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Stability of excited Bose stars*

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ABSTRACT

I discuss the problem of the dynamical stability of the equilibrium solutions for the bosonic stellar configurations in the framework of general relativity. In particular I show that the eigenvalues of the pulsation equation, which governs the time evolution of the infinitesimal radial oscillations, are real for Bose star configurations corresponding to excited states. This fact, together with the theorem that for a configuration with critical central density ρ_{crit} the pulsation equation has a zero mode and that for small central densities one can show that the lowest eigenvalue is positive, leads to the conclusion that the excited Bose star configurations are classically stable for central densities up to ρ_{crit} .

1 Introduction

Recent developments in particle physics and cosmology suggest that evolving scalar fields may have played an important role in the evolution of the early universe, for instance in primordial phase transitions, and that they may make up the missing dark matter. This latter possibility raises naturally the question whether cold, gravitational equilibrium configurations of massive scalar fields - Bose stars - may exist and whether they are dynamically stable [1-3]. Spherically symmetric gravitational equilibrium configurations in asymptotically flat space-times for massive complex fields (interacting only gravitationally) have been found by Kaup [4] as well as Ruffini and Bonazzola [5] by solving the coupled system of Einstein-Klein-Gordon equations. They analyzed only the nodeless radial scalar solutions, corresponding to the ground state. The results obtained in Refs. [4,5] have been confirmed and extended later on, see Refs. [6,7]. In particular Friedberg, Lee and Pang [7] have also computed the radial solutions with nodes, which correspond to excited states. It has been found that there is always a critical mass and a critical particle number above which mass gravitational collapse occurs. These quantities for the nodeless solution are respectively given by $M_{crit} = 0.633 M_{Pl}/m$ and $N_{crit} = 0.653 M_{Pl}^2/m^2$, where $M_{Pl} = G_N^{-1/2}$ is the Planck mass and m is the mass of the scalar particle. For the excited states the critical mass as well as the critical number grow approximately linearly with the number of nodes [7].

The extension to the self-interacting case, for a potential $V(\phi) = \frac{1}{4}(\phi^* \phi)^2 (\lambda > 0)$, has been considered in the work of Colpi, Shapiro and Wasserman [8]. The consequence of switching on a quartic self-interaction between the scalar particles is to increase the above limits on M_{crit} and N_{crit} . If one plots the mass for the equilibrium configurations of the Bose

star against the central density, one finds, except for the scales, a behaviour remarkably similar to that of neutron stars: the mass quickly rises to a maximum (for $\rho = \rho_{crit}$), drops a little, oscillates and approaches an asymptotic value at large central densities, the same happening for the particle number.

A similar behaviour is also found when taking scalars coupled to a $U(1)$ gauge field [9]. The value of the maximum mass increases, with increasing value of the gauge coupling constant e . At the same time the binding energy per particle decreases. If $e^2/4\pi \geq G_N m^2$ it is no more possible to have bounded equilibrium solutions, because now the Coulomb repulsion between the particles is bigger than the gravitational attraction.

For the above cases one can show that the particle number and the mass have their extrema, in particular their maximum, at the same value of the central density. From this fact it follows, as pointed out in Ref. [2], that the pulsation equation which governs the time evolution of the oscillations around the equilibrium configurations has a zero eigenvalue where M and N have their extrema. This is the same as in the case of a star described by a perfect fluid. For the case without quartic self-interaction and gauge coupling it has been shown [2] that in the non-relativistic limit, corresponding to a very small central density, the lowest eigenvalue of the pulsation equation is positive also for equilibrium configurations with nodes. For the nodeless case it is easily proved that all the eigenvalues are real [1,2]; thus with the above facts it leads to the conclusion that the equilibrium solutions up to the critical central density ρ_{crit} are classically stable. Above ρ_{crit} the equilibrium solutions are unstable. For $\rho > \rho_{crit}$ there are other values of ρ for which M and N are stationary and where there is therefore a zero mode. The fact that for these points there is a negative upper bound for the lowest eigenvalue [1] establishes that it must be a higher eigenvalue which becomes zero.

Here I discuss the question whether in the case of an equilibrium solution with nodes the eigenvalues of the pulsation equation are still real. In fact for equilibrium solutions with nodes the corresponding pulsation equation has singularities. Nevertheless it can still be proved, by analyzing the behaviour of the solution at the singularities, that the eigenvalues are real, which means also that the corresponding operator is symmetric as in the nodeless case. Therefore like in the nodeless case we conclude that the excited Bose star configurations are classically stable for central densities up to ρ_{crit} .

2 Equilibrium configurations

The system we consider is a complex scalar field coupled to gravity, whose action is given by

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G_N} + \frac{g^{\mu\nu}}{2} \phi_{;\mu}^* \phi_{;\nu} - \frac{m^2}{2} |\phi|^2 - \frac{\lambda}{4} |\phi|^4 \right). \quad (1)$$

This action is invariant under a global $U(1)$ phase transformation, $\phi \rightarrow e^{i\theta} \phi$, which implies the conservation of its generator N , the number of particles minus the number of antiparticles. For our considerations it is more convenient to write the complex scalar field as $\phi = (\phi_1 + i\phi_2)e^{-i\omega t}$, where ϕ_1 and ϕ_2 are two real fields, which are functions of r and t alone since we consider only spherically symmetric equilibrium configurations. We express

the metric in Schwarzschild co-ordinates

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

where ν and λ are functions of r and t only ($g^{00} = -e^{-\nu}$ and $g^{rr} = e^{-\lambda}$). For $r \rightarrow \infty$, we require $e^{\lambda(r,t)} \rightarrow 1$ as well as $e^{\nu(r,t)} \rightarrow 1$. At the equilibrium configuration, the functions ν, λ are time-independent. In order to recover the equilibrium solutions found in Refs. [4-8] we set $\phi_2 = 0$ as well as $\phi_1(r, t) = \phi_0(r)$. We will denote all the equilibrium quantities by a subscript 0. The equilibrium solutions are determined by two Einstein equations

$$(\pi e^{-\lambda_0})' = 1 - 4\pi G_N r^2 [(m^2 + e^{-\nu_0} \omega^2 + \frac{\tilde{\lambda}}{2} \phi_0^2) \phi_0^2 + e^{-\lambda_0} \phi_0'^2], \quad (3)$$

$$\frac{e^{-\lambda_0}}{r} - \nu_0' = \frac{1}{r^2} (1 - e^{-\lambda_0}) + 4\pi G_N [(e^{-\nu_0} \omega^2 - m^2 - \frac{\tilde{\lambda}}{2} \phi_0^2) \phi_0^2 + e^{-\lambda_0} \phi_0'^2], \quad (4)$$

and the scalar wave equation

$$\phi_0'' + \left(\frac{2}{r} + \frac{\nu_0'}{r}\right) \phi_0' - e^{\lambda_0} (m^2 - e^{-\nu_0} \omega^2 + \tilde{\lambda} \phi_0^2) \phi_0 = 0. \quad (5)$$

The total particle number is given by

$$N_0 = 4\pi\omega \int_0^\infty dr r^2 \phi_0^2 e^{(\lambda_0 - \nu_0)/2}. \quad (6)$$

As boundary conditions we impose $\phi_0(0) = \text{const}$, $\phi_0'(0) = 0$ and $\phi_0(\infty) = \phi_0'(\infty) = 0$ in order to have a localized particle distribution. We allow the possibility of ϕ_0 to have nodes, giving rise to excited states. The condition $e^{\nu_0(\infty)} = 1$ is imposed to get the ordinary Minkowski metric at asymptotia and the condition $e^{\lambda_0(0)} = 1$ is a regularity condition. With these boundary conditions the equations (3)-(5) form an eigenvalue equation for ω^2 .

We can also define an effective radius for the Bose star as follows [9]:

$$R = \frac{4\pi}{N_0} \int_0^\infty dr r^3 \omega \phi_0^2 e^{(\lambda_0 - \nu_0)/2}. \quad (7)$$

3 Pulsation equation and variational principle

Consider now the situation where the equilibrium configuration is perturbed in a way such that the spherical symmetry is still preserved. These perturbations will give rise to motions in the radial direction. The equations governing the small perturbations are obtained by expanding all functions to first order ($\lambda(r, t) = \lambda_0(r) + \delta\lambda(r, t)$, $\nu(r, t) = \nu_0(r) + \delta\nu(r, t)$, $\phi_1(r, t) = \phi_0(r) + \delta\phi_1(r, t)$, $\phi_2(r, t) = \delta\phi_2(r, t)$) and by linearizing the equations. We obtain the following set of Einstein equations for the perturbed quantities

$$\begin{aligned} (\pi e^{-\lambda_0} \delta\lambda)' &= -8\pi G_N r^2 \left[\frac{1}{2} \delta\nu e^{-\nu_0} \omega^2 \phi_0^2 + \frac{1}{2} \delta\lambda e^{-\lambda_0} \phi_0'^2 - e^{-\nu_0} (\omega^2 \phi_0 \delta\phi_1 + \omega \phi_0 \delta\phi_2) \right. \\ &\quad \left. - m^2 \phi_0 \delta\phi_1 - \tilde{\lambda} \phi_0^3 \delta\phi_1 - e^{-\lambda_0} \phi_0' \delta\phi_1' \right], \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{e^{-\lambda_0}}{r} (\delta\nu' - \nu_0' \delta\lambda) &= \frac{e^{-\lambda_0}}{r^2} \delta\lambda + 8\pi G_N \left[-\frac{1}{2} \delta\nu e^{-\nu_0} \omega^2 \phi_0^2 - \frac{1}{2} \delta\lambda e^{-\lambda_0} \phi_0'^2 \right. \\ &\quad \left. + e^{-\nu_0} (\omega^2 \phi_0 \delta\phi_1 + \omega \phi_0 \delta\phi_2) - m^2 \phi_0 \delta\phi_1 - \tilde{\lambda} \phi_0^3 \delta\phi_1 + e^{-\lambda_0} \phi_0' \delta\phi_1' \right], \end{aligned} \quad (9)$$

and

$$\delta\dot{\lambda} = 8\pi G_N r (\phi_0' \delta\phi_1 + \omega \phi_0 \delta\phi_2' - \omega \phi_0' \delta\phi_2). \quad (10)$$

The linearized equations for $\delta\phi_1$ and $\delta\phi_2$ are

$$\begin{aligned} \delta\phi_1'' + \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{r}\right) \delta\phi_1' + \frac{1}{2} \phi_0' (\delta\nu' - \delta\lambda') + \delta\lambda e^{\lambda_0} (\omega^2 e^{-\nu_0} \phi_0 - m^2 \phi_0 - \tilde{\lambda} \phi_0^3) \\ + e^{\lambda_0} (\omega^2 e^{-\nu_0} - m^2 - 3\tilde{\lambda} \phi_0^2) \delta\phi_1 - e^{\lambda_0 - \nu_0} \phi_0 \omega^2 \delta\nu - e^{\lambda_0 - \nu_0} \delta\phi_1 + 2e^{\lambda_0 - \nu_0} \omega \delta\phi_2 = 0, \end{aligned} \quad (11)$$

and

$$\begin{aligned} \delta\phi_2'' + \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{r}\right) \delta\phi_2' + e^{\lambda_0} (\omega^2 e^{-\nu_0} - m^2 - \tilde{\lambda} \phi_0^2) \delta\phi_2 - e^{\lambda_0 - \nu_0} \delta\phi_2 \\ + \frac{1}{2} e^{\lambda_0 - \nu_0} \omega [\phi_0 (\delta\nu' - \delta\lambda') - 4\delta\phi_1'] = 0. \end{aligned} \quad (12)$$

Not all these equations are independent, for instance, one can show that Eq.(12) is a consequence of the previous equations [1]. The basic system of four equations (8)-(11) can be reduced to a system of two coupled equations involving only $\delta\phi_1$ and $\delta\phi_2$. Notice that Eq.(10) integrates directly in time if we write $\delta\phi_2(r, t) = \phi_0(r) \dot{g}(r, t)$. In the following we denote $f_1(r, t) = \delta\phi_1(r, t)$ and $f_2(r, t) = g(r, t)$ (prime and dot denote differentiation with respect to r and t respectively). We also suppose for f_1 and f_2 a time dependence of the form $e^{i\omega t}$. This is possible since all coefficients entering the equations are time-independent. We get thus the following eigenvalue or pulsation equation (f_1 and f_2 denote from now on only the radial part)

$$L_{ij} f_j = \sigma^2 M_{ij} f_j \quad i, j = 1, 2 \quad (13)$$

where

$$M_{ij} = e^{\lambda_0 - \lambda_0} \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \quad (14)$$

with $G_1 = r^2 e^{(\nu_0 - \lambda_0)/2}$ and $G_2 = r^2 \phi_0^2 e^{(\nu_0 - \lambda_0)3/2}$,

$$L_{ij} = \begin{pmatrix} -\frac{\partial}{\partial r} G_1 \frac{\partial}{\partial r} + G_1 C_1(r) & -2\omega \frac{\partial}{\partial r} (G_1 \phi_0) + G_1 C_3(r) \\ 2\omega G_1 \phi_0 \frac{\partial}{\partial r} + G_1 C_2(r) & -\frac{\partial}{\partial r} G_2 \frac{\partial}{\partial r} + G_2 C_2(r) \end{pmatrix}, \quad (15)$$

$$\begin{aligned} C_1(r) &= [e^{\lambda_0} (3\omega^2 e^{-\nu_0} + m^2 + 3\tilde{\lambda} \phi_0^2) + 16\pi G_N e^{\lambda_0} \tau \phi_0' (m^2 \phi_0 + \tilde{\lambda} \phi_0^3) \\ &\quad - 4\pi G_N \tau \phi_0'^2 (\nu_0' - \lambda_0' + \frac{2}{r})], \end{aligned} \quad (16a)$$

interval (a,b) of the positive real axis, which does not contain singular points. We get

$$2i(Im\sigma^2) \int_a^b e^{\nu_0 - \lambda_0} (G_1 |f_1|^2 + G_2 |f_2|^2) dr = G_1 [f_1 f_1' - f_1' f_1]_a^b + G_2 [f_2 f_2' - f_2' f_2]_a^b + 2\omega G_1 \phi_0 [f_1 f_2 - f_1' f_2']_a^b. \quad (20)$$

For convenience, we consider the interval $(0, \tau_0)$, where τ_0 is the position of the first node of $\phi_0(r)$. From the above Eq.(20) we see that the behaviour of the solution of Eq.(13) at the origin and at the point τ_0 puts constraints on the value of $(Im\sigma^2)$. The behaviour of the solution $f = (f_1, f_2)$ at the origin can be studied by inserting in Eq.(13) the following expansion:

$$Re f_1 = \sum_{n=0}^{\infty} a_n r^{\alpha+n}, \quad Re f_2 = \sum_{n=0}^{\infty} b_n r^{\beta+n}$$

and similarly for $Im f_1$ and $Im f_2$. It turns out that $Re f_1$ as well as $Im f_1$ behave as r^{-1} or are regular (constant or r^2); whereas $Re f_2$ and $Im f_2$ behave as r^{-2} or are regular (like r). The singular solution, like r^{-1} , for $Re f_1$ and $Im f_1$ has to be excluded due to the boundary condition on f_1 , which is for $r \rightarrow 0$ $f_1 = \text{const}$. Similarly a behaviour as r^{-2} at the origin for $Re f_2$ and $Im f_2$ is also excluded by the boundary condition coming from the requirement that $\delta N = 0$. f_1 and f_2 are thus regular at the origin and therefore in Eq.(20) the right-hand side contribution from the origin vanishes due to the fact that both G_1 and G_2 are proportional to r^2 .

To study the behaviour of the solution of Eq.(13) at τ_0 , the position of the first node (clearly the results are the same at every node position), it is convenient to use the following expansion:

$$Re f_1 = \sum_{n=0}^{\infty} a_n \phi_0^{\alpha+n}, \quad Re f_2 = \sum_{n=0}^{\infty} b_n \phi_0^{\beta+n}$$

and

$$Im f_1 = \sum_{n=0}^{\infty} A_n \phi_0^{\alpha+n}, \quad Im f_2 = \sum_{n=0}^{\infty} B_n \phi_0^{\beta+n}.$$

We notice that $\phi_0(r)$ has a simple zero at the nodes, otherwise also $\phi_0'(r)$ vanishes at τ_0 and due to the zero-order Eq.(5) ϕ_0'' and thus all higher derivatives would also vanish at the node, which is clearly inconsistent. Inserting the above expansion into Eq.(13), one gets

$$\begin{aligned} \sum_{n=0}^{\infty} (a_n \phi_0^{\alpha+n-2} (\alpha+n)(\alpha+n-1) \phi_0'^2 + a_n \phi_0^{\alpha+n-1} [(\alpha+n) \phi_0'' + (\frac{2}{\tau_0} + \frac{(\nu_0' - \lambda_0')}{2}) (\alpha+n) \phi_0']) \\ + a_n \phi_0^{\alpha+n} [e^{\lambda_0} (-3\omega^2 e^{-\lambda_0} - m^2) + 4\pi G_N \tau_0 \phi_0'^2 (\nu_0' - \lambda_0' + \frac{2}{\tau_0}) + (Re \sigma^2) e^{\lambda_0 - \nu_0}] \\ + b_n \phi_0^{\beta+n} [2\omega(\beta+n) \phi_0' + 4\omega \phi_0''] - (Im \sigma^2) A_n \phi_0^{\beta+n} e^{\lambda_0 - \nu_0} = 0 \end{aligned} \quad (21)$$

$$C_2(r) = \left[\frac{2(\lambda_0' - \nu_0')}{r} + \frac{2}{r^2} - \frac{(\nu_0'' - \lambda_0'')}{2} - 2 \frac{\phi_0''}{\phi_0} + 2 \left(\frac{\phi_0'}{\phi_0} \right)^2 \right. \\ \left. + \frac{(\lambda_0' - \nu_0')}{2} [(\nu_0' - \lambda_0') + 4 \frac{\phi_0'}{\phi_0}] - 4\pi G_N \tau \omega^2 \phi_0^2 e^{\lambda_0 - \nu_0} (\nu_0' - \lambda_0' + \frac{2}{r}) \right], \quad (16b)$$

$$C_3(r) = [-2\phi_0' \omega + 8\pi G_N \tau e^{\lambda_0} (m^2 \phi_0 + \lambda_0 \phi_0^3) \omega \phi_0^2 - 4\pi G_N \tau \phi_0' (\nu_0' - \lambda_0' + \frac{2}{r}) \omega \phi_0^2]. \quad (16c)$$

We expect the radial perturbations not to change the total particle number N_0 . In fact it turns out that the integrand of δN is a total derivative with respect to r [1], so that

$$\delta N = 8\pi (f_2 r^2 \phi_0^2 e^{(\nu_0 - \lambda_0)/2})|_0^{\infty}. \quad (17)$$

Therefore the requirement $\delta N = 0$ translates into conditions on the boundary values for f_2 . For $r \rightarrow \infty$ we require $r^2 \phi_0^2 f_2 \rightarrow 0$ and for $r \rightarrow 0$ we have $r^2 f_2 \rightarrow 0$. The boundary conditions for f_1 are: $r \rightarrow \infty, f_1 \rightarrow 0$; and $r \rightarrow 0, f_1 = \text{const}$. With these boundary conditions the operators L_{ij} and M_{ij} are symmetric in the case that $\phi_0(r)$ is nodeless [1], and therefore the corresponding eigenvalues are real.

From the form of Eq.(13), with definitions (14)-(16), one can also find the following variational principle

$$\sigma^2 \int_0^{\infty} \frac{1}{2} e^{(\lambda_0 - \nu_0)} (G_1 f_1^2 + G_2 f_2^2) dr = \int_0^{\infty} \left[\frac{1}{2} G_1 f_1'^2 + \frac{1}{2} G_1 f_1^2 C_1(r) + \frac{1}{2} G_2 f_2'^2 \right. \\ \left. + \frac{1}{2} G_2 f_2^2 C_2(r) + f_2 f_1' (2\phi_0 \omega G_1) + f_2 f_1 G_1 C_3(r) \right] dr. \quad (18)$$

Eigenfunctions related to different eigenvalues satisfy the following orthogonality relation

$$\int_0^{\infty} \frac{1}{2} e^{(\lambda_0 - \nu_0)} (G_1 f_{1i} f_{1j} + G_2 f_{2i} f_{2j}) dr = \delta_{ij}. \quad (19)$$

Equation (18) is a minimal principle, in fact the integral on the right-hand side of Eq.(18) is bounded from below. From this, it is clear that a sufficient condition for dynamical instability to occur is that the right-hand side of Eq.(18) vanishes (or becomes negative) for some chosen pair of trial functions f_1, f_2 , which satisfy the appropriate boundary conditions as discussed above.

4 Symmetry of L_{ij} for the excited configurations

Consider now the case where $\phi_0(r)$ has one or more nodes. Eq.(13) is then singular at the points where $\phi_0(r)$ vanishes. This complicates the analysis of the spectrum of the operator L_{ij} . Suppose that there is also a complex eigenvalue σ^2 with corresponding eigenfunction $f = (f_1, f_2)$, then also the complex conjugate function $\bar{f}$ is a solution with eigenvalue $\bar{\sigma}^2$, since all the coefficients of the equation are real. An equation involving the imaginary part of the complex eigenvalue σ^2 can be obtained by evaluating the integral $\int_a^b dr (f_i L_{ij} \bar{f}_j - \bar{f}_i L_{ij} f_j) = (\bar{\sigma}^2 - \sigma^2) \int_a^b dr (f_i M_{ij} \bar{f}_j)$ ($i, j = 1, 2$) over an arbitrarily chosen

apply when the scalars are coupled to a $U(1)$ gauge field [10].

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Figure Captions

Fig. 1: Bosc star mass in units of M_{Pl}^2/m (solid line) and particle number in units of M_{Pl}^2/m^2 (broken line) as a function of $\phi_0(0)$. The charge e is given in units of $M_{Pl}/(\sqrt{8\pi m})$. No quartic self-coupling is present. Also the lines going through the maxima of mass and particle number are drawn.

Fig. 2: The particle number in units of M_{Pl}^2/m^2 as a function of the effective radius R (in units of $1/m$) for the $\lambda = 0$ case for two different values of the charge e (in units of $M_{Pl}/(\sqrt{8\pi m})$).

$$\begin{aligned}
 & \text{and similarly for } Im f_1 \text{ by interchanging } (a_n, \alpha), (b_n, \beta) \text{ by } (A_n, \gamma), (B_n, \delta) \\
 & \sum_{n=0}^{\infty} (b_n \phi_0^{\beta+n-2} [(\beta+n+1) - 2\phi_0'^2 + b_n \phi_0^{\beta+n-1} [(\beta+n+2)\phi_0'' \\
 & + (\frac{2}{\tau_0} + \frac{3}{2}(\nu_0' - \lambda_0'))(\beta+n)\phi_0' + 2(\nu_0' - \lambda_0')\phi_0] + b_n \phi_0^{\beta+n} [\frac{2(\nu_0' - \lambda_0')}{\tau_0} - \frac{2}{\tau_0^2} \\
 & + \frac{(\nu_0'' - \lambda_0'')}{2} + \frac{(\nu_0' - \lambda_0')^2}{2} + (Re\sigma^2)e^{\lambda_0 - \nu_0}] + a_n \phi_0^{\alpha+n-2} [(1 - \alpha - n)2\phi_0' e^{\lambda_0 - \nu_0} \\
 & + a_n \phi_0^{\alpha+n} [4\pi G_N \tau_0 \phi_0' (\nu_0' - \lambda_0' + \frac{2}{\tau_0})\omega e^{\lambda_0 - \nu_0}] - (Im\sigma^2)B_n \phi_0^{\delta+n} e^{\lambda_0 - \nu_0}] = 0
 \end{aligned} \tag{22}$$

and similarly for $Im f_2$ by interchanging $(b_n, \beta), (a_n, \alpha)$ by $(B_n, \delta), (A_n, \gamma)$. Each power of ϕ_0 , in the above equations, has to vanish separately. This gives constraints which allow us to determine $\alpha, \beta, \gamma, \delta$ as well as the coefficients a_n, b_n, A_n and B_n . From the constraints it follows that $Re f_1$ and $Im f_1$ are regular at τ_0 , where they behave like const, ϕ_0 or ϕ_0^2 . For $Re f_2$, and similarly for $Im f_2$, the coefficient in front of the lowest power of ϕ_0 in Eq.(22) is: $b_0(\beta^2 + \beta - 2)$, which suggests that $\beta = -2$ could be a possible solution. Notice that $f_2 = g'$ and that the "physical" quantity is $\delta\phi_2 = \phi_0 g$ and so if $f_2 \sim \phi_0^{-2}$ then $\delta\phi_2 \sim \text{const}$. But if we take $\beta = -2$, then from Eq.(21) it follows that the coefficient b_1 in the expansion has to vanish ($\alpha \geq 0$ otherwise $a_0 = 0$). This implies that the coefficient in front of ϕ_0^{-3} in Eq.(22), which has to vanish, is given by: $b_0[-\phi_0'(\nu_0' - \lambda_0') - (4/\tau_0)\phi_0']$. Using the zero-order Eqs.(3) and (4) one can write it also as: $-2b_0[1/\tau_0(e^{\lambda_0} + 1)]$, and since $e^{\lambda_0} \geq 0$ it follows that $b_0 = 0$ itself. The singular solution with $\beta = -2$ is therefore excluded, the same being true for ϕ_0^{-1} . In conclusion also $Re f_2$ and $Im f_2$ are regular at the node and behave like ϕ_0 or ϕ_0^2 . Therefore the second and third terms on the right-hand side of Eq.(20) vanish at τ_0 , since G_2 is proportional to ϕ_0^2 . A non-vanishing contribution on the right-hand side of Eq.(20) can come only from the first term. But since f_1 and f_2 have no singularities at the nodes one can consider as integration limits in Eq.(20) the interval $(0, \infty)$, and at infinity all the functions f_1, f_2 and their derivatives vanish. Therefore it follows that $Im\sigma^2 = 0$ and thus that the operator, also in the node case, has only real eigenvalues.

In conclusion we have established that for the case with no quartic self-interaction the equilibrium solutions with nodes up to the critical central density ρ_{crit} are classically stable. Above ρ_{crit} the system becomes unstable at least up to the next value of ρ for which M and N are stationary. It seems very likely that also above this point the system like in the nodeless case is unstable. This fact is also suggested from the behaviour of the particle number (or mass) versus radius diagram [9]. In fact the shape of the particle number versus radius diagram is bent counterclockwise at the critical points. We notice that a rigorous proof, that from this behaviour it follows that the system stays unstable, exists only if the pulsation equation is a 1-component equation. In the cases where the pulsation equation consists of two or more coupled differential equations, this is still an open point. We expect similar conclusions to hold also for the case where there is a quartic self-interacting term or even for a more general scalar potential, since the proof that the operator L_{ij} has only real eigenvalues does not depend upon the form of the self-interaction. The same does also

FIGURE 1

FIGURE 2