CM-P00064391

Kinetic Description of the Beam Relaxation caused by Incoherent Synchrotron Radiation in High-Energy Electron Storage Rings*

F. Ruggiero and E. Picasso, CERN

L.A. Radicati

Scuola Normale Superiore, Pisa, Italy

Geneva, Switzerland

27 July 1989

Abstract

The asymptotic beam-envelope matrix for an ideal electron-positron storage ring is shown to be the unique fixed point of a mapping, derived from the linearized Fokker-Planck equation, which includes the integrated effect of radiation damping and quantum excitation over one ring revolution. The corresponding beam distribution has azimuthal periodicity and is Gaussian in all the phase-space variables.

In the limit of weak dissipation and far from linear resonances, this fixed point can be approximated by a linear combination of three periodic Twiss matrices, associated with the normal modes of the system. To first order in the relative energy loss per turn, the coefficients of the linear combination are constant along the ring and coincide with the equilibrium beam emittances.

A general formula is derived for the equilibrium emittances, which reduces to well known expressions in terms of radiation integrals for a storage ring with no horizontal-vertical coupling and with vanishing dispersion in the RF-cavities. In case of linear resonance, it is shown that two further 'generalized equilibrium emittances' are required to specify the beam-envelope matrix. It is also shown that, in the limit of weak dissipation, the asymptotic beam-envelope matrix can be derived from a variational principle.

*Submitted to Annals of Physics (N.Y.)

Contents

1	Introduction	3
I	Asymptotic solution of the linearized Fokker–Planck equation	5
2	Matrix formalism and linearized Fokker–Planck equation	5
3	Beam-envelope matrix R and asymptotic solution	7
4	Single-turn mapping and algebraic equation for R	9
II	Weak dissipation and equilibrium beam emittances	11
5	Approximate algebraic equation R	11
6	Hamiltonian evolution in the matrix formalism	13
6.1	Invariant scalar product	14
6.2	Eigenvectors of M and Twiss matrices	14
7	Solution of the approximate algebraic equation for R	18
8	The case of linear resonance	21
9	Variational principle for the asymptotic beam-envelope matrix	24
10	Example of an idealized storage ring without coupling	27
10.1	Single-particle Hamiltonian dynamics	27
10.2	Radiation effects and equilibrium emittances	31
11	Summary and conclusions	34
	References	36
	Appendices	38
A	Properties of the scalar product	38
B	Properties of the Twiss matrices	38
C	Damping constants in the limit of weak dissipation	40

1 Introduction

Particle beams in accelerators have sometimes been described as systems close to thermodynamic equilibrium [1–4] and, in particular, the concepts of entropy and temperature have been introduced. In the case of an electron beam undergoing uniform, linear acceleration and emitting synchrotron radiation, an equilibrium state does exist: the corresponding temperature is well defined and coincides with the Unruh temperature [5,6]. In the case of circular accelerators and storage rings with strong focusing, where synchrotron radiation remains the dominant relaxation mechanism, the situation is less clear and the thermodynamic approach does not seem to lead to consistent results.

As a preliminary step to the understanding of the difficulties associated with the thermodynamic approach, we shall discuss in this paper, from the point of view of the kinetic theory, the relaxation of the beam envelope in the single-particle phase space, caused by incoherent synchrotron radiation in high-energy electron-positron storage rings [7,8]. We shall further simplify the problem, by considering the evolution of the beam only for small *deviations* $x = (\mathbf{q}, \mathbf{p})$ of the particle coordinates and momenta from those of a ‘synchronous particle’, moving on a closed, equilibrium reference orbit. The kinetic approach is of course not new and a number of results concerning the asymptotic state of the beam have already been derived [9–16]. However, to our knowledge, the very existence of a unique asymptotic state, in case of strong focusing, has never been proved and we believe that a compact characterization of this state is still lacking.

A rigorous description of the dynamics of a particle accelerator would require the solution of some 10^{12} coupled, stochastic differential equations. The electromagnetic forces acting on the beam particles are due to external electromagnetic fields and to the interactions of the particles among themselves and with their environment. Obviously this approach is unpractical and therefore one has to resort to a statistical description. In principle, this should be based on the introduction of an N -particle distribution function f . The function f allows the calculation of the time-dependent average values of any physical observable and provides a complete probabilistic description of the system. Such a description, however, is still too complicated and one is often forced, in practice, to integrate the function f over the variables of $(N - 1)$ particles, thus introducing a distribution function $\psi(x, \tau)$, giving the average particle density in the six-dimensional phase space at point x and at time $\tau = s/c$. Here s is the curvilinear abscissa along the reference orbit and c is the speed of light.

In order that the product $\psi d\Omega$ represent the average number of particles in the phase-space volume element $d\Omega$, the distribution function ψ must be non negative and normalized to the total number of particles N in the beam

$$\int d\Omega \psi = N. \quad (1.1)$$

The evolution of ψ is governed by a kinetic equation of the form [17,18]

$$\frac{\partial \psi}{\partial \tau} + [\psi, \mathcal{H}(\mathbf{q}, \mathbf{p}, \tau)] = -\frac{\partial}{\partial \mathbf{p}} \cdot \mathcal{I}, \quad (1.2)$$

where $\mathcal{H}(\mathbf{q}, \mathbf{p}, \tau)$ is the single-particle Hamiltonian, the symbol $[\dots, \dots]$ denotes Poisson brackets with respect to the canonical variables $\mathbf{q}$ and $\mathbf{p}$ and the vector $\mathcal{I}$ is the particle

flux in momentum space, associated with irreversible processes. We have neglected a possible dependence of $\mathcal{H}$ on ψ , because we are not considering collective effects, such as the beam-beam interaction or the beam-wall interaction through wake-fields.

In high-energy electron-positron storage rings, the irreversible flux $\mathcal{I}$ is mainly caused by incoherent synchrotron radiation. Since the emission of an individual photon takes place within an azimuthal angle of order $1/\gamma$ and since the Lorentz factor γ of the particles is usually very large, the correlation time between two emissions is negligible with respect to all the other dynamical periods of the storage ring. Therefore the irreversible flux $\mathcal{I}$ takes the form

$$\mathcal{I}_k = - \left(\mathcal{A}_k \psi + \frac{1}{2} \frac{\partial}{\partial p_j} (\mathcal{B}_{kj} \psi) \right), \quad k = 1, 2, 3 \quad (1.3)$$

where summation over repeated indices is implied. The vector $\mathcal{A}_k(x, \tau)$ is associated with dissipation and the symmetric tensor $\mathcal{B}_{kj}(x, \tau)$ with diffusion in momentum space. In the last section, we shall give an explicit expression for $\mathcal{I}$ in terms of the radiation reaction force experienced by a relativistic electron moving in a magnetic field; the origin of the diffusive term will then be identified with quantum fluctuations of the radiation field. When $\mathcal{I}$ depends only on ψ and on its first derivatives, as in Eq. (1.3), the kinetic equation (1.2) is usually referred to as the Fokker-Planck equation [17,19,20].

The linearization of the Fokker-Planck equation leads naturally to a six-dimensional matrix formalism that will be outlined in Sec. 2. Then, in Sec. 3, we will show that any initial distribution of the beam relaxes, under reasonable assumptions, to a unique asymptotic distribution $\psi_\infty(x, \tau)$ which is Gaussian in x and periodic in τ . Therefore, the asymptotic state of the beam is characterized by the symmetric 6×6 matrix of the mean square values of the particle coordinates and momenta, R_∞ , having $6 \times (6+1)/2 = 21$ independent elements that are periodic functions of τ : R_∞ is called the asymptotic beam-envelope matrix.

In Sec. 4, by integrating the evolution equation for R over one turn, we define a mapping whose unique fixed point is R_∞ . In Sec. 5, we consider this mapping in the limit of *weak dissipation*, i.e. when the mean value δ of the relative energy loss per turn due to synchrotron radiation is much smaller than unity. Then, since the effect of radiation represents a small perturbation to the purely Hamiltonian evolution of R , we are led to split R_∞ as follows:

$$R_\infty = R_H + Z, \quad (1.4)$$

where R_H is a fixed point of the purely Hamiltonian mapping associated with the symplectic matrix $\bar{M}$, known as the single-turn transfer matrix. Therefore, R_H is a solution of the linear algebraic equation

$$R_H = \bar{M} R_H {}^t\bar{M}, \quad (1.5)$$

while the residual matrix Z is orthogonal to R_H with respect to a scalar product (defined in Sec. 6) invariant under the Hamiltonian mapping. In Sec. 6 we will show that, under the assumption that $\bar{M}$ has non-degenerate eigenvalues of unit norm, R_H can be parametrized by three coefficients ε_1 , ε_2 and ε_3 and can be written as

$$R_H = \sum_{k=1}^3 \varepsilon_k \sigma_k, \quad (1.6)$$

where the three independent, symmetric matrices σ_1 , σ_2 and σ_3 (known as Twiss matrices) form an orthonormal basis with respect to the same scalar product.

In Sec. 7, we will prove that the residual matrix Z is of order δ and we will calculate the coefficients ε_k : to first order in δ , these coefficients are constant and coincide with their average values along the ring, $\bar{\varepsilon}_k$, known as the *equilibrium beam emittances*. These will be expressed in terms of the σ 's and of two matrices, associated with radiation damping and quantum fluctuations. In Sec. 8 we consider the case when the eigenvalues of $\bar{M}$ are degenerate; the number of independent Twiss matrices is shown to be five rather than three and, correspondingly, the asymptotic beam-envelope matrix R_∞ must be parametrized by five 'generalized emittances'.

In Sec. 9 we show, again in the limit of weak dissipation, that the asymptotic beam-envelope matrix can be derived from a *variational principle*. This result is still valid when $\bar{M}$ has degenerate eigenvalues and may suggest some connection with the principle of 'minimum entropy production', that applies to systems whose steady state slightly deviates from the state of thermodynamic equilibrium as a consequence of external constraints [21]. We plan to come back to this point in a subsequent paper.

In the last section we apply our formalism to an idealized storage ring, with no horizontal-vertical coupling and with vanishing dispersion in the RF-cavities. This allows us to provide an explicit expression for the matrices σ_k and for the two fundamental matrices associated with radiation damping and quantum noise: in this case our formula for the equilibrium emittances reduces to well known expressions in terms of radiation integrals [10].

Part I

Asymptotic solution of the linearized Fokker–Planck equation

2 Matrix formalism and linearized Fokker–Planck equation

For small deviations $x = (\mathbf{q}, \mathbf{p})$ of the particle coordinates and momenta from those of a synchronous particle, moving on a closed equilibrium orbit, we retain only quadratic terms in the Hamiltonian $\mathcal{H}(x, \tau)$ [we recall that $\tau = s/c$] and the lowest order terms in the irreversible flux $\mathcal{I}$. In this *linear approximation*, the Fokker–Planck equation takes a more compact form if one adopts a six-dimensional notation, where both the Hamiltonian evolution and the effects of synchrotron radiation are described by matrices.

Indeed the Hamiltonian $\mathcal{H}(x, \tau)$ can be written in the form

$$\mathcal{H} = \frac{1}{2} {}^t x H x, \quad (2.1)$$

where ${}^t x$ denotes the transpose of the vector x and the symmetric 6×6 matrix $H(\tau)$ is periodic in τ with the ring revolution period T . Since the Poisson brackets $[f, g]$ can be expressed through the unit 6×6 symplectic matrix J

$$[f, g] = \frac{\partial f}{\partial {}^t x} J \frac{\partial g}{\partial x}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (2.2)$$

with I denoting the 3×3 identity matrix, the corresponding single-particle equation of motion, i.e. Hamilton's equation, then becomes

$$\frac{dx}{d\tau} = [x, \mathcal{H}] = J \frac{\partial \mathcal{H}}{\partial x} = J H x \equiv L x. \quad (2.3)$$

The trace-less matrix $L(\tau)$, related to the Hamiltonian matrix $H(\tau)$, satisfies

$$L = JH, \quad {}^tL = J LJ. \quad (2.4)$$

The irreversible flux $\mathcal{I}_k$ in three-dimensional momentum space can be generalized to a six-dimensional flux $\mathcal{I}_\alpha$ (where $\alpha = 1, \dots, 6$) by adding zero's in the appropriate places. In linear approximation, the generalization of Eq. (1.3) reads (summation over repeated indices is implied)

$$\mathcal{I}_\alpha = - \left(A_{\alpha\beta} x_\beta + \frac{1}{2} B_{\alpha\beta} \frac{\partial}{\partial x_\beta} \right) \psi, \quad (2.5)$$

where the 6×6 matrices $A(\tau)$ and $B(\tau)$ are again periodic in τ with period T and are associated with classical radiation damping and with noise due to quantum fluctuations, respectively; both of them are proportional to the mean value δ of the relative energy loss per turn due to synchrotron radiation. In the last section, we will show that the symmetric noise matrix $B(\tau)$ has non-negative eigenvalues while the trace of the dissipation matrix $A(\tau)$ has the following general property [22]:

$$\oint d\tau \operatorname{Tr} [A(\tau)] = 4\delta. \quad (2.6)$$

The assumption that the synchronous particle (characterized by $x = 0$) moves along a closed equilibrium orbit implies that the zeroth-order term in the expansion of $\mathcal{A}_k(x, \tau)$ [see Eqs. (1.3) and (2.5)] in powers of x vanishes, as it will appear later.

With this notation, the linearized Fokker-Planck equation can be written in the form

$$\frac{\partial \psi}{\partial \tau} + \frac{\partial}{\partial x_\alpha} \left[(L_{\alpha\beta} - A_{\alpha\beta}) x_\beta \psi - \frac{1}{2} B_{\alpha\beta} \frac{\partial \psi}{\partial x_\beta} \right] = 0. \quad (2.7)$$

The dissipation matrix A can be split as follows:

$$A = \frac{1}{2} (A + J {}^t A J) + \frac{1}{2} (A - J {}^t A J) \equiv A_H + A_D, \quad (2.8)$$

where the trace-less matrix A_H has the same properties as L and gives rise to corrections of the Hamiltonian particle motion (e.g. slight modifications of the machine tunes), while the damping matrix A_D is associated with a pure contraction of the phase-space volume occupied by the beam (see Eq. (2.6))

$${}^t A_H = J A_H J, \quad {}^t A_D = -J A_D J. \quad (2.9)$$

The decomposition (2.8) is equivalent to writing the matrix JA as the sum of its symmetric and antisymmetric parts; the former can be absorbed in H , thus obtaining a renormalized Hamiltonian matrix.

3 Beam-envelope matrix R and asymptotic solution

Owing to the relatively simple structure of the linearized Fokker-Planck equation (2.7), we can give the explicit form of the asymptotic solution $\psi_\infty(x, \tau)$ and prove its uniqueness; this is an extension of a well known result discussed by Chandrasekhar [19] when L , A and B are independent of τ .

To this aim, we begin by defining the first and second order moments of ψ with respect to x by

$$\bar{x}_\alpha = \frac{1}{N} \int d\Omega x_\alpha \psi, \quad (3.1)$$

$$R_{\alpha\beta} = \frac{1}{N} \int d\Omega (x_\alpha - \bar{x}_\alpha)(x_\beta - \bar{x}_\beta) \psi. \quad (3.2)$$

As a consequence of Eq. (2.7), these moments satisfy the following evolution equations:

$$\frac{d\bar{x}}{d\tau} = (L - A) \bar{x}, \quad (3.3)$$

$$\frac{dR}{d\tau} = (L - A)R + R(L - A) + B. \quad (3.4)$$

These two equations are decoupled from each other and from the evolution equations of higher order moments. It is therefore straightforward to verify that any function of the form

$$G(x, \tau) = \frac{1}{(2\pi)^3 \sqrt{\det R}} \exp \left\{ -\frac{1}{2} (x - \bar{x}) R^{-1} (x - \bar{x}) \right\} \quad (3.5)$$

is a solution of the linearized Fokker-Planck equation (2.7).

This distribution is normalized to unity for all τ 's and, when R becomes small, it approaches a six-dimensional Dirac function $\delta^6(x - \bar{x})$. Thus if one chooses $\bar{x}(\tau)$ and $R(\tau)$ such that, at time τ_o , they satisfy the following initial conditions:

$$\bar{x}(\tau_o) = x_o, \quad R(\tau_o) = 0, \quad (3.6)$$

the corresponding Gaussian distribution of Eq. (3.5) coincides with the Green function $G(x, \tau|x_o, \tau_o)$ of the linearized Fokker-Planck equation (2.7). It expresses the transition probability from the initial point x_o , at time τ_o , to the final point x , at a subsequent time τ . Since $G(x, \tau|x_o, \tau_o)$ is a particular solution of Eq. (2.7) with initial condition

$$G(x, \tau_o|x_o, \tau_o) = \delta^6(x - x_o), \quad (3.7)$$

the general solution of Eq. (2.7) with arbitrary initial condition $\psi(x_o, \tau_o)$ can be written as

$$\psi(x, \tau) = \int d\Omega_o G(x, \tau|x_o, \tau_o) \psi(x_o, \tau_o), \quad (3.8)$$

where $d\Omega_o$ is the six-dimensional phase-space volume element at x_o . Therefore the general solution of the linearized Fokker-Planck equation can be derived from the evolution of the first two moments $\bar{x}$ and R , with initial conditions (3.6).

The vector $\bar{x}$ represents the center of mass of the beam distribution in phase space; from Eq. (3.3) we see that its motion is independent of quantum fluctuations. Moreover,

owing to the absence of a zeroth-order term in the expansion of the dissipative term $\mathcal{A}_k$, Eq. (3.3) is homogeneous in $\bar{x}$. Therefore the solution $\bar{x} = 0$ exists, so that the reference orbit is an equilibrium orbit as we had assumed. The evolution of $\bar{x}$ can be expressed by means of a 6×6 matrix $U(\tau|\tau_o)$, defined as follows¹:

$$\frac{dU}{d\tau} = (L - A)U \quad \text{with initial condition} \quad U(\tau_o|\tau_o) = I, \quad (3.9)$$

which describes the deterministic particle motion under the effect of restoring forces (derivable from a Hamiltonian) and of radiation damping. Then, the general solution of Eq. (3.3) reads

$$\bar{x}(\tau) = U(\tau|\tau_o) \bar{x}(\tau_o). \quad (3.10)$$

In order to insure the asymptotic stability of the equilibrium, reference orbit, we now introduce the following assumption

$$\lim_{\tau \rightarrow \infty} U(\tau|\tau_o) = 0, \quad (3.11)$$

which is equivalent to requiring that, for any initial condition $\bar{x}(\tau_o)$, the oscillations of the beam center of mass around the reference orbit are always damped as a consequence of synchrotron radiation, i.e.

$$\bar{x}_\infty = \lim_{\tau \rightarrow \infty} \bar{x}(\tau) = 0. \quad (3.12)$$

The matrix $U(\tau|\tau_o)$ also governs the evolution of the symmetric matrix R , to be called in the following the beam-envelope matrix, that can be interpreted as the matrix of the mean square values of the beam sizes in phase space. From the definition (3.2), it follows that R has non-negative eigenvalues. Equation (3.4) is non homogeneous in R , with the noise matrix B being the driving term. Its general solution is given by the general solution of the associated homogeneous equation, that includes only Hamiltonian and damping terms, plus a special solution which takes into account the effect of noise

$$R(\tau) = U(\tau|\tau_o)R(\tau_o)U(\tau|\tau_o) + \int_{\tau_o}^{\tau} d\tau' U(\tau|\tau')B(\tau')U(\tau|\tau'). \quad (3.13)$$

As a consequence of assumption (3.11), the first term in the r.h.s. of Eq. (3.13) tends to zero as $\tau \rightarrow \infty$, while, as shown in the next section, the second term tends to a unique, asymptotic matrix $R_\infty(\tau)$. Therefore, both $\bar{x}_\infty$ and $R_\infty(\tau)$ are independent of the initial conditions $\bar{x}(\tau_o)$ and $R(\tau_o)$, respectively: we shall thus refer to Eq. (3.11) as the *memory loss assumption*.

From the definition (3.9) of $U(\tau|\tau_o)$ and from the periodicity of the matrices $L(\tau)$ and $A(\tau)$, it follows that

$$U(\tau + T|\tau_o + T) = U(\tau|\tau_o). \quad (3.14)$$

Therefore, since the noise matrix $B(\tau)$ is also periodic in τ , the asymptotic beam-envelope matrix $R_\infty(\tau)$ can be obtained by Eq. (3.13) in the limit $\tau_o \rightarrow -\infty$ and reads

$$R_\infty(\tau) = \int_{-\infty}^{\tau} d\tau' U(\tau|\tau')B(\tau')U(\tau|\tau'). \quad (3.15)$$

¹Here I denotes the 6×6 identity matrix.

Thus $R_\infty(\tau)$ is a periodic solution of Eq. (3.4); the convergence of the integral appearing in Eq. (3.15) shall be proved in the next section, where we will also show that $R_\infty(\tau)$ is the *unique* periodic solution of Eq. (3.4).

As we have seen, our ‘memory loss’ assumption (3.11) implies that the asymptotic values $\bar{x}_\infty = 0$ and $R_\infty(\tau)$ are independent of the initial conditions $\bar{x}(\tau_o)$ and $R(\tau_o)$, respectively. Correspondingly, the Green function $G(x, \tau|x_o, \tau_o)$ has a unique asymptotic form, independent of x_o and τ_o and given by Eq. (3.5) with $\bar{x} = 0$ and $R = R_\infty$. Since, according to condition (1.1), the initial value $\psi(x_o, \tau_o)$ of the beam distribution must be normalized to N , from Eq. (3.8) we conclude that the linearized Fokker-Planck equation (2.7) has a unique asymptotic solution, $\psi_\infty(x, \tau)$, independent of the initial condition and whose explicit form is

$$\psi_\infty(x, \tau) = \frac{N}{(2\pi)^3 \sqrt{\det R_\infty(\tau)}} \exp \left\{ -\frac{1}{2} {}^t x R_\infty^{-1}(\tau) x \right\}. \quad (3.16)$$

4 Single-turn mapping and algebraic equation for R

In this section we will show that the beam-envelope matrix $R(\tau)$, whose evolution is governed by the differential equation (3.4), relaxes to a unique, finite asymptotic matrix $R_\infty(\tau)$, periodic in τ and given by Eq. (3.15). To this end, we adopt the following strategy:

- First we integrate the evolution equation for R over one turn. This yields a single-turn mapping $R_n(\tau) \rightarrow R_{n+1}(\tau)$, from turn n to turn $n+1$.
- Then we show that the ‘memory loss’ assumption, Eq. (3.11), is equivalent to requiring that the single-turn mapping is a *contraction*. Therefore there exists a unique fixed point $R_\infty(\tau)$ of the mapping, satisfying an algebraic equation with periodic coefficients.
- Finally, we show that this fixed point coincides with the asymptotic solution of Eq. (3.4), i.e. that the integral in Eq. (3.15) is convergent.

By integrating the evolution equation for R over one turn (or, equivalently, by considering Eq. (3.13) for $\tau = \tau_o + T$), we obtain the following mapping:

$$R_{n+1} = \bar{U} R_n {}^t \bar{U} + \bar{B}, \quad (4.1)$$

where

$$R_n(\tau) = R(\tau + nT), \quad \bar{U}(\tau) = U(\tau + T|\tau), \quad (4.2)$$

$$\bar{B}(\tau) = \int_\tau^{\tau+T} d\tau' U(\tau + T|\tau') B(\tau') {}^t U(\tau + T|\tau'). \quad (4.3)$$

The matrix $\bar{B}$ represents the integrated effect of quantum fluctuations over one turn; it contains the local contribution $B(\tau')$ of quantum noise at any intermediate time τ' , between τ and $\tau + T$, propagated downstream to the final time $\tau + T$ by the matrix $U(\tau + T|\tau')$. Using Eq. (3.14), we see that both $\bar{U}(\tau)$ and $\bar{B}(\tau)$ are periodic in τ with period T .

To prove that the mapping (4.1) is a contraction, we first observe that from the ‘memory loss’ assumption on $U(\tau|\tau_o)$, Eq. (3.11), it follows

$$\lim_{n \rightarrow \infty} \bar{U}^n = 0, \quad (4.4)$$

i.e. the complex eigenvalues of the single-turn matrix $\bar{U}$ have a norm smaller than unity. Indeed, from the definition (3.9) of $U(\tau|\tau_o)$, we see that U satisfies the composition law

$$U(\tau|\tau_o) = U(\tau|\tau')U(\tau'|\tau_o) \quad \text{where} \quad \tau_o \leq \tau' \leq \tau \quad (4.5)$$

and thus we obtain

$$\lim_{n \rightarrow \infty} \bar{U}^n(\tau) = \lim_{n \rightarrow \infty} U(\tau + nT|\tau) = \lim_{\tau' \rightarrow \infty} U(\tau'|\tau) = 0. \quad (4.6)$$

The same is true for the linear operator $\mathcal{L}(R) = \bar{U}R\bar{U}$, appearing in the mapping (4.1), since we have

$$\lim_{n \rightarrow \infty} \mathcal{L}^n(R) = \lim_{n \rightarrow \infty} \bar{U}^n R \bar{U}^n = 0. \quad (4.7)$$

Therefore there exists a Euclidean norm $\|R\|$ and an associated Euclidean metric $\rho(R, R') = \|R - R'\|$, defined on the linear space of the 6×6 symmetric matrices R , such that $\mathcal{L}$ is a contraction², i.e.

$$\|\bar{U}R\bar{U}\| \leq \lambda \|R\| \quad \text{with} \quad 0 < \lambda < 1, \quad (4.8)$$

for any R . This also implies that the mapping (4.1) is a contraction, because

$$\rho(\bar{U}R_n\bar{U} + \bar{B}, \bar{U}R_m\bar{U} + \bar{B}) = \|\bar{U}(R_n - R_m)\bar{U}\| \leq \lambda \|R_n - R_m\| = \lambda \rho(R_n, R_m). \quad (4.9)$$

Since the corresponding Euclidean space (of dimension $6 \times (6 + 1)/2 = 21$) is a complete metric space, for each value of τ the single-turn mapping (4.1) has a unique fixed point $R_\infty(\tau)$, which satisfies the following algebraic equation:

$$R = \bar{U}R\bar{U} + \bar{B}. \quad (4.10)$$

Moreover, since both $\bar{U}(\tau)$ and $\bar{B}(\tau)$ are periodic with period T , also $R_\infty(\tau)$ has the same periodicity.

To prove that the fixed point R_∞ coincides with the asymptotic solution of the differential equation (3.4), we observe that it can also be obtained by iterating the mapping (4.1). This gives

$$R_n(\tau) = \bar{U}^n R_o(\tau) \bar{U}^n + \sum_{k=0}^{n-1} \bar{U}^k \bar{B} \bar{U}^k. \quad (4.11)$$

In the limit $n \rightarrow \infty$, recalling Eq. (4.7), the asymptotic matrix $R_\infty(\tau)$ becomes independent of the initial value $R_o(\tau)$ and can be written

$$R_\infty(\tau) = \lim_{n \rightarrow \infty} R_n(\tau) = \sum_{k=0}^{\infty} \bar{U}^k \bar{B} \bar{U}^k. \quad (4.12)$$

²The proof remains valid even when $\bar{U}$ can not be diagonalized: the Euclidean metric ρ is called Ljapunov function and its construction is based on the possibility of reducing any matrix to triangular form, with arbitrarily small off-diagonal elements [23].

The convergence of the sum over k is insured by condition (4.8), indeed

$$\left\| \sum_{k=0}^{\infty} \bar{U}^k \bar{B} \bar{U}^k \right\| \leq \sum_{k=0}^{\infty} \left\| \bar{U}^k \bar{B} \bar{U}^k \right\| \leq \sum_{k=0}^{\infty} \lambda^k \left\| \bar{B} \right\| = \frac{\left\| \bar{B} \right\|}{1 - \lambda} < +\infty. \quad (4.13)$$

From the definitions (4.2) and (4.3) of $\bar{U}$ and $\bar{B}$, using the properties (3.14) and (4.5) of the matrix U , it is simple to show that the convergent series of Eq. (4.12) coincides with the integral of Eq. (3.15) and thus that $R_{\infty}(\tau)$ is the asymptotic solution of Eq. (3.4).

We conclude this section by remarking that the asymptotic beam-envelope matrix is the unique periodic solution of the differential equation (3.4). Indeed, from expression (3.15), we know that $R_{\infty}(\tau)$ is a solution of Eq. (3.4). At the same time, it is the only solution of the algebraic equation (4.10) and, from Eq. (3.13), this is also the necessary and sufficient condition for a solution of Eq. (3.4) to be periodic.

Part II

Weak dissipation and equilibrium beam emittances

Since we consider the Fokker–Planck equation in *linear approximation*, our results apply only if the particles are close to the phase-space origin $x = 0$. In practice, this means that any source of non-linearity (such as sextupole magnets or sinusoidal deviations of the RF-voltage from linearity) should have a negligible effect within the statistically most populated region of phase space, namely a six-dimensional ellipsoid characterized by the asymptotic beam-envelope matrix R_{∞} . The linear approximation is justified only for small storage rings, such as ADONE, where the natural chromaticity is small and there is no need to introduce sextupoles. However, even though non linear effects often play an important role, a linear analysis is always the first step of machine design (the same is true when collective effects are considered). On the other hand, any electron storage ring must satisfy our *memory-loss* assumption Eq. (3.11), since otherwise the particles would not circulate in the machine for an appreciable time. In the following sections, we shall discuss the asymptotic beam-envelope matrix under another, rather general assumption, namely that the mean value δ of the relative energy loss per turn due to synchrotron radiation be much smaller than unity. This assumption of *weak dissipation* is very well satisfied even in large storage rings such as LEP, where $\delta \simeq 2.4 \times 10^{-2}$ at a beam energy of 95 GeV, and the opposite would indicate that the machine radius has not been optimized for the design energy.

5 Approximate algebraic equation R

The matrix U defined by Eq. (3.9) can be expanded in powers of δ and, in the limit of weak dissipation, the expansion can be truncated to first order: inserting the corresponding expansions for $\bar{U}$ and $\bar{B}$ into Eq. (4.10), yields an approximate algebraic equation for the asymptotic beam-envelope matrix.

Since the dissipation matrix A is of order δ [see Eq. (2.6)], we can rewrite Eq. (3.9) as follows:

$$\frac{dU}{d\tau} = (L - \delta \tilde{A})U \quad \text{with} \quad U(\tau_o|\tau_o) = I, \quad (5.1)$$

where we have defined $A = \delta \tilde{A}$, so that the elements of the matrix $\tilde{A}(\tau)$, integrated in τ over the ring, are of order unity.

Let us now expand $U(\tau|\tau_o)$ in powers of δ

$$U(\tau|\tau_o) = \sum_{n=0}^{\infty} U_n(\tau|\tau_o) \delta^n. \quad (5.2)$$

Inserting (5.2) into the evolution equation (5.1) and equating terms containing the same powers of δ , we see that the zeroth order term $M(\tau|\tau_o) \equiv U_o(\tau|\tau_o)$ represents the transfer matrix associated with purely Hamiltonian motion

$$\frac{dM}{d\tau} = L M \quad \text{with} \quad M(\tau_o|\tau_o) = I, \quad (5.3)$$

while the subsequent terms of the expansion satisfy the following recursive equations

$$\frac{dU_n}{d\tau} = L U_n - \tilde{A} U_{n-1} \quad \text{with} \quad U_n(\tau_o|\tau_o) = 0, \quad \text{for } n \geq 1. \quad (5.4)$$

Let us remark that, from Eq. (5.3) and from the property (2.4) of the matrix L , it follows that $M(\tau|\tau_o)$ is a symplectic matrix, i.e.

$${}^t M J M = J. \quad (5.5)$$

Using Eq. (5.3), the subsequent terms can be expressed recursively in terms of M and $\tilde{A}$

$$U_n(\tau|\tau_o) = - \int_{\tau_o}^{\tau} d\tau' M(\tau|\tau') \tilde{A}(\tau') U_{n-1}(\tau'|\tau_o). \quad (5.6)$$

In the limit of weak dissipation ($\delta \ll 1$), we can truncate expansion (5.2) to first order in δ thus obtaining

$$U(\tau|\tau_o) = M(\tau|\tau_o) - \int_{\tau_o}^{\tau} d\tau' M(\tau|\tau') \tilde{A}(\tau') M(\tau'|\tau_o) + O(\delta^2). \quad (5.7)$$

Here the ‘exact propagator’ $U(\tau|\tau_o)$ is expressed as the sum of the ‘Hamiltonian propagator’ $M(\tau|\tau_o)$ plus a small perturbation term, obtained by the Hamiltonian evolution from τ_o to all the intermediate times τ' , multiplied by the local effect of radiation damping $A(\tau') = \delta \tilde{A}(\tau')$ and then again propagated to τ by $M(\tau|\tau')$.

By inserting this truncated expansion of U into Eq. (4.10), we get the following *approximate* algebraic equation for the beam-envelope matrix R :

$$R = (\bar{M} - \bar{A}) R (\bar{M} - \bar{A}) + \bar{B}, \quad (5.8)$$

where $\bar{M}$ denotes the symplectic, single-turn transfer matrix corresponding to M , while $\bar{A}$ and $\bar{B}$ represent the integrated effect over one turn of radiation damping and quantum

noise, respectively, to the lowest order in δ

$$\begin{aligned}\bar{M}(\tau) &= M(\tau + T|\tau), \\ \bar{A}(\tau) &= \int_{\tau}^{\tau+T} d\tau' M(\tau + T|\tau') A(\tau') M(\tau'|\tau), \\ \bar{B}(\tau) &= \int_{\tau}^{\tau+T} d\tau' M(\tau + T|\tau') B(\tau') {}^tM(\tau + T|\tau').\end{aligned}\tag{5.9}$$

It is worth noting that, again, all the matrix coefficients of the algebraic equation (5.8) are periodic in τ , so that its solution $R(\tau)$ is also periodic in τ and can be identified with the asymptotic beam-envelope matrix (to first order in δ). The advantage of the approximate algebraic equation (5.8) over the exact initial equation (4.10) is that we only need the symplectic matrix $\bar{M}$, corresponding to the Hamiltonian particle motion, instead of the matrix $\bar{U}$ which includes the integrated effect of radiation damping. By means of the recurrence relation (5.6), this integrated effect of radiation damping can be computed to any desired accuracy.

6 Hamiltonian evolution in the matrix formalism

In the limit of weak dissipation, the effect of synchrotron radiation can be considered as a small perturbation to the Hamiltonian particle motion. The latter dominates the short-term evolution of the beam distribution, i.e. the evolution over time intervals of the order of a few betatron or synchrotron periods, while the irreversible effect of radiation damping and quantum excitation leads to a slow relaxation towards the asymptotic beam distribution. Once this asymptotic state is eventually reached, the corresponding periodic distribution of the beam particles is the result of a dynamical equilibrium between their single-turn Hamiltonian evolution, governed by the symplectic matrix $\bar{M}$, and the integrated effect of damping and noise, described by the two matrices $\bar{A}$ and $\bar{B}$ of Eq. (5.9).

Since the asymptotic distribution is completely determined by the periodic beam-envelope matrix [see Eq. (3.16)], we are led to solve the approximate algebraic equation (5.8), whose physical meaning can be summarized as follows: starting at a given point of the storage ring (corresponding to a given value of τ), propagate the beam-envelope matrix R through one turn by means of the symplectic matrix $\bar{M}$ (which describes betatron and synchrotron oscillations), add the small integrated effect of radiation damping and quantum noise (associated with the two matrices $\bar{A}$ and $\bar{B}$, both of order δ) and require that the final value of R be equal to its initial value. Since the integrated effect of synchrotron radiation over one machine revolution is of order δ and thus very small, in general the asymptotic beam-envelope matrix will be close to a solution of Eq. (5.8) without damping and noise, i.e. to a fixed point of the purely Hamiltonian single-turn mapping

$$R = \bar{M} R {}^t\bar{M}.\tag{6.1}$$

In this section, therefore, we focus our attention on the properties of the symplectic matrix

$\bar{M}$, which governs the purely Hamiltonian single-turn evolution of the beam and allows us to define the three normal modes of the system.

6.1 Invariant scalar product

In contrast with the algebraic equation (5.8), Eq. (6.1) is homogeneous in R and, for a fixed value of τ , the set Σ_τ of its symmetric solutions is a linear space. We investigate the connection between the solutions of Eq. (6.1) corresponding to different values of τ , i.e. to different azimuthal positions along the storage ring.

The single-turn transfer matrix $\bar{M}(\tau)$ is periodic in τ and, for any pair of values τ_1 and τ_2 , the following similarity transformation holds:

$$\bar{M}(\tau_2) = M(\tau_2|\tau_1)\bar{M}(\tau_1)M^{-1}(\tau_2|\tau_1). \quad (6.2)$$

Therefore, the continuous Hamiltonian evolution from τ_1 to τ_2 induces a linear transformation $\mathcal{M}_{\tau_2\tau_1}$ from Σ_{τ_1} to Σ_{τ_2} , that associates to any symmetric matrix R_{τ_1} in Σ_{τ_1} a symmetric matrix $\mathcal{M}_{\tau_2\tau_1}(R_{\tau_1})$ in Σ_{τ_2}

$$\mathcal{M}_{\tau_2\tau_1}(R_{\tau_1}) = M(\tau_2|\tau_1)R_{\tau_1}{}^t M(\tau_2|\tau_1). \quad (6.3)$$

Since the symplectic matrix $M(\tau_2|\tau_1)$ is invertible, $\mathcal{M}_{\tau_2\tau_1}$ is an isomorphism between Σ_{τ_1} and Σ_{τ_2} . This isomorphism preserves the scalar product (R, S) defined as follows

$$(R, S) \equiv -\frac{1}{2} \text{Tr}(RJSJ), \quad (6.4)$$

where the factor $-1/2$ has been introduced for future convenience: for a symmetric 2×2 matrix R , the scalar product (R, R) coincides with the determinant of R and therefore it is *not* positive definite. In Appendix A, however, we show that the scalar product (R, S) between two symmetric, non-negative matrices R and S is also non negative.

Let us remark that the symmetric matrices $\mathcal{M}_{\tau\tau_0}(R_{\tau_0})$, obtained by transforming any element R_{τ_0} of Σ_{τ_0} , are periodic solutions of the purely Hamiltonian evolution equation

$$\frac{dR}{d\tau} = LR + R^t L. \quad (6.5)$$

Given a base in the linear space Σ_{τ_0} , the isomorphism $\mathcal{M}_{\tau\tau_0}$ generates a base in Σ_τ and hence the most general symmetric solution $R(\tau)$ of Eq. (6.1) is a linear combination, with *periodic coefficients*, of the symmetric, periodic solutions of the differential equation (6.5).

6.2 Eigenvectors of $\bar{M}$ and Twiss matrices

As we have seen, our memory-loss assumption (3.11) implies that all the complex eigenvalues of the matrix $\bar{U}$, defined by Eq. (4.2), have a norm smaller than unity. Therefore, after expanding $\bar{U}$ in powers of δ , in order to fulfill the memory-loss assumption for arbitrarily small δ we must require that all the eigenvalues of the matrix $\bar{M}$, representing the zeroth order term of the expansion, have a norm smaller or at most equal to one. On the other hand, $\bar{M}$ is a real symplectic matrix and if λ is an eigenvalue of $\bar{M}$ the same is true for

its inverse $1/\lambda$ [24]. The conclusion is that $\bar{M}$ must be stable, i.e. all its eigenvalues must have a norm equal to one.

We shall restrict our discussion to the case when the matrix $\bar{M}$ can be diagonalized. This means that any phase-space vector can be expressed as a linear combination of the six complex eigenvectors of $\bar{M}$; the tensor products between these eigenvectors provide a natural basis in the linear space of the 6×6 matrices and allows a simple parametrization of the general solution of Eq. (6.1). In this section, we assume that all the six eigenvalues $\lambda_k = e^{i\mu_k}$ and $\lambda_k^* = e^{-i\mu_k}$ of $\bar{M}$ be distinct from one another and we then prove that the most general symmetric solution $R_{II} = {}^t R_{II}$ of Eq. (6.1) can be parametrized by three independent coefficients. The real numbers μ_k ($k = 1, 2, 3$) are the so-called phase advances associated with the three normal modes of the system and the non-degeneracy of the eigenvalues of $\bar{M}$ corresponds to

$$\begin{aligned} \mu_k &\neq n\pi, \\ \mu_k \pm \mu_r &\neq 2n\pi, \quad \text{for } k \neq r \end{aligned} \tag{6.6}$$

where n is an integer number. These conditions can be summarized by saying that the machine tunes $\nu_k = \mu_k/2\pi$ must be chosen so as to avoid *half-integer resonances* (corresponding to $\lambda_k = \lambda_k^*$) and *sum or difference coupling resonances* (corresponding to $\lambda_k = \lambda_r^*$ or to $\lambda_k = \lambda_r$, respectively, for $k \neq r$)³.

Although the matrix $\bar{M}(\tau)$ depends periodically on the azimuthal position along the storage ring, its eigenvalues and thus also the phase advances μ_k are independent of τ . Indeed, for any couple of values τ_1 and τ_2 , the corresponding matrices $\bar{M}(\tau_1)$ and $\bar{M}(\tau_2)$ are related by the similarity transformation Eq. (6.2). Let us denote by $|e_k\rangle$ and $|e_k^*\rangle$ the complex eigenvectors of $\bar{M}$ corresponding to the six non-degenerate eigenvalues λ_k and λ_k^* ; the real and imaginary parts of these eigenvectors form a complete set, i.e. a basis, for the six-dimensional, single-particle phase space. Here and in the following we adopt Dirac's notation: the 'ket' vectors $|e_k\rangle$ are column vectors with six complex components and the associated 'bra' vectors $\langle e_k|$ are row vectors consisting of the complex conjugate elements⁴. Therefore we have

$$\bar{M}|e_k\rangle = e^{i\mu_k}|e_k\rangle, \quad \langle e_k|\bar{M} = \langle e_k|e^{-i\mu_k}. \tag{6.7}$$

³It is interesting to remark that conditions (6.6) imply the so-called strong stability of $\bar{M}$, namely the stability of $\bar{M}$ under small symplectic perturbations such as those associated with unavoidable imperfections in the magnetic lattice of a storage ring. This is because, in order to leave the complex unit circle, two eigenvalues of a stable symplectic matrix have first to become equal [24]. The converse is not true, however, since the 'collision' between two eigenvalues of $\bar{M}$ does not necessarily lead to an instability (see, for example, the discussion of difference resonances in Ref. [25]). Therefore our non-degeneracy condition for the eigenvalues of $\bar{M}$, which allows us to characterize the asymptotic beam distribution with three equilibrium emittances, is more stringent than the usual condition of strong stability required for proton machines, where the effect of synchrotron radiation is negligible.

⁴The corresponding scalar product $\langle \xi|\eta\rangle$ is the usual Hermitian product between complex vectors.

From the symplecticity of $\bar{M}$, it follows

$$\begin{aligned}\langle e_k | J | e_r \rangle &= \langle e_k | \bar{M} J \bar{M} | e_r \rangle = e^{i(\mu_r - \mu_k)} \langle e_k | J | e_r \rangle, \\ \langle e_k^* | J | e_r \rangle &= \langle e_k^* | \bar{M} J \bar{M} | e_r \rangle = e^{i(\mu_r + \mu_k)} \langle e_k^* | J | e_r \rangle.\end{aligned}\tag{6.8}$$

Then, using the non-degeneracy conditions (6.6) and adopting a proper normalization for $|e_k\rangle$, we obtain (possibly after swopping between $|e_k\rangle$ and $|e_k^*\rangle$, i.e. after replacing μ_k by $-\mu_k$)

$$\begin{aligned}\langle e_k | J | e_r \rangle &= i\delta_{kr}, \\ \langle e_k^* | J | e_r \rangle &= 0.\end{aligned}\tag{6.9}$$

We see that the complex eigenvectors of $\bar{M}(\tau)$ are orthogonal with respect to the unit symplectic matrix J (and the fact that $\langle e_k | J | e_k \rangle$ is purely imaginary follows from the antisymmetry of J); this property is independent of the azimuthal position along the storage ring, since both the symplecticity and the non-degeneracy of $\bar{M}(\tau)$ are preserved by the transformation (6.2).

To each of the three pairs $|e_k\rangle$, $|e_k^*\rangle$ we can associate the plane of the coordinate and momentum of each normal mode. The mutually orthogonal projectors onto these three planes are given by

$$P_k = i(|e_k^*\rangle\langle e_k^*| - |e_k\rangle\langle e_k|)J.\tag{6.10}$$

Indeed, from conditions (6.9) it follows

$$P_k^2 = P_k, \quad P_k P_r = 0 \quad \text{for } k \neq r.\tag{6.11}$$

Moreover, the completeness of the eigenvectors $|e_k\rangle$ and $|e_k^*\rangle$ implies that the sum of the three projectors P_k be the 6×6 identity matrix.

We are now ready to discuss the general solution of Eq. (6.1). Any real 6×6 matrix R can be expanded in the basis of the tensor products between the eigenvectors of $\bar{M}$ as follows:

$$R = \sum_{k,r=1}^3 \left(c_{kr} |e_k\rangle\langle e_r| + c_{kr}^* |e_k^*\rangle\langle e_r^*| + d_{kr} |e_k\rangle\langle e_r^*| + d_{kr}^* |e_k^*\rangle\langle e_r| \right),\tag{6.12}$$

where, thanks to conditions (6.9), the complex coefficients c_{kr} and d_{kr} are given by

$$c_{kr} = -\langle e_k | J R J | e_r \rangle, \quad d_{kr} = \langle e_k | J R J | e_r^* \rangle.\tag{6.13}$$

For a symmetric matrix $R = {}^t R$, we have

$$c_{kr} = c_{rk}^*, \quad d_{kr} = d_{rk}.\tag{6.14}$$

Therefore the 3×3 coefficient matrix c_{kr} is Hermitian and has 9 real, independent parameters, while the symmetric 3×3 coefficient matrix d_{kr} has 12 independent parameters: the total number of parameters is 21, as it should be for a real, symmetric 6×6 matrix R .

From the symplecticity of $\bar{M}$ and from the definition (6.7) of the eigenvectors $|e_k\rangle$, it follows

$$\begin{aligned}\langle e_k | J(\bar{M} R {}^t \bar{M}) J | e_r \rangle &= \langle e_k | {}^t \bar{M}^{-1} (J R J) \bar{M}^{-1} | e_r \rangle = -c_{kr} e^{i(\mu_k - \mu_r)}, \\ \langle e_k | J(\bar{M} R {}^t \bar{M}) J | e_r^* \rangle &= \langle e_k | {}^t \bar{M}^{-1} (J R J) \bar{M}^{-1} | e_r^* \rangle = d_{kr} e^{i(\mu_k + \mu_r)}.\end{aligned}\tag{6.15}$$

Therefore the matrix R is a solution of Eq. (6.1) provided its ‘components’ c_{kr} and d_{kr} are such that

$$c_{kr} = c_{kr} e^{i(\mu_k - \mu_r)}, \quad d_{kr} = d_{kr} e^{i(\mu_k + \mu_r)}.\tag{6.16}$$

Recalling the non-degeneracy conditions (6.6), this implies

$$c_{kr} = \varepsilon_k \delta_{kr}, \quad d_{kr} = 0.\tag{6.17}$$

Since the diagonal elements ε_k of the Hermitian coefficient matrix c_{kr} are real numbers, from expansion (6.12), it follows that the most general solution R_H of Eq. (6.1) can be written in the form

$$R_H = \sum_{k=1}^3 \varepsilon_k \sigma_k,\tag{6.18}$$

where the three real, symmetric matrices σ_k are defined by

$$\sigma_k = |e_k\rangle\langle e_k| + |e_k^*\rangle\langle e_k^*|.\tag{6.19}$$

In Appendix B, we show that the matrices σ_k , referred to as the Twiss matrices, have non-negative eigenvalues. Since each projector P_k has a trace equal to two,

$$\text{Tr}(P_k) = i(\langle e_k^* | J | e_k^* \rangle - \langle e_k | J | e_k \rangle) = 2,\tag{6.20}$$

it follows from the definition (6.4) of the scalar product in Σ that the σ ’s form an orthonormal basis

$$(\sigma_k, \sigma_r) = -\frac{1}{2} \text{Tr}(\sigma_k J \sigma_r J) = \frac{1}{2} \text{Tr}(P_k) \delta_{kr} = \delta_{kr}.\tag{6.21}$$

Moreover, from conditions (6.9), we have

$$\sigma_k J \sigma_r = i(|e_k\rangle\langle e_k| - |e_k^*\rangle\langle e_k^*|) \delta_{kr} = P_k J \delta_{kr}.\tag{6.22}$$

This property of the Twiss matrices expresses the fact that the corresponding three normal modes are decoupled. Indeed, as shown in Appendix B, the single-turn transfer matrix $\bar{M}$ can be uniquely written in the following exponential form:

$$\bar{M}(\tau) = \exp \left\{ \sum_{k=1}^3 \mu_k \sigma_k(\tau) J \right\}.\tag{6.23}$$

Let us remark that the choice of the normalization (6.9) fixes each eigenvector $|e_k\rangle$ only up to an arbitrary phase factor⁵. This phase factor can be a function of τ and, in particular,

⁵This reflects the non uniqueness of the corresponding canonical transformation from the original phase-space variables to normal-mode variables.

it can be chosen in such a way that $|e_k(\tau)\rangle$ be periodic in τ with the ring revolution period T . Indeed, if $|e_k(\tau_1)\rangle$ is an eigenvector of $\bar{M}(\tau_1)$ corresponding to the eigenvalue λ_k , from the similarity transformation (6.2), it follows that $M(\tau_2|\tau_1)|e_k(\tau_1)\rangle$ is an eigenvector of $\bar{M}(\tau_2)$ corresponding to the same eigenvalue. Therefore, the most general transformation law for the eigenvectors of $\bar{M}$, compatible with the normalization (6.9), reads

$$|e_k(\tau_2)\rangle = e^{i(\phi_k(\tau_2) - \phi_k(\tau_1))} M(\tau_2|\tau_1)|e_k(\tau_1)\rangle. \quad (6.24)$$

By choosing the phases $\phi_k(\tau)$ such that

$$\phi_k(\tau + T) = \phi_k(\tau) - \mu_k + 2n\pi, \quad (6.25)$$

the eigenvectors $|e_k(\tau)\rangle$ become periodic in τ , since

$$|e_k(\tau + T)\rangle = e^{i(\phi_k(\tau+T) - \phi_k(\tau))} M(\tau + T|\tau)|e_k(\tau)\rangle = e^{-i\mu_k} M(\tau)|e_k(\tau)\rangle = |e_k(\tau)\rangle. \quad (6.26)$$

Even restricting the choice of the phases $\phi_k(\tau)$ by condition (6.25), as we shall always assume in the following, the periodic eigenvectors $|e_k(\tau)\rangle$ are not uniquely fixed, since there can be local as well as global variations of the ϕ 's still compatible with (6.25). However, since the 'bra' vectors $\langle e_k(\tau)|$ are obtained by taking the complex conjugate of the corresponding 'ket' vectors $|e_k(\tau)\rangle$, the tensor products $|e_k(\tau)\rangle\langle e_k(\tau)|$ are independent of the arbitrary phases ϕ_k appearing in Eq. (6.24). In particular, from the definition (6.19), the Twiss matrices are independent of the phases ϕ_k and transform as follows:

$$\sigma_k(\tau_2) = M(\tau_2|\tau_1)\sigma_k(\tau_1)M(\tau_2|\tau_1). \quad (6.27)$$

Therefore they are also periodic solutions of the purely Hamiltonian evolution equation (6.5). Recalling Eq. (2.3), we see that the matrix $x^t x$ satisfies the same equation (6.5) and thus that the scalar products

$$J_k = (x^t x, \sigma_k) \quad (6.28)$$

are constants of motion. These three quadratic functions of the single-particle dynamical variables represent a generalization of the well known Courant-Snyder betatron invariants [25].

7 Solution of the approximate algebraic equation for R

In the limit of weak dissipation, the asymptotic beam-envelope matrix $R_\infty(\tau)$ is the unique solution of the approximate algebraic equation (5.8): it is a symmetric matrix, periodic in τ and, in the following, we shall drop the subscript ' ∞ ' and denote it by $R(\tau)$

Using the decomposition (6.12) and splitting the coefficient matrix c_{kr} into the sum of its diagonal and off-diagonal parts, respectively, the asymptotic beam-envelope matrix R can be uniquely written as follows:

$$R = R_H + Z, \quad (7.1)$$

where R_H is a fixed point of the purely Hamiltonian, single-turn mapping and can be parametrized by three periodic coefficients $\varepsilon_k(\tau)$, as in Eq. (6.18), while the residual $Z(\tau)$ is a symmetric matrix, periodic in τ and orthogonal to the σ 's in the sense of the scalar product (6.4)

$$(Z, \sigma_k) = 0 \quad \text{for} \quad k = 1, 2, 3. \quad (7.2)$$

Inserting the decomposition (7.1) into the approximate algebraic equation (5.8) and assuming that the machine tunes are far from linear resonances, we will show that Z is of order δ and we will calculate the coefficients ε_k , which are constant along the ring up to $O(\delta)$ terms.

As we have seen, the Twiss matrices form a basis in the space Σ of the symmetric solutions of Eq. (6.1). Therefore any symmetric, non-null matrix Z orthogonal to the σ 's can not be a fixed point of the purely Hamiltonian mapping, i.e.

$$Z \neq 0 \quad \implies \quad \bar{M} Z {}^t\bar{M} \neq Z. \quad (7.3)$$

On the other hand, from the orthonormality condition (6.21), we obtain

$$\varepsilon_k = (R, \sigma_k). \quad (7.4)$$

In Appendix A, we show that the scalar product (6.4) between two symmetric, non-negative matrices is a non-negative number. Therefore, recalling that both the beam-envelope matrix R and the Twiss matrices σ_k are symmetric and non negative (see Appendix B), we see that also the coefficients ε_k must be non negative: they are the mean values of the single-particle Hamiltonian invariants (6.28).

Inserting the decomposition (7.1) into the approximate algebraic equation (5.8), we obtain

$$Z - \bar{M} Z {}^t\bar{M} = - \sum_{k=1}^3 \varepsilon_k (\bar{A} \sigma_k {}^t\bar{M} + \bar{M} \sigma_k {}^t\bar{A}) + \bar{B} - (\bar{A} Z {}^t\bar{M} + \bar{M} Z {}^t\bar{A}). \quad (7.5)$$

The r.h.s. of this equation is $O(\delta)$, since it depends on the matrices $\bar{A}$ and $\bar{B}$, representing the integrated effect over one turn of radiation damping and quantum noise, respectively. Therefore, in the case when $\bar{M}$ has non degenerate eigenvalues, from Eq. (7.3) it follows that the residual matrix Z is also of order δ and the last term in the r.h.s. of Eq. (7.5) is $O(\delta^2)$.

As a consequence of the symplecticity of the transfer matrix $\bar{M}$ and of the definition of the scalar product (6.4), we have

$$(\bar{M} Z {}^t\bar{M}, \sigma_k) = (Z, \bar{M}^{-1} \sigma_k {}^t\bar{M}^{-1}) = (Z, \sigma_k) = 0, \quad (7.6)$$

while from Eq. (6.22) it follows

$$(\bar{A} \sigma_k {}^t\bar{M}, \sigma_r) = (\bar{M} \sigma_k {}^t\bar{A}, \sigma_r) = 0 \quad \text{for} \quad k \neq r. \quad (7.7)$$

Therefore, taking the scalar product of both sides of Eq. (7.5) with σ_k yields

$$2\varepsilon_k (\bar{A} \sigma_k {}^t\bar{M}, \sigma_k) = (\bar{B}, \sigma_k) + O(\delta^2). \quad (7.8)$$

Expanding ε_k in powers of δ and retaining only the zeroth order term $\bar{\varepsilon}_k$, we obtain

$$\bar{\varepsilon}_k = \frac{1}{2} \frac{(\bar{B}, \sigma_k)}{(\bar{A}\sigma_k, \sigma_k)}. \quad (7.9)$$

Using the definitions (5.9) of $\bar{A}$ and $\bar{B}$ and the transformation properties (6.27) of the matrices σ_k , it is simple to show that

$$\begin{aligned} (\bar{B}, \sigma_k) &= \oint d\tau (B, \sigma_k), \\ (\bar{A}\sigma_k, \sigma_k) &= \oint d\tau (A\sigma_k, \sigma_k). \end{aligned} \quad (7.10)$$

Thus the three coefficients $\bar{\varepsilon}_k$, known as equilibrium emittances, are independent of τ , while the deviations $\varepsilon_k(\tau) - \bar{\varepsilon}_k$ are of order δ .

Recalling the properties of the components A_H and A_D of the dissipation matrix A (see Eqs. (2.8) and (2.9)), we have that

$$(A\sigma_k, \sigma_k) = (A_D\sigma_k, \sigma_k) \quad (7.11)$$

and thus the final formula for the equilibrium beam emittances $\bar{\varepsilon}_k$ reads

$$\bar{\varepsilon}_k = \frac{1}{2} \frac{\oint d\tau (B, \sigma_k)}{\oint d\tau (A_D\sigma_k, \sigma_k)}, \quad (7.12)$$

where the integral appearing in the denominator is the damping constant associated with each normal mode. Since we know that the emittances can not be negative (because they coincide to order δ with the average values of the non-negative functions $\varepsilon_k(\tau)$) and since the numerator in Eq. (7.12) is also non negative (because it is the integral of the scalar product between two symmetric, non-negative matrices), we conclude that the three damping constants must be positive. The opposite would indicate that, as a consequence of synchrotron radiation, at least one normal mode becomes ‘anti-damped’ and there is no equilibrium value for the corresponding emittance; as shown in Appendix C, this would violate our ‘memory loss’ assumption Eq. (3.11).

We can now compute the residual matrix Z , orthogonal to the σ ’s. Denoting by W the r.h.s. of Eq. (7.5) to first order in δ , i.e.

$$W = - \sum_{k=1}^3 \bar{\varepsilon}_k (\bar{A}\sigma_k, \sigma_k) + \bar{B}, \quad (7.13)$$

and neglecting $O(\delta^2)$ terms, Eq. (7.5) becomes

$$Z - \bar{M}Z\bar{M} = W. \quad (7.14)$$

Thanks to the completeness of the eigenvectors $|e_k\rangle$ and $|e_k^*\rangle$ of $\bar{M}$, we can expand the symmetric matrix Z as in Eq. (6.12), with the complex coefficients $c_{k\tau}$ and $d_{k\tau}$ given by

$$c_{k\tau} = -\langle e_k | JZJ | e_\tau \rangle, \quad d_{k\tau} = \langle e_k | JZJ | e_\tau^* \rangle. \quad (7.15)$$

The orthogonality between Z and the three σ_k implies that the diagonal coefficients c_{kk} must vanish. Therefore the Hermitian, 3×3 coefficient matrix c_{kr} has 6 independent parameters, while the symmetric 3×3 coefficient matrix d_{kr} has 12 independent parameters: the residual Z can thus be described by 18 independent parameters. This is true also for the other two matrices, $\bar{M}Z^t\bar{M}$ and W , appearing in Eq. (7.14), because they are both symmetric and orthogonal to the σ 's.

Recalling Eqs. (6.15), we see that the components of $\bar{M}Z^t\bar{M}$ are $c_{kr}e^{i(\mu_k-\mu_r)}$ and $d_{kr}e^{i(\mu_k+\mu_r)}$, respectively. Therefore, from Eq. (7.14), we obtain

$$\begin{aligned} c_{kr} &= -\frac{\langle e_k | JWJ | e_r \rangle}{1 - e^{i(\mu_k - \mu_r)}}, \quad \text{for } k \neq r \\ d_{kr} &= \frac{\langle e_k | JWJ | e_r^* \rangle}{1 - e^{i(\mu_k + \mu_r)}}. \end{aligned} \tag{7.16}$$

Since W is of order δ , also the residual matrix Z is $O(\delta)$ and can be neglected provided the machine tunes are far from linear resonances.

Starting from the original algebraic equation (4.10) and using the recurrence relation (5.6) for the integrated effect of radiation damping, one could now compute the deviations $\varepsilon_k(\tau) - \bar{\varepsilon}_k$ from the equilibrium beam emittances along the storage ring. Similarly to the residual Z , however, these deviations are of order δ and can be neglected provided the machine tunes are far from linear resonances.

8 The case of linear resonance

Until now, we have assumed that the machine tunes $\nu_k = \mu_k/2\pi$ are far from linear resonances, i.e. that the eigenvalues $e^{\pm i\mu_k}$ of the single-turn transfer matrix $\bar{M}$ are non degenerate (see Eq. (6.6)) and such that the denominators in Eqs. (7.16) have an absolute value much larger than δ . In order to discuss the divergent behaviour of the coefficients c_{kr} and d_{kr} corresponding to a degenerate transfer matrix, we start by considering the following case of a difference resonance:

$$\mu_1 - \mu_2 = 2n\pi. \tag{8.1}$$

Then, the three matrices σ_1 , σ_2 and σ_3 are no longer the only independent, symmetric solutions of Eq. (6.1), since the degenerate eigenvectors $|e_1\rangle$ and $|e_2\rangle$ (together with their complex conjugates) can be combined to construct two further independent solutions, which we denote by σ_4 and σ_5

$$\begin{aligned} \sigma_4 &= \frac{1}{\sqrt{2}} (|e_1\rangle\langle e_2| + |e_2^*\rangle\langle e_1^*| + |e_2\rangle\langle e_1| + |e_1^*\rangle\langle e_2^*|), \\ \sigma_5 &= \frac{i}{\sqrt{2}} (|e_1\rangle\langle e_2| + |e_2^*\rangle\langle e_1^*| - |e_2\rangle\langle e_1| - |e_1^*\rangle\langle e_2^*|). \end{aligned} \tag{8.2}$$

Indeed, under the resonance condition (8.1), Eqs. (6.7) imply that also these two symmetric matrices satisfy

$$\bar{M}\sigma_4^t\bar{M} = \sigma_4, \quad \bar{M}\sigma_5^t\bar{M} = \sigma_5, \tag{8.3}$$

while, from Eqs. (6.9) and (6.19), they have the following properties:

$$\begin{aligned} (\sigma_4, \sigma_4) &= (\sigma_5, \sigma_5) = 1, & (\sigma_4, \sigma_5) &= 0, \\ (\sigma_k, \sigma_4) &= (\sigma_k, \sigma_5) = 0 & \text{for } k &= 1, 2, 3. \end{aligned} \tag{8.4}$$

Moreover, provided the arbitrary phases $\phi_k(\tau)$, appearing in Eqs. (6.24) and (6.25), are chosen such that $\phi_1 = \phi_2$, the transformation properties of σ_4 and σ_5 are the same as those of the three matrices σ_k , given by Eq. (6.27). This implies the existence of two additional single-particle invariants, J_4 and J_5 , defined by a generalization of Eq. (6.28). However, Eqs. (6.22) can not be extended to the five matrices σ_α ($\alpha = 1, \dots, 5$); we now have, instead, (for α or β equal to 4 or 5)

$$\sigma_\alpha J \sigma_\beta \neq \sigma_\beta J \sigma_\alpha \neq 0 \quad \text{for } \alpha \neq \beta. \tag{8.5}$$

The origin of the resonant denominators in Eqs. (7.16) is now clear: in case of resonance, the residual matrix Z is no longer of order δ , since the behaviour of its components proportional to σ_4 and σ_5 is similar to that of the other three matrices σ_k . Then the original decomposition (7.1) must be replaced by

$$R = \sum_{\alpha=1}^5 \varepsilon_\alpha \sigma_\alpha + Z', \tag{8.6}$$

where the new residual matrix Z' , orthogonal to the five matrices σ_α , can be parametrized by 16 independent coefficients of order δ . We should now recall that, close to resonance, the Hamiltonian contribution of synchrotron radiation to the machine tunes can not be neglected. Therefore, we shall include this contribution in the single-turn transfer matrix $\bar{M}$ and in the associated matrices σ_α , thus assuming that the resonance condition (8.1) refers to the tunes *in presence of radiation*. On the other hand, the matrix $\bar{A}$ appearing in the approximate algebraic equation (5.8) should be replaced by a matrix $\bar{A}_D$, defined only in terms of the damping matrix A_D of Eq. (2.9).

Inserting the new decomposition given by Eq. (8.6) into Eq. (5.8), taking the scalar product with the σ 's and neglecting $O(\delta)$ terms, we obtain the following system of coupled, linear equations for the five coefficients $\bar{\varepsilon}_\alpha$, which represent a generalization of the three equilibrium emittances in case of resonance:

$$2 \sum_{\alpha=1}^5 \bar{\varepsilon}_\alpha (\bar{A}_D \sigma_\alpha {}^t \bar{M}, \sigma_\beta) = (\bar{B}, \sigma_\beta). \tag{8.7}$$

It is simple to show that, similarly to Eqs. (7.10), we have

$$\begin{aligned} (\bar{B}, \sigma_\beta) &= \oint d\tau (B, \sigma_\beta), \\ \bar{\Gamma}_{\alpha\beta} &\equiv (\bar{A}_D \sigma_\alpha {}^t \bar{M}, \sigma_\beta) = \oint d\tau (A_D \sigma_\alpha, \sigma_\beta). \end{aligned} \tag{8.8}$$

Thanks to the property (2.9) of the damping matrix A_D , the 5×5 matrix $\bar{\Gamma}_{\alpha\beta}$ is symmetric: indeed

$$\begin{aligned} (A_D \sigma_\alpha, \sigma_\beta) &= -\frac{1}{2} \text{Tr}(A_D \sigma_\alpha J \sigma_\beta J) = -\frac{1}{2} \text{Tr}(J \sigma_\beta J \sigma_\alpha A_D) \\ &= -\frac{1}{2} \text{Tr}(\sigma_\beta J \sigma_\alpha J A_D) = -\frac{1}{2} \text{Tr}(A_D \sigma_\beta J \sigma_\alpha J) \\ &= (A_D \sigma_\beta, \sigma_\alpha). \end{aligned} \quad (8.9)$$

In the next section, we shall see that this symmetry allows us to derive the asymptotic beam-envelope matrix from a variational principle, even in case of linear resonance.

Let us remark that the generalized Twiss matrices σ_α form a basis in the space Σ of the symmetric solutions of Eq. (6.1) and, as a consequence of Eqs. (6.21) and (8.4), this basis is orthonormal with respect to the scalar product (6.4). Moreover, since the eigenvalues λ_1 and λ_2 of the transfer matrix are degenerate, any linear combination of the corresponding eigenvectors $|e_1\rangle$ and $|e_2\rangle$ is still an eigenvector belonging to the same eigenvalue; therefore we can perform an ‘orthogonal’ transformation of the associated matrices σ_α to diagonalize the symmetric matrix $\bar{\Gamma}_{\alpha\beta}$, which expresses the integrated effect of radiation damping in the space Σ . As shown in Appendix C, our ‘memory loss’ assumption (3.11) implies that the matrix $\bar{\Gamma}_{\alpha\beta}$ has positive eigenvalues and thus it can be inverted. Hence the system of linear equations (8.7) has a unique solution given by

$$\bar{\epsilon}_\alpha = \frac{1}{2} \sum_{\beta=1}^5 (\bar{\Gamma}^{-1})_{\alpha\beta} (\bar{B}, \sigma_\beta) \quad (8.10)$$

and the five generalized emittances $\bar{\epsilon}_\alpha$ are independent of τ .

If, instead of the resonance condition (8.1), we consider the case of a sum resonance

$$\mu_1 + \mu_2 = 2n\pi, \quad (8.11)$$

there are again two further Twiss matrices σ_4 and σ_5 , satisfying Eqs. (8.3) and given by

$$\begin{aligned} \sigma_4 &= \frac{1}{\sqrt{2}} (|e_1\rangle\langle e_2^*| + |e_2\rangle\langle e_1^*| + |e_2^*\rangle\langle e_1| + |e_1^*\rangle\langle e_2|), \\ \sigma_5 &= \frac{i}{\sqrt{2}} (|e_1\rangle\langle e_2^*| + |e_2\rangle\langle e_1^*| - |e_2^*\rangle\langle e_1| - |e_1^*\rangle\langle e_2|). \end{aligned} \quad (8.12)$$

Similarly, in the case of a half-integer resonance, i.e. for

$$\mu_1 = n\pi, \quad (8.13)$$

the new Twiss matrices are

$$\begin{aligned} \sigma_4 &= (|e_1\rangle\langle e_1^*| + |e_1^*\rangle\langle e_1|), \\ \sigma_5 &= i(|e_1\rangle\langle e_1^*| - |e_1^*\rangle\langle e_1|). \end{aligned} \quad (8.14)$$

The asymptotic beam-envelope matrix R can be again parametrized as in Eq. (8.6) and the five generalized emittances $\bar{\epsilon}_\alpha$ are solutions of the system of linear equations (8.7).

The generalized Twiss matrices σ_α still form an orthogonal basis in the space Σ , but contrary to the case of a difference resonance (see Eqs. (8.4)), the new matrices σ_4 and σ_5 are now normalized to -1

$$(\sigma_4, \sigma_4) = (\sigma_5, \sigma_5) = -1. \quad (8.15)$$

Therefore, in the general case of linear resonance, the symmetric matrix $g_{\alpha\beta}$ associated with the scalar product (6.4),

$$g_{\alpha\beta} = (\sigma_\alpha, \sigma_\beta), \quad (8.16)$$

is *not* positive definite and it is impossible to diagonalize simultaneously both $g_{\alpha\beta}$ and the symmetric matrix $\bar{\Gamma}_{\alpha\beta}$ of the damping coefficients. However, the determinant of g is different from zero and thus the inverse matrix g^{-1} exists: this is a consequence of our assumption that the transfer matrix $\bar{M}$ can be diagonalized⁶.

As shown in Appendix C, our ‘memory loss’ assumption (3.11) implies that the complex eigenvalues of the product $g^{-1}\bar{\Gamma}$ have a positive real part; hence

$$\det(g^{-1}\bar{\Gamma}) = \det(g^{-1})\det(\bar{\Gamma}) \neq 0 \quad (8.17)$$

and, since $\det(g) \neq 0$, also $\bar{\Gamma}$ has a non-zero determinant and can be inverted. Therefore, the system of linear equations (8.7) has again a unique solution given by Eq. (8.10) and, provided the phases ϕ_1 and ϕ_2 are suitably chosen⁷, the five generalized emittances $\bar{\epsilon}_\alpha$ are independent of τ .

9 Variational principle for the asymptotic beam-envelope matrix

In this section, we derive the asymptotic beam-envelope matrix from a variational principle. To this end, we shall find a functional of the beam-envelope matrix R which, under suitable constraints, decreases monotonically at each turn and reaches its minimum when R relaxes to the asymptotic matrix R_∞ . In the case of half-integer or sum resonances, R_∞ will only be a stationary point of the functional (i.e. the latter will not necessarily reach a minimum).

The single-turn evolution of the beam-envelope matrix R is governed by the mapping (4.1) and, in the limit of weak dissipation, this can be written

$$R' = \bar{U}R\bar{U} + \bar{B}, \quad \bar{U} = \bar{M} - \bar{A}_D, \quad (9.1)$$

⁶As a counter-example, the (degenerate and non-diagonalizable) transfer matrix of a drift space of length l and the associated, unique σ -matrix are

$$\bar{M} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In this case we have $\det(g) = (\sigma, \sigma) = 0$.

⁷We must choose $\phi_1 + \phi_2 = 0$ for the sum resonance (8.11) and $\phi_1 = 0$ for the half-integer resonance (8.13). This is not always compatible with Eq. (6.25) and can lead to double-valued eigenvectors $|e_k(\tau)\rangle$. However the corresponding Twiss matrices are single-valued functions of τ , since they oscillate at twice the frequency of the eigenvectors. An equivalent prescription consists in using the definitions (8.12) or (8.14) only to obtain a starting value of σ_4 and σ_5 at a given azimuth τ_1 : the value at any other azimuth τ_2 is then defined by a generalization of Eq. (6.27).

where R stands for R_n and R' for R_{n+1} . The effect of the Hamiltonian component A_H of the dissipation matrix has been included in the symplectic matrix $\bar{M}$. We can now split both R and R' as follows:

$$\begin{aligned} R &= R_H + Z, \quad R_H = \sum_{\alpha} \varepsilon_{\alpha} \sigma_{\alpha}, \quad (Z, \sigma_{\alpha}) = 0, \\ R' &= R'_H + Z', \quad R'_H = \sum_{\alpha} \varepsilon'_{\alpha} \sigma_{\alpha}, \quad (Z', \sigma_{\alpha}) = 0, \end{aligned} \quad (9.2)$$

The number of coefficients ε_{α} (and ε'_{α}), required to parametrize the Hamiltonian component R_H (and R'_H), need not be specified; these are usually the three beam emittances, but in case of linear resonance five (or more) parameters are required.

Contrary to R_H , the dynamics of the residual Z is dominated by the symplectic matrix $\bar{M}$, i.e. by betatron and synchrotron oscillations. Therefore, provided Z is of order δ at the beginning, it remains $O(\delta)$ forever and we only have to consider the evolution of the Hamiltonian component R_H , i.e. the slow relaxation of the generalized emittances ε_{α} towards their equilibrium values. The fact that Z remains of order δ is a consequence of the averaging effect produced by the phase factors $e^{i(\mu_k \pm \mu_r)}$, appearing in the single-turn evolution of its ‘components’ c_{kr} and d_{kr} [see Eq. (6.15) and the discussion following Eq. (7.15)]. Since the components of Z keep oscillating at each turn, while the corresponding driving terms associated with the noise matrix $\bar{B}$ have the same periodicity of the storage ring, there can be no resonant build up of these oscillations.

From the definition (8.16) of the metric $g_{\alpha\beta}$ and from Eqs. (9.2), it follows

$$\varepsilon_{\alpha} = g_{\alpha\beta}^{-1}(R, \sigma_{\beta}), \quad \varepsilon'_{\alpha} = g_{\alpha\beta}^{-1}(R', \sigma_{\beta}) \quad (9.3)$$

and thus, using Eq. (9.1),

$$\varepsilon'_{\alpha} = g_{\alpha\beta}^{-1}[(\bar{U} \varepsilon_{\gamma} \sigma_{\gamma} {}^t \bar{U}, \sigma_{\beta}) + (\bar{B}, \sigma_{\beta}) + (\bar{U} Z {}^t \bar{U}, \sigma_{\beta})], \quad (9.4)$$

where summation over repeated indices is implied. Having assumed that Z is $O(\delta)$, we have

$$(\bar{U} Z {}^t \bar{U}, \sigma_{\beta}) = (\bar{M} Z {}^t \bar{M}, \sigma_{\beta}) + O(\delta^2) = O(\delta^2) \quad (9.5)$$

and, as a consequence of Eqs. (8.8) and (8.16),

$$(\bar{U} \sigma_{\alpha} {}^t \bar{U}, \sigma_{\beta}) = g_{\alpha\beta} - 2\bar{\Gamma}_{\alpha\beta}. \quad (9.6)$$

Therefore, neglecting $O(\delta^2)$ terms, we obtain the following single-turn mapping for the generalized emittances ε_{α}

$$\varepsilon'_{\alpha} = g_{\alpha\beta}^{-1}[\varepsilon_{\gamma}(g_{\gamma\beta} - 2\bar{\Gamma}_{\gamma\beta}) + (\bar{B}, \sigma_{\beta})]. \quad (9.7)$$

Denoting by $\Delta\varepsilon_{\alpha} \equiv \varepsilon'_{\alpha} - \varepsilon_{\alpha}$ the single-turn variation of the emittances, this can finally be written

$$\Delta\varepsilon_{\alpha} = -g_{\alpha\beta}^{-1}[2\bar{\Gamma}_{\beta\gamma}\varepsilon_{\gamma} - (\bar{B}, \sigma_{\beta})]. \quad (9.8)$$

The term proportional to the ε 's, in the r.h.s. of this equation, arises from radiation damping, whereas the second term represents the emittance growth induced by quantum excitation.

Recalling that $\bar{\Gamma}$ is a symmetric matrix, the previous equation can be associated with the following 'potential':

$$F(\varepsilon) = \bar{\Gamma}_{\alpha\beta} \varepsilon_\alpha \varepsilon_\beta - (\bar{B}, \sigma_\alpha) \varepsilon_\alpha. \quad (9.9)$$

Indeed we have

$$\Delta \varepsilon_\alpha = -g_{\alpha\beta}^{-1} \frac{\partial F(\varepsilon)}{\partial \varepsilon_\beta} \quad (9.10)$$

and the single-turn variation of the emittances is thus related to the derivatives of $F(\varepsilon)$. When the beam relaxes to its asymptotic state, i.e. when the generalized emittances ε_α reach their equilibrium values $\bar{\varepsilon}_\alpha$, we have

$$\Delta \bar{\varepsilon}_\alpha = 0 \quad \Longleftrightarrow \quad \left[\frac{\partial F(\varepsilon)}{\partial \varepsilon_\alpha} \right]_{\bar{\varepsilon}} = 0 \quad (9.11)$$

and $F(\varepsilon)$ is stationary at $\varepsilon_\alpha = \bar{\varepsilon}_\alpha$. Since $\Delta \varepsilon_\alpha$ is of order δ , we can approximate the variation of the potential over one turn by

$$\Delta F(\varepsilon) \simeq \frac{\partial F(\varepsilon)}{\partial \varepsilon_\alpha} \Delta \varepsilon_\alpha = -g_{\alpha\beta} \Delta \varepsilon_\alpha \Delta \varepsilon_\beta, \quad (9.12)$$

where Eq. (9.10) has been used. In the non-degenerate case, or in the case of a difference resonance, we know that the metric g is positive definite and thus the single-turn variation $\Delta F(\varepsilon)$ is always negative: then the potential $F(\varepsilon)$ is *minimum* at $\varepsilon_\alpha = \bar{\varepsilon}_\alpha$ ⁸.

Inserting Eq. (9.2) into the definition (9.9) of the potential, recalling that Z is assumed to be $O(\delta)$ and using Eq. (8.8) for $\bar{\Gamma}_{\alpha\beta}$, we can express F directly in terms of the beam-envelope matrix R

$$F(R) = (\bar{A}_D R^t \bar{M}, R) - (\bar{B}, R). \quad (9.13)$$

Then, up to $O(\delta)$ terms, the asymptotic beam-envelope matrix can be characterized by requiring that the functional (9.13) be stationary, under the constraint that R be a symmetric solution of Eq. (6.1).

Let us remark that the definition (9.9) of the potential $F(\varepsilon)$ is somewhat arbitrary. Indeed, after multiplying Eq. (9.8) by any function of ε_α , one could introduce different potentials that are all stationary at $\varepsilon_\alpha = \bar{\varepsilon}_\alpha$. However, the possibility of expressing the potential (9.9) directly in terms of the beam envelope matrix R , even at resonance, makes its choice quite natural.

As we have seen in the previous sections, the asymptotic beam-envelope matrix is characterized by periodic emittances $\varepsilon_\alpha(\tau)$ which coincide, up to order δ terms, with their average values $\bar{\varepsilon}_\alpha$ along the ring. Therefore, when looking for a stationary point of the functional $F(R)$, we can restrict the variation of R to linear combinations of the generalized Twiss matrices σ_α , with *constant* coefficients. We recall that the generalized Twiss matrices are symmetric, periodic solutions of Eq. (6.5) and that their evolution must include the

⁸Since at equilibrium we have $(\bar{B}, \sigma_\alpha) \bar{\varepsilon}_\alpha = 2 \bar{\Gamma}_{\alpha\beta} \bar{\varepsilon}_\alpha \bar{\varepsilon}_\beta$ and since $\bar{\Gamma}$ has positive eigenvalues, the corresponding value of the potential $F(\bar{\varepsilon}) = -\bar{\Gamma}_{\alpha\beta} \bar{\varepsilon}_\alpha \bar{\varepsilon}_\beta$ is negative.

(small) effect of the Hamiltonian component A_H of the dissipation matrix. This means that the constraint based on Eq. (6.1) can be replaced by the requirement that R be a symmetric, periodic solution of the following differential equation:

$$\frac{dR}{d\tau} = (L - A_H) R + R(L - A_H). \quad (9.14)$$

Correspondingly, using Eqs. (8.8), the functional (9.13) becomes independent of τ and takes the form

$$F(R) = \oint d\tau [(A_D R, R) - (B, R)]. \quad (9.15)$$

Then, up to $O(\delta)$ terms, the following variational principle holds:

The asymptotic beam-envelope matrix R_∞ is a stationary point of the functional (9.15), under the constraint that R be a symmetric, periodic solution of the purely Hamiltonian evolution equation (9.14).

This variational principle defines the asymptotic beam-envelope matrix R_∞ in terms of known quantities, namely L , $A = A_H + A_D$ and B : for a non-degenerate system, or in the case of a difference resonance, it becomes a principle of minimum F .

10 Example of an idealized storage ring without coupling

In this section we apply our formalism to an idealized storage ring, with no horizontal-vertical coupling and with vanishing dispersion in the RF-cavities. To simplify the discussion, we also neglect the effects of synchrotron radiation on the reference orbit [26]. This allows us to provide an explicit expression for the Twiss matrices σ_k and for the matrices A and B associated with radiation damping and quantum excitation. The general formula (7.12) for the equilibrium emittances $\bar{\epsilon}_k$ then reduces to well known expressions in terms of radiation integrals.

10.1 Single-particle Hamiltonian dynamics

Let (x, y, s) be the curvilinear coordinates of a particle with respect to a reference equilibrium orbit, lying in the horizontal plane $y = 0$ and characterized by a curvature $G(s) = 1/\rho(s)$. A ‘synchronous particle’ moves with constant velocity v_o , corresponding to the nominal energy of the machine $E_o = \gamma_o m c^2$, along the reference orbit and is associated with a ‘synchronous frame’. In this frame, we define the longitudinal displacement z and the conjugate momentum p_z

$$z = s - v_o t, \quad p_z = \frac{E - E_o}{v_o}, \quad (10.1)$$

where E is the particle energy and p_z denotes the deviation from the nominal energy E_o , divided by v_o . Let us stress that p_z does *not* coincide with the longitudinal mechanical momentum of the particle. We shall use $\mathbf{q} = (x, y, z)$ and $\mathbf{p} = (p_x, p_y, p_z)$ as dynamical variables and $\tau = s/c$ as the new independent variable, instead of time t . Moreover, we choose the Coulomb gauge and make the following assumptions:

- i) The magnetic lattice of the storage ring does not contain skew elements, giving rise to x-y coupling.
- ii) The transverse components A_x , A_y , E_x and E_y of the vector potential and of the electric field vanish.
- iii) The end fields associated with the magnet edges and with the edges of the RF-cavities can be neglected.
- iv) The RF-cavities are placed along the straight sections of the machine, where the curvature $G(s)$ vanishes.
- v) The energy of the synchronous particle remains constant.
- vi) The velocity v_o of the synchronous frame is close to c .

Under these conditions, the canonical momenta p_x , p_y and p_z become *gauge invariant* quantities and the single-particle Hamiltonian to second order in the dynamical variables (corresponding to linearized motion) can be cast in the form [27]

$$\begin{aligned} \mathcal{H}(x, y, z, p_x, p_y, p_z; s/c) = & \frac{1}{2} \left\{ \left[\frac{p_x^2}{m_\perp} + k_x(s)x^2 \right] + \left[\frac{p_y^2}{m_\perp} + k_y(s)y^2 \right] + \left[\frac{p_z^2}{m_\parallel} + k_z(s)z^2 \right] \right\} \\ & - G(s)cxp_z. \end{aligned} \quad (10.2)$$

The Hamiltonian (10.2) describes a three-dimensional, parametric, anisotropic oscillator, whose degrees of freedom are characterized by different masses and restoring forces. The radial and longitudinal degrees of freedom are coupled through the last term $-Gcxp_z$, which depends on the curvature G . The periodic functions $k_x(s) = m_\perp c^2 [G(s)^2 - K(s)]$ and $k_y(s) = m_\perp c^2 K(s)$ represent the transverse focusing associated with magnetic dipoles and quadrupoles (the function $K(s)$ is known as the quadrupole strength [25]), while $k_z(s)$ expresses the periodic restoring effect towards the synchronous particle provided by the RF-cavities. For a single cluster of cavities, located at $s = 0$, the latter can be written $k_z(s) = (e\dot{V}_o/c)\delta(s \bmod L)$, where $\dot{V}_o$ denotes the slope of the RF-voltage at the synchronous phase and $L = cT$ is the length of the reference orbit. The relativistic (transverse) mass $m_\perp$ and the so-called longitudinal mass $m_\parallel$, which measures the inertia of the particle in the direction of motion, are given by

$$m_\perp = \gamma_o m, \quad m_\parallel = \gamma_o^3 m. \quad (10.3)$$

The radial and longitudinal degrees of freedom can be decoupled by going over to normal mode variables. The corresponding action variables J_k can be identified with the invariants of Eq. (6.28) and allow us to compute the Twiss matrices σ_k . The Hamilton equations for the radial variables x and p_x can be combined to obtain

$$m_\perp \ddot{x} + k_x(s)x = G(s)cp_z. \quad (10.4)$$

We shall consider solutions of the form

$$x = x_\beta + D(s) \frac{p_z}{m_\perp c}, \quad (10.5)$$

where x_β is a solution of the homogeneous equation associated with (10.4) and the periodic function $D(s)$, known as the horizontal dispersion, satisfies the equation

$$D'' + [G(s)^2 - K(s)]D = G(s) \quad (10.6)$$

and the conditions

$$D(s)k_z(s) = D'(s)k_z(s) = 0. \quad (10.7)$$

These conditions are equivalent to assuming that both the dispersion $D(s)$ and its derivative $D'(s)$ vanish in correspondence with the RF-cavities.

The closed trajectory defined by $x_\beta = y = 0$ is the equilibrium orbit corresponding to a given, fixed p_z . When $p_z = 0$, i.e. when the particle energy E is equal to the nominal energy E_o , the equilibrium orbit coincides with the reference orbit. From the Hamiltonian (10.2), it follows that on the equilibrium orbit

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m_\parallel} - G(s)cx = \frac{p_z}{\gamma_o^3 m} [1 - \gamma_o^2 G(s)D(s)], \quad (10.8)$$

i.e.

$$p_z = \frac{\gamma_o^3 m}{1 - \gamma_o^2 G(s)D(s)} \dot{z} \equiv m_z(s) \dot{z}. \quad (10.9)$$

Contrary to the longitudinal mass $m_\parallel$, the mass $m_z(s)$ defined by Eq. (10.9) can become negative when γ_o is high enough: this is a consequence of the coupling term $-Gcxp_z$ appearing in the Hamiltonian (10.2), which in turn arises from the energy-dependence of the equilibrium orbit. When $m_z(s)$ is negative, an increase of particle energy, corresponding to a positive p_z , is accompanied by an azimuthal slowing down, which means a negative $\dot{z}$. The reason is that the particle velocity can not exceed c , while an energy increase leads to a lengthening of the equilibrium orbit.

To simplify the discussion, in the following we will replace the s -dependent mass $m_z(s)$ by the constant ‘synchrotron mass’ m_s , defined as⁹

$$m_s = \frac{\gamma_o^3 m}{1 - \gamma_o^2 / \gamma_t^2}. \quad (10.10)$$

This amounts to replacing the product $G(s)D(s)$ by its average value $\alpha_c = 1/\gamma_t^2$ over the ring circumference, known as momentum compaction. The Lorentz factor γ_t corresponds to the so-called transition energy and m_s is negative for $\gamma_o > \gamma_t$: in this case, also V_o must be negative, in order to insure the longitudinal stability of the beam.

The Hamiltonian $\mathcal{H}$ can be reduced to the sum of three independent terms by going over to the variables of the normal modes $(x_\beta, p_{x\beta})$ and (z_s, p_{zs}) , describing radial betatron

⁹This approximation does not affect the final results.

oscillations and synchrotron oscillations, respectively. They are defined by the following canonical transformation:

$$\begin{aligned} x_\beta &= x - D(s) \frac{p_z}{m_\perp c}, & p_{x\beta} &= p_x - D'(s) p_z, \\ z_s &= z - D(s) \frac{p_x}{m_\perp c} + D'(s) x, & p_{zs} &= p_z. \end{aligned} \quad (10.11)$$

Then the new Hamiltonian $\tilde{\mathcal{H}}$ for the normal modes reads

$$\tilde{\mathcal{H}}(x_\beta, y, z_s, p_{x\beta}, p_y, p_{zs}; s/c) = \frac{1}{2} \left\{ \left[\frac{p_{x\beta}^2}{m_\perp} + k_x(s) x_\beta^2 \right] + \left[\frac{p_y^2}{m_\perp} + k_y(s) y^2 \right] + \left[\frac{p_{zs}^2}{m_s} + k_z(s) z_s^2 \right] \right\}. \quad (10.12)$$

The action variables $\mathbf{J}$ can now be written in terms of the normal mode variables $\mathbf{Q} = (x_\beta, y, z_s)$ and $\mathbf{P} = (p_{x\beta}, p_y, p_{zs})$ as follows [27]:

$$J_k = \frac{1}{2} \left\{ \gamma_k |m_k| c Q_k^2 + 2\alpha_k Q_k P_k + \frac{\beta_k}{|m_k| c} P_k^2 \right\}, \quad k = 1, 2, 3 \quad (10.13)$$

where the three masses m_k are $m_\perp$, $m_\perp$ and m_s , respectively. The periodic Twiss parameters α_k , β_k and γ_k are defined by

$$\alpha_k = -\frac{1}{2} \frac{m_k}{|m_k|} \beta'_k, \quad \gamma_k = \frac{1 + \alpha_k^2}{\beta_k}, \quad (10.14)$$

$$\frac{1}{2} \beta_x \beta_x'' - \frac{1}{4} \beta_x'^2 + [G(s)^2 - K(s)] \beta_x^2 = 1,$$

$$\frac{1}{2} \beta_y \beta_y'' - \frac{1}{4} \beta_y'^2 + K(s) \beta_y^2 = 1, \quad (10.15)$$

$$\frac{1}{2} \frac{m_s}{|m_s|} \beta_z \beta_z'' - \frac{1}{4} \beta_z'^2 + \frac{e \dot{V}_o}{m_s c^3} \delta(s \bmod L) \beta_z^2 = 1.$$

The action variables J_k can be identified with the single-particle invariants of Eq. (6.28). Since they depend only on the variables $x_k = (Q_k, P_k)$ of the corresponding normal mode, we can introduce 2×2 Twiss matrices $\tilde{\sigma}_k$ and write Eq. (10.13) as a scalar product

$$J_k = (x_k \, {}^t x_k, \tilde{\sigma}_k), \quad (10.16)$$

where

$$\tilde{\sigma}_k = \begin{pmatrix} \bar{\beta}_k & -\alpha_k \\ -\alpha_k & \bar{\gamma}_k \end{pmatrix}, \quad \bar{\beta}_k = \frac{\beta_k}{|m_k| c}, \quad \bar{\gamma}_k = \gamma_k |m_k| c. \quad (10.17)$$

Similarly, expressing the quadratic invariants J_k in terms of the original (physical) variables, we obtain the associated 6×6 Twiss matrices σ_k . We recall, however, that only the radial and the longitudinal degrees of freedom are coupled, through the horizontal dispersion D . Moreover, neglecting the aperture of the radiation cone (of order $1/\gamma$), there

is no quantum excitation of the vertical betatron oscillations. Therefore it is sufficient to consider a reduced, four-dimensional phase space with coordinates¹⁰ (x, p_x, z, p_z) . Using Eqs. (10.11) and (10.17), we can thus write the 4×4 Twiss matrices $\sigma_{x\beta}$ and σ_{zs} , corresponding to the betatron and synchrotron invariants $J_{x\beta}$ and J_{zs} , in terms of measurable quantities, namely the periodic Twiss parameters, the masses $m_\perp$ and m_s , the horizontal dispersion and its derivative:

$$\sigma_{x\beta} = \begin{pmatrix} \bar{\beta}_x & -\alpha_x & -(\alpha_x \bar{D} + \bar{\beta}_x D') & 0 \\ -\alpha_x & \bar{\gamma}_x & (\gamma_x D + \alpha_x D') & 0 \\ -(\alpha_x \bar{D} + \bar{\beta}_x D') & (\gamma_x D + \alpha_x D') & \bar{\mathcal{W}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (10.18)$$

$$\sigma_{zs} = \begin{pmatrix} \bar{\gamma}_z \bar{D}^2 & \bar{\gamma}_z \bar{D} D' & -\alpha_z \bar{D} & \bar{\gamma}_z \bar{D} \\ \bar{\gamma}_z \bar{D} D' & \bar{\gamma}_z D'^2 & -\alpha_z D' & \bar{\gamma}_z D' \\ -\alpha_z \bar{D} & -\alpha_z D' & \bar{\beta}_z & -\alpha_z \\ \bar{\gamma}_z \bar{D} & \bar{\gamma}_z D' & -\alpha_z & \bar{\gamma}_z \end{pmatrix}, \quad (10.19)$$

where

$$\bar{D} = \frac{D}{m_\perp c}, \quad \bar{\mathcal{W}} = \frac{\mathcal{W}}{m_\perp c}, \quad \mathcal{W} = \frac{D^2 + (\alpha_x D + \beta_x D')^2}{\beta_x}. \quad (10.20)$$

10.2 Radiation effects and equilibrium emittances

To obtain an explicit expression for the irreversible flux $\mathcal{I}$ of Eq. (1.3), we start from the radiation reaction force $\mathbf{R}$ experienced by an electron moving with relativistic velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$, orthogonal to $\mathbf{v}$:

$$\mathbf{R} = -\frac{W(\tau)}{c^2} \mathbf{v}, \quad (10.21)$$

where $W(\tau)$ is the instantaneous radiated power, which fluctuates owing to quantum effects. Expression (10.21) amounts to neglecting angular deviations of the emitted photons, of order $1/\gamma$, from the direction of $\mathbf{v}$. The stochastic variable $W(\tau)$ can be assumed to be Gaussian and to satisfy the following conditions [14]:

$$\overline{W(\tau)} = \bar{W}, \quad (10.22)$$

$$\overline{W(\tau)W(\tau') - \bar{W}(\tau)\bar{W}(\tau')} = \frac{55}{24\sqrt{3}} \bar{W} \varepsilon_{ph} \delta(\tau - \tau'),$$

where $\bar{W}$ is the mean instantaneous radiated power and ε_{ph} is the critical energy of the emitted photons. Their expressions, in terms of the Lorentz factor γ and of the magnetic field intensity B , are

$$\bar{W} = \frac{2}{3} (r_e \gamma B)^2 c, \quad (10.23)$$

$$\varepsilon_{ph} = 3\gamma^2 \mu B.$$

¹⁰For the sake of clarity, in this section we modify the ordering of the phase-space variables. This also affects the definition of the unit symplectic matrix J .

Here $r_e = e^2/mc^2$ is the classical electron radius and $\mu = eh/2mc$ the Bohr magneton.

In presence of radiation reaction, the single particle equations of motion can be written

$$\frac{d\mathbf{q}}{d\tau} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}, \quad \frac{d\mathbf{p}}{d\tau} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}} + \mathcal{F}, \quad (10.24)$$

where the generalized force $\mathcal{F}$ represents the non-Hamiltonian rate of change of the momenta $\mathbf{p}$ as a function of the independent variable τ . Therefore it is related to the radiation reaction force $\mathbf{R}$ as follows:

$$\begin{aligned} \mathcal{F}_x &= \frac{dt}{d\tau} \mathbf{R} \cdot \mathbf{e}_x = -\frac{W}{c} x', \\ \mathcal{F}_y &= \frac{dt}{d\tau} \mathbf{R} \cdot \mathbf{e}_y = -\frac{W}{c} y', \\ \mathcal{F}_z &= eV_o \delta(s \bmod L) + \frac{1}{c} \frac{dt}{d\tau} \mathbf{R} \cdot \mathbf{v} = eV_o \delta(s \bmod L) - \frac{W}{c} (1 - z'), \end{aligned} \quad (10.25)$$

with $\mathbf{e}_x$ and $\mathbf{e}_y$ denoting the unit vectors in the x and y directions, respectively. We have included in the component $\mathcal{F}_z$ of the generalized force also the energy gain, eV_o , experienced by the synchronous particle in the RF-cavities at $s = 0$: this is because we neglected this term in the Hamiltonian $\mathcal{H}$.

The generalized force $\mathcal{F}$ is, as W , a Gaussian δ -correlated stochastic variable characterized by a mean value $\bar{\mathcal{F}}$ and by a fluctuating part with spectral density $\hat{\mathcal{F}}$. In terms of these quantities, the dissipative vector $\mathcal{A}_k$ and the diffusive tensor $\mathcal{B}_{kj}$, appearing in the irreversible flux $\mathcal{I}$ of Eq. (1.3) read [19,20]

$$\mathcal{A}_k = \bar{\mathcal{F}}_k - \frac{1}{2} \frac{\partial \hat{\mathcal{F}}_k}{\partial p_j} \hat{\mathcal{F}}_j \quad (10.26)$$

$$\mathcal{B}_{kj} = \hat{\mathcal{F}}_k \hat{\mathcal{F}}_j.$$

The second equation shows that the symmetric tensor $\mathcal{B}_{kj}$ has non-negative eigenvalues.

Using Eq. (10.1), the particle energy E and the corresponding Lorentz factor γ can be expressed in terms of the canonical momentum p_z . Furthermore, in a storage ring consisting only of bending magnets and quadrupoles, the magnetic field intensity is given by $B = (E_o/e)[K^2 y^2 + (G - Kx)^2]^{1/2}$. Then, from the assumptions on W and from the relation between $\mathcal{F}$ and $\mathbf{R}$, in linear approximation we can write

$$\bar{\mathcal{F}} = a \left(p_x, p_y, 2p_z + m_\perp c \frac{G^2 - 2K}{G} x \right), \quad (10.27)$$

$$\hat{\mathcal{F}} = (0, 0, \sqrt{b}),$$

where

$$a = \frac{2}{3} r_e c \gamma_o^3 G(s)^2, \quad b = \frac{55}{24\sqrt{3}} e^2 \hbar \gamma_o^7 |G(s)|^3. \quad (10.28)$$

The coefficients a and b , which depend on $\tau = s/c$ through the curvature $G(s)$, are associated with classical radiation damping and with quantum fluctuations, respectively.

The relative energy loss per turn, δ , experienced by the synchronous particle is

$$\delta = \oint d\tau a(\tau). \quad (10.29)$$

It is globally balanced by the relative energy gain eV_o/E_o in the RF-cavities. However, consistently with our previous assumption of neglecting the effects of synchrotron radiation on the reference orbit, we have also assumed a *local* energy balance in order to simplify the expression of the average force $\bar{\mathcal{F}}$. From Eq. (10.27), we see that the divergence of the average force in momentum space, integrated over one ring revolution, is

$$\oint d\tau \frac{\partial}{\partial \mathbf{p}} \cdot \bar{\mathcal{F}} = 4\delta. \quad (10.30)$$

This is equivalent to Eq. (2.6).

As a consequence of Eqs. (10.26) and (10.27), the 4×4 matrices A and B , associated with dissipation and with quantum excitation in the reduced, four-dimensional phase space with coordinates (x, p_x, z, p_z) , are given by

$$A = a \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m_{\perp} c \frac{G^2 - 2K}{G} & 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}. \quad (10.31)$$

Therefore, using the Twiss matrices of Eqs. (10.18) and (10.19) and the definition (6.4) of the scalar product, we obtain

$$(A\sigma_{x\beta}, \sigma_{x\beta}) = \frac{a}{2} \left[1 - \frac{D}{G}(G^2 - 2K) \right], \quad (A\sigma_{zs}, \sigma_{zs}) = \frac{a}{2} \left[2 + \frac{D}{G}(G^2 - 2K) \right], \quad (10.32)$$

$$(B, \sigma_{x\beta}) = \frac{b}{2} \bar{\mathcal{W}}, \quad (B, \sigma_{zs}) = \frac{b}{2} \bar{\beta}_z.$$

The general formula Eq. (7.12) thus reduces to the following expressions for the (radial) betatron equilibrium emittance, $\bar{\epsilon}_{x\beta}$, and for the synchrotron equilibrium emittance $\bar{\epsilon}_{zs}$:

$$\bar{\epsilon}_{x\beta} = \frac{1}{2m_{\perp}c} \frac{\oint ds b \mathcal{W}}{\oint ds a \left[1 - \frac{D}{G}(G^2 - 2K) \right]}, \quad (10.33)$$

$$\bar{\epsilon}_{zs} = \frac{1}{2|m_s|c} \frac{\oint ds b \beta_z}{\oint ds a \left[2 + \frac{D}{G}(G^2 - 2K) \right]}. \quad (10.34)$$

Since the synchrotron phase advance $\mu_s = \oint ds / \beta_z$ is usually smaller than unity, the function $\beta_z(s)$ can be approximated by a constant [27] and, using the last of Eqs. (10.15), we have

$$\beta_z \simeq \frac{L}{\mu_s} \simeq \sqrt{\frac{m_s c^3 L}{e V_o}}. \quad (10.35)$$

Then the equilibrium emittances become

$$\begin{aligned}\bar{\varepsilon}_{x\beta} &= \frac{55}{32\sqrt{3}} \hbar \gamma_o^3 \frac{I_5}{I_2 - I_4}, \\ \bar{\varepsilon}_{zs} &= \frac{55}{32\sqrt{3}} \hbar \gamma_o^3 \frac{I_3}{2I_2 + I_4} \sqrt{\left(\frac{1}{\gamma_o^2} - \frac{1}{\gamma_t^2}\right) \frac{E_o}{e\dot{V}_o} Lc},\end{aligned}\tag{10.36}$$

where the radiation integrals I_2 to I_5 are defined as follows [10]:

$$\begin{aligned}I_2 &= \oint ds G(s)^2, \\ I_3 &= \oint ds |G(s)|^3, \\ I_4 &= \oint ds D(s)G(s)[G(s)^2 - 2K(s)], \\ I_5 &= \oint ds |G(s)|^3 \mathcal{W}(s).\end{aligned}\tag{10.37}$$

11 Summary and conclusions

In this paper we have given a unified description, based on the linearized Fokker-Planck equation, of the Hamiltonian beam evolution in presence of the irreversible effects associated with synchrotron radiation in electron storage rings. Under the ‘memory loss’ assumption, Eq. (3.11), in Part I we have proved the existence and uniqueness of an asymptotic beam distribution, whose explicit form is given in Eq. (3.16). This distribution is Gaussian in all the phase-space variables and is therefore characterized by the asymptotic beam-envelope matrix $R_\infty(\tau)$. The latter is the fixed point of the single-turn mapping (4.1), which we have proved to be a contraction. It is worth noticing that the mapping (4.1) describes the transient behaviour of the beam-envelope matrix and can also be used to evaluate radiation effects in the final focus of a linear collider [28].

In Part II, we have found the explicit form of R_∞ under the assumption of weak dissipation ($\delta \ll 1$) and in the case when the symplectic transfer matrix M can be diagonalized. The matrix R_∞ has been split as in Eq. (1.4), where the two parts R_H and Z are orthogonal with respect to the scalar product (6.4), defined on the space of the symmetric matrices of even dimension. The Hamiltonian component R_H is a fixed point of the single-turn mapping (4.1) without damping and noise. The residual matrix Z has been shown to be of order δ .

Far from resonances, R_H is a linear combination of three periodic Twiss matrices σ_k with coefficients ε_k . Up to terms of order δ , these coefficients are constant along the ring and coincide with the equilibrium emittances $\bar{\varepsilon}_k$, given by Eq. (7.12). At resonance, two further Twiss matrices and two corresponding coefficients are needed to parametrize R_H : in this case the five generalized, equilibrium emittances $\bar{\varepsilon}_\alpha$ are given by Eq. (8.10).

Finally we have shown that R_H can be derived from a variational principle, namely by requiring that the functional F , defined by Eq. (9.15), be stationary under the constraint (9.14). The result is valid also in case of linear resonance. This variational principle

is reminiscent of the principle of ‘minimum entropy production’ that applies in linear, non-equilibrium thermodynamics.

References

- [1] H. Bruck, *Accélérateurs circulaires de particules* (Presses Univ. France, Paris, 1966).
- [2] R.C. Davidson, *Theory of non-neutral plasmas* (W.A. Benjamin, London, 1974).
- [3] J.S. Bell and J.M. Leinaas, *Nucl. Phys. B* **212**, 131 (1983).
- [4] J.D. Lawson, CERN 76-09 (1976).
- [5] W.G. Unruh, *Phys. Rev. D* **14**, 870 (1976).
- [6] N. Myhrvold, *Ann. Phys.* **160**, 102 (1985).
- [7] A.A. Sokolov and I.M. Ternov, *Synchrotron radiation* (Pergamon Press, New York, 1968).
- [8] M. Sands, Stanford report SLAC-121 (1970).
- [9] C. Bernardini and C. Pellegrini, *Ann. Phys.* **46**, 174 (1968).
- [10] R.H. Helm, M.J. Lee, P.L. Morton and M. Sands, *IEEE Trans. Nucl. Sci.* **NS-20**, 900 (1973).
- [11] A.W. Chao and M.J. Lee, *J. Appl. Phys.* **47**, 4453 (1976).
- [12] A.W. Chao, *J. Appl. Phys.* **50**, 595 (1979) and *Nucl. Instrum. Methods* **180**, 29 (1981).
- [13] G. Guignard, CERN ISR-BOM/79-30 and LEP note 154 (1979).
- [14] J.M. Jowett, *Proc. Joint US-CERN School on Particle Accelerators, Santa Margherita di Pula, Sardinia, 1985*, ed. J.M. Jowett, M. Month and S. Turner (*Lecture Notes in Physics* **247**, Springer-Verlag, Berlin, 1986), p. 343.
- [15] G. Guignard and Y. Marti, CERN ISR-BOM-TH/81-23 (1981).
- [16] K. Hirata, *Proc. Second Advanced ICFA Beam Dynamics Workshop, Lugano, 1988*, ed. J. Hagel and E. Keil (CERN 88-04), p. 62.
- [17] L.D. Landau and E.M. Lifshitz, *Kinetic physics*, (Pergamon Press, London, 1981).
- [18] L.A. Radicati, E. Picasso and F. Ruggiero, same Proc. as Ref. [14], p. 1.
- [19] N. Wax (ed.), *Selected papers on noise and stochastic processes* (Dover, New York, 1954).
- [20] R.L. Stratonovich, *Theory of random noise* (Gordon & Breach, New York, 1963).
- [21] C. Kittel, *Elementary statistical physics* (J. Wiley, New York, 1958).
- [22] K.W. Robinson, *Phys. Rev.* **111**, 373 (1958).

- [23] V.I. Arnold, Ordinary differential equations (MIT Press, Cambridge, Mass., 1973).
- [24] V.I. Arnold, Mathematical methods of classical mechanics (Springer-Verlag, New York, 1979).
- [25] E.D. Courant and H.S. Snyder, Ann. Phys. **3**, 1 (1958).
- [26] M. Bassetti, Proc. 11th Int. Conf. on High Energy Acc. (1980), p. 650.
- [27] F. Ruggiero, CERN 86-06 (1986).
- [28] F. Ruggiero and B. Zotter, CERN LEP-TII/88-33 (1988), presented at the EPAC Conference, Rome, 1988.

Appendices

A Properties of the scalar product

In this appendix we prove that the scalar product (R, S) , defined by Eq. (6.4), between two symmetric, non-negative matrices of even dimension is also non-negative.

Indeed, denoting by λ_α and μ_β the real eigenvalues of R and S and by $|\xi_\alpha\rangle$ and $|\eta_\beta\rangle$ the corresponding eigenvectors, which can be chosen real and orthonormal, we have

$$R = \sum_{\alpha} \lambda_{\alpha} |\xi_{\alpha}\rangle \langle \xi_{\alpha}|, \quad S = \sum_{\beta} \mu_{\beta} |\eta_{\beta}\rangle \langle \eta_{\beta}| \quad (\text{A.1})$$

and therefore

$$\text{Tr}(RJSJ) = \sum_{\alpha, \beta} \lambda_{\alpha} \mu_{\beta} \langle \xi_{\alpha} | J | \eta_{\beta} \rangle \langle \eta_{\beta} | J | \xi_{\alpha} \rangle = - \sum_{\alpha, \beta} \lambda_{\alpha} \mu_{\beta} \left(\langle \xi_{\alpha} | J | \eta_{\beta} \rangle \right)^2. \quad (\text{A.2})$$

If R and S are non negative, all their eigenvalues are non-negative numbers, i.e.

$$\lambda_{\alpha}, \mu_{\beta} \geq 0, \quad (\text{A.3})$$

and also the products $\lambda_{\alpha} \mu_{\beta}$ are non negative. This implies

$$(R, S) = -\frac{1}{2} \text{Tr}(RJSJ) = \frac{1}{2} \sum_{\alpha, \beta} \lambda_{\alpha} \mu_{\beta} \left(\langle \xi_{\alpha} | J | \eta_{\beta} \rangle \right)^2 \geq 0. \quad (\text{A.4})$$

Let us remark that the same result still holds if both R and S are non positive (i.e. if all the eigenvalues λ_{α} and μ_{β} are negative or zero), while the scalar product (R, S) is negative or zero when the matrices R and S have opposite sign.

B Properties of the Twiss matrices

In this appendix we show that the single-turn transfer matrix $\bar{M}$ can be uniquely written in the exponential form (6.23) and that the three Twiss matrices σ_k , defined by Eq. (6.19), are non negative.

As in Sec. 6.2, we assume that $\bar{M}$ has non-degenerate complex eigenvalues $e^{\pm i\mu_k}$ and that the corresponding eigenvectors $|e_k\rangle$, $|e_k^*\rangle$ form a basis in the six-dimensional complex phase space. Then, from Eqs. (6.7) and (6.9), it follows that the matrix $\bar{M}$ can be uniquely written in terms of the Hermitian operators $|e_k\rangle \langle e_k|$ and of their complex conjugates $|e_k^*\rangle \langle e_k^*|$

$$\bar{M} = i \sum_{k=1}^3 \left(|e_k^*\rangle \langle e_k^*| e^{-i\mu_k} - |e_k\rangle \langle e_k| e^{i\mu_k} \right) J. \quad (\text{B.1})$$

As already remarked, since the ‘bra’ vectors $\langle e_k(\tau)|$ are obtained by taking the complex conjugate of the corresponding ‘ket’ vectors $|e_k(\tau)\rangle$, the three complex matrices $|e_k(\tau)\rangle \langle e_k(\tau)|$ are independent of the arbitrary phases ϕ_k appearing in Eq. (6.24).

Using the definitions (6.10) and (6.19) for the projectors P_k and for the matrices σ_k , respectively, expression (B.1) for $\bar{M}$ becomes

$$\bar{M} = \sum_{k=1}^3 \left(P_k \cos \mu_k + \sigma_k J \sin \mu_k \right). \quad (\text{B.2})$$

This is equivalent to the exponential form (6.23) thanks to the completeness of the eigenvectors $|e_k\rangle$, $|e_k^*\rangle$ and to conditions (6.9). Indeed, from Eq. (6.22) and from the properties of the projectors P_k , it follows

$$\sum_{k=1}^3 P_k = I, \quad P_k^2 = P_k, \quad (\sigma_k J)^2 = -P_k, \quad (\sigma_k J)^3 = -\sigma_k J. \quad (\text{B.3})$$

We now show that the three matrices σ_k are non negative as a consequence of the stability of $\bar{M}$, i.e. of our assumption that all its eigenvalues have a norm equal to one¹¹. We start from the last two of Eqs. (B.3), which can be combined as follows:

$$P_k \sigma_k J = -(\sigma_k J)^3 = \sigma_k J. \quad (\text{B.4})$$

Then, after multiplication by J on the right, we obtain

$$P_k \sigma_k = \sigma_k. \quad (\text{B.5})$$

If $|\xi\rangle$ is an eigenvector of σ_k with eigenvalue λ , i.e.

$$\sigma_k |\xi\rangle = \lambda |\xi\rangle, \quad (\text{B.6})$$

from Eq. (B.5) it follows

$$P_k \sigma_k |\xi\rangle = P_k \lambda |\xi\rangle = \sigma_k |\xi\rangle = \lambda |\xi\rangle. \quad (\text{B.7})$$

Therefore, we obtain

$$P_k |\xi\rangle = |\xi\rangle \quad \text{for} \quad \lambda \neq 0. \quad (\text{B.8})$$

Thus the eigenvectors of σ_k with eigenvalues different from zero belong to the two-dimensional eigenspace $|e_k\rangle$, $|e_k^*\rangle$ of the projector P_k , corresponding to the eigenvalue 1 of P_k . Since σ_k is a symmetric 6×6 matrix, it has six real eigenvalues and the associated eigenvectors can be chosen real and orthonormal. As we have seen, only two of the six eigenvalues can be different from zero: let us denote them by λ and μ , respectively, and let $|\xi\rangle$ and $|\eta\rangle$ be the corresponding real and orthonormal eigenvectors, belonging to the conjugate phase-space plane generated by $|e_k\rangle$ and $|e_k^*\rangle$. The matrix σ_k can thus be written

$$\sigma_k = \lambda |\xi\rangle \langle \xi| + \mu |\eta\rangle \langle \eta|. \quad (\text{B.9})$$

¹¹It is simple to show that the following 2×2 symplectic matrix, associated to a σ with eigenvalues of opposite sign,

$$\exp \left\{ \mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} J \right\} = \begin{pmatrix} \cosh \mu & \sinh \mu \\ \sinh \mu & \cosh \mu \end{pmatrix}$$

corresponds to a hyperbolic rotation and is therefore unstable.

From the antisymmetry of J and the reality of the eigenvectors, it follows that $\langle \xi | J | \xi \rangle = \langle \eta | J | \eta \rangle = 0$ and therefore

$$\sigma_k J \sigma_k J = \lambda \mu \left(|\xi\rangle \langle \xi | J | \eta \rangle \langle \eta | + |\eta\rangle \langle \eta | J | \xi \rangle \langle \xi | \right) J. \quad (\text{B.10})$$

By taking the trace, we obtain

$$\text{Tr}(\sigma_k J \sigma_k J) = 2 \lambda \mu \langle \xi | J | \eta \rangle \langle \eta | J | \xi \rangle = -2 \lambda \mu \left(\langle \xi | J | \eta \rangle \right)^2. \quad (\text{B.11})$$

Recalling Eq. (6.21), we know that this trace is equal to -2 and, since the real factor in brackets is squared and thus non negative, this implies that the product $\lambda \mu$ be positive¹²

$$\lambda \mu > 0. \quad (\text{B.12})$$

On the other hand, the sum $\lambda + \mu$ coincides with the trace of σ_k and, from Eq. (6.19), we see that it must also be positive

$$\lambda + \mu = \text{Tr}(\sigma_k) = \langle e_k | e_k \rangle + \langle e_k^* | e_k^* \rangle = 2 \langle e_k | e_k \rangle > 0. \quad (\text{B.13})$$

Therefore both λ and μ are positive numbers: this proves that the Twiss matrices σ_k , associated with the three normal modes of the beam, are non negative.

C Damping constants in the limit of weak dissipation

The deterministic particle motion under the effect of radiation damping is described by the matrix $\bar{U}$ and, as discussed in Sec. 4, our ‘memory loss’ assumption (3.11) is equivalent to requiring that $\bar{U}$ be a contraction. In the limit of weak dissipation, i.e. to first order in δ , this matrix becomes

$$\bar{U} = \bar{M} - \bar{A}. \quad (\text{C.1})$$

In this appendix, we first consider the case when the symplectic single-turn transfer matrix $\bar{M}$ has non-degenerate eigenvalues $e^{\pm i\mu_k}$ of unit norm and show that the matrix $\bar{U}$ is a contraction if and only if the damping constants of the three normal modes are positive: these damping constants can be expressed in terms of $\bar{A}$ and of the Twiss matrices σ_k . In the case when $\bar{M}$ has degenerate eigenvalues, we show that, as a consequence of our ‘memory loss’ assumption (3.11), the product $g^{-1} \bar{\Gamma}$ between the inverse of the metric g , defined by Eq. (8.16), and the matrix (8.8) of the generalized damping constants has eigenvalues with positive real part. In particular, this implies that $\bar{\Gamma}$ can be inverted.

¹²In general, the product $\lambda \mu$ is different from 1 since the square of the real factor in brackets, in Eq. (B.11), is not unity. In the case of 2×2 matrices, however, any proper orthogonal transformation (i.e. any rotation) is also symplectic (i.e. area preserving). In particular, the orthogonal transformation $O = (|\xi\rangle|\eta\rangle)$ which diagonalizes σ is symplectic (up to a possible exchange between $|\xi\rangle$ and $|\eta\rangle$) and the product $\langle \xi | J | \eta \rangle$ between the eigenvectors of σ is ± 1 : therefore $\lambda \mu = 1$ and the Twiss matrix σ for a system with only one degree of freedom has unit determinant.

Since the dissipation matrix $\bar{A}$ is of order δ , if the symplectic matrix $\bar{M}$ is non degenerate, the eigenvalues λ_k and eigenvectors $|f_k\rangle$ of the matrix $\bar{U}$ are close to the eigenvalues and eigenvectors of $\bar{M}$

$$\begin{aligned}\bar{U}|f_k\rangle &= \lambda_k|f_k\rangle, & \lambda_k &= e^{i\mu_k} + \Delta\lambda_k, \\ |f_k\rangle &= |e_k\rangle + |\Delta e_k\rangle, & \Delta\lambda_k, |\Delta e_k\rangle &\sim O(\delta).\end{aligned}\tag{C.2}$$

Moreover, by a proper choice of normalization, we can always assume that the variation $|\Delta e_k\rangle$ of the k^{th} eigenvector be orthogonal to $|e_k\rangle$ with respect to J

$$\langle e_k|J|\Delta e_k\rangle = 0.\tag{C.3}$$

Using Eq. (C.2), we now obtain

$$\begin{aligned}\bar{U}|f_k\rangle\langle f_k|\bar{U} &= |\lambda_k|^2 |f_k\rangle\langle f_k| = \\ &= \bar{M}|f_k\rangle\langle f_k|^t\bar{M} - (\bar{A}|e_k\rangle\langle e_k|^t\bar{M} + \bar{M}|e_k\rangle\langle e_k|^t\bar{A}) + O(\delta^2).\end{aligned}\tag{C.4}$$

By adding the complex conjugate, we get

$$|\lambda_k|^2(|f_k\rangle\langle f_k| + |f_k^*\rangle\langle f_k^*|) = \bar{M}(|f_k\rangle\langle f_k| + |f_k^*\rangle\langle f_k^*|)^t\bar{M} - (\bar{A}\sigma_k^t\bar{M} + \bar{M}\sigma_k^t\bar{A}) + O(\delta^2).\tag{C.5}$$

We now take the scalar product of this equation by σ_k and use the following identities

$$\begin{aligned}(\bar{A}\sigma_k^t\bar{M}, \sigma_k) &= (\bar{M}\sigma_k^t\bar{A}, \sigma_k), \\ (|f_k\rangle\langle f_k| + |f_k^*\rangle\langle f_k^*|, \sigma_k) &= (\sigma_k, \sigma_k) = 1, \\ (\bar{M}(|f_k\rangle\langle f_k| + |f_k^*\rangle\langle f_k^*|)^t\bar{M}, \sigma_k) &= (\sigma_k, \sigma_k) = 1,\end{aligned}\tag{C.6}$$

[the last two equations follow from (C.3)] to show that the norm of the eigenvalues of $\bar{U}$ can be written

$$|\lambda_k|^2 = 1 - 2(\bar{A}\sigma_k^t\bar{M}, \sigma_k) + O(\delta^2).\tag{C.7}$$

Therefore, to first order in δ , the scalar product $(\bar{A}\sigma_k^t\bar{M}, \sigma_k)$ represents the damping constant of the k^{th} normal mode. It is independent of τ because

$$(\bar{A}\sigma_k^t\bar{M}, \sigma_k) = \oint d\tau (\bar{A}\sigma_k, \sigma_k).\tag{C.8}$$

Let us remark that, as a consequence Eq. (7.11), only the component A_D of the dissipation matrix contributes to the damping constants; the effect of the Hamiltonian component A_H can always be included in the symplectic matrix $\bar{M}$ and leads to small variations of its eigenvalues and eigenvectors.

The matrix $\bar{U}$ is a contraction if and only if all its eigenvalues λ_k have a norm smaller than unity; from Eq. (C.7) we see that, neglecting $O(\delta^2)$ terms,

$$|\lambda_k|^2 < 1 \iff (\bar{A}\sigma_k^t\bar{M}, \sigma_k) > 0.\tag{C.9}$$

Thus $\bar{U}$ is a contraction if and only if all the three damping constants $(\bar{A}\sigma_k^t\bar{M}, \sigma_k)$ are positive.

In the case when $\bar{M}$ has degenerate eigenvalues, we consider the evolution induced by $\bar{U}$ on a matrix R belonging to the linear space Σ , i.e. on a linear combination of the generalized Twiss matrices σ_α

$$R = \sum_{\alpha} \varepsilon_{\alpha} \sigma_{\alpha}, \quad (C.10)$$

$$\bar{U} R \bar{U} = \sum_{\alpha} \varepsilon'_{\alpha} \sigma_{\alpha} + Z,$$

where Z is orthogonal to all the σ_{α} and, as a consequence of Eq. (C.1), is of order δ . Since the matrix g defined in Sec. 8 has a determinant different from zero, we have

$$\varepsilon'_{\alpha} = g_{\alpha\beta}^{-1}(\bar{U} R \bar{U}, \sigma_{\beta}) = g_{\alpha\beta}^{-1}(\bar{U} \varepsilon_{\gamma} \sigma_{\gamma} \bar{U}, \sigma_{\beta}), \quad (C.11)$$

where summation over repeated indices is implied. Using Eqs. (8.8) and (8.16) and replacing $\bar{A}$ by $\bar{A}_D$ in Eq. (C.1), we can write

$$(\bar{U} \sigma_{\alpha} \bar{U}, \sigma_{\beta}) = g_{\alpha\beta} - 2\bar{\Gamma}_{\alpha\beta}. \quad (C.12)$$

Thus, recalling that both g and $\bar{\Gamma}$ are symmetric matrices, we obtain the following mapping for the generalized emittances ε_{α} :

$$\begin{aligned} \varepsilon'_{\alpha} &= g_{\alpha\beta}^{-1} \varepsilon_{\gamma} (g_{\gamma\beta} - 2\bar{\Gamma}_{\gamma\beta}) = \\ &= (\delta_{\alpha\gamma} - 2g_{\alpha\beta}^{-1} \bar{\Gamma}_{\beta\gamma}) \varepsilon_{\gamma}. \end{aligned} \quad (C.13)$$

If $\bar{U}$ is a contraction, so is the mapping (C.13), which represents the projection of $\bar{U} R \bar{U}$ onto Σ . Therefore the matrix $I - 2g^{-1}\bar{\Gamma}$, appearing in (C.13), must have eigenvalues with norm smaller than unity. Since $\bar{\Gamma}$ is $O(\delta)$, this implies that the real part of the eigenvalues of $g^{-1}\bar{\Gamma}$ must be positive. In particular, since g has non-zero determinant, also $\bar{\Gamma}$ has non-zero determinant and can thus be inverted.

Let us remark that, if the metric g is positive definite, as in the case of a difference resonance, it is possible to diagonalize simultaneously both g and $\bar{\Gamma}$, without affecting the sign of their eigenvalues. Then the metric g can be reduced to the identity matrix (corresponding to orthonormality of the σ 's) and $\bar{U}$ is a contraction provided $\bar{\Gamma}$ has positive eigenvalues.