.(2.11) when restricted to A and by the forms Q_S or Q_A when restricted to $\text{Mat}_n(\mathbb{C})$, will be denoted as G_S and G_A respectively. The automorphisms of corresponding higher spin Lie superalgebras, which are constructed from G_S and G_A by means of eq.(2.10), are denoted as $\mathcal{P}_S$ and $\mathcal{P}_A$ respectively.

Before analyzing various subalgebras of $\text{shs}_u(n|2k)$ which can be extracted with the aid of the (anti)automorphisms discus-

sed above, let us discuss the simplest example of the Lie algebra $u(n)$ originating from $\text{Mat}_n(\mathbb{C})$. First of all let us note that those subalgebras, which are extracted by the conditions of the form

$$\mu_U(\mathcal{G}) = \mathcal{G} \quad (2.19)$$

with unitary U , decompose into the direct sums of several subalgebras if $\mu_U \neq I$. Actually, let us diagonalize the matrix U_i^j . By setting $U_i^j = u_i \delta_i^j$, one reduces the condition (2.19) to the form

$$(u_i u_j^{-1} - 1) \mathcal{G}_i^j = 0 \quad (2.20)$$

(no summation over i and j). The general solution of this equation is described by the block-diagonal matrix b_i^j , the size of each block coinciding with the degree of degeneracy for the corresponding eigenvalue u_i . For example, if

$$U_{\kappa}^{\ell} = (-1)^{\mathcal{A}(\kappa)} \delta_{\kappa}^{\ell} \quad (2.21)$$

with $\mathcal{A}(\kappa) = 0(1)$ for $1 \leq \kappa \leq m$ ($m < \kappa \leq n$), then the general solution of eq.(2.20) is described by those matrices b_i^j which are allowed to have non-vanishing elements only when either $1 \leq i \leq m$ and $1 \leq j \leq m$ or $m < i \leq n$ and $m < j \leq n$. Obviously, this implies that the corresponding subalgebra of $\text{Mat}_n(\mathbb{C})$ ($u(n)$) reduces to $\text{Mat}_m(\mathbb{C}) \oplus \text{Mat}_{n-m}(\mathbb{C})$ ($u(m) \oplus u(n-m)$). Since μ_U acts trivially on the algebra $\mathcal{A}$, ~~($\mathcal{A} = \text{Mat}_n(\mathbb{C})$)~~ we conclude that the subalgebra of $\text{shs}_u(n; n|2k)$, extracted by the condition (2.19), (2.21), coincides with $\text{shs}_u(m; m|2k) \oplus \text{shs}_u(n-m; n-m|2k)$. However, in practice, there is no need in using automorphisms in order to build direct sums of known higher spin superalgebras (one can take any sum he wants afterwards). Consequently, we shall not use conditions (2.19) in what follows.

Remarkably enough, this conclusion simultaneously implies that one can disregard those antiautomorphisms of $\text{Mat}_n(\mathbb{C})$ which originate from bilinear forms which are neither symmetric nor antisymmetric. Actually, according to eq.(2.15), $\mathcal{G}^2$ reduces to the automorphism μ_U (2.9) with

$$U_i^k = \eta_{ip}^{-1} \eta^{kp}. \quad (2.22)$$

If $U_i^k \neq \lambda \delta_i^k$ for some $\lambda \neq 0$, the automorphism μ_U (2.9) turns out to be nontrivial. Since the subalgebra extracted by the antiautomorphism $\mathcal{G}$ belongs to the subalgebra extracted by the automorphism $\mu = \mathcal{G}^2$, it should decompose into the direct sum of several subalgebras if $\mathcal{G}^2 \neq I$. According to eq.(2.22), the condition $\mathcal{G}^2 = I$ implies that

$$\eta^{kp} = \lambda \eta^{pk}. \quad (2.23)$$

As a result we are left with only two distinguishing possibilities, $\lambda = 1$ and $\lambda = -1$, which correspond respectively to $\mathcal{G}_S$ and $\mathcal{G}_A$. Because the restriction of $\mathcal{G}$ to the algebra A , which is defined by eq.(2.11), is involutive (possesses the unit square), this implies that our analysis is valid for the case of $\text{shq}(n; n|2k)$ as well. Note that the subalgebras of $u(n)$, extracted by the conditions $\mathcal{P}_{S(A)}(b) = b$ with $\mathcal{P}_{S(A)} = -\mathcal{G}_{S(A)}$ (cf. eq.(2.10) in which $\mathcal{H}(\mathcal{G}) = 0$ due to the absence of the fermion sector), coincide with $o(n)$ and $\text{usp}(n)$ respectively because these conditions merely mean that the bilinear forms $Q_{S(A)}$ remain invariant.

To conclude this section, let us mention that, in addition to automorphisms μ_U (2.9) with $U \in \text{GL}(n)$, one can build automorphisms of A^n as similarity transformations generated by various invertible elements $U \in A^n$. However, in practice, the condition that automorphisms under consideration should trivialize when

restricted to the subalgebra of quadratic polynomials $\text{sp}_{\text{dS}}(4)$, leaves us with the only possibility to generalize μ_U by allowing matrices U_i^j to depend on the Klein operator K . This is because the condition above implies in particular that the automorphisms under consideration should be Lorentz covariant while it follows from eq.(2.1a) that no Lorentz singlet can be constructed from the operators q_α and r_β which does not reduce to some constant, i.e. K is the only nontrivial Lorentz invariant element of A .

As shown in ref./10/, using K -dependent similarity transformations one can derive equivalence relations between some of automorphisms of $\text{shs}_U(1;1|4)$. Specifically, acting along the lines of ref./10/ one can easily prove the equivalence relation

$$\int_{S'}(A) \sim f \circ \int_S(A) \quad (2.24)$$

by showing that

$$f \circ \int_S(A) = \tau_{\pi/4} \circ \int_S(A) \circ \tau_{\pi/4}^{-1} \quad (2.25)$$

where τ_φ is the similarity automorphism defined by the relation

$$\tau_\varphi(a) = e^{i\varphi K} a e^{-i\varphi K} \quad (2.26)$$

and φ is an ambiguous real parameter. Also let us mention the simple fact that $f = \tau_{\pi/2}$.

One can also build antiautomorphisms of A^n with the aid of eq.(2.14) with K -dependent "metric tensors" η_Q . According to eq.(2.16), these antiautomorphisms can be thought of as combinations of usual (K -independent) antiautomorphisms with K -dependent similarity automorphisms. Although K -dependent (anti)auto-

morphisms may be useful in several respects, it will be sufficient for us in what follows to use K-independent automorphisms described previously.

3. Extended Higher Spin Superalgebras

In this section, we discuss various types of extended higher spin superalgebras which can be extracted from $shs_u(n;n/2k)$ with the aid of the automorphisms constructed in Sect.2. It is very likely that superalgebras described below exhaust all possibilities for $d=4$ extended higher spin superalgebras which give rise to consistent dynamics of massless higher spins.

As mentioned in sect.2, the automorphism μ_U (2.9) alone does not lead to interesting subalgebras. Let us however consider subalgebras extracted by the condition

$$f \circ \mu_U (b) = b \quad , \quad (3.1)$$

where f is the boson-fermion automorphism (2.2) while the matrix U is assumed to be of the form (2.21) (the latter condition ensures that μ_U is involutive, $\mu^2=I$, thus making trivial the condition $(\mu_U)^2(b)=b$ which follows from eq.(3.1)). Assuming that indices $k', \ell', \dots$ run from 1 to m while indices $k'', \ell'', \dots$ run from $m+1$ to n , one finds that the subalgebra of $shs_u(n;n/2k)$ extracted by eq.(3.1) gives rise to non-vanishing bosonic fields

$$\omega_{k'}^{\ell'}(q,r,K) \text{ and } \omega_{k''}^{\ell''}(q,r,K) \text{ as well as fermionic fields } \omega_{k'}^{\ell''}(q,r,K) \text{ and } \omega_{k''}^{\ell'}(q,r,K).$$

The very important point consists of the fact that, for constructed subalgebras, the matrix U_k^{ℓ} (2.21) possesses all the properties of the Klein operator K . Namely, U (anti)commutes with (fermions) bosons, and $U^2=I$. Taking into account that $KU=UK$ be-

cause U is bosonic operator and K does not carry internal indices i, j , one observes that the operator $P=KU$ commutes with all elements of the subalgebra under consideration. Since $P^2=I$, this subalgebra decomposes into the direct sum of two further subalgebras which are projected out by the projection operators $\Pi_{\pm} = \frac{1}{2}(1 \pm P)$. It is essential that these projection operators are well-defined both for higher spin Lie superalgebras and for free differential algebras of ref./2/ based on these superalgebras (this is simply because, with respect to the Klein operator K and matrix indices $i, j, \dots$, all products of differential forms, which emerge in the free differential algebras of ref./2/, reduce to the ordinary products in the original associative algebra $A \otimes \text{Mat}_n(\mathbb{C})$). Therefore, using the projectors $\Pi_{\pm}$ one can extract out subalgebras corresponding to consistent truncations of the original higher spin free differential algebra.

Let us denote the subalgebras of $\text{shs}_u(n; n|2k)$, which are extracted by the condition (3.1) and the projection operator Π_{+} , as $\text{shs}_u(m; n-m|2k)$. The non-vanishing combinations of the gauge fields of the original superalgebra $\text{shs}_u(n; n|2k)$, which correspond to the gauge fields of $\text{shs}_u(m; n-m|2k)$, are

$$\omega_{\kappa'}^{\ell'}(q, z) = \frac{1}{2} (\omega_{\kappa'}^0 \ell'(q, z) + \omega_{\kappa'}^1 \ell'(q, z)), \quad (3.2a)$$

$$\omega_{\kappa''}^{\ell''}(q, z) = \frac{1}{2} (\omega_{\kappa''}^0 \ell''(q, z) - \omega_{\kappa''}^1 \ell''(q, z)), \quad (3.2b)$$

for bosons ($\omega(q, r) = \omega(-q, -r)$), and

$$\omega_{\kappa'}^{\ell''}(q, z) = \frac{1}{2} (\omega_{\kappa'}^0 \ell''(q, z) + \omega_{\kappa'}^1 \ell''(q, z)), \quad (3.3a)$$

$$\omega_{\kappa''}^{\ell'}(q, z) = \frac{1}{2} (\omega_{\kappa''}^0 \ell'(q, z) - \omega_{\kappa''}^1 \ell'(q, z)) \quad (3.3b)$$

for fermions ($\omega(-q, -r) = -\omega(q, r)$). Let us recall that the labels

0 and 1 on the right hand side of eqs.(3.2), (3.3) correspond to specific values of the label A in the decomposition $\omega_k^q(q,r,K) = \sum_{A=0,1} (K)^A \omega_k^A(q,r)$. Note also that the replacement $\Pi_+ \leftrightarrow \Pi_-$ is equivalent to the replacement $m \leftrightarrow n-m$, $k', \ell', \dots \leftrightarrow k'', \ell'' \dots$

Spin-1 gauge fields are described by those parts of 1-forms $w_{k'}^{\ell'}(q, \tau)$ and $w_{k''}^{\ell''}(q, \tau)$ which do not depend on q and r. Taking into account the hermiticity conditions (2.8), we see that the spin-1 fields behave as Yang-Mills fields for the gauge group $U(m) \times U(n-m)$. This is the reason for denoting the superalgebra under consideration as $shs_u(m; n-m | 2k)$. Comparing the structure of the gauge fields (3.2) and (3.3) with the structure of the sets of the higher spin gauge fields associated in ref./15/ with spin-s massless particles (see also preceding section), one determines the spectra of spins in the higher spin gauge theories based on $shs_u(m; \ell | 4)$. Namely, one finds that boson gauge fields of $shs_u(m; \ell | 4)$ describe the infinite chain of integer spins 1, 2, 3, ... in which every spin emerges in the adjoint representation of $u(m) \oplus u(\ell)$ (i.e., with the multiplicity $m^2 + \ell^2$). Fermion gauge fields form the infinite chain of spins 3/2, 5/2... , every spin emerges in the representation $m \otimes \bar{\ell} + \ell \otimes \bar{m}$ of $u(m) \oplus u(\ell)$ (i.e. fermions have the multiplicity $2\ell m$).

It is a simple matter to make sure that the higher spin superalgebras $shs_u(m; n | 2k)$ can independently be defined as the Lie superalgebras formed by $(m+n) \times (m+n)$ matrices of the form

$$X_k^\ell = \begin{bmatrix} a_{k'}^{\ell'}(q, \tau) & b_{k'}^{\ell''}(q, \tau) \\ c_{k''}^{\ell'}(q, \tau) & d_{k''}^{\ell''}(q, \tau) \end{bmatrix} \quad (3.4)$$

(From now on, indices $k', \ell', \dots$ and $k'', \ell'', \dots$ take on values from 1 to m and from 1 to n respectively!) Here $a_{\ell'}^{\ell'}(q, r)$, $b_{\ell'}^{\ell''}(q, r)$, $c_{\ell''}^{\ell'}(q, r)$ and $d_{\ell''}^{\ell''}(q, r)$ are arbitrary polynomials of the operators q_α and r_β (2.1a) at the condition that a and d are even,

$$a_{\ell'}^{\ell'}(-q, -r) = a_{\ell'}^{\ell'}(q, r), \quad d_{\ell''}^{\ell''}(-q, -r) = d_{\ell''}^{\ell''}(q, r) \quad (3.5)$$

while b and c are odd,

$$b_{\ell'}^{\ell''}(-q, -r) = -b_{\ell'}^{\ell''}(q, r), \quad c_{\ell''}^{\ell'}(-q, -r) = -c_{\ell''}^{\ell'}(q, r). \quad (3.6)$$

Since q_α and r_β possess spinorial indices, the normal relation between spin and statistics requires that matrices of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ be even ($\pi=0$) elements of $\text{shs}_u(m; n|2k)$ while matrices of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ be odd ($\pi=1$) elements of $\text{shs}_u(m; n|2k)$. Evidently, the matrix $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ obeys all properties of the Klein operator. Note that K -dependent automorphisms and antiautomorphisms, discussed at the end of the preceding section reduce in terms of matrices (3.4) to usual similarity automorphisms and antiautomorphisms based on bilinear forms.

The complexification of $\text{shs}_u(m; n|2k)$ is constructed with the aid of the standard formula (2.3) from the complex associative algebra of matrices (3.4). The real Lie superalgebra $\text{shs}_u(n; m|2k)$ is formed by the matrices (3.4) which are anti-hermitian in the sense that

$$\begin{bmatrix} a(q, r) & b(q, r) \\ c(q, r) & d(q, r) \end{bmatrix}^\dagger = - \begin{bmatrix} a(q, r) & i b(q, r) \\ i c(q, r) & d(q, r) \end{bmatrix} \quad (3.7)$$

where $\dagger$ denotes the combination of the ordinary hermitian conju-

gation of matrices with the involution $\dagger$ (2.6) for the operators q_α and r_β (let us recall that $\dagger$ reverses an order of operators in products of q 's and r 's; for more details about extracting real forms of complex Lie superalgebras with the aid of involutions see ref./7/).

Equivalently, one can say /7/ that $shs_u(m;n|2k)$ is such superalgebra which gives rise to the gauge fields $\omega_\kappa^\ell(q,r)$ with $k, \ell = 1 \dots m+n$ obeying the antihermiticity conditions

$$(\omega_\kappa^\ell(q,r))^\dagger = -\omega_\kappa^\ell(q,r) \quad (3.8)$$

at the condition $(q_\alpha)^\dagger = r_\alpha$, $(r_\alpha)^\dagger = q_\alpha$. In accordance with the definition above, $\omega_{\kappa'}^{\ell'}(q,r)$ and $\omega_{\kappa''}^{\ell''}(q,r)$ are assumed to be bosons (even polynomials in q and r) while $\omega_{\kappa'}^{\ell''}(q,r)$ and $\omega_{\kappa''}^{\ell'}(q,r)$ are fermions (odd polynomials).

Superalgebras $shs_u(m;n|2k)$ at $n=m$ coincide with the original superalgebras $shs_u(n;n|2k)$ used in ref./2/ (at $k=2$). On the other hand, Lie algebras $shs_u(0;n|4) \sim shs_u(n;0|4)$ coincide with purely bosonic higher spin superalgebras discussed in refs./6-8/ for $n=1$ and in ref./2/ for arbitrary $n>0$. The case of $shs_u(n;n|2k)$ is special in that respect that corresponding higher spin theories are supersymmetric in the usual sense /8/, that manifests itself in equal numbers of bosons and fermions of every spin. On the other hand, theories based on $shs_u(m;n|4)$ with $n \neq m$ are not supersymmetric with respect to the usual anti-de Sitter supersymmetry (the numbers of bosons, n^2+m^2 , exceed the numbers of fermions, $2nm$, when $n \neq m$). Note also that the isomorphism takes place $shs_u(n;m|2k) \sim shs_u(m;n|2k)$ and that $shs_u(m;n|2k)$ contains the one-dimensional center originating from the (q,r) -independent unit matrix. The factor of $shs_u(n;m|2k)$ over this center is simple.

In order to construct consistent higher spin free differential algebras corresponding to $\text{shs}_u(m; n|4)$, one can replace gauge 1-forms and Weyl 0-forms of ref./2/ by respectively 1- and 0-forms of the form (3.4) simultaneously replacing the Klein operator K , which emerges explicitly in the free differential algebra of ref./2/, by the matrix $U = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. In fact, as mentioned in Sect. 2, the free differential algebra of ref./2/ contains the pair of operators, Q and R , which serve as Klein-type operators separately for undotted and dotted spinorial indices, and describes both massless and auxiliary fields. The gauge 1-forms describing massless fields depend on Q and R only via $K=QR$. In order to carry out the procedure outlined above, one first of all should redefine the Weyl 0-forms, $C \rightarrow C'=QC$, and then truncate the system by allowing to be non-vanishing only those fields which depend on Q and R via $K=QR$.

Now let us discuss further higher spin subalgebras of $\text{shs}_u(m; n|2k)$. An attempt to use the conditions of the type (3.1) once again does not lead to new superalgebras, and we consider those subalgebras of $\text{shs}_u(m; n|2k)$ which can be extracted with the aid of the automorphisms $\mathcal{P}_S$ and $\mathcal{P}_A$ constructed from the antiautomorphisms of the original associative algebra. In practice, we require the quadratic form Q (2.13), which defines the antiautomorphism $\mathcal{G}^a$ via eq.(2.14), to obey the property that the spaces of primed and double primed indices be orthogonal with respect to Q . This implies that only the components $\gamma_a^{ij'}$ and $\gamma_a^{i''j''}$ of the metric tensor γ_a^{ij} are allowed to be non-vanishing. This requirement ensures the invariance of the Klein operator $\delta_{ij'} - \delta_{i''j''}$ with respect to the action of the antiautomorphism $\mathcal{G}^a$ that is necessary because the Klein operator emerges explicitly in the

equations of motion of massless fields of ref./2/.

As follows from eqs.(2.10),(2.11),(2.14), the condition $\rho(\mathcal{X})=\mathcal{X}$ for elements of $\text{shs}_u(m;n|2k)$ of the form (3.4) reads

$$-i^{\pi(x_{\kappa}^e)} \eta^{\ell u} x_u^v(iq,iz) \eta_{v\kappa}^{-1} = x_{\kappa}^{\ell}(q,z). \quad (3.9)$$

Note that we assume the Weyl ordering of operators q and r . In terms of corresponding gauge fields, we get

$$w_{\kappa'}^{\ell'} \alpha_{\dots} \alpha_{u\beta} \dots \beta_v = -i^{u+v} \eta^{\ell'p'} \eta_{q'\kappa'}^{-1} w_{p'}^{\ell'} \alpha_{\dots} \alpha_{u\beta} \dots \beta_v, \quad (3.10a)$$

$$w_{\kappa''}^{\ell''} \alpha_{\dots} \alpha_{u\beta} \dots \beta_v = -i^{u+v} \eta^{\ell''p''} \eta_{q''\kappa''}^{-1} w_{p''}^{\ell''} \alpha_{\dots} \alpha_{u\beta} \dots \beta_v \quad (3.10b)$$

for boson fields ($u+v$ is even) and

$$w_{\kappa'}^{\ell'} \alpha_{\dots} \alpha_{u\beta} \dots \beta_v = i^{u+v-1} \eta^{\ell'p'} \eta_{q'\kappa'}^{-1} w_{p'}^{\ell'} \alpha_{\dots} \alpha_{u\beta} \dots \beta_v, \quad (3.11a)$$

$$w_{\kappa''}^{\ell''} \alpha_{\dots} \alpha_{u\beta} \dots \beta_v = i^{u+v-1} \eta^{\ell'p'} \eta_{q''\kappa''}^{-1} w_{p'}^{\ell''} \alpha_{\dots} \alpha_{u\beta} \dots \beta_v \quad (3.11b)$$

for fermion fields ($u+v$ is odd). Let us note that eq.(3.11a) is equivalent to eq.(3.11b) if and only if $\eta^{\ell'p'} = \lambda \eta^{p''\ell''}$ and $\eta^{\ell'p'} = \lambda \eta^{p'\ell'}$ with the same λ . Otherwise eqs.(3.11) will imply that fermions vanish, that agrees with the conclusion obtained in the preceding section where it was argued that Q should possess a definite symmetry type.

Setting $u=v=0$ in eq.(3.10), one finds that spin-1 fields $w^{p'q'} = \eta^{p'\ell'} w_{\ell'}^{q'}$ and $w^{p''q''} = \eta^{p''\ell''} w_{\ell''}^{q''}$ (w 's are 1-forms) are antisymmetric if η^{pq} is symmetric ($p = p_S$) and symmetric if η^{pq} is antisymmetric ($p = p_A$). In other words, if $p = p_S$, then spin-1 fields belong to the adjoint representation of $\mathfrak{o}(n) \oplus \mathfrak{o}(m)$, while if $p = p_A$ then spin-1 fields belong

to the adjoint representation of $\text{usp}(n) \oplus \text{usp}(m)$ (in the latter case, the requirement that η^{pq} be nondegenerate implies that both n and m are even, which property is assumed to hold in what follows for the symplectic case). Consequently, we denote subalgebras of $\text{shs}_u(m; n|2k)$, which are extracted by the automorphisms $\int_S$ and $\int_A$ with the aid of eq.(3.9), as $\text{shs}_o(m; n|2k)$ and $\text{shs}_{\text{usp}}(m; n|2k)$ respectively.

According to eqs.(3.10),(3.11), one finds that these superalgebras lead to the following sets of massless gauge fields (let us recall that massless spin $s \geq 1$ is described /15/ by the set of fields $w^{\dot{\alpha}_1} \dots \dot{\alpha}_s \dot{\beta}_1 \dots \dot{\beta}_s$ with various u and v such that $u+v = 2(s-1)$):

(i) All odd integer spins in the adjoint representation of $o(m) \oplus o(n)$ for $\text{shs}_o(m; n|2k)$ and of $\text{usp}(m) \oplus \text{usp}(n)$ for $\text{shs}_{\text{usp}}(m; n|2k)$.

(ii) All even integer spins in the symmetric representation of $o(m) \oplus o(n)$ ($w^{\dot{\alpha}\dot{\alpha}'} = w^{\dot{\alpha}'}\dot{\alpha}$, $w^{\dot{\alpha}\dot{\alpha}''} = w^{\dot{\alpha}''}\dot{\alpha}$) for $\text{shs}_o(m; n|2k)$ and in the antisymmetric representation of $\text{usp}(m) \oplus \text{usp}(n)$ for $\text{shs}_{\text{usp}}(m; n|2k)$.

(iii) Fermions of all spins, every spin in the representation $n \times m$ (for superalgebras of both kinds).

It is worth mentioning that the representations of even spins are reducible containing the singlet representation for both types of superalgebras. This implies that the gravitational field, which is identified with the singlet spin-2 field, survives among gauge fields of the higher spin superalgebras above. Also, let us mention that analogously to the case of $\text{shs}_u(m; n|4)$, higher spin theories based on $\text{shs}_o(m; n|4)$ and $\text{shs}_{\text{usp}}(m; n|4)$ are not supersymmetric in the usual sense when $n \neq m$. Once again, this

fact manifests itself in different numbers of bosons and fermions. The reason is that, when $n \neq m$, the higher spin superalgebras constructed above contain the anti-de Sitter subalgebra $sp_{dS}(4) \sim o(3,2)$, formed by polynomials quadratic in q_α and r_β which are proportional to the identity element of $Mat_n(\mathbb{C})$, but do not contain any its supersymmetric extension $osp(1; 4)$. On the other hand, all superalgebras $shs_u(n; n|4)$, $shs_o(n; n|4)$ and $shs_{usp}(n; n|4)$ are supersymmetric containing the subalgebra $osp(1; 4)$ which gives rise to singlet spin-2 fields $w_i^j \sim \delta_i^j$ and spin-3/2 fields $w_i^{j''} \sim \delta_i^{j''}$ (which form singlets with respect of the diagonal subalgebra of semisimple Lie algebras $u(n) \oplus u(n)$, $o(n) \oplus o(n)$ or $usp(n) \oplus usp(n)$). In this case the infinite towers of massless fields can be decomposed in ordinary supermultiplets.

Now we would like to make a few comments about higher spin superalgebras proposed in ref./7/. These superalgebras are constructed starting from Clifford algebras C_N in place of $Mat_n(\mathbb{C})$. Since Clifford algebras are particular cases of matrix algebras ($C_N = Mat_{2^{N/2}}(\mathbb{C})$ for even N and $C_N = Mat_{2^{(N-1)/2}}(\mathbb{C}) \oplus Mat_{2^{(N-1)/2}}(\mathbb{C})$ for odd N) they in fact need no special analysis. However, the role of the Klein operator became clear in refs./8,2/ after completing the works /6,7/. Therefore, generally, the procedure described above can be consistently applied to higher spin superalgebras based on Clifford algebras if the Klein operator is added to the set of the generating elements of ref./7/. The less trivial point, which we would like to discuss now, is that some of superalgebras of ref./7/ need no additional Klein operator because this is already contained among the elements of these superalgebras. In other words, some of the higher spin superalgebras of

ref./7/ belong to the sequences of higher spin superalgebras described above and therefore can be used for building fully consistent theories of interacting massless fields of all spins.

In the notations of ref./7/, these are superalgebras $\text{shs}^E(N,4)$ with even N which can be constructed as follows. Consider the associative algebra B with generating elements q_a and r_a obeying the commutation relations (2.1a) supplemented by the generating elements ψ_a ($a=1\dots N=2p$) of the Clifford algebra C_N .

$$\{\psi_a, \psi_b\} = 2\delta_{ab}, [\psi_a, q_a] = 0, [\psi_a, r_a] = 0. \quad (3.12)$$

(We have not introduced the Klein operator here!) After that we define the subalgebra $B^E \subset B$ formed by various even polynomials $\mathcal{P}(q, r, \psi)$,

$$\mathcal{P}(-q, -r, -\psi) = \mathcal{P}(q, r, \psi), \quad (3.13)$$

Finally, $\text{shs}^E(N,4)$ is the Lie superalgebra constructed from B^E by means of the formula (2.3) in which the grading $\bar{G}$ is defined in accordance with the normal relation between spin and statistics,

$$\mathcal{P}(-q, -r, \psi) = (-1)^{\bar{G}(\mathcal{P})} \mathcal{P}(q, r, \psi) \quad (3.14)$$

(in fact, we are discussing the real superalgebra originating from the conditions $(q_a)^\dagger = r_a$, $(r_a)^\dagger = q_a$, $(\psi_i)^\dagger = \psi_i$).

For the superalgebra constructed in such a way, one can identify the Klein operator with the " γ_5 " element of C_N by setting

$$K = i \frac{N(N-1)}{2} \psi_1 \cdot \psi_2 \cdot \dots \cdot \psi_N, \quad (3.15)$$

Actually, the operator K defined in such a way obeys the property $K^2=1$ and (anti)commutes with (fermions) bosons as conse-

quence of eqs.(3.13),(3.14) (recall that N is assumed to be even). It is not difficult to verify that the following isomorphism takes place

$$shs^E(N,4) \sim shs_u(2^{\frac{N}{2}-1}; 2^{\frac{N}{2}-1} | 4) . \quad (3.16)$$

In addition, the subalgebras of $shs^E(N,4)$ were considered in ref./7/, denoted as $shs^E(N,4|\nu)$, $\nu=0$ or 1. These are extracted from $shs^E(N,4)$ with the aid of the antiautomorphism $\mathcal{G}$ defined by the relations

$$\mathcal{G}(\Psi_a) = (-1)^\nu \Psi_a, \quad \mathcal{G}(q_\alpha) = i q_\alpha, \quad \mathcal{G}(\tau_{\dot{\beta}}) = i \tau_{\dot{\beta}} \quad (3.17)$$

(Let us note that superalgebras $shs^E(N,4|\nu)$ can be shown to be isomorphic at $\nu=0$ and $\nu=1$ *.) However, the antiautomorphism $\mathcal{G}$, defined in such a way, leaves the operator K (3.15) invariant only if $N=4r$ (for even N). Consequently, the superalgebras $shs^E(N,4|\nu)$ lead to consistent higher spin equations of motion along the lines of ref./2/ only when $N=4r$. It turns out that the following isomorphisms take place:

$$shs^E(8p, 4|\nu) \sim shs_o(2^{4p-1}; 2^{4p-1} | 4) , \quad (3.18a)$$

$$shs^E(8p+4, 4|\nu) \sim shs_{u,sp}(2^{4p+1}; 2^{4p+1} | 4) . \quad (3.18b)$$

The simplest way to see this is to count the dimensionalities of appropriate finite dimensional subalgebras. In order to do this, we take into account that, as shown in ref./7/, the gauge fields of odd spins originating from $shs^E(N,4|\nu)$ are described by various fully antisymmetric internal tensors $w_{a_1 \dots a_k}$ with $k=4q+2$. Let M_α denote dimensionalities of the linear spa-

* This isomorphism originates from the appropriate counterpart of the equivalence relation (2.24).

ces of various fully antisymmetric tensors of ranks $4q+\alpha$ ($\alpha = 0, 1, 2$ or 3 , $q=0, 1, 2, \dots$, tensorial indices run from 1 to N).

Taking into account that

$$\operatorname{Re}(1+i)^N = M_0 - M_2, \quad \operatorname{Im}(1+i)^N = M_1 - M_3; \quad (3.19a)$$

$$(1+1)^N = M_0 + M_1 + M_2 + M_3, \quad (1-1)^N = M_0 + M_2 - M_1 - M_3 \quad (3.19b)$$

one can easily find for $N=4r$ that

$$M_0 = 2^{\frac{N}{2}-1} \cdot (2^{\frac{N}{2}-1} + (-1)^{N/4}) \quad , \quad (3.20a)$$

$$M_2 = 2^{\frac{N}{2}-1} (2^{\frac{N}{2}-1} - (-1)^{N/4}) \quad , \quad (3.20b)$$

$$M_1 = M_3 = 2^{N-2} \quad , \quad (3.20c)$$

Evidently, M_2 is twice the dimensionality of $O(2^{\frac{N}{2}-1})$ when $N=8p$ and M_0 is twice the dimensionality of $usp(2^{\frac{N}{2}-1})$ when $N=8p+4$. *(For complete proof of the isomorphisms (3.18), one can also take into account the multiplicity of even spins (3.22a).)*

To conclude this section, we would like to mention that we confined ourselves to the consideration of higher spin superalgebras with compact subalgebras of internal symmetries such as $u(n) \oplus u(m)$, $o(n) \oplus \tilde{o}(n)$ and $usp(n) \oplus usp(m)$ because only in this case one can hope to build meaningful (unitary) physical theories based on these superalgebras. Formally, one can modify the hermiticity conditions (3.7) and define a number of "higher spin" superalgebras with non-compact internal (spin-1) symmetries, which can in self-evident fashion be denoted as $shs_u(p, q; s, t | 2k)$, $shs_o(p, q; s, t | 2k)$ and so on.

4. Massless Unitary Representations of Higher Spin

Superalgebras

Here we construct massless unitary representations of extended higher spin superalgebras acting along the lines of the approach used in ref./10/ for the simplest case of ordinary (non-ex-

tended) superalgebras $shs_u(m;n|4)$ and $shs_o(m;n|4)$ at $m+n=1$ or 2 . Our analysis is based heavily on the remarkable observation of Flato and Fronsdal /11/ that all massless unitary representations of the anti-de Sitter algebra $O(3,2) \sim sp(4)$ are contained in tensor products of pairs of singleton representations discovered by Dirac /12/. From the viewpoint of the analysis of higher spin superalgebras, the singleton representations are special in that respect that they form representations of both higher spin Lie superalgebras and the underlying associative algebras.

Consider the associative algebra A with generating elements a, a^+, b and b^+ obeying the commutation relations (2.5a). The usual Fock space corresponding to these creation (a^+, b^+) and annihilation (a, b) operators forms some representation S of A . Its basis vectors are of the form

$$e^{pq} = (a^+)^p (b^+)^q |0\rangle \quad (4.1)$$

where $|0\rangle$ denotes the Fock vacuum obeying the conditions

$$a|0\rangle = b|0\rangle = 0 \quad (4.2)$$

Evidently, S simultaneously forms some representation of the Lie algebra $O(3,2) \sim sp(4)$ whose generators are identified with various quadratic polynomials constructed from a, a^+, b and b^+ (for more detail about this oscillator realization of $sp(4)$ see ref. /10/ and references therein). However S forms the reducible representation of $sp(4)$ decomposing into the sum of two irreducible representations

$$S = Rac \oplus Di \quad (4.3)$$

where Di and Rac denote subspaces of S spanned by the vectors

e^{pq} with odd and even $p+q$ respectively. Di and Rac are just

the two elementary singleton representations of $o(3,2)$ discovered by Dirac /12/ (for more detailed discussion of the representations of $o(3,2)$ see ref./10/ and references therein).

In terms of the elementary singletons, one can easily construct representations of the complex associative algebra constituted by the $(m+n) \times (m+n)$ matrices of the form (3.4) in which the diagonal blocks a and d contain even powers of q and r , (equivalently, a, a^+, b and b^+) while the off-diagonal blocks b and c contain odd powers of a, a^+, b and b^+ . There exist two types of representations,

$$S_+^{mn} = \begin{bmatrix} (Rac)_{i'j'} \\ (Di)_{i''j''} \end{bmatrix}, \quad S_-^{mn} = \begin{bmatrix} (Di)_{j'i'} \\ (Rac)_{j''i''} \end{bmatrix} \quad (4.4)$$

(recall that $i', j' \dots = 1 \dots m$ and $i'', j'' \dots = 1 \dots n$). Let us note that $S_{\pm}^{11}$ coincide with the singleton representations $S_{\pm}$ of the algebra A , which were introduced in ref./10/.

Since $S_{\pm}^{mn}$ form representations of the associative algebra of matrices (3.4), they simultaneously form representations of the Lie superalgebra constructed from it by means of eq.(2.3). So far we discussed complex representations of the complex associative algebras and Lie superalgebras. Now we discuss unitary (hermitian) representations of the real Lie superalgebras $shs_u(m;n|4)$ originating from the hermiticity conditions (3.7). This implies that the representation spaces are assumed to be endowed with some positive definite hermitian scalar product*. For the singleton representations, this scalar product can be de-

* This scalar product is assumed to be invariant with respect to infinitesimal supertransformations $\exp(i \varepsilon_i T_i)$ where T_i are (super)generators while ε_i are (anti)commuting parameters obeying such "hermiticity conditions" which ensure $\varepsilon_i T_i$ to be hermitian.

defined in the standard fashion

$$(\psi, \varphi) = \sum_{i=1}^{m+n} \langle \psi^i | \varphi_i \rangle \quad (4.5)$$

where $\langle, \rangle$ denotes the usual scalar product in the Fock space S . Let us note that subspaces $S_{\pm}^{nm}$ and $S_{\mp}^{nm}$ are orthogonal with respect to the hermitian form (4.5) because their scalar product reduces to averages of odd numbers of the operators $a^{(+)}$ and $b^{(+)}$.

It is important that for the real case of $\text{shs}_u(m;n|4)$, the number of inequivalent singleton representations is doubled due to the complex conjugation. The reason is just the same as for the representations of $\text{su}(n)$. Namely, let the element $x_{\pm}^j$ (3.4) be represented in the form

$$x_{\pm}^j = \sum_{p,q,f,g} C_{\pm}^j{}_{pqfg} (\alpha^*)^p (\beta^*)^q (\alpha)^f (\beta)^g \quad (4.6)$$

with some complex coefficients $C_{\pm}^j{}_{pqfg}$. Then, define the complex conjugate matrix $\bar{x}_{\pm}^j$ by the relation

$$\bar{x}_{\pm}^j = \sum_{p,q,f,g} \bar{C}_{\pm}^j{}_{pqfg} (\alpha^*)^p (\beta^*)^q (\alpha)^f (\beta)^g \quad (4.7)$$

where the coefficients $\bar{C}_{\pm}^j{}_{pqfg}$ are complex conjugate to $C_{\pm}^j{}_{pqfg}$. The representations $S_{\pm}^{mn}$ of $\text{shs}_u(m;n|4)$ originate from the action of $x_{\pm}^j$ (4.6) on the columns (4.4). Let us define the conjugate singleton representations $\bar{S}_{\pm}^{mn}$ by requiring that for any element $x_{\pm}^j$ (4.6) of $\text{shs}_u(m;n|4)$, its action on $\bar{S}_{\pm}^{mn}$ reduces to the action of $\bar{x}_{\pm}^j$ (4.7) on the columns (4.4). It is not difficult to make sure that $\bar{S}_{\pm}^{mn}$, defined in such a way, actually form some representations of $\text{shs}_u(m;n|4)$. In order to prove that $S_{\pm}^{mn}$ are inequivalent to $\bar{S}_{\pm}^{mn}$, it suffices to consider these representations as the representations of the subalgebra $u(m) \oplus u(n)$ of $\text{shs}_u(m;n|4)$.

Thus, we have four types of singleton representations of $\text{shs}_u(m; n|4)$ (these are $S_{\pm}^{mn}$ and $\bar{S}_{\pm}^{mn}$). Now one can build various representations of $\text{shs}_u(m; n|4)$ as tensor products of elementary singleton representations which are in many respects analogous to the fundamental representations n and $\bar{n}$ of $\text{su}(n)$. Let us emphasize that the operation of tensor product of representations is well-defined for Lie superalgebras but makes no sense for the original associative algebras.

Generally, the action of elements u of some superalgebra U on the tensor product $X \otimes Y$ of two its representations can be defined by the relation

$$u(x \otimes y) = u(x) \otimes y + (-1)^{\pi(u)\pi(x)} x \otimes u(y) \quad (4.8)$$

which is assumed to hold for all $x \in X$, $y \in Y$ and $u \in U$. Here $\pi(x) = 0$ or 1 is the grading of $x \in X$ defined in such a way that $\pi(u(x)) = |\pi(u) + \pi(x)|_{\text{mod } 2}$ for every $x \in X$ and $u \in U$. For the case of singleton representations, we define π by setting

$$\pi(e^{pq}) = |p+q|_{\text{mod } 2} \quad (4.9)$$

for basis elements (4.1) (for every element of the columns (4.4)). This means that

$$\pi(D_i) = 1, \quad \pi(R_{ac}) = 0. \quad (4.10)$$

If representations X and Y are equivalent, then the representation $X \otimes Y$ decomposes into the sum of the symmetric and antisymmetric representations,

$$X \otimes Y = (X \otimes Y)_S \oplus (X \otimes Y)_A \quad (4.11)$$

where

$$(x \otimes y)_{S(A)} = x \otimes y + \varepsilon (-1)^{\pi(x)\pi(y)} y \otimes x \quad (4.12)$$

with $\varepsilon = 1$ and -1 for $(X \otimes Y)_S$ and $(X \otimes Y)_A$ respectively.

As shown by Flato and Fronsdal /11/, massless representations of $o(3,2)$ are contained in tensor products of pairs of the elementary singletons. The spin spectra of these tensor products correspond to the reduction of the representations under investigation with respect to the subalgebra $o(3,2) \sim sp(4)$ formed by various quadratic polynomials constructed from $a^{(+)}$ and $b^{(+)}$. All representations $S_{\pm}^{mn}$ and $\bar{S}_{\pm}^{mn}$ of $shs_u(m;n|4)$ behave as some sets of elementary singletons Di and Rac (cf. eq.(4.4)). Therefore, in order to determine spectra of spins for various pair products of $S_{\pm}^{mn}$ and $\bar{S}_{\pm}^{mn}$, it suffices to use the following formulae /10/:

$$(Rac \otimes Rac)_S = (0^1, 2, 4, 6, \dots), (Di \otimes Di)_S = (0^2, 2, 4, 6, \dots); \quad (4.13a)$$

$$(Rac \otimes Rac)_A = (1, 3, 5, \dots), (Di \otimes Di)_A = (1, 3, 5, \dots); \quad (4.13b)$$

$$(Di \otimes Rac) = (1/2, 3/2, 5/2, \dots). \quad (4.13c)$$

The right hand sides of these relations contain spectra of spins for massless representations of $sp(4)$. Here 0^1 and 0^2 denote two inequivalent scalar massless representations $(1,0)$ and $(2,0)$ of $sp(4)$. (For more detail see refs./11,10/ and references therein.

Also, let us mention that in ref./11/ the spectra of spins are presented for non-symmetrized tensor products of singletons which are described by eq.(4.13c) and the sums of the right hand sides of eqs.(4.13a) and (4.13b) for $Di \otimes Di$ and $Rac \otimes Rac$.)

Let us emphasize that tensor products of pairs of singletons contain all massless representations and do not contain massive representations of $o(3,2)$, the fact /11/ which enables one to associate these representations with some purely gauge (=massless) theories of higher spins.

Now, we are ready to analyze whether extended higher spin superalgebras constructed in the preceding section are admissible in the sense that they possess massless unitary representations with just the same spectra of spins as predicted in sect.3 with the aid of the arguments based on the dynamical analysis of corresponding free higher spin theories along the lines of ref./15/. Let us begin with $\text{shs}_u(m;n|4)$. In this case, there exist two appropriate representations, $S_+^{mn} \oplus \bar{S}_+^{mn}$ and $S_-^{mn} \oplus \bar{S}_-^{mn}$. Actually, with the use of eqs.(4.13), it is easily seen that these massless representations contain integer spins in the adjoint representation $m \oplus \bar{m} \oplus n \oplus \bar{n}$ of $u(m) \oplus u(n)$ and half-integer spins in the representation $n \oplus \bar{m} \oplus m \oplus \bar{n}$. Comparing these results with the spectra of gauge fields of $\text{shs}_u(m;n|4)$ presented in sect.3, one finds that $\text{shs}_u(m;n|4)$ obeys the admissibility condition.

Note that the representations $S_{\pm}^{mn} \oplus S_{\pm}^{mn}$ and $\bar{S}_{\pm}^{mn} \oplus \bar{S}_{\pm}^{mn}$, which formally lead to the same spectra of spins as $S_{\pm}^{mn} \oplus \bar{S}_{\pm}^{mn}$, cannot nevertheless correspond to some higher spin gauge theory based on $\text{shs}_u(m;n|4)$ because spin-1 (Yang-Mills) states should belong to the adjoint representation of $u(m) \oplus u(n)$, while the aforementioned representations give rise to integer spins in the representations $m \oplus m \oplus n \oplus n$ and $\bar{m} \oplus \bar{m} \oplus \bar{n} \oplus \bar{n}$.

On the other hand, since $S_{\pm}^{mn} \oplus S_{\pm}^{mn}$ form representations of $\text{shs}_u(m;n|4)$, they simultaneously form some representations of its subalgebras $\text{shs}_0(m;n|4)$ and $\text{shs}_{\text{usp}}(m;n|4)$ constituted by matrices obeying the condition (3.9) with symmetric and antisymmetric bilinear form η respectively. Note that with respect to these subalgebras, the representations $S_{\pm}^{mn}$ and $\bar{S}_{\pm}^{mn}$ are equivalent. (The reason is the same as for the representations with

upper and lower indices of $o(n)$ and $usp(n)$). The representations $S_{\pm}^{mn} \otimes S_{\pm}^{mn}$ are reducible decomposing into the symmetric and anti-symmetric parts according to eqs.(4.11).^(4.12) Using eqs.(4.13), one can readily verify that the spectra of spins of the massless representations $(S_{\pm}^{mn} \otimes S_{\pm}^{mn})_S$ and $(S_{\pm}^{mn} \otimes S_{\pm}^{mn})_A$ correspond precisely to the spectra of spins of gauge fields of $shs_o(n;m|4)$ and $shs_{usp}(n;m|4)$ respectively.

Thus all higher spin superalgebras $shs_u(n;m|4)$, $shs_o(n;m|4)$ and $shs_{usp}(n;m|4)$ are shown to obey the admissibility condition. Let us emphasize the remarkable fact that the analysis of unitary representations leads to just the same list of algebras as the analysis of possible consistent truncations of the higher spin equations of motion of ref./2/. In the authors' opinion, this fact provides a highly nontrivial support of the approach of ref./2/. Note also that the admissibility of the higher spin superalgebra $shs_o(8;8|4)$ was discovered previously by Bergshoeff, Salam, Sezgin and Tani in ref./13/ where it was explicitly shown that the symmetric tensor square of the $N=8$ supersingleton leads to just the same spectrum of massless fields as predicted by the structure of gauge fields of the higher spin superalgebra $shs^E(8,4|0)$ of ref./7/ which has been shown in the preceding section to be isomorphic to $shs_o(8;8|4)$. (Unfortunately, the authors of ref./13/ denoted this superalgebra as $shs(8,4)$, which designation was reserved in ref./7/ for another superalgebra which is ruled out by the admissibility condition).

Finally, let us emphasize the important fact that the representations $S_{+}^{mn} \otimes S_{+}^{(-)mn}$ and $S_{-}^{mn} \otimes S_{-}^{(-)mn}$ are not equivalent to each other at least when $n \neq m$. Actually, as is seen from eqs.(4.4) the transition from S_{+}^{mn} to S_{-}^{mn} reduces to the replacement $Di \rightarrow Rac$, $Rac \rightarrow Di$. According to eq.(4.13a), this replacement

interchanges two inequivalent scalar representations O^1 and O^2 ($S_+^{mn} \otimes \bar{S}_+^{mn}$ contains m^2 scalars O^1 and n^2 scalars O^2 while $S_-^{mn} \otimes \bar{S}_-^{mn}$ contains n^2 scalars O^1 and m^2 scalars O^2). It seems natural to suppose that there exist two inequivalent versions of higher spin gauge theories corresponding to $S_+ \otimes S_+$ and $S_- \otimes S_-$ type representations. Perhaps, this possibility is connected with the fact that higher spin equations of motion proposed in ref./2/ contain an ambiguous phase parameter $\eta = e^{i\varphi}$ which cannot be absorbed by field redefinitions (see Appendix of ref./2/). It is interesting to investigate which values of the parameter η in the equations of motion of ref./2/ do really correspond to unitary representations of $shs_u(n;m|4)$ (for example, the unitarity will be guaranteed after construction of a ghost-free hermitian action leading to these equations). It is possible that this unitarity requirement leaves only some discrete ambiguity in the choice of η (say, $\eta = \pm 1$) which corresponds to the two types of representations mentioned above.

4. Conclusion

Thus, it is shown that there exist three sequences of infinite dimensional extended higher spin superalgebras, $shs_u(n;m|4)$, $shs_o(n;m|4)$, and $shs_{usp}(n;m|4)$ with various integers $n \geq 0$, $m \geq 0$, $n+m > 0$ (both n and m are assumed to be even for the case of $shs_{usp}(n;m|4)$). These superalgebras give rise to consistent equations of motion for interacting $d=4$ massless fields of all spins in the framework of the approach of ref./2/ and possess massless unitary representations with appropriate spin spectra. In the spin-1 sector, the proposed higher spin superalgebras reduce to the Yang-Mills subalgebras $u(n) \oplus u(m)$, $o(n) \oplus o(m)$, and $usp(n) \oplus usp(m)$ respectively.

Superalgebras $shs_0(n;m|4)$ and $shs_{usp}(n;m|4)$ are simple. As for $shs_u(n;m|4)$, it decomposes as follows: $shs_u(n;m|4) = shs'_u(n;m|4) \oplus u(1)$ where the one-dimensional center $u(1)$ belongs to the Yang-Mills subalgebra $u(n) \oplus u(m)$, while the superalgebra $shs'_u(n;m|4)$ is simple. In principle, one can consider higher spin theories based on direct sums of the simple higher spin superalgebras. Naively, this leads to some set of mutually non-interacting worlds, each containing its own matter and gravitational fields. Nevertheless, semi-simple higher spin superalgebras can have interesting physical applications if some additional "matter" fields are present which realize nontrivial representations containing tensor products of the representations corresponding to different simple higher spin subalgebras. This situation is analogous to that familiar for ordinary gauge theories of the type of the electroweak model and may correspond to some spontaneous breaking of a broader symmetry. Specifically the additional massive (higher spin) fields may result from compactifications of massless higher spin theories in higher dimensions.

Let us note that one can consider higher spin superalgebras $shs_{\infty}(n;m|4)$ with $n = \infty$ or/and $m = \infty$, that lead to higher spin theories with infinite numbers of fields of any spin. Also one can speculate that superalgebras $shs_{\infty}(n;m|2k)$ with $k \neq 2$ may serve as higher spin superalgebras when the space-time dimension d is different from four. (For non-extended higher spin superalgebras, such a possibility was suggested in ref./16/ and investigated in some details by Blencowe /17/ for $d=3$ ($k=1$).)

The results of the present paper indicate unambiguously that there exists a broad class of higher spin gauge theories with various internal symmetries corresponding to classical sequences of

Lie algebras. Let us emphasize that the method of constructing unitary representations of higher spin superalgebras via tensor products of pairs of singletons provides strong arguments in favour of the conclusion that the higher spin superalgebras proposed in the paper exhaust all possibilities in $d=4$ (every meaningful type of tensor products of pairs of singletons corresponds to some higher spin superalgebra and vice versa).

Nevertheless it would be interesting to solve the problem of the classification of admissible higher spin superalgebras conclusively. It is very likely that this classification is equivalent to the classification of consistent truncations of equations of motion of massless higher spins proposed in ref./2/. The fact that this problem may be nontrivial enough is manifested by the existence of N -extended supergravities which are expected to correspond to some consistent truncations of the higher spin equations of ref./2/ as well as the ordinary gravity and lower spin massless theories containing spins $s \leq 1$. All these theories correspond to truncations with finite numbers of fields which cannot be constructed by virtue of some automorphisms of $shs_u(m;n|4)$. Although the truncation to Yang-Mills theories is very simple, this is not the case for supergravities as can immediately be seen from the well-known restriction $N \leq 8$ on the number of gravitinos. In fact, the question is whether there exist other truncations which cannot be obtained by means of automorphisms of $shs_u(m;n|4)$. In this context it seems interesting to investigate whether it is possible to construct higher spin theories based on exceptional Lie algebras. Perhaps, the restriction $N \leq 8$ indicates that the exceptional Lie algebras may be closely related to the supergravitational truncations.

References

- /1/ E.S.Fradkin and M.A.Vasiliev, Phys. Lett. 189B (1987) 89; Nucl. Phys. B291 (1987) 141.
- /2/ M.A.Vasiliev, Lebedev Institute preprint (1988) N°58 (Ann. of Phys. (1989) ~~192 (1989) 59~~); Phys. Lett. 209B (1988) 491.
- /3/ C.Aragone and S.Deser, Phys. Lett. 86B (1979) 161.
S.Christensen and M.Duff, Nucl. Phys. B154 (1979) 301.
B. de Wit and D.Z.Freedman, Phys. Rev. D21 (1980) 358.
F.A.Berends, J.W. van Holten, P.van Nieuwenhuizen and B. de Wit, J. Phys. A13 (1980) 1643.
- /4/ A.K.H.Bengtsson, I.Bengtsson and L.Brink, Nucl. Phys. B227 (1983) 31; 41.
F.A.Berends, G.J.H.Burgers and H. van Dam, Z. Phys. C24 (1984) 247; Nucl. Phys. B260 (1985) 295; B271 (1986) 429.
A.K.H.Begtsson and I.Bengtsson, Class. Quant. Grav. 3 (1986) 927; Phys. Lett. 174B (1986) 294.
I.G.Koh and S.Ouvry, Phys. Lett. 179B (1986) 115.
A.K.H.Bengtsson, Phys. Lett. B182 (1986) 321; B189 (1987) 337; Class. Quant. Grav. 5 (1988) 437.
A.K.H.Bengtsson, I.Bengtsson and N.Linden, Class. Qaunt. Grav. 4 (1987) 1333.
- /5/ P.van Nieuwenhuizen, Phys. Rep. 68 (1981) 189.
- /6/ E.S.Fradkin and M.A.Vasiliev, Ann.Phys.177 (1987) 63.
- /7/ M.A.Vasiliev, Fortschr. Phys. 36 (1988) 33.
- /8/ E.S.Fradkin and M.A.Vasiliev, Int. J. Mod. Phys. A 3 (1988) 2983.
- /9/ S.W.MacDowell and F.Mansouri, Phys. Rev. Lett. 38 (1977) 739.
- /10/ S.E.Konstein and M.A.Vasiliev, Lebedev Institute Preprint (1988) N°94; Nucl. Phys. B 312 (1989) 402.
- /11/ M.Flato and C.Fronsdal, Lett. Math. Phys. 2 (1978) 421; Phys. Lett. 97B (1980) 236.
- /12/ P.A.M.Dirac, J. Math. Phys. 4 (1963) 901.
- /13/ E.Bergshoeff, A.Salam, E.Sezgin and Y.Tanii, *Nucl. Phys.* B305 [F523](1988) 497.
- /14/ M.A.Vasiliev, Nucl. Phys. B307 (1988) 319.
- /15/ M.A.Vasiliev, Fortschr. Phys. 35 (1987) 741.
- /16/ M.A.Vasiliev, Nucl. Phys. B301 (1988) 26.
- /17/ M.P.Blencowe, Imperial College preprint (1988) TP-87-88/30.

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