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TYPE II SUPERSTRINGS ON TWISTED GROUP-MANIFOLDS AND THEIR HETEROTIC COUNTERPARTS *

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Abstract

We construct type II superstrings in four space-time dimensions compactified on a twisted Wess-Zumino-Witten model based on the group $SU(2)^3$. It is shown that within this framework, it is possible to obtain models with $N = 6, 5$ and 3 space-time supersymmetries, in addition to the usual models with $N = 8, 4$ and 2 . The map of these models into the corresponding heterotic superstrings with $N = 4, 2$ and 1 space-time supersymmetries is also derived: in complete analogy to the compactifications on six-dimensional manifolds also in this case this map from type II to heterotic superstrings corresponds geometrically to the embedding of the 9-dimensional compact manifold spin-connection into the gauge connection. The superstring compactifications we discuss are equivalent to asymmetric orbifold constructions in six dimensions with no need, however, of introducing chiral bosons.

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1 - Introduction

In the construction of four dimensional superstring models [1-9], it has become evident that the use of different conformal field-theories to describe the dynamics of the internal degrees of freedom may lead sometimes to different formulations of the same string theory. Examples of this phenomenon are the orbifold models [10] and the free fermion constructions [3,5,7], the Gepner models [9], which use rational conformal field theories [11] and the σ -models on Calabi-Yau manifolds [12].

Recently four dimensional heterotic superstrings were constructed [13,14], in which the internal conformal field-theory with central charge $c = 9$ is based on a twisted Wess-Zumino-Witten model on a group-manifold G_T [15]. The value of the central charge and the existence of massless target fermions requires the group G_T , if semisimple, to be $SU(2)^3$. If the group is not semisimple, other solutions are possible. Furthermore massless target fermions do exist if and only if the group G_T is twisted. This avoids the no-go theorem of Friedan, Qiu and Shenker [16] on the existence of massless fermions in superstrings propagating on a non-Abelian Lie group-manifold.

At the quantum level the internal dynamics of these models is described by a system of free left-moving and right-moving fermions, which turns out to be the same as the system considered in the free fermion constructions of ref.[3,5,7]. The world-sheet supercurrent, however, has a very specific structure, similar to, yet different from the structure of the current considered in [3,5,7]. This supercurrent yields a more restricted set of solutions to the problem of constructing modular invariant theories that preserve world-sheet supersymmetry. Its structure is a direct consequence of the geometric interpretation of the internal system as describing a non-linear σ -model on a (twisted) group-manifold G_T .

In this paper we extend the $SU(2)^3$ approach to type II superstrings.

The motivation for this study is the existence of a canonical map [4,9,17], hereafter called the heterotic h-map, which associates two different modular invariant heterotic superstring models to every consistent modular invariant type II superstring theory. As emphasized by Gepner [9], this map is the analogue, at the level of $2D$ -conformal field-theories, of the embedding of the target-space spin-connection into the gauge-connection coupled to the heterotic fermions. As we are going to show, the heterotic h-map admits a particularly simple and elegant formulation in the case of free fermion superstrings.

This opens the possibility of classifying $SU(2)^3$ heterotic superstrings according to their parent type II superstring. Such a viewpoint is advantageous for the following reason. In type II superstrings space-time supersymmetries emerge from both the left and the right movers and their number N is related to the numbers n_L and n_R of world-sheet supersymmetries in the left and right sector, which, as we are going to see, can be different. Therefore there is a set of internal superconformal theories with $c = 9$ and (n_L, n_R) supersymmetries that can be alternatively utilized for heterotic or type II superstrings (this choice of alternatives is the true meaning of the heterotic c-map). The moduli space of these theories is clearly independent of their use in either type of superstring constructions so that what we learn about its structure from type II models holds true also in heterotic models and viceversa. In particular, when we look at the theory from the type II vantage point we can utilize the constraints imposed by

extended space-time supersymmetry. This is a philosophy already fully developed in the case of Calabi-Yau and orbifold compactifications on left-right symmetric conformal theories [19-23]: we just extend it to the left-right asymmetric cases provided by twisted group-manifolds.

Specifically in this paper we get the following results.

- i) We show how one can obtain $D = 4$ type II models with $N=8,6,5,4,3$ and 2 space-time supersymmetries, respectively associated with $(6,6)$, $(4,6)$, $(2,6)$, $(4,4)$ or $(6,0)$, $(4,2)$ and $(2,0)$ superconformal $c = 9$ field theories. This is just a confirmation of a result already obtained in [24,25]. What we do more is that we get the geometric interpretation of these models in terms of group-manifolds with bilateral twisting. Such an interpretation leads to a σ -model formulation in terms of non-chiral bosons.
- ii) We relate in a precise way the bilaterally twisted group-manifolds to symmetric and asymmetric Z_2 or $Z_2 \otimes Z_2$ orbifolds, showing, in this way, that the $SU(2)^3$ approach corresponds to a description of these orbifolds in terms of non-chiral bosons.
- iii) We discuss the moduli spaces of the twisted group-manifolds, pointing out that they do indeed provide equivalent compactifications to those on asymmetric orbifolds, but at very specific points of their moduli spaces, corresponding to enhanced gauge symmetries. As a consequence, although the spectrum is the same, the moduli spaces of the twisted group-manifolds are smaller than those of the corresponding orbifolds, since one has to consider only those deformations that preserve the enhanced symmetries.
- iv) We derive the general rules of the heterotic c-map in free fermion language and we study a few examples. In particular we show how, given the boundary vectors and the free signs defining a type II model, we can immediately write down the boundary vectors and the GSO projectors of the corresponding heterotic model.

2 - Twisted group-manifolds and boundary vectors in type II superstrings

Geometrically, twisting means that the target group G_T is modded out by a discrete subgroup $B \subset G_T$. In terms of group elements we write the following equivalence relation:

$$g_1 \sim g_2 \text{ iff } g_1 = g_2 b \text{ with } b \in B \subset G_T \quad (2.1)$$

The corresponding variety $M_{\text{compact}} = G_T/B$ is a smooth manifold since it is a coset space of a Lie Group. In general non-smooth varieties (conifolds) can be introduced, if one considers bilateral cosets, i.e. if points in the group-manifold are identified through the following relation:

$$g_1 \sim g_2 \text{ iff } g_1 = \bar{b} g_2 b \text{ with } (\bar{b}, b) \in B \subset G_T \otimes G_T \quad (2.2)$$

On such manifolds the WZW field $g(z, \bar{z})$ has as many twisted sectors as there are elements $(b, \bar{b})$ in the homotopy group B of the compact manifold $M_{\text{compact}} = G_T/B$. In each of these sectors the WZW field $g(z, \bar{z})$ obeys the following boundary conditions:

$$g(z e^{2\pi i}, \bar{z} e^{-2\pi i}) = \bar{b} g(z, \bar{z}) b \quad (2.3)$$

As a consequence of eq. (2.3) the conserved left- and right-moving Kač-Moody currents of the target group $G_T = SU(2)^3$ *,

$$J(z) = -2i g^{-1}(z, \bar{z}) \partial_z g(z, \bar{z}) = J_i^A(z) t_i^A \quad (2.4a)$$

$$\bar{J}(\bar{z}) = -2i \partial_{\bar{z}} g(z, \bar{z}) g^{-1}(z, \bar{z}) = \bar{J}_i^A(z) t_i^A \quad (2.4b)$$

which, locally, fulfill the $SU(2)^3$ affine Lie algebra, encoded in the following OPEs:

$$J_i^A(z) J_j^B(w) = \delta_{ij} \left(\frac{2\delta^{AB}}{(z-w)^2} + \frac{i\sqrt{2}\epsilon^{ABC}}{z-w} J_i^C(w) \right) + \text{reg. terms} \quad (2.5)$$

globally, are subject, in each sector, to the following twisted boundary conditions:

$$J(z e^{2\pi i}) = b^{-1} J(z) b \quad ; \quad \bar{J}(\bar{z} e^{-2\pi i}) = \bar{b} \bar{J}(\bar{z}) \bar{b}^{-1} \quad (2.6)$$

which eventually break the symmetry to $U(1)^3$ and give rise to massless fermion states. In the case of a coset manifold we have $\bar{b} = 1$ and only the left currents are twisted. This situation can be considered only in the case of heterotic superstrings. For type II superstrings, massless fermions should arise from both the left and the right sectors, so that we are led to consider bilateral cosets in order to introduce twisting on both sets of currents.

In a recent paper [14] it has been shown how to relate the choice of the generators b to the boundary vectors one considers in the construction of modular invariant free fermion theories. To make this statement clear we must recall that, at the quantum level, the group-manifold construction is reduced to a free fermion one through the fermionization of the currents (2.4) [13]. Explicitly we set:

$$J_i^A(z) = 2\sqrt{2}\epsilon^{ABC} \chi_i^B(z) \chi_i^C(z) \quad (2.7a)$$

$$\bar{J}_i^A(\bar{z}) = 2\sqrt{2}\epsilon^{ABC} \bar{\chi}_i^B(\bar{z}) \bar{\chi}_i^C(\bar{z}) \quad (2.7b)$$

where $\chi_i^A(z)$ and $\bar{\chi}_i^B(\bar{z})$ are $9 \oplus 9$ free fermions in the adjoint representation of $SU(2)^3$ (the *fake* fermions, in the language of refs. [13-14]), that are to be added to the set of $9 \oplus 9$ free fermions $\lambda_i^A(z)$ and $\bar{\lambda}_i^B(\bar{z})$ (the *true* fermions in the same language) already present in the σ -model lagrangian and corresponding to the fermionic superpartners of the Kač-Moody currents (2.4). Upon use of this fermionization, we see that the internal degrees of freedom of the Type II superstring are described by a set of 18 left-moving free fermions:

$$\chi_i^A(z) \quad , \quad \lambda_i^A(z) \quad (A = 1, 2, 3 \quad ; \quad i = 1, 2, 3) \quad (2.8a)$$

* t_i^A are the generators of $SU(2)^3$, $A=1,2,3$ enumerating the generators within each $SU(2)$ and $i=1,2,3$ enumerating the $SU(2)$ factors

$$\lambda_i^A(z) \lambda_j^B(w) = -\frac{i}{4} \frac{\delta^{AB} \delta_{ij}}{z-w} + \text{reg.} \quad (2.8b)$$

$$\chi_i^A(z) \chi_j^B(w) = -\frac{i}{4} \frac{\delta^{AB} \delta_{ij}}{z-w} + \text{reg.} \quad (2.8c)$$

$$\chi_i^A(z) \lambda_j^B(w) = \text{regular} \quad (2.8d)$$

plus an identical set of 18 right-moving free fermions:

$$\bar{\chi}_i^A(\bar{z}) \quad , \quad \bar{\lambda}_i^A(\bar{z}) \quad (A = 1, 2, 3 \quad ; \quad i = 1, 2, 3) \quad (2.9a)$$

$$\bar{\lambda}_i^A(\bar{z}) \bar{\lambda}_j^B(\bar{w}) = -\frac{i}{4} \frac{\delta^{AB} \delta_{ij}}{\bar{z}-\bar{w}} + \text{reg.} \quad (2.9b)$$

$$\bar{\chi}_i^A(\bar{z}) \bar{\chi}_j^B(\bar{w}) = -\frac{i}{4} \frac{\delta^{AB} \delta_{ij}}{\bar{z}-\bar{w}} + \text{reg.} \quad (2.9c)$$

$$\bar{\chi}_i^A(\bar{z}) \bar{\lambda}_j^B(\bar{w}) = \text{regular} \quad (2.9d)$$

The explicit form of the stress-energy tensor and local supercurrent of this internal free fermion system is obtained via substitution of eq.(2.7) into the following formulae:

$$G_{int}(z) = \sqrt{2} e^{i\pi/4} : \sum_{i=1}^3 \left(J_i^A \lambda_i^A + \frac{2\sqrt{2}}{3} \lambda_i^A \lambda_i^B \lambda_i^C \epsilon_{ABC} \right) : , \quad (2.10a)$$

$$T_{int}(z) = : \sum_{i=1}^3 \left(\frac{1}{8} J_i^A J_i^A - 2i \lambda_i^A \partial_z \lambda_i^A \right) : \quad (2.10b)$$

$$\tilde{G}_{int}(\bar{z}) = \sqrt{2} e^{i\pi/4} : \sum_{i=1}^3 \left(\bar{J}_i^A \bar{\lambda}_i^A + \frac{2\sqrt{2}}{3} \bar{\lambda}_i^A \bar{\lambda}_i^B \bar{\lambda}_i^C \epsilon_{ABC} \right) : \quad (2.10c)$$

$$\tilde{T}_{int}(\bar{z}) = : \sum_{i=1}^3 \left(\frac{1}{8} \bar{J}_i^A \bar{J}_i^A - 2i \bar{\lambda}_i^A \partial_{\bar{z}} \bar{\lambda}_i^A \right) : \quad (2.10d)$$

that correspond to the Noether currents derived through variation of the classical (1,1) locally supersymmetric WZW action with respect to the two-dimensional gravitinos and vielbein. In this paper we do not discuss the structure of this action, since the only information we need for the construction of type II superstring models is the form of the superconformal algebra. Clearly this latter is nothing else but the repetition, also in the right sector, of the algebra obtained in the left sector, whose derivation was given in [14] for the case of the heterotic superstrings.

The complete conformal superalgebra is obtained by summing to the generators of the internal one those corresponding to flat Minkowski space:

$$G_{Mink}(z) = \sqrt{2} e^{i\pi/4} \psi^\mu(z) P^\mu(z) \quad (2.11a)$$

$$T_{Min\kappa}(z) = \frac{1}{2} P^\mu(z) P^\mu(z) - i \psi^\mu(z) \partial_z \psi^\mu(z) \quad (2.11b)$$

$$\tilde{G}_{Min\kappa}(z) = \sqrt{2} e^{i\pi/4} \tilde{\psi}^\mu(\bar{z}) \tilde{P}^\mu(\bar{z}) \quad (2.11c)$$

$$\tilde{T}_{Min\kappa}(z) = \frac{1}{2} \tilde{P}^\mu(\bar{z}) \tilde{P}^\mu(\bar{z}) - i \tilde{\psi}^\mu(\bar{z}) \partial_{\bar{z}} \tilde{\psi}^\mu(\bar{z}) \quad (2.11d)$$

As previously stated, in ref.[14] the available choices for the generators of the group B appearing in eq.(2.1), were completely determined from the requirements of space-time supersymmetry, multiloop modular invariance and world-sheet local supersymmetry, as it is generated by the supercurrent $G_{Min\kappa}(z) = \psi^\mu(z) \partial_z \psi^\mu(z)$. Furthermore the correspondence of these generators with suitable *left boundary vectors* of free fermion boundary conditions was given.

These results can be utilized to derive the available choices for the generators of the group B appearing in eq.(2.2). It suffices to take pairs of allowed generators, interpreting the second element of the pair as corresponding to the boundary conditions of the left-moving fermions and the first element of the pair, as corresponding to the boundary conditions of the right-moving ones.

More specifically the elements $\hat{b}, \hat{\bar{b}} \in SU(2)^3$ out of which we can construct the pairs $(\hat{b}, \hat{\bar{b}}) \in SU(2)^3 \otimes SU(2)^3$ are elements of the discrete subgroup $\hat{\Delta} = \hat{\Delta}_1 \otimes \hat{\Delta}_2 \otimes \hat{\Delta}_3$, where $\hat{\Delta}_i \subset SU(2)$, is a $Z_4 \otimes Z_4$. In the adjoint representation of $SU(2)$ (which is two-to-one), the elements of $\hat{\Delta}_i$ are generated by rotations of π , denoted e_{A_i} , around the three coordinate axes ($A = 1, 2, 3$) of $SO(3) = SU(2)/Z_2$. As such these 3×3 diagonal matrices e_A generate a $Z_2 \otimes Z_2$ subgroup of $SO(3)$ and have the following explicit form:

$$(e_A)^{BC} = \delta^{BC} (\delta^{AB} + \delta^{AC} - 1) \quad (2.12)$$

Furthermore they satisfy the obvious relations:

$$e_A e_B = 1 \delta_{AB} + i \epsilon_{ABC} e_C \quad (2.13)$$

In terms of these matrices the explicit form of eq.(2.6) is:

$$J_i^A(z e^{2\pi i}) = J_i^B(z) e_{U_i}^{BA} \quad (2.14a)$$

$$\tilde{J}_i^A(\bar{z} e^{-2\pi i}) = \tilde{J}_i^B(\bar{z}) \tilde{e}_{U_i}^{BA} \quad (2.14b)$$

and, in order to specify an element $(\hat{b}, \hat{\bar{b}}) \in B$ we just have to give two triplets of matrices e_{U_i} ($i = 1, 2, 3$) and $\tilde{e}_{U_i}$ ($i = 1, 2, 3$).

In the free fermion description of the type II superstrings, the χ 's, the λ 's and the ψ_T 's ($T = 1, 2$) form the 20 left-moving (LM) world-sheet fermions, while the 20 right-moving (RM) fermions are given by the same objects with a tilde. A sector of the string is identified by the set of boundary conditions (BC) of the above 20+20 fermions. In full analogy with the formalism adopted for heterotic superstrings [3,5,7,13,14] and following the conventions of [13,14] it is convenient to represent the BC of the world-sheet fermions in a given string sector by a 40-component *boundary vector* whose components are 0 or 1

$$a = [\alpha_{\psi^T}, \alpha_{\chi^T}, \alpha_{\lambda^T}; \alpha_{\tilde{\chi}^T}, \alpha_{\tilde{\lambda}^T}, \alpha_{\tilde{\psi}^T}], \quad (2.15)$$

according to the rule:

$$\alpha_\varphi = \begin{cases} 0 & \text{if } \varphi \text{ is a NS fermion;} \\ 1 & \text{if } \varphi \text{ is a R fermion.} \end{cases} \quad (2.16)$$

Relying on eq.(2.7) we see that every choice of a boundary vector corresponds to a choice of two triplets of matrices e_{U_i} ($i = 1, 2, 3$), $\tilde{e}_{U_i}$ ($i = 1, 2, 3$) and hence, as explained above, to the choice of a generator $(\hat{b}, \hat{\bar{b}}) \in B \subset SU(2)^3 \otimes SU(2)^3$.

Defining the scalar product of two boundary vectors as it follows:

$$a \cdot b \stackrel{\text{def}}{=} \sum_{i=1}^{20} \alpha_i \beta_i - \sum_{j=21}^{20} \alpha_j \beta_j \quad (2.17)$$

modular invariance imposes the following constraints on the set Ξ of boundary vectors defining a superstring model. First of all Ξ should be an Abelian group under the vector addition (each component is summed modulo two): this group is just the homomorphic image of B under the previously discussed correspondence. Secondly we should have:

$$\forall a \in \Xi \quad a^2 = a \cdot a \equiv 0 \pmod{8} \quad (2.18a)$$

$$\forall a, b \in \Xi \quad a \cdot b \equiv 0 \pmod{4} \quad (2.18b)$$

$$\forall a, b, c \in \Xi \quad a \cdot b \cdot c = \frac{1}{2} (a \cdot c + b \cdot c - (a + b) \cdot c) \equiv 0 \pmod{2} \quad (2.18c)$$

Thirdly the group Ξ must contain the universal boundary vector γ_0 given below:

$$[1|1] = \gamma_0 \in \Xi \quad (2.19a)$$

where 1 denotes the vector in which all the 20 fermions have Ramond boundary conditions, that is, all the entries are equal to one. This vector corresponds to no $SU(2)^3$ -twisting:

$$\gamma_0 \sim (\bar{1} \otimes \bar{1} \otimes \bar{1}, 1 \otimes 1 \otimes 1) \quad (2.19b)$$

The analysis of [14] reveals that, if we focus on the left boundary vectors (LBL), then there are just three sets of generators compatible among themselves, with world-sheet supersymmetry and with multiloop modular invariance. They are:

$$\mathcal{A} = [1, b, b', s, s', s''] \quad (2.20a)$$

$$\mathcal{B} = [1, b, b', s, s', c'] \quad (2.20b)$$

$$\mathcal{C} = [1, b, b', s, c, c'] \quad (2.20c)$$

where the explicit form of the above vectors is completely specified by listing the set of Ramond fields from which listing we also read off the corresponding B generators:

$$s = \{\psi^T(T=1,2), \chi_i^1, \lambda_i^1(i=1,2,3)\} \sim e_1 \otimes e_1 \otimes e_1, \quad (2.21a)$$

$$b = \{\chi_i^2 \chi_j^3 \lambda_k^1 \lambda_l^2 \lambda_m^3 \mid (i=2,3)\} \sim 1 \otimes e_1 \otimes e_1 \quad (2.21b)$$

$$b' = \{\chi_i^2 \chi_j^3 \lambda_k^1 \lambda_l^2 \lambda_m^3 \mid (i=1,2)\} \sim e_1 \otimes e_1 \otimes 1 \quad (2.21c)$$

$$s' = \{\psi^T, \chi_i^3 \lambda_j^1 \lambda_k^2 \mid (i=1,2) \chi_3^1 \lambda_3^1\} \sim e_3 \otimes e_3 \otimes e_1 \quad (2.21d)$$

$$s'' = \{\psi^T, \chi_i^3 \lambda_j^1 \lambda_k^2 \mid (i=1,3) \chi_2^1 \lambda_2^1\} \sim e_3 \otimes e_1 \otimes e_3 \quad (2.21e)$$

$$c = \{\psi^T, \chi_1^3 \lambda_1^1, \chi_2^3 \lambda_2^1, \chi_3^3 \lambda_3^1\} \sim e_1 \otimes e_3 \otimes e_2 \quad (2.21f)$$

$$c' = \{\psi^T, \chi_1^3 \lambda_1^1, \chi_2^3 \lambda_2^1, \chi_3^3 \lambda_3^1\} \sim e_2 \otimes e_1 \otimes e_3 \quad (2.21g)$$

As already stated, these results can be utilized to construct the generators of the boundary group Ξ in the case of type II superstrings. These generators will have the form:

$$\gamma_\Lambda \in \Xi \implies \gamma_\Lambda = [x | \bar{y}] \quad (2.22)$$

where $x, (\bar{y})$ are taken from the above sets of consistent left (respectively right) boundary vectors. In the next section we show how this is explicitly done.

3 - Construction of modular invariants and GSO projectors

The theories we will consider are based on a group Ξ generated by the following set of boundary vectors:

$$\gamma_\Lambda = \gamma_0, \gamma_+, \gamma_-, \gamma_i \quad (i=1, \dots, K) \quad (3.1)$$

where K is some integer number. (For the time being we do not attempt a complete classification of these models, which, however, is clearly feasible). In all theories γ_0 is given by eq.(2.19) and furthermore we set:

$$\gamma_+ = [s | \bar{0}] \sim (\bar{1} \otimes \bar{1} \otimes \bar{1}, e_1 \otimes e_1 \otimes e_1) \quad (3.2a)$$

$$\gamma_- = [0 | \bar{s}] \sim (\bar{e}_1 \otimes \bar{e}_1 \otimes \bar{e}_1, 1 \otimes 1 \otimes 1) \quad (3.2b)$$

The remaining generators vary instead from model to model. We note that γ_+ and γ_- are needed to get massless gravitinos in both the left- and right-moving sectors of the spectrum.

We utilize the following convention for the boundary vector indices. The $K+3$ generators are enumerated by capital Greek indices $\Lambda, \Sigma, \Gamma = 0, +, -, i (=1, \dots, K)$. The generators from $+$ to K are enumerated by capital Latin indices:

$$\gamma_\Lambda = \gamma_+, \gamma_-, \gamma_i \quad (3.3)$$

while the lower case indices i, j, k label the non-universal generators whose choice depends on the model.

Once the generators have been selected the modular invariant still depends on a certain number of free signs which is given by:

$$\text{number of free signs} = 4 + \frac{K(K+5)}{2} \quad (3.4)$$

The assignment of these signs can be understood as the assignment of a $\mathbf{Z}_2$ cocycle on the boundary group $\Xi \sim (\mathbf{Z}_2)^{K+3}$.

To be precise, we recall that the modular invariant partition function has the following structure:

$$Z(\tau) = \sum_{a,b \in \Xi} c \begin{bmatrix} a \\ b \end{bmatrix} Z \begin{bmatrix} a \\ b \end{bmatrix} \quad (3.5)$$

where a, b is any pair of elements of Ξ defining a spin structure of the free fermion system and where the coefficients are $\mathbf{Z}_2$ elements:

$$\left(c \begin{bmatrix} a \\ b \end{bmatrix} \right)^2 = 1 \quad (3.6)$$

The relations imposed by modular invariance are:

$$c \begin{bmatrix} a \\ b \end{bmatrix} = \delta_{\gamma_0} c \begin{bmatrix} a \\ a+b+\gamma_0 \end{bmatrix} e^{i\pi a^2/8} \quad (3.7a)$$

$$c \begin{bmatrix} a \\ b \end{bmatrix} = e^{i\frac{3}{4}a \cdot b} c \begin{bmatrix} b \\ a \end{bmatrix} \quad (3.7b)$$

$$c \begin{bmatrix} a \\ b \end{bmatrix} c \begin{bmatrix} a \\ c \end{bmatrix} = \delta_a c \begin{bmatrix} a \\ b+c \end{bmatrix} \quad (3.7c)$$

where δ_a is the statistics of the sector corresponding to the boundary vector a (δ_a is $+1$ if a represents a boson, while it is -1 if a represents a fermion). The relations (3.7) are true for any type of fermionic string, both heterotic and type II.

Eqs. (3.7) allow the determination of all the coefficients $c \begin{bmatrix} a \\ b \end{bmatrix}$ for any pair of Ξ group elements if we assign the following data. Let $\epsilon_{\Lambda\Sigma}$ be a $\mathbf{Z}_2$ -valued tensor ($\epsilon_{\Lambda\Sigma}^2 = 1$) with the following symmetry:

$$\epsilon_{\Lambda\Sigma} = e^{i\frac{\pi}{4} \gamma_\Lambda \cdot \gamma_\Sigma} \epsilon_{\Sigma\Lambda} \quad (3.8)$$

and fulfilling the additional condition:

$$\epsilon_{A0} = \epsilon_{0A} = \epsilon_{AA} e^{i\pi \gamma_A^2/8} \quad (3.9)$$

Clearly the tensor $\epsilon_{\Lambda\Sigma}$ has a number of independent components equal to the one given by eq.(3.4). For instance as independent components we can choose the following ones:

a six-vector of left-movers plus a six-vector of right-movers according to the following rules. Splitting the range of the $SO(6)$ -vector index as follows:

$$X = \begin{cases} i & = 1, 2, 3 \\ i^* & = 1^*, 2^*, 3^* \end{cases} \quad (4.2)$$

we set:

$$\psi^X(z) = \begin{cases} \psi^i(z) = \sqrt{\frac{k}{2}} \lambda_{(i)}^1 & (i=1,2,3) \\ \psi^{i^*}(z) = \sqrt{\frac{k}{2}} \chi_{(i)}^1 & (i=1,2,3) \end{cases} \quad (4.3a)$$

similar definitions for $\tilde{\psi}^X(\bar{z})$ (4.3b)

Then it is easy to check that the $6 \oplus 6$ currents (4.1) and the $6 \oplus 6$ fermions (4.3) obey the OPEs corresponding to the direct product of a $U(1)^6$ left-moving times a $U(1)^6$ right-moving super Kac-Moody algebra:

$$P^X(z) P^Y(w) = \frac{\delta^{XY}}{(z-w)^2} + \text{reg} \quad (4.4a)$$

$$P^X(z) \psi^Y(w) = \text{reg} \quad (4.4b)$$

similar equations for tilded quantities (4.4c, d)

Relying on eqs. (4.4) we can postulate the following bosonization rules:

$$\mathcal{P}^i(z) = P^i(z) + i P^{i^*}(z) = -i \partial_z \mathcal{Z}_L^i(z, \bar{z}) \quad (4.5a)$$

$$\tilde{\mathcal{P}}^i(\bar{z}) = \tilde{P}^i(\bar{z}) + i \tilde{P}^{i^*}(\bar{z}) = -i \partial_{\bar{z}} \mathcal{Z}_R^i(z, \bar{z}) \quad (4.5b)$$

where $\mathcal{Z}_{L/R}^i(z, \bar{z})$ are two triplets of complex coordinates describing two different T^6 tori (the left and the right ones). In general we cannot identify $\mathcal{Z}_L^i$ with $\mathcal{Z}_R^i$ because of the boundary conditions fulfilled by $\mathcal{P}^i(z)$ and $\tilde{\mathcal{P}}^i(\bar{z})$ are different as a consequence of the non-equality of b with $\bar{b}$ in the elements $(\bar{b}, b)$ of the group B . Henceforth we conclude that $\mathcal{Z}_L^i$ and $\mathcal{Z}_R^i$ are in general chiral free bosons describing what in the literature is named an asymmetric orbifold. In those special cases where $\mathcal{P}^i(z)$ and $\tilde{\mathcal{P}}^i(\bar{z})$ are given equal boundary conditions we can set $\mathcal{Z}_L^i = \mathcal{Z}_R^i = \mathcal{Z}^i$ and we obtain a symmetric orbifold.

Let us now consider table I, where we have listed a few examples of type II superstring models. In each case we have written the explicit form of the additional γ_i generators and the associated element $(\bar{b}, b)$ of the B group. The corresponding orbifold is immediately read off by comparison of eq.s (4.1) with (2.20). To see how this works let us first note that in the γ_0 , γ_+ and γ_- sectors the currents (4.5) are all periodic:

$$\mathcal{P}^i(z e^{2\pi i}) = \mathcal{P}^i(z) \quad (4.6a)$$

$$\tilde{\mathcal{P}}^i(z e^{2\pi i}) = \tilde{\mathcal{P}}^i(z) \quad (4.6b)$$

$$\varepsilon_{00} \varepsilon_{++} \varepsilon_{--} \varepsilon_{+-} \varepsilon_{-+} \varepsilon_{-i} \varepsilon_{ij} \quad (i \leq j) \quad (3.10)$$

Given the tensor $\varepsilon_{A\Sigma}$ we set:

$$c \begin{bmatrix} \gamma_A \\ \gamma_\Sigma \end{bmatrix} = \varepsilon_{A\Sigma} \quad (3.11)$$

and

$$c \begin{bmatrix} \gamma_A \\ 0 \end{bmatrix} = \delta_{\gamma_A} \quad (3.12)$$

Inserting these relations into formulae (3.7) we can work out the value of the coefficients $c \begin{bmatrix} a \\ b \end{bmatrix}$ for any pair of boundary vectors. In turn this implies that we can write the GSO projector for any sector of the theory, the projector being given by the following general formula:

$$P_a = \prod_{A=0}^K \frac{1}{2} \left(1 + \delta_{a,c} \begin{bmatrix} a \\ \gamma_A \end{bmatrix} (-1)^{\gamma_A \cdot F} \right) \quad (3.13)$$

where, for any boundary vector a , we have $a \cdot F = \sum \alpha_i F^i$, F^i being the fermion number of the i^{th} world-sheet fermion φ^i . More precisely, we can write

$$(-1)^{F^i} = \begin{cases} (-1)^{N_{NS}^i} & , N_{NS}^i = \sum_{r=\frac{1}{2}}^{\infty} \varphi_r^{i\dagger} \varphi_r^i \quad \text{for } \alpha_i = 0 \text{ (NS)} ; \\ (-1)^{N_R^i} \sqrt{2} \varphi_0^i & , N_R^i = \sum_{n=1}^{\infty} \varphi_n^{i\dagger} \varphi_n^i \quad \text{for } \alpha_i = 1 \text{ (R)} . \end{cases} \quad (3.14)$$

where the zero mode φ_0^i of the Ramond field is proportional to a suitable gamma matrix.

4 - Bilaterally twisted $SU(2)^3$ group-manifolds and the corresponding (asymmetric) orbifold

In order to make contact with the six-dimensional orbifold picture we follow ref.[14], and we introduce the six bilinear combinations of the internal free fermions that we list below:

$$\begin{aligned} P^i(z) &= 2\sqrt{2} \left(\chi_{(i)}^2(z) \chi_{(i)}^3(z) + \lambda_{(i)}^2(z) \lambda_{(i)}^3(z) \right) \\ &= \frac{1}{2} \left(J_{(i)}^1(z) + 2\sqrt{2} \varepsilon^{1BC} \lambda_{(i)}^B(z) \lambda_{(i)}^C(z) \right) ; \quad (i = 1, 2, 3) \end{aligned} \quad (4.1a)$$

$$P^{i^*}(z) = 2\sqrt{2} \left(\chi_{(i)}^2(z) \lambda_{(i)}^3(z) - \chi_{(i)}^3(z) \lambda_{(i)}^2(z) \right) ; \quad (i = 1, 2, 3) \quad (4.1b)$$

similar definitions for $\tilde{P}^i(\bar{z})$ and $\tilde{P}^{i^*}(\bar{z})$ (4.1b, c)

As the reader can notice eqs. (4.1) involve only a $12 \oplus 12$ subset of the $18 \oplus 18$ fermions introduced by eqs. (2.8) and (2.9). The remaining $6 \oplus 6$ fermions can be arranged into

the same way as it happens in the 0-sector. Therefore if we do not introduce any γ_i boundary vector (case $K = 0$), then the orbifold is just an untwisted T^6 torus. This is consistent with the fact that the target space SUSY can be chosen to be $N=8$ by appropriately fixing the $\varepsilon_{\Lambda\Sigma}$ signs.

Therefore we just need to analyse the twisting introduced by the additional generators γ_i . Consider for instance the second case of table I. In the γ_1 sector we find:

$$\mathcal{P}^i(z e^{2\pi i}) = -\mathcal{P}^i(z) \quad (i = 1, 2) \quad (4.7a)$$

$$\mathcal{P}^3(z e^{2\pi i}) = \mathcal{P}^3(z) \quad (4.7b)$$

$$\tilde{\mathcal{P}}^i(\bar{z} e^{-2\pi i}) = \tilde{\mathcal{P}}^i(z) \quad (i = 1, 2, 3) \quad (4.7c)$$

Since $\mathcal{P}^i(z)$ and $\tilde{\mathcal{P}}^i(\bar{z})$ obey different boundary conditions they cannot be regarded as the holomorphic and antiholomorphic derivatives of the same free field $\mathcal{Z}^i(z, \bar{z})$. For this reason we get an asymmetric orbifold. The right-movers are the coordinates of an ordinary untwisted torus T^6 . The left-movers, instead, correspond to an orbifold $T^6/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ identification group acts as follows:

$$\mathbb{Z}_2 : \begin{cases} \mathcal{Z}_1^i \\ \mathcal{Z}_2^i \end{cases} \Rightarrow \begin{cases} -\mathcal{Z}_1^i \\ -\mathcal{Z}_2^i \end{cases} \quad (i = 1, 2) \quad (4.8)$$

With a similar analysis we see that, in the fourth case of table I, we have a $\mathbb{Z}_2$ which acts on both the left and the right coordinates according to eq. (4.8):

$$\mathbb{Z}_2 : \begin{cases} \mathcal{Z}_L^i \\ \mathcal{Z}_R^i \end{cases} \Rightarrow \begin{cases} -\mathcal{Z}_L^i \\ -\mathcal{Z}_R^i \end{cases} \quad \mathcal{Z}_L^i \Rightarrow -\mathcal{Z}_R^i \quad (i = 1, 2) \quad (4.9a)$$

plus an additional $\mathbb{Z}_2'$ that acts only on the left coordinates as it follows:

$$\mathbb{Z}_2' : \begin{cases} \mathcal{Z}_L^i \\ \mathcal{Z}_R^i \end{cases} \Rightarrow \begin{cases} -\mathcal{Z}_L^i \\ -\mathcal{Z}_R^i \end{cases} \quad (i = 1, 3) \quad (4.9b)$$

In this way one can proceed and work out all the correspondences listed in the third column of table I. The general rule is easily obtained. The boundary vectors s' and s'' correspond, respectively, to the $\mathbb{Z}_2$ and $\mathbb{Z}_2'$ twisting of eq.s (4.8) and (4.9b). The boundary vectors b and b' , on the other hand do not introduce additional twistings but modify the zero-mode spectrum: they correspond to a translation in the moduli space of the orbifold.

To understand the fourth column of table I we observe that the local world-sheet supercurrent (2.9a) can be rewritten in terms of the orbifold currents (4.1) and of the free fermions (4.3) as follows:

$$G^{local}(z) = e^{i\pi/4} \sqrt{2} \psi^X(z) P^X(z) \quad (4.10)$$

Introducing the $SO(6)$ Kač-Moody currents:

$$J^{XY}(z) = 2\sqrt{2} : \psi^X(z) \psi^Y(z) : \quad (4.11a)$$

$$\begin{aligned} J^{XY}(z) J^{ZW}(w) &= \frac{2}{(z-w)^2} (\delta^{XZ} \delta^{YW} - \delta^{XW} \delta^{YZ}) \\ &- i \frac{\sqrt{2}}{z-w} (\delta^{XZ} J^{YW}(w) - \delta^{XW} J^{YZ}(w) + \delta^{YW} J^{XZ}(w) - \delta^{YZ} J^{XW}(w)) \\ &+ reg. \end{aligned} \quad (4.11b)$$

under which $\psi^X(z)$ transforms as a vector

$$J^{XY}(z) \psi^Z(w) = \frac{i\sqrt{2}}{(z-w)} (\psi^X(w) \delta^{YZ} - \psi^Y(w) \delta^{XZ}) + reg \quad (4.12)$$

and $P^X(z)$ is inert

$$J^{XY}(z) P^Z(w) = reg. \quad (4.13)$$

we see that $G^{local}(z)$ is just one element of a multiplet of 6 supercurrents transforming as a vector under $SO(6)$. These currents are part of a chiral superalgebra that we name $n=6$: it is not properly a superconformal algebra since it includes generators of weight higher than $w=2$, yet it is characterized by 6 generators of weight $w=3/2$ and for this reason the name $n=6$ is justified.

If we do not introduce additional generators γ_i , all the 6 supercurrents are global symmetries of the internal conformal field-theory which is, therefore, properly named a (6,6) theory. The addition of the boundary vectors s' and s'' breaks, successively the superconformal symmetry to $n=4$ and to $n=2$. How this happens was described in detail for the left-movers in [14]. We just have to repeat the same analysis in the same way also for the right-movers and we arrive at the results listed in the fourth column of table I.

Finally the number of space-time supersymmetries is given by the sum of those emerging from the left-movers plus those emerging from the right-movers: in each sector the relation with the number of world-sheet supercharges is the usual one and this explains the last column of table I.

5 - The GSO projectors in the universal sectors and the moduli spaces

Since we want to classify our models according to the number of space-time supersymmetries we are primarily concerned with the GSO projectors in the gravitino sectors, namely in the sectors corresponding to the boundary vectors γ_+ and γ_- . In addition to these fermionic sectors all the models contain also two other universal massless sectors, that is the identity sector $\bar{o}$ and the Ramond-Ramond sector $\gamma_+ + \gamma_-$.

In the present section we want to write the GSO projectors in these universal sectors. If no additional γ_i generators are introduced there the massless states contained in the universal sectors fill up the spectrum of $N=8$ supergravity. Adding γ_i boundary vectors yields a consistent truncation of the $N=8$ theory to theories with fewer space-time supercharges ($N < 8$). Occasionally some of the new sectors introduced by the γ_i

generators are also massless. In this case we say that there are twisted massless multiplets and the effective Lagrangian cannot be viewed as a pure truncation of the $N = 8$ theory. We will see examples of this phenomenon in the sequel. For the time being we analyse the universal sectors. Let us begin with the GSO projectors in the gravitino sectors γ_+ and γ_- .

a) Gravitino projectors

Utilizing eq.s (3.13) and (3.9) we can write

$$\mathcal{P}_{\gamma_+} = \mathcal{P}_+ U_+ \quad (5.1a)$$

$$\mathcal{P}_{\gamma_-} = \mathcal{P}_- U_- \quad (5.1b)$$

where $U_{\pm}$ are universal projectors (those of the $N = 8$ theory) and $\mathcal{P}_{\pm}$ are the additional projectors due the extra generators γ_i .

Explicitly we have:

$$U_+ = \frac{1}{2} (1 + \epsilon_{++} (-) \gamma_0 \cdot F) \frac{1}{2} (1 - \epsilon_{++} (-) \gamma_+ \cdot F) \frac{1}{2} (1 - \epsilon_{+-} (-) \gamma_- \cdot F) \quad (5.2a)$$

$$U_- = \frac{1}{2} (1 + \epsilon_{--} (-) \gamma_0 \cdot F) \frac{1}{2} (1 - \epsilon_{--} (-) \gamma_- \cdot F) \frac{1}{2} (1 - \epsilon_{+-} (-) \gamma_+ \cdot F) \quad (5.2b)$$

The additional projectors are:

$$\mathcal{P}_+ = \prod_{i=1}^k \frac{1}{2} (1 - \epsilon_{+i} (-) \gamma_i \cdot F) \quad (5.3a)$$

$$\mathcal{P}_- = \prod_{i=1}^k \frac{1}{2} (1 - \epsilon_{-i} (-) \gamma_i \cdot F) \quad (5.3b)$$

where δ_i denotes the statistic of the δ_i generator.

Prior to the projection the massless emission vertices in the gravitino sectors have the following form in the $(\frac{1}{2}, 1)$ and $(1, \frac{1}{2})$ pictures, respectively:

$$V_{(\frac{1}{2}, 1)}^{\dot{\alpha} \dot{P} \mu}(z, \bar{z}, k) = e^{\frac{1}{2} \phi(z)} S^{\dot{\alpha}}(z) \Sigma^{\dot{P}}(z) 2e^{i\pi/4} \phi^{\mu}(z) \tilde{\psi}^{\mu}(\bar{z}) e^{i k \cdot X(z, \bar{z})} \quad (5.4a)$$

$$V_{(1, \frac{1}{2})}^{\dot{\alpha} \dot{P} \mu}(z, \bar{z}, k) = 2e^{\phi(z)} \psi^{\mu}(z) \tilde{S}^{\dot{\alpha}}(\bar{z}) \Sigma^{\dot{P}}(\bar{z}) 2e^{i\pi/4} \phi^{\mu}(\bar{z}) e^{i k \cdot X(z, \bar{z})} \quad (5.4b)$$

where $\phi(z)$ and $\tilde{\phi}(\bar{z})$ denote the scalar fields obtained in the bosonization of the left-moving (respectively right-moving) superghosts and $\Sigma^{\dot{P}}(z)$, $\tilde{\Sigma}^{\dot{P}}(\bar{z})$ are the spin fields of the left moving (respectively right-moving) $SO(6)$ current algebra (4.11), while $S^{\dot{\alpha}}(z)$

and $\tilde{S}^{\dot{\alpha}}(\bar{z})$ are the space-time spin fields. The conventions utilized here are those of ref. [14].

The action of the fermion numbers on these vertices can be worked out by utilizing the following relations that fix our gamma matrix conventions:

$$\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6 = \Gamma_7 i \quad ; \quad \gamma_0 \gamma_1 \gamma_2 \gamma_3 = -i \gamma_5 \quad (5.5)$$

with this information we easily see that the fermion numbers act as follows:

$$(-)^{\gamma_+ \cdot F} V_{(\frac{1}{2}, 1)}^{\mu} = \gamma_5 \Gamma_7 V_{(\frac{1}{2}, 1)}^{\mu} \quad (5.6a)$$

$$(-)^{\gamma_- \cdot F} V_{(\frac{1}{2}, 1)}^{\mu} = -V_{(\frac{1}{2}, 1)}^{\mu} \quad (5.6b)$$

$$(-)^{\gamma_0 \cdot F} V_{(\frac{1}{2}, 1)}^{\mu} = -\gamma_5 \Gamma_7 V_{(\frac{1}{2}, 1)}^{\mu} \quad (5.6c)$$

$$(-)^{\gamma_+ \cdot F} V_{(1, \frac{1}{2})}^{\mu} = -V_{(1, \frac{1}{2})}^{\mu} \quad (5.6d)$$

$$(-)^{\gamma_- \cdot F} V_{(1, \frac{1}{2})}^{\mu} = \gamma_5 \Gamma_7 V_{(1, \frac{1}{2})}^{\mu} \quad (5.6e)$$

$$(-)^{\gamma_0 \cdot F} V_{(1, \frac{1}{2})}^{\mu} = -\gamma_5 \Gamma_7 V_{(1, \frac{1}{2})}^{\mu} \quad (5.6f)$$

By comparison of eqs. (5.6) with eqs. (5.2) we conclude that, in order to have space-time supersymmetry, we must fix

$$\epsilon_{+-} = 1 \quad (5.7)$$

Otherwise we obtain an $N = 0$ theory since all the gravitinos are projected out.

We also understand the meaning of the other universal coefficients. From eqs. (5.2) we see that ϵ_{++} and ϵ_{--} correspond to the choices of the chiralities for the left-moving and the right-moving gravitinos. Clearly the only thing which is significant and not a matter of convention is the relative chirality. Therefore we fix our chirality conventions by setting:

$$\begin{cases} \epsilon_{++} = -1 & \text{1 Type II B} \\ \epsilon_{--} = -\eta & \text{-1 Type II A} \end{cases} \quad \eta = \begin{matrix} \nearrow \\ \searrow \end{matrix} \quad (5.8)$$

As indicated by equation (5.8) the sign η distinguishes between type II A and type II B $N = 8$ superstrings.

The remaining universal parameter ϵ_{00} is relevant only to the massive sectors of the superstring.

Let us now consider the additional projectors.

$\mathcal{P}_+$ and $\mathcal{P}_-$ in the cases of interest to us.

We classify the theories according to the number K of additional projections.

As examples we shall consider the second, the third and the fourth models of table I, corresponding, respectively to $N = 6$, $N = 4$ and $N = 3$ space-time supersymmetries.

Explicitly we have:

$$\mathcal{P}_+^{(N=6)} = \frac{1}{2} \left(1 - \epsilon_{+1}^{(N=6)} \right) \gamma_1^{(N=6), F} \quad (5.9a)$$

$$\mathcal{P}_-^{(N=6)} = \frac{1}{2} \left(1 - \epsilon_{-1}^{(N=6)} \right) \gamma_1^{(N=6), F} \quad (5.9b)$$

$$\mathcal{P}_+^{(N=4)} = \frac{1}{2} \left(1 - \epsilon_{+1}^{(N=4)} \right) \gamma_1^{(N=4), F} \quad (5.9c)$$

$$\mathcal{P}_-^{(N=4)} = \frac{1}{2} \left(1 - \epsilon_{-1}^{(N=4)} \right) \gamma_1^{(N=4), F} \quad (5.9d)$$

$$\mathcal{P}_+^{(N=3)} = \mathcal{P}_+^{(N=4)} \frac{1}{2} \left(1 - \epsilon_{+2}^{(N=3)} \right) \gamma_2^{(N=3), F} \quad (5.9e)$$

$$\mathcal{P}_-^{(N=3)} = \mathcal{P}_-^{(N=4)} \frac{1}{2} \left(1 - \epsilon_{-2}^{(N=3)} \right) \gamma_2^{(N=3), F} \quad (5.9f)$$

The signs in eqs. (5.9) are dictated by the statistics of the generators γ_1 and γ_2 which is immediately read off from table I ($\gamma_1^{(N=6)} = \text{fermion}$; $\gamma_1^{(N=4)} = \text{boson}$; $\gamma_2^{(N=3)} = \text{fermion}$).

Furthermore the structure of eqs. (5.9) shows that the $N = 3$ theory is obtained as a truncation of the $N = 4 \oplus$ possibly twisted states.

The action of the various number operators on the vertices (5.6) is the following one:

$$(-)^{\gamma_1^{(N=6), F}} V_{(\frac{1}{2}, 1)}^\mu = -i\gamma_5 \Gamma_{33} \cdot V_{(\frac{1}{2}, 1)}^\mu \quad (5.10a)$$

$$(-)^{\gamma_1^{(N=4), F}} V_{(1, \frac{1}{2})}^\mu = -V_{(1, \frac{1}{2})}^\mu \quad (5.10b)$$

$$(-)^{\gamma_1^{(N=4), F}} V_{(\frac{1}{2}, 1)}^\mu = i\gamma_5 \Gamma_{33} \cdot V_{(\frac{1}{2}, 1)}^\mu \quad (5.10c)$$

$$(-)^{\gamma_1^{(N=4), F}} V_{(1, \frac{1}{2})}^\mu = i\gamma_5 \Gamma_{33} \cdot V_{(1, \frac{1}{2})}^\mu \quad (5.10d)$$

$$(-)^{\gamma_2^{(N=3), F}} V_{(\frac{1}{2}, 1)}^\mu = -i\gamma_5 \Gamma_{22} \cdot V_{(\frac{1}{2}, 1)}^\mu \quad (5.10e)$$

$$(-)^{\gamma_2^{(N=3), F}} V_{(1, \frac{1}{2})}^\mu = -V_{(1, \frac{1}{2})}^\mu \quad (5.10f)$$

where we have utilized the gamma matrix conventions of ref.[14].

Comparing eqs. (5.10) with (5.9) we see that, if we do not want to project out all the right-moving gravitinos, then we have to set

$$\epsilon_{-1}^{(N=6)} = 1 \quad (5.11)$$

Similarly we can fix the chiralities of the $N = 6$ theory by setting

$$\epsilon_{+1}^{(N=6)} = -1 \quad (5.12)$$

and we can fix the chiralities of the $N = 4$ theory by setting

$$\epsilon_{+1}^{(N=4)} = \epsilon_{-1}^{(N=4)} = -1 \quad (5.13)$$

From the point of view of the massless sector, eqs. (5.12) and (5.13) correspond to an irrelevant choice of conventions. At the massive level, however, flipping these signs may involve a structural change of the spectrum and of the interaction.

In the $N = 8$ case we just distinguish between the type II A and type II B superstrings.

In the lower N theories there are more chirality choices and each of them corresponds to a different model.

In refs. [20,21] the meaning of the sign flip between type II A and type II B superstrings compactified on a (2,2) conformal field theory was analysed at the level of the effective Lagrangian and it was seen to correspond to the remarkable c-map between restricted Kähler manifolds and quaternionic manifolds [21].

Clearly flipping the signs of $\epsilon_{+1}^{(N=6)}$ or $\epsilon_{\pm}^{(N=4)}$ leads to generalized c-maps whose structure for the moment we do not investigate, since for $N \geq 3$ the massless effective Lagrangian is anyhow uniquely fixed by supersymmetry at the level of two derivative terms.

Hence continuing our analysis we go over to the $N = 3$ theory and we note that if we do not want to project out all the right-moving gravitinos we have to choose:

$$\epsilon_{-2}^{(N=3)} = 1 \quad (5.14)$$

Furthermore one can fix the chirality choices by setting.

$$\epsilon_{+2}^{(N=3)} = -1 \quad (5.15)$$

With these choices the GSO projectors in the three theories under consideration have the following action on the gravitino vertices (5.6)

$$\mathbf{P}_{\gamma_+}^{(N=6)} V_{(\frac{1}{2}, 1)}^\mu = \frac{1}{2} (1 + \gamma_5 \Gamma_7) \frac{1}{2} (1 - i\gamma_5 \Gamma_{33}) V_{(\frac{1}{2}, 1)}^\mu \quad (5.16a)$$

$$\mathbf{P}_{\gamma_-}^{(N=6)} V_{(1, \frac{1}{2})}^\mu = \frac{1}{2} (1 + \eta\gamma_5 \Gamma_7) V_{(1, \frac{1}{2})}^\mu \quad (5.16b)$$

$$\mathbf{P}_{\gamma_+}^{(N=4)} V_{(\frac{1}{2}, 1)}^\mu = \frac{1}{2} (1 + \gamma_5 \Gamma_7) \frac{1}{2} (1 + i\gamma_5 \Gamma_{33}) V_{(\frac{1}{2}, 1)}^\mu \quad (5.16c)$$

$$\mathbf{P}_{\gamma_-}^{(N=4)} V_{(\frac{1}{2}, \frac{1}{2})}^\mu = \frac{1}{2} (1 + \eta \gamma_5 \Gamma_7) \frac{1}{2} (1 + i \gamma_5 \Gamma_{33}) V_{(\frac{1}{2}, \frac{1}{2})}^\mu \quad (5.16d)$$

$$\mathbf{P}_{\gamma_+}^{(N=3)} V_{(\frac{1}{2}, 1)}^\mu = \frac{1}{2} (1 + \gamma_5 \Gamma_7) \frac{1}{2} (1 + i \gamma_5 \Gamma_{33}) \frac{1}{2} (1 - i \gamma_5 \Gamma_{22}) V_{(\frac{1}{2}, 1)}^\mu \quad (5.16e)$$

$$\mathbf{P}_{\gamma_-}^{(N=3)} V_{(\frac{1}{2}, \frac{1}{2})}^\mu = \frac{1}{2} (1 + \eta \gamma_5 \Gamma_7) \frac{1}{2} (1 + i \gamma_5 \Gamma_{33}) V_{(\frac{1}{2}, \frac{1}{2})}^\mu \quad (5.16f)$$

and following ref.[14] we conclude that the number of surviving gravitinos is indeed $N = 6$, $N = 4$ and $N = 3$ as claimed.

b) Graviton Projectors

Next we construct the GSO projectors in the graviton sector. Here we can write

$$\mathbf{P}_{(0)} = U_{(0)} \mathcal{P}_{(0)} \quad (5.17)$$

where

$$U_{(0)} = \frac{1}{2} (1 + (-)^{\gamma_0 F}) \frac{1}{2} (1 - (-)^{\gamma_+ F}) \frac{1}{2} (1 - (-)^{\gamma_- F}) \quad (5.18)$$

is the universal part of the projector, corresponding to the $N = 8$ theory.

The model dependent part is, instead, given by:

$$\mathcal{P}_{(0)} = \prod_{i=1}^4 \frac{1}{2} (1 + \delta_i (-)^{\gamma_i F}) \quad (5.19)$$

Let us write its form explicitly in the three cases we have chosen to consider

$$\mathcal{P}_{(0)}^{(N=6)} = \frac{1}{2} (1 - (-)^{\gamma_1^{(N=6)} F}) \quad (5.20a)$$

$$\mathcal{P}_{(0)}^{(N=4)} = \frac{1}{2} (1 + (-)^{\gamma_1^{(N=4)} F}) \quad (5.20b)$$

$$\mathcal{P}_{(0)}^{(N=3)} = \frac{1}{2} (1 + (-)^{\gamma_1^{(N=3)} F}) \frac{1}{2} (1 - (-)^{\gamma_2^{(N=3)} F}) \quad (5.20c)$$

We can now determine the action of these projection operators on the massless states appearing in the 0-sector. We study the problem in the $(1, 1)$ -picture which is the easiest to work with.

In this picture the massless vertex must have conformal weights $\frac{1}{2}, \frac{1}{2}$ (apart from the ghosts) and it is therefore made out of fermions.

The universal projector $U_{(0)}$ also requires the following fermion numbers

$$\gamma_0 \cdot F = 0 \bmod 2, \gamma_+ \cdot F = 1 \bmod 2 \quad (5.21a)$$

$$\gamma_- \cdot F = 1 \bmod 2 \quad (5.21b)$$

so that we are left with the following vertices corresponding to the $N = 8$ theory:

Graviton vertex

$$V_{(1,1)}^{\mu\nu/mu}(z, \bar{z}; k) = 2e^{i\pi/4} e^{\phi(z)} \psi^\mu(z) 2e^{i\pi/4} e^{\phi(\bar{z})} \bar{\psi}^\mu(\bar{z}) e^{ik \cdot X(z, \bar{z})} \quad (5.22a)$$

Graviton vertices

$$V_{(1,1)}^{\mu\nu X}(z, \bar{z}; k) = 2e^{i\pi/4} e^{\phi(z)} \psi^\mu(z) 2e^{i\pi/4} e^{\phi(\bar{z})} \bar{\psi}^\mu(\bar{z}) e^{ik \cdot X(z, \bar{z})} \quad (5.22b)$$

$$V_{(1,1)}^{X/mu}(z, \bar{z}; k) = 2e^{i\pi/4} e^{\phi(z)} \psi^X(z) 2e^{i\pi/4} e^{\phi(\bar{z})} \bar{\psi}^\mu(\bar{z}) e^{ik \cdot X(z, \bar{z})} \quad (5.22c)$$

Graviscalar Vertex

$$V_{(1,1)}^{XY}(z, \bar{z}; k) = 2e^{i\pi/4} e^{\phi(z)} \psi^X(z) 2e^{i\pi/4} e^{\phi(\bar{z})} \bar{\psi}^Y(\bar{z}) e^{ik \cdot X(z, \bar{z})} \quad (5.22d)$$

The physical states created by these vertices are the graviton $g_{\mu\nu}(X)$, the axion $b_{\mu\nu}(X)$, the dilaton $\Phi(X)$, $6 + 6 = 12$ graviphotons $A_\mu^X(X)$, $\bar{A}_\mu^X(X)$ and $6 \times 6 = 36$ scalars parametrize the coset manifold (see also [2,24])

$$\mathcal{M}_{(\text{moduli})}^{(N=8)} = \frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,6)}{SO(6) \times SO(6)} \quad (5.23)$$

which is just the moduli space of the toroidal compactification on T^6 . Clearly the manifold does not contain all the $N = 8$ massless scalars that are known to be 70. As we are going to see in a moment the remaining $70 - 38 = 32$ scalars come from the Ramond-Ramond sector and parametrize the complex Kaehler manifold (see [24]):

$$\mathcal{M}_{(\text{charged scalars})}^{(N=8)} = \frac{SU(4,4)}{SU(4) \otimes SU(4) \otimes U(1)} \quad (5.24)$$

The manifolds (5.23) and (5.24) are the two submanifolds in which we can split the $N=8$ scalar manifold:

$$\mathcal{M}_{(\text{complete})}^{(N=8)} = \frac{E_{7(7)}}{SU(8)} \quad (5.25)$$

when we decompose the $SU(8)$ irreducible representations with respect to the maximal subgroup:

$$SU(4) \otimes SU(4) \otimes SU(4) \otimes U(1) \sim SO(6) \otimes SO(6) \otimes U(1) \subset SU(8) \quad (5.26)$$

Recalling eqs. (4.3) and (5.22) and following the nomenclature of ref [14], we see that the non-solitonic submanifold of the moduli space (5.23) is given by:

$$\mathcal{M}_{(\text{non-soliton})}^{(N=8)} = \frac{SU(1,1)}{U(1)} \otimes \frac{SO(3,3)}{SO(3) \otimes SO(3)} \quad (5.27)$$

and it is spanned by those scalars whose emission vertices involve only the true fermions $\psi^i, \bar{\psi}^i$ and not the fake ones $\psi^*, \bar{\psi}^*$.

The manifold (5.27) is the moduli space of the $SU(2)^3$ compactification that is smaller than the moduli space of the corresponding T^6 torus. This is explained by the fact the $SU(2)^3$ compactification is equivalent to the toroidal one at special points of the toroidal moduli space.

Let us now see which of these massless states survive the additional projections (5.20). In all cases the graviton, the axion and the dilaton are unaffected by the additional projectors. For the graviphotons and the graviscalars the situation is the following. In the $N=6$ case we must retain only those states for which

$$\gamma_1^{(N=6)} \cdot F = 1 \bmod 2 \quad (5.28)$$

Comparing the explicit form of the generator $\gamma_1 = [s|b]$ with the vertices (5.22) we conclude that the surviving graviphotons are the $4 \oplus 2$ created by $V_{(1,1)}^{\mu X}$ with $X = 1, 2, 3, 1^*, 2^*, 3^*$ and $V_{(1,1)}^{X\mu}$ with $X = 3, 3^*$, while the surviving graviscalars are the $2 \oplus 6 = 12$ created by $V_{(1,1)}^{XY}$ with $X = 3, 3^*$ and $Y = 1, 2, 3, 1^*, 2^*, 3^*$.

Henceforth the moduli space (5.23) is truncated to:

$$\mathcal{M}_{(\text{moduli})}^{(N=6)} = \frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,6)}{SO(2) \otimes SO(6)} \quad (5.29)$$

The non-soliton submanifold of (5.29), yielding the moduli space of the appropriate bilaterally twisted $SU(2)^3$ group-manifold, is easily seen to be the following:

$$\mathcal{M}_{(\text{non-soliton})}^{(N=6)} = \frac{SU(1,1)}{U(1)} \otimes \frac{SO(1,3)}{SO(3)} \quad (5.30)$$

With a similar argument we see that the states surviving the $N=4$ projection, characterized by the fermion number

$$\gamma_1^{(N=4)} \cdot F = [s'|\bar{s}'] \cdot F = 0 \bmod 2 \quad (5.31)$$

are the $2 \oplus 2$ graviphotons respectively created by $V_{(1,1)}^{X\mu}$ and by $V_{(1,1)}^{\mu X}$ with $X = 3, 3^*$ and the $2 \oplus 2 \oplus 4 \oplus 4$ graviscalars created by $V_{(1,1)}^{X\mu}$ with X, Y respectively taken in the range $(3, 3^*)$ and in the range $(1, 2, 1^*, 2^*)$. Hence we get:

$$\mathcal{M}_{(\text{moduli})}^{(N=4)} = \frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2)}{SO(2) \otimes SO(2)} \otimes \frac{SO(4,4)}{SO(4) \otimes SO(4)} \quad (5.32)$$

The corresponding non-soliton submanifold is

$$\mathcal{M}_{(\text{non-soliton})}^{(N=4)} = \frac{SU(1,1)}{U(1)} \otimes SO(1,1) \otimes \frac{SO(2,2)}{SO(2) \otimes SO(2)} \quad (5.33)$$

Finally if we add the projection leading to the $N=3$ theory, which enforces the constraint:

$$\gamma_2^{(N=3)} \cdot F = [s''|\bar{b}'] \cdot F = 1 \bmod 2 \quad (5.34)$$

we are left with the 2 graviphotons created by $V_{(1,1)}^{\mu X}$ with $X = 3, 3^*$ and with the $2 \oplus 4$ graviscalars created by $V_{(1,1)}^{XY}$ with $X = 2, 2^*$ and $Y = 1, 2, 1^*, 2^*$. Therefore we obtain:

$$\mathcal{M}_{(\text{moduli})}^{(N=3)} = \frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,4)}{SO(2) \otimes SO(4)} \quad (5.35a)$$

$$\mathcal{M}_{(\text{non-soliton})}^{(N=3)} = \frac{SU(1,1)}{U(1)} \otimes \frac{SO(1,2)}{SO(2)} \quad (5.35b)$$

c) Universal Ramond-Ramond sector

In the $\gamma_+ + \gamma_-$ sector the GSO projector can be written as follows

$$P_{\gamma_+ + \gamma_-} = P_{+-} U_{+-} \quad (5.36)$$

where by inserting eqs. (5.7), (5.8) into eqs. (3.7) and (3.13) we obtain:

$$U_{+-} = \frac{1}{2} (1 + \eta(-)^{\gamma_0 \cdot F}) \frac{1}{2} (1 + (-)^{\gamma_+ \cdot F}) \frac{1}{2} (1 + \eta(-)^{\gamma_- \cdot F}) \quad (5.37)$$

and

$$P_{+-} = \prod_{i=1}^K \frac{1}{2} (1 + \delta_i \epsilon_+ - \epsilon_- (-)^{\gamma_i \cdot F}) \quad (5.38)$$

In the three cases we consider, utilizing eqs. (5.12), (5.13), (5.14) and (5.15) we get

$$P_{+-}^{(N=6)} = \frac{1}{2} (1 + (-)^{\gamma_1^{(N=6)} \cdot F}) \quad (5.39a)$$

$$P_{+-}^{(N=4)} = \frac{1}{2} (1 + (-)^{\gamma_1^{(N=4)} \cdot F}) \quad (5.39b)$$

$$P_{+-}^{(N=3)} = \frac{1}{2} (1 + (-)^{\gamma_1^{(N=3)} \cdot F}) \frac{1}{2} (1 + (-)^{\gamma_2^{(N=3)} \cdot F}) \quad (5.39c)$$

In the $(\frac{1}{2}, \frac{1}{2})$ picture the unprojected massless vertex of the $\gamma_+ \oplus \gamma_-$ sector has the following form

$$V^{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}} \hat{Q}(z, \bar{z}; k) = e^{\frac{1}{2}\theta(z)} \Sigma^{\dot{\beta}}(z) e^{\frac{1}{2}\theta(\bar{z})} \bar{\Sigma}^{\dot{\gamma}}(\bar{z}) e^{i(k \cdot X(z, \bar{z}))} \quad (5.40)$$

From this vertex we obtain both massless vectors and massless scalars, the vector and scalar vertices being respectively given by

$$V_{(\frac{1}{2}, \frac{1}{2})}^{\mu\hat{P}\hat{Q}}(z, \bar{z}; k) = i(\gamma^{\mu\nu})_{\hat{\alpha}\hat{\beta}} V^{\hat{\alpha}\hat{\beta}\hat{P}\hat{Q}}(z, \bar{z}; k) \quad (5.41a)$$

$$V_{(\frac{1}{2}, \frac{1}{2})}^{\hat{P}\hat{Q}}(z, \bar{z}; k) = i(\gamma^{\hat{P}\hat{Q}})_{\hat{\alpha}\hat{\beta}} V^{\hat{\alpha}\hat{\beta}\hat{P}\hat{Q}}(z, \bar{z}; k) \quad (5.41b)$$

Utilizing the gamma-matrix basis $(\gamma^{\hat{P}\hat{Q}})_{\hat{\alpha}\hat{\beta}}, (\gamma^{\hat{P}\hat{Q}})^{\hat{\alpha}\hat{\beta}}$ we immediately conclude that

$$V^{\alpha\hat{\alpha}\hat{P}\hat{Q}} \quad ; \quad V^{\dot{\alpha}\hat{\beta}\hat{P}\hat{Q}}$$

contribute to the vector states, while

$$V^{\alpha\hat{\alpha}\hat{P}\hat{Q}} \quad ; \quad V^{\dot{\alpha}\hat{\beta}\hat{P}\hat{Q}}$$

contribute to the scalar states.

On the unprojected vertex (5.40) and in an obvious notation, the universal projector (5.37) acts as follows:

$$\begin{aligned} \left[U_+ V_{(\frac{1}{2}, \frac{1}{2})} \right]^{\hat{\alpha}\hat{\beta}\hat{P}\hat{Q}} &= \\ &= \frac{1}{2} (1 + \gamma_5 \Gamma_7)^{\hat{\alpha}\hat{\beta}\hat{P}\hat{Q}} \frac{1}{2} (1 + \gamma_5 \Gamma_7)^{\hat{\beta}\hat{\beta}'\hat{Q}\hat{Q}'} V^{\hat{\alpha}'\hat{\beta}'\hat{P}'\hat{Q}'}_{(\frac{1}{2}, \frac{1}{2})} \quad (5.42) \\ &\equiv \frac{1}{2} (1 + \gamma_5 \Gamma_7) \otimes \frac{1}{2} (\bar{1} + \gamma_5 \bar{\Gamma}_7) V_{(\frac{1}{2}, \frac{1}{2})} \end{aligned}$$

If we choose, for instance, $\eta = 1$, then the states surviving the projection are $4 \times 4 = 16$ real vectors created by $V^{\hat{\alpha}\hat{\beta}\hat{A}\hat{B}}$ plus its complex conjugate $V^{\hat{\alpha}\hat{\beta}\hat{A}\hat{B}}$ and $4 \times 4 = 16$ complex scalars created by $V^{\hat{\alpha}\hat{\beta}\hat{A}\hat{B}}$ and its complex conjugate $V^{\hat{\alpha}\hat{\beta}\hat{A}\hat{B}}$.

These 16 complex scalars parametrize the coset manifold (5.24). As already pointed out, the union of the 38-dimensional coset (5.23) with the 32-dimensional coset (5.24) makes the 70-dimensional coset (5.25) of $N = 8$ supergravity. Similarly the 16 sectors of the $\gamma_+ + \gamma_-$ sector complete, together with the 12 vectors of the 0-sector the set of 28 spin-one fields contained in the $N = 8$ supermultiplet.

On the massless vertex (5.40) the additional projectors (5.39) take the following form

$$\mathcal{P}_{+-}^{(N=6)} = \frac{1}{2} (1 - i\gamma_5 \Gamma_{33}) \otimes \bar{1} \quad (5.43)$$

$$\mathcal{P}_{+-}^{(N=4)} = \frac{1}{2} (1 \otimes \bar{1} - \gamma_5 \Gamma_{33} \otimes \gamma_5 \Gamma_{33}) \quad (5.44a)$$

$$\mathcal{P}_{+-}^{(N=3)} = \frac{1}{2} (1 \otimes \bar{1} - \gamma_5 \Gamma_{33} \otimes \gamma_5 \Gamma_{33}) \frac{1}{2} (1 - i\gamma_5 \Gamma_{22} \otimes \bar{1}) \quad (5.44b)$$

The result of the projections is the following one:

i) **The $N = 6$ theory**

In this case the surviving states are $2 \times 4 = 8$ vector fields created by $V^{\hat{\alpha}\hat{\beta}\hat{A}\hat{B}}_{(\frac{1}{2}, \frac{1}{2})}$, $V^{\hat{\alpha}\hat{\beta}\hat{A}\hat{B}}_{(\frac{1}{2}, \frac{1}{2})}$ and $2 \times 4 = 8$ complex scalars created by $V^{\hat{\alpha}\hat{\beta}\hat{A}\hat{B}}_{(\frac{1}{2}, \frac{1}{2})}$, $V^{\hat{\alpha}\hat{\beta}\hat{A}\hat{B}}_{(\frac{1}{2}, \frac{1}{2})}$. The scalar fields parametrize the following submanifold of the coset (5.24):

$$\mathcal{M}_{\text{scalars}}^{(N=6)} = \frac{SU(2, 4)}{SU(2) \times SU(4) \times U(1)} \quad (5.45)$$

The union the manifold (5.29) with the manifold (5.45) makes the complete scalar manifold $SO^*(12)/SU(6) \times U(1)$ of $N = 6$ supergravity.

As explained in [24] the three submanifolds $SU(1, 1)/U(1)$, $SO(2, 6)/SO(2) \times SO(6)$ and $SU(2, 4)/SU(2) \times SU(4) \times U(1)$ correspond to the coset of scalars of the $N = 4$ supermultiplets contained in $N = 6$ with maximal helicities $\Lambda_{\text{max}} = 2, 1$ and $3/2$, respectively. Furthermore the 8 vector fields of the Ramond-Ramond sector together with the 2 + 6 graviphotons of the 0-sector complete the set of 16 spin-one states contained in the $N = 6$ supermultiplet.

ii) **The $N = 4$ theory**

In this case the projector (5.44a) leaves us with 8 vector fields and 16 scalars parametrizing the coset

$$\mathcal{M}_{\text{scalars}}^{(N=4)} = \frac{SU(2, 2)}{SU(2) \times SU(2) \times U(1)} \otimes \frac{SU(2, 2)}{SU(2) \times SU(2) \times U(1)} \quad (5.46)$$

Putting together the manifold (5.32) we obtain $2 + 6 \times 6 = 38$ scalars parametrizing the manifold

$$\frac{SU(1, 1)}{U(1)} \otimes \frac{SO(6, 6)}{SO(6) \times SO(6)} \quad (5.47)$$

which is the correct scalar manifold for an $N = 4$ supergravity coupled to 6 vector multiplets. Indeed, of the $12 = 4 + 8$ vector fields 6 belong to the $N = 4$ graviton multiplet and 6 are the highest helicity states of 6 vector multiplets.

This shows that the addition of the generator $\gamma_1 = [s', s']$ truncates the $N = 8$ theory to an $N = 4$ supergravity coupled to 6 vector multiplets.

However as we shall see in the next section, the complete massless sector is not exhausted by the states contained in this truncation. Indeed there are 8 additional vector multiplets contributed by the twisted sectors.

iii) **The $N = 3$ theory**

In this case the states surviving the projector (5.44b) are 4 vectors and 8 scalars parametrizing the coset

$$\mathcal{M}_{\text{scalars}}^{(N=3)} = \frac{SU(1, 2)}{SU(2) \times U(1)} \otimes \frac{SU(1, 2)}{SU(2) \times U(1)} \quad (5.48)$$

so that we have a total of $4 + 2 = 6$ vectors and a total of $8 + 2 + 8 = 18$ scalars, corresponding to the union of the manifold (5.48) with the manifold (5.35a). Together these 18 scalars parametrize the coset

$$SU(3,3)/SU(3) \times SU(3) \times U(1) \quad (5.49)$$

which is the correct scalar manifold of 3 $N = 3$ vector multiplets.

Indeed of the 6 massless vectors 3 belong to the $N = 3$ graviton multiplet, while the other 3 belong to vector multiplets.

This shows that the addition of the $\gamma_2 = [s''|b']$ boundary vector truncates the previous $N = 4$ theory with 6 vector multiplets to 3 vector multiplets.

Clearly the additional projection acts also on the 8 twisted multiplets and, in principle, it can introduce new twisted multiplets. We discuss this point in the next section.

6 - Twisted massless sectors

In this section we briefly consider the twisted sectors. In the $N=6$ case, since the complete supergravity multiplet is already contained in the universal sectors, it follows that no other massless states should emerge from the twisted sectors. This is easily checked. Due to space-time supersymmetry we can confine our attention to bosons and look for the additional sectors that contain massless bosonic states before the GSO projection. The only such sector is the following one:

$$\gamma_+ + \gamma_- = [s + s' | \bar{b}] \quad (6.1)$$

and the corresponding projector, with the sign choices (5.7), (5.8), (5.11) and (5.12) is given by:

$$\begin{aligned} \mathbf{P}_{\gamma_+ + \gamma_-}^{N=6} &= \frac{1}{2} (1 - \epsilon_{11}^{N=6} (-)^{\gamma_0 \cdot F}) \frac{1}{2} (1 + (-)^{\gamma_+ \cdot F}) \mathbf{x} \\ &\quad \frac{1}{2} (1 - (-)^{\gamma_- \cdot F}) \frac{1}{2} (1 + \epsilon_{11}^{N=6} (-)^{\gamma_1^{A \cdot F}} \cdot F) \end{aligned} \quad (6.2)$$

In the sector (6.1) the massless states are given by the product of the $s + s'$ Ramond vacuum times the b Ramond vacuum. On this state we have $\gamma_- \cdot F = 0$ so that the whole state is projected out and no additional massless bosons are introduced, as we anticipated.

Let us now check the case of the $N=4$ theory. Here the additional massless bosonic states are located in the sectors:

$$\gamma_1 = [s' | \bar{s}'] \quad (6.3a)$$

$$\gamma_1 + \gamma_+ + \gamma_- = [s + s' | \bar{s} + \bar{s}'] \quad (6.3b)$$

Since the only available $N=4$ matter multiplets are the vector multiplets, it suffices to count the massless vectors. These latter can come only from the sector (6.3a), since the sector (6.3b) is made out of scalars. In the γ_1 sector the projector is:

$$\begin{aligned} \mathbf{P}_{\gamma_1}^{N=6} &= \frac{1}{2} (1 + \epsilon_{11}^{N=6} (-)^{\gamma_0 \cdot F}) \frac{1}{2} (1 + (-)^{\gamma_+ \cdot F}) \mathbf{x} \\ &\quad \frac{1}{2} (1 + (-)^{\gamma_- \cdot F}) \frac{1}{2} (1 + \epsilon_{11}^{N=6} (-)^{\gamma_1^{A \cdot F}} \cdot F) \end{aligned} \quad (6.4)$$

where we have used eqs. (5.13). The vertex in the twisted sector (6.4) is given by:

$$V_{(\frac{1}{2}, \frac{1}{2})}^{\alpha \dot{\beta} \dot{\gamma} \dot{\delta}} \hat{Q}^{(twisted)}(z, \bar{z}, k) = e^{\frac{1}{2} \phi(z)} S^{\dot{\alpha}}(z) \Lambda^{\dot{\beta}}(z) e^{\frac{1}{2} \bar{\phi}(\bar{z})} \bar{S}^{\dot{\gamma}}(\bar{z}) \bar{\Lambda}^{\dot{\delta}}(\bar{z}) e^{i k \cdot X(z, \bar{z})} \quad (6.5)$$

where $\Lambda^{\dot{\beta}}(z)$ is a spin field for the $SO(6)$ current algebra generated by the six internal fermions:

$$\psi_{twisted}^X = (\chi_i^3, \lambda_i^3 \ (i = 1, 2), \chi_{31}^1, \lambda_3^1) \quad (6.6)$$

having Ramond boundary conditions in s' . (A similar definition applies to $\bar{\Lambda}^{\dot{\delta}}$). Using notations similar to those utilized in the s -sector, the projector (6.4) becomes identical in form to the projector (5.36) (with $P_{+-}^{N=4}$ given by eq. (5.39b)) that operates on the universal $\gamma_+ \gamma_-$ sector; that is we have:

$$\mathbf{P}_{\gamma_1} = \frac{1}{2} (1 \otimes \bar{1} + \epsilon_{11} \gamma_5 \Gamma_7 \otimes \bar{\gamma}_5 \bar{\Gamma}_7) \frac{1}{2} (1 + \gamma_5 \Gamma_{33}) \frac{1}{2} (\bar{1} + \bar{\gamma}_5 \bar{\Gamma}_{33}) \quad (6.7)$$

Hence the result is the same and we conclude that there are 8 additional massless vector multiplets, coming from the twisted sector.

Summarizing, the effective Lagrangian of the third string model listed in table I, with the sign choices (5.13) is $N=4$ supergravity coupled to 14 vector multiplets. The corresponding scalar manifold is clearly $SU(1,1)/U(1) \otimes SO(6,14)/SO(6) \otimes SO(14)$.

A similar analysis could be performed for the twisted sectors of the $N=3$ example but we leave it as an exercise for the reader.

7 - The h-map

Gepner [9] has pointed out that, given a modular invariant type II superstring model, there is a general procedure that associates with it two modular invariant heterotic superstring models, respectively based on the Kač-Moody algebra of the group $SO(26)$ and on the Kač-Moody algebra of the group $SO(10) \otimes E_8$. The idea is that of substituting the $c = 13$ conformal system composed by the superghosts ($c = 11$) plus the space-time fermions $\psi^\mu(z)$ ($c = 2$) with a $c = 13$ conformal system of matter fields whose characters transform under the modular group in an isomorphic way to the characters of the $SO(2)$ Kač-Moody algebra spanned by the transverse $\psi^\mu(z)$ fields ($\mu = 3, 4$).

The reason why $SO(26)$ and $SO(10) \otimes E_8$ are selected is twofold. On one hand they have both $\text{rank} = 13$ and therefore, at the lowest level $k/\theta^2 = 1$, they yield a $c = 13$ conformal system. On the other hand, as we show below, their characters transform isomorphically to the characters of $SO(2)$ and, as a consequence, they are interchangeable with these latter without spoiling modular invariance [4,9,17].

A more geometrical way of understanding these two particular groups was pointed out by Gepner and relies on the interpretation of the $d = 4$ superstring models as compactifications of the $d = 10$ model on a suitable 6-dimensional manifold. The argument is that on the same 6-manifold M_6 (leading to the same internal conformal field theory with $c = 9$) we can compactify both the type II superstring and the heterotic superstring. In both cases the left-sector contains the six internal fermions $\psi^X(z)$ ($X = 1, 2, 3, 4, 5, 6$) that couple to the background spin-connection $\omega^{XY}(X)$. The difference resides only in the right-sector where, for the type II case we have the six internal fermions $\psi^X(\bar{z})$, also coupling to the spin-connection, while in the heterotic case we have the 32 heterotic fermions $\xi^p(\bar{z})$ coupling to the gauge field $A^{pq}(X)$ or $SO(32)$ or $SO(16) \otimes SO(16)$ (in the second case the group $SO(16) \otimes SO(16)$ is promoted to $E_8 \otimes E_8$ by the twisted sectors).

In the heterotic case, in order to cancel the σ -model anomalies one has to introduce the Lorentz Chern-Simons term into the definition of the axion field-strength $\mathcal{H}$ and to fulfil the Bianchi identity:

$$d\mathcal{H} = \text{Tr} R \wedge R - \text{Tr} \mathcal{F} \wedge \mathcal{F} \quad (7.1)$$

A general way to do this is to set $\mathcal{H} = 0$ and to identify the background spin-connection ω with the background gauge field A , so that one gets:

$$\text{Tr} R \wedge R = \text{Tr} \mathcal{F} \wedge \mathcal{F} \quad (7.2)$$

This is the geometrical counterpart of the heterotic h-map from type II superstrings to heterotic ones. Indeed, identifying the background spin connection ω with the background gauge field means that six of the thirty-two heterotic fermions are assimilated to the six internal fermions $\psi^X(\bar{z})$. The remaining 26 heterotic fermions couple to the gauge field of either $SO(26)$ or $SO(10) \otimes E_8$, which is the normalizer of $SO(6)$ (the internal holonomy group) in either $SO(32)$ or $E_8 \otimes E_8$. This normalizer is the residual massless gauge group in the $d = 4$ model.

In the case of special internal manifolds the holonomy group can be a proper subgroup of $SO(6)$:

$$\mathcal{H}ol(M_6) \subset SO(6) \quad (7.3)$$

In these cases the normalizer in $SO(32)$ or $E_8 \otimes E_8$ is larger and we have an enhanced gauge group. A typical case is provided by Calabi-Yau compactifications [12], where $\mathcal{H}ol(M_6) = SU(3)$ and, as a consequence $SO(10) \otimes E_8$ is promoted to $E_6 \otimes E_8$. Other examples are provided by the orbifold compactifications where $\mathcal{H}ol(M_6)$ becomes a discrete group and the gauge group is even larger. In particular on Z_3 orbifolds it becomes $SU(3) \otimes E_6 \otimes E_8$.

What we want to point out is that this geometrical interpretation of the h-map holds true also in the $SU(2)^3$ -approach, where the internal manifold is 9-dimensional, rather than 6-dimensional. For a generic 9-manifold which is an exact zero of the beta function (not necessarily the group-manifold $SU(2)^3$) the holonomy group is $\mathcal{H}ol(M_9) = SO(9)$. The gauge group coupling to the heterotic fermions, on the other hand, is either $SO(35)$ or $SO(19) \otimes SO(16)$ (possibly promoted to $SO(19) \otimes E_8$ by the twisted states). After identification of the spin connection ω with the gauge connection A , the massless gauge fields are associated with the normalizer of the holonomy group $SO(9)$ in either $SO(35)$ or $SO(19) \otimes E_8$. This normalizer is once more respectively equal to $SO(26)$ and to $SO(10) \otimes E_8$.

The actual gauge group is enhanced if the holonomy group $\mathcal{H}ol(M_9)$ is a proper subgroup of $SO(9)$ (possibly a discrete subgroup as in the case of the orbifolds).

As in the previous case, what we are actually doing by identifying $\omega = A$ is to assimilate 9 of the heterotic fermions to the 9 internal fermions $\psi^I(\bar{z})$ ($I=1, \dots, 9$) of the type II string.

Let us now describe the h-map in technical terms. As we are going to see, in the fermionizable models we have been discussing, the h-map admits a particularly simple and elegant formulation directly in terms of the boundary vectors and the $c \left[\frac{a}{b} \right]$ coefficients.

Denoting by:

$$Z^{(2n)} \left[\frac{a}{b} \right] (\tau) = \left[\frac{\theta \left[\frac{a}{b} \right] (0|\tau)}{\eta(\tau)} \right]^n \quad (7.4)$$

the partition function of $2n$ free fermions endowed with the same spin structure $\left[\frac{a}{b} \right]$, the characters of the level one $SO(2n)$ Kač-Moody algebra can be written as follows [26]:

$$B_0^{(2n)}(\tau) = Z^{(2n)} \left[\frac{0}{0} \right] (\tau) + Z^{(2n)} \left[\frac{0}{1} \right] (\tau) \quad (7.5a)$$

$$B_v^{(2n)}(\tau) = Z^{(2n)} \left[\frac{0}{0} \right] (\tau) - Z^{(2n)} \left[\frac{0}{1} \right] (\tau) \quad (7.5b)$$

$$B_s^{(2n)}(\tau) = Z^{(2n)} \left[\frac{1}{0} \right] (\tau) + Z^{(2n)} \left[\frac{1}{1} \right] (\tau) \quad (7.5c)$$

$$B_{\bar{s}}^{(2n)}(\tau) = Z^{(2n)} \left[\frac{1}{0} \right] (\tau) - Z^{(2n)} \left[\frac{1}{1} \right] (\tau) \quad (7.5d)$$

where the index $j = 0, v, s, \bar{s}$ denotes the singlet, the vector, the spinor and the conjugate spinor representation respectively.

Under the one-loop modular group, the characters transform as follows:

$$B_i^{(2n)}(\tau + 1) = T_{ij}^{(2n)} B_j^{(2n)}(\tau) \quad (7.6a)$$

$$B_i^{(2n)}(-\frac{1}{\tau}) = S_{ij}^{(2n)} B_j^{(2n)}(\tau) \quad (7.6b)$$

where

$$T_{ij}^{(2n)} = \text{diag} \left(1, -1, e^{i\pi n/4}, e^{i\pi n/4} \right) e^{i\pi n/12} \quad (7.7a)$$

$$S_{ij}^{(2n)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & e^{i\pi n/2} & -e^{i\pi n/2} \\ 1 & -1 & -e^{i\pi n/2} & e^{i\pi n/2} \end{pmatrix} \quad (7.7b)$$

The algebraic basis for the $SO(26)$ h-map is contained in the following identity pointed out by Gepner [9]:

$$T^{(2n)} = M T^{(2n+24)} M \quad (7.8a)$$

$$S^{(2n)} = M S^{(2n+24)} M \quad (7.8b)$$

where the idempotent matrix M ($M^2 = 1$) is:

$$M_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (7.9)$$

From eq.(7.8a) it follows that, if in the expression for the modular invariant partition function of a type II superstring we make the following replacement:

$$B_i^{(2)}(\tau) \Rightarrow M_{ij} B_j^{(26)}(\tau) \quad (7.10)$$

we obtain the modular invariant partition function of a heterotic string model. In eq. (7.10) $B_i^{(2)}(\tau)$ are the characters of the $SO(2)$ Kač-Moody algebra spanned by the transverse space-time fermions, while $B_i^{(26)}(\tau)$ are the characters of the $SO(26)$ Kač-Moody algebra.

Combining eq.(7.10) with eq.(7.5) we can also write the above substitution in the following equivalent way:

$$Z^{(2)} \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) \Rightarrow Z^{(26)} \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) \quad (7.11a)$$

$$Z^{(2)} \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau) \Rightarrow -Z^{(26)} \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau) \quad (7.11b)$$

$$Z^{(2)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tau) \Rightarrow -Z^{(26)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tau) \quad (7.11c)$$

$$Z^{(2)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}(\tau) \Rightarrow -Z^{(26)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}(\tau) \quad (7.11d)$$

Eqs. (7.11) can be formalized at the level of boundary vectors in the following way. Let us first subdivide the 44 right-moving fermions into the following subsets:

$$\tilde{\chi}_i^A \tilde{\lambda}_i^A \quad (i = 1, 2, 3), \quad \tilde{\theta}^p \quad (p = 1, \dots, 10), \quad \tilde{\varphi}^p \quad (p = 1, \dots, 16)$$

where $\tilde{\chi}_i^A$ are the *fake* fermions fermionizing the $SU(2)^3$ bosons, while the 9 $\tilde{\lambda}_i^A$ plus the 10 $\tilde{\theta}^p$, plus the 16 $\tilde{\varphi}^p$ complete the set of 35 heterotic fermions. To the right-moving boundary vectors $\tilde{s}, \tilde{s}', \tilde{s}''$, $\tilde{b}$ and $\tilde{b}'$, defined in eqs. (2.20) and utilized in the construction of the type II generators γ_Λ , we can associate the following right-moving heterotic boundary vectors:

$$\tilde{s} \Rightarrow \sigma = \left\{ \tilde{\chi}_i^1, \tilde{\lambda}_i^1 \quad (i = 1, 2, 3), \quad \tilde{\theta}^p \quad (p = 1, \dots, 10), \quad \tilde{\varphi}^p \quad (p = 1, \dots, 16) \right\} \quad (7.12a)$$

$$\tilde{s}' \Rightarrow \sigma' = \left\{ \tilde{\chi}_i^3, \tilde{\lambda}_i^3 \quad (i = 1, 2), \quad \tilde{\chi}_3^1, \tilde{\lambda}_3^1, \quad \tilde{\theta}^p \quad (p = 1, \dots, 10), \quad \tilde{\varphi}^p \quad (p = 1, \dots, 16) \right\} \quad (7.12b)$$

$$\tilde{s}'' \Rightarrow \sigma'' = \left\{ \tilde{\chi}_i^3, \tilde{\lambda}_i^3 \quad (i = 1, 3), \quad \tilde{\chi}_2^1, \tilde{\lambda}_2^1, \quad \tilde{\theta}^p \quad (p = 1, \dots, 10), \quad \tilde{\varphi}^p \quad (p = 1, \dots, 16) \right\} \quad (7.12c)$$

$$\tilde{b} \Rightarrow \beta = \left\{ \tilde{\chi}_i^2, \tilde{\lambda}_i^2, \tilde{\lambda}_i^3 \quad (i = 2, 3) \right\} \quad (7.12d)$$

$$\tilde{b}' \Rightarrow \beta' = \left\{ \tilde{\chi}_i^2, \tilde{\lambda}_i^2, \tilde{\lambda}_i^3 \quad (i = 1, 2) \right\} \quad (7.12d)$$

Given the generators $\gamma_\Lambda^{\text{type II}}$ of the Ξ group in a type II model, we immediately obtain the Ξ generators $\gamma_\Lambda^{\text{heterotic}}$ in the corresponding heterotic model by performing the substitutions (7.12). Once defined on the generators, the correspondence extends trivially to all the Ξ -group-elements. Furthermore the heterotic $c_{[a]}^{[a]}$ coefficients are related to the type II ones by the following simple rule:

$$c \begin{bmatrix} a^{\text{het}} \\ b^{\text{het}} \end{bmatrix} = c \begin{bmatrix} a \\ b \end{bmatrix} \exp \left[\frac{i\pi}{4} (2a \cdot \psi + 2b \cdot \psi + (a \cdot \psi)(b \cdot \psi)) \right] \quad (7.13)$$

where the vector ψ is defined below:

$$\psi = [0 | \psi^T \quad (T = 3, 4)] \quad (7.14)$$

In this way, given the sectors and the GSO projectors of a type II superstring, we can immediately write down the corresponding sectors and GSO projectors of the associated heterotic theory based on the $SO(26)$ algebra.

The extension of our rules to the $SO(10) \otimes E_8$ case is quite simple. The characters of this Kač-Moody algebra can be written as follows:

$$B_i^{[SO(10) \otimes E_8]}(\tau) = B_i^{(10)}(\tau) B^{(E_8)}(\tau) \quad (i = 0, v, s, \bar{s}) \quad (7.15)$$

where we have defined

$$B^{(E_8)}(\tau) = Z^{(16)} \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) + Z^{(16)} \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau) + Z^{(16)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tau) + Z^{(16)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}(\tau) \quad (7.16)$$

and it can be immediately checked that they transform under the modular group as the $SO(26)$ characters, that is with the matrices $T^{(26)}$ and $S^{(26)}$. This result follows from:

$$B^{(E_8)}(\tau + 1) = e^{-2i\pi/3} B^{(E_8)}(\tau) \quad (7.17a)$$

$$B^{(E_8)}(-\frac{1}{\tau}) = B^{(E_8)}(\tau) \quad (7.17b)$$

In terms of fermion partition functions the replacement

$$B_i^{(2)}(\tau) \Rightarrow M_{ij} B_j^{(SO(10) \oplus E_8)}(\tau) \quad (7.18)$$

corresponds to the substitution

$$Z^{(2)} \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) \Rightarrow Z^{(10)} \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) \cdot B^{(E_8)}(\tau) \quad (7.19a)$$

$$Z^{(2)} \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau) \Rightarrow -Z^{(10)} \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau) \cdot B^{(E_8)}(\tau) \quad (7.19b)$$

$$Z^{(2)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tau) \Rightarrow -Z^{(10)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tau) \cdot B^{(E_8)}(\tau) \quad (7.19c)$$

$$Z^{(2)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}(\tau) \Rightarrow -Z^{(10)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}(\tau) \cdot B^{(E_8)}(\tau) \quad (7.19d)$$

Eqs (7.19) can be interpreted in terms of a two-step process. First we map the type II model into the $SO(26)$ model as described above. Then we extend the Ξ group through the addition of a boundary vector

$$\Delta = [0 | \bar{\delta}] \quad (7.20a)$$

$$\bar{\delta} = \left\{ \varphi^{p''} (p'' = 1, \dots, 16) \right\} \quad (7.20b)$$

and we determine the additional $c_b^{[a]}$ coefficients via the following rule:

$$c \begin{bmatrix} a^{het} + \Delta \\ b^{het} \end{bmatrix} = c \begin{bmatrix} a^{het} \\ b^{het} + \Delta \end{bmatrix} = c \begin{bmatrix} a^{het} + \Delta \\ b^{het} + \Delta \end{bmatrix} = c \begin{bmatrix} a^{het} \\ b^{het} \end{bmatrix} \quad (7.21)$$

As an example we consider the h-map of the N=6 theory described in the previous sections. In this case the h-map of the Ξ generators yields:

$$\gamma_0^{het} = [1 | 1] \quad (7.22a)$$

$$\gamma_+^{het} = [s | 0] \quad (7.22b)$$

$$\gamma_-^{het} = [0 | \bar{s}] \quad (7.22c)$$

$$\gamma_1^{het} = [s' | \beta] \quad (7.22d)$$

In both the $SO(26)$ and the $SO(10) \otimes E_8$ case the space-time supersymmetry is N=2 since we have a surviving gravitino sector γ_+^{het} where the GSO projector is given by:

$$\begin{aligned} P_{\gamma_+^{het}} &= \frac{1}{2} \left(1 - \varepsilon_{++}^{N=6} (-) \gamma_0^{het, F} \right) \frac{1}{2} \left(1 - \varepsilon_{++}^{N=6} (-) \gamma_+^{het, F} \right) x \\ &\quad \frac{1}{2} \left(1 + \varepsilon_{+-}^{N=6} (-) \gamma_-^{het, F} \right) \frac{1}{2} \left(1 - \varepsilon_{+1}^{N=6} (-) \gamma_1^{het, F} \right) \end{aligned} \quad (7.23)$$

Using eqs. (5.11), (5.12) and (5.8) from (7.23) we obtain:

$$\begin{aligned} P_{\gamma_+^{het}} &= \frac{1}{2} \left(1 + (-) \gamma_0^{het, F} \right) \frac{1}{2} \left(1 + (-) \gamma_+^{het, F} \right) x \\ &\quad \frac{1}{2} \left(1 + (-) \gamma_-^{het, F} \right) \frac{1}{2} \left(1 + (-) \gamma_1^{het, F} \right) \end{aligned} \quad (7.24)$$

so that on the heterotic gravitino vertex:

$$V_{\left(\frac{1}{4}\right)}^{\hat{\alpha}\hat{\rho}\mu}(z, \bar{z}, k) = e^{\frac{1}{2}\phi(z)} S^{\hat{\alpha}}(z) \Sigma^{\hat{\rho}}(z) \bar{P}^{\mu}(\bar{z}) e^{ik \cdot X(z, \bar{z})} \quad (7.25)$$

reduces to the standard N=2 projector

$$P_{GSO}^{(N=2)} = \frac{1}{2} (1 + \gamma_5 \Gamma_7) \frac{1}{2} (1 - i\gamma_5 \Gamma_{3^*}) \quad (7.26)$$

The gauge group is also easily determined. Indeed the only sector containing massless vectors is the identity sector 0. Here the GSO projector is given by:

$$\begin{aligned} P_{0^{het}} &= \frac{1}{2} \left(1 - (-) \gamma_0^{het, F} \right) \frac{1}{2} \left(1 + (-) \gamma_+^{het, F} \right) x \\ &\quad \frac{1}{2} \left(1 - (-) \gamma_-^{het, F} \right) \frac{1}{2} \left(1 - (-) \gamma_1^{het, F} \right) \end{aligned} \quad (7.27)$$

and we easily see that the resulting gauge group is the following:

$$G^{gauge} = U(1) \otimes SO(8) \otimes SO(4) \otimes SO(32) \quad (7.28)$$

The $U(1)$ factor corresponds to the universal vector multiplet containing the dilaton and the axion. The factor $SO(8) \otimes SO(4)$ can be understood as the Kaluza-Klein $U(1)^8$ isometry group of the torus T^8 , enhanced by solitonic states. Finally the factor $SO(32)$ has the following origin. Since the internal conformal field-theory on which we have

compactified our superstring is a (4,6) theory we have an additional $SO(6)$ Kač-Moody algebra that, together with the $SO(26)$ algebra introduced by the h-map defines a $rank = 16$ algebra $SO(6) \otimes SO(26)$. In addition the $weight = \frac{1}{2}$ primary fields of the $SO(6)$ system (i.e. the free fermions $\tilde{\psi}^X(\bar{z})$) can be coupled with the $weight = \frac{1}{2}$ primary fields of the $SO(26)$ system (i.e. the free fermions $\tilde{\psi}^{\mu}(\bar{z})$) to make a vertex operator spanning the coset $SO(32)/SO(6) \otimes SO(26)$. In this way we reconstruct the full $SO(32)$ Kač-Moody algebra.

This mechanism is fully analogous to the way in which the E_8 group is obtained in the case of (2,2) compactifications. In that case one extends the $SO(10)$ algebra by means of the $U(1)$ contained in the $n=2$ superconformal algebra and then one adjoins the massless vertices of the coset $E_8/U(1) \otimes SO(10)$ which are given as suitable tensor products of primary fields of the internal (2,2) system times primary fields of the $SO(10) \otimes E_8$ system.

In addition to the 0-sector, in the h-map of the $N=6$ theory there is also another bosonic massless sector:

$$\gamma_+^{het} \oplus \gamma_-^{het} = [s + s' | \beta] \quad (7.29)$$

that contains only hypermultiplets (the twisted sector).

Finally we note that if we rather do the $SO(10) \otimes E_8$ h-map, by adding the Δ -boundary vector (7.20) to the previous system, then the gauge group (7.28) is turned into:

$$G_{gauge} = U(1) \otimes SO(8) \otimes SO(4) \otimes E_8 \otimes E'_8 \quad (7.30)$$

8 - Conclusions

In this paper we have derived four-dimensional superstring-theories with $8 \geq N \geq 2$ extended space-time supersymmetry from type II strings compactified on twisted group-manifolds. Group-manifolds used as internal conformal field-theories allow us to discuss chiral (or in general non-left-right symmetric) theories without the need to introduce chiral bosons in the framework of $2d$ sigma-models.

This formulation is equivalent to some asymmetric orbifold compactifications of $10D$ -superstrings and has been used to discuss (p,q) superconformal field-theories with $p \neq q$ giving rise, in the type II context, to extended space-time supergravities with a different number of gravitinos emerging from the left- and right-moving sectors.

In this way one may obtain models with $N=6,5$ and 3 space-time supersymmetries, which are not possible in the context of Calabi-Yau compactifications.

For all these models it is possible to associate two types of maps to heterotic counterpart theories (h-maps) by means of a straightforward generalization of an analogous map used for left-right symmetric internal superconformal field-theories. In so doing one gets exceptional heterotic theories with $N=1$ and $N=2$ space-time supersymmetry coming from the h-map of type II theories with $N=3,5,6$ space-time supersymmetries.

The mapping between heterotic and type II strings suggests a new way of classifying heterotic string models.

From an effective field-theory point of view, the low-energy Lagrangian of such theories is expected to be severely constrained from the parent type II string. Such types of constraints have been proven very powerful for heterotic compactifications on Calabi-Yau manifolds and should play a similar role in this class of models.

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Table I

Examples of models with $K = 0, 1, 2$

γ_1	γ_2	Corresponding Orbifold	World-sheet and target SUSY
		T^6	$(6, 6) \Rightarrow N = 8$
$\begin{bmatrix} s' \bar{b}' \end{bmatrix} \sim (\bar{1} \otimes \bar{e}_1 \otimes \bar{e}_1, e_3 \otimes e_3 \otimes e_1)$		$\left(\frac{T^6}{Z_2}\right)_L \otimes (T^6)_R$	$(4, 6) \Rightarrow N = 6$
$\begin{bmatrix} s' \bar{s}' \end{bmatrix} \sim (\bar{e}_3 \otimes \bar{e}_3 \otimes \bar{e}_1, e_3 \otimes e_3 \otimes e_1)$		$\frac{T^6}{Z_2}$	$(4, 4) \Rightarrow N = 4$
$\begin{bmatrix} s' \bar{s}' \end{bmatrix} \sim (\bar{e}_3 \otimes \bar{e}_3 \otimes \bar{e}_1, e_3 \otimes e_3 \otimes e_1)$	$\begin{bmatrix} s'' \bar{b}' \end{bmatrix} \sim (\bar{e}_1 \otimes \bar{e}_1 \otimes \bar{1}, e_3 \otimes e_1 \otimes e_3)$	$\left(\frac{T^6}{Z_2 \otimes Z_2}\right)_L \otimes \left(\frac{T^6}{Z_2}\right)_R$	$(2, 4) \Rightarrow N = 3$
$\begin{bmatrix} s' \bar{b}' \end{bmatrix} \sim (\bar{1} \otimes \bar{e}_1 \otimes \bar{e}_1, e_3 \otimes e_3 \otimes e_1)$	$\begin{bmatrix} s'' \bar{b}' \end{bmatrix} \sim (\bar{e}_1 \otimes \bar{e}_1 \otimes \bar{1}, e_3 \otimes e_1 \otimes e_3)$	$\left(\frac{T^6}{Z_2 \otimes Z_2}\right)_L \otimes (T^6)_R$	$(2, 6) \Rightarrow N = 5$
$\begin{bmatrix} s' \bar{s}' \end{bmatrix} \sim (\bar{e}_3 \otimes \bar{e}_3 \otimes \bar{e}_1, e_3 \otimes e_3 \otimes e_1)$	$\begin{bmatrix} s'' \bar{s}' \end{bmatrix} \sim (\bar{e}_3 \otimes \bar{e}_1 \otimes \bar{e}_3, e_3 \otimes e_1 \otimes e_3)$	$\frac{T^6}{Z_2 \otimes Z_2}$	$(2, 2) \Rightarrow N = 2$

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