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REAL WEIGHT b - c SYSTEMS AND CONFORMAL
FIELD THEORY IN HIGHER GENUS

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ABSTRACT

We define bases for differentials with real weight λ on higher genus Riemann surfaces and the corresponding $b - c$ systems. We generalize them by defining real weight bases and $b - c$ systems describing spin field or vertex insertions at specified points of the Riemann surface. We use this formalism to calculate various kinds of correlation functions of λ -spin fields, for rational λ . We compute also the basic correlation functions of chiral vertex insertions in minimal models.

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1. INTRODUCTION

Duality between bosons and fermions in two dimensions is a well established property. Free fermion theories can be bosonized on any Riemann surface ^[1]. The natural generalization of this is to express free chiral anticommuting $b - c$ systems and commuting $\beta - \gamma$ systems in terms of bosonic ones. While this bosonization has been carried out for $b - c$ systems (or $\beta - \gamma$ systems) with integral or half integral weight, little is known so far for systems of generic weight. The problem is not merely academic. Take for example, the rational conformal field theories corresponding to the series of minimal models. It is well-known from the work of Feigin-Fuks ^[2] and in a more explicit way from Dotsenko-Fateev's ^[3] that such models can be formulated as bosonic field theories with a (in general irrational) charge at infinity. This fact brought about an important improvement in the study of minimal models as it allowed to calculate correlation functions through an integral procedure instead of as solutions of complicated differential equations.

It is evident from the work of Feigin and Fuks that there is another formulation beside the bosonic one. In ref.[2] it appears in terms of semiinfinite forms constructed starting from bases of real (or complex) weight tensors. Now these bases can be envisaged as bases of $b - c$ systems. Therefore one expects that there exists also a second field theory formulation (beside the Coulomb gas one). This was indeed proved, for the genus zero case, in ref.[4], where we introduced a $b - c$ system of suitable (in general irrational) weight, to describe any given minimal model, and we eventually arrived at the same results as with the Coulomb gas method.

We notice that, in the light of recent works on Wess-Zumino-Novikov-Witten ^[5] and coset models in which the latter are formulated in terms of suitable bosonic fields, the latter models too can probably be formulated in terms of suitable $b - c$ and $\beta - \gamma$ systems.

The aim of the present paper is to extend the formalism and the results of ref.[4] to higher genus Riemann surfaces, by generalizing the global operator approach of ref.[6]. We deal first of all (section 2) with the problem of generalizing bases of real weight λ differentials to arbitrary genus. Then, inspired by the calculations of the spin field correlation functions, we apply these results to calculate correlation functions of field insertions whose

interpretation depends on the particular $b - c$ system we are considering. To this end in section 3 we introduce suitable $b - c$ systems, essentially characterized by the bases over which we expand them. The bases contain the information concerning the field insertions we want to describe.

In section 4 we test our formalism by rederiving known results of spin field correlation functions and generalize them, by calculating correlation functions of spin fields for rational λ . In this case field insertions represent exactly spin fields. Finally in section 5 we apply our method to bases of $b - c$ systems ("fat" bases) such that the corresponding field insertions represent chiral vertex operators in minimal models. Finally, Appendix A and Appendix B contain details relevant to Section 2 and Section 5, respectively. Appendix C is devoted to our notation and conventions concerning theta function theory.

A few distinctive features of our approach deserve to be pointed out. Our bases allow us to define bra and ket vacua both depending on the Riemann surface. This is why we can generalize in a natural way our previous approach on the sphere ^[4]. Furthermore we obtain the conservation of the total charge in the correlation functions, eq.(3.20) below, as a consequence of geometrical consistency on the basis (the total charge corresponds to the higher genus "charge at infinity" of the Coulomb gas approach). This is an example of the connection between conformal field theory and geometry exposed by the $b - c$ system approach.

2. REAL WEIGHT DIFFERENTIALS

In this section we introduce and give the explicit form of bases for real weight differentials. This generalizes the work of refs.[6,7].

Real weight differentials in general are well defined only on a covering (in general with infinite sheets) of a Riemann surface Σ . However for brevity we will refer to them as λ -differentials on Σ . Let $P_+, P_- \in \Sigma$ be two distinguished points in general position which for $g = 0$ can be identified with $0, \infty$. In the following, $z_{\pm}$ will denote local coordinates vanishing at $P_{\pm}$. For $\lambda \in \mathbb{R}$ we define a λ -differential holomorphic outside $P_{\pm}$ with the

following behaviour in a neighborhood of these points:

$$f_j^{(\lambda,l)}(z_{\pm}) = a_j^{(\lambda)\pm}(l) z_{\pm}^{\pm j - s(\lambda)} (1 + O(z_{\pm})) (dz_{\pm})^{\lambda} = h_j^{(\lambda,l)}(z_{\pm}) (dz_{\pm})^{\lambda}, \quad (2.1)$$

$$s(\lambda) \equiv \frac{g}{2} - \lambda(g-1),$$

where $j \in \mathbb{Z} + s(\lambda) + \lambda(l-1)$, $l \in \mathbb{Z}^+$ and $a_j^{(\lambda)\pm}(l)$ are constants. When $\lambda = \frac{m}{n}$, with m and n relatively prime numbers, there are only n distinct sectors, i.e. $l=1, \dots, n$.

When we go once around P_+ along a trivial homological cycle separating P_+ and P_- , $h_j^{(\lambda,l)}$ picks up a phase:

$$h_j^{(\lambda,l)} \rightarrow e^{2\pi i \lambda (l \mp 1 - g \pm g)} h_j^{(\lambda,l)}. \quad (2.2)$$

For instance when $\lambda \in \mathbb{Z} + \frac{1}{2}$ the Neveu - Schwarz ($l=1$) and the Ramond ($l=2$) sectors are recovered [6].

We remark that $f_j^{(\lambda,l)}(z)$ is a well defined λ - differential on Σ only if $l=1$ (single-valuedness of $h_j^{(\lambda,1)}$ in P_+), and

$$Q(\lambda) \equiv -2s(\lambda) + 1 \equiv (2\lambda - 1)(g-1)$$

is an integer (single-valuedness in P_-). Notice that the same information can be gotten from the Riemann-Roch theorem. Indeed when $\lambda > 1$, $g > 1$, the λ - form $f_j^{(\lambda,1)}(z)$ is holomorphic for $s(\lambda) \leq j \leq -s(\lambda)$, therefore eq.(2.1) gives all the zero modes of the Cauchy - Riemann operator $\bar{\partial}$ coupled to λ - differentials and therefore $Q(\lambda)$ must be an integer.

Due to the Riemann - Roch theorem eq.(2.1) must be modified in a few specific cases which are listed and treated in detail in Appendix A.

For $\lambda \in \frac{\mathbb{Z}}{2}$ the Riemann - Roch theorem guarantees the existence and uniqueness of $f_j^{(\lambda,l)}(z)$ up to the multiplicative constants $a_j^{(\lambda)+}(l)$ ($a_j^{(\lambda)-}(l)$ is completely determined once the $a_j^{(\lambda)+}(l)$ is chosen). For arbitrary λ we will show the existence of $f_j^{(\lambda,l)}(z)$ by explicit construction. The uniqueness follows from the fact that given two λ - differentials satisfying eq.(2.1), their quotient is a meromorphic function with g poles in general position; therefore, by the Weierstrass (Noether) gap theorem [8], this function is a constant.

On Σ the euclidean time can be defined to be the harmonic function

$$\tau = \text{Re} \int_{Q_0}^Q f_{-\frac{g}{2}}^{(1)}(z), \quad (2.3)$$

where $f_{-\frac{g}{2}}^{(1)}(z)$ is the third kind differential with simple poles in $P_{\pm}$ with residues ± 1 , normalized in such a way that the periods along the homology cycles be purely imaginary; its explicit expression is given in Appendix A. The level lines

$$C_{\tau} = \{Q \in \Sigma | \text{Re} \int_{Q_0}^Q f_{-\frac{g}{2}}^{(1)}(z) = \tau\}, \quad (2.4)$$

for $\tau \rightarrow \pm\infty$ become small circles around $P_{\mp}$.

The dual $f_{(1-\lambda,l)}^j(z)$ of $f_j^{(\lambda,l)}(z)$ is defined by means of

$$\frac{1}{2\pi i} \oint_{C_{\tau}} f_i^{(\lambda,l)}(z) f_{(1-\lambda,l)}^j(z) = \delta_i^j. \quad (2.5)$$

Let us start now with the construction of the λ - differential $f_j^{(\lambda,l)}(z)$ in terms of theta functions and prime forms. The expression

$$\frac{E(z, P_+)^{j+s(\lambda)-2\lambda}}{E(z, P_-)^{j+s(\lambda)}},$$

is a (multivalued) λ - differential with degree in P_+ exceeding that of $f_j^{(\lambda,l)}(z)$ by $2s(\lambda) - 2\lambda = g(1 - 2\lambda)$. Since by the Riemann vanishing theorem^[9] the divisor of $\theta(P - gP_+ + \Delta)$ is gP_+ , we put

$$f_j^{(\lambda,l)}(z) = \frac{E(z, P_+)^{j+s(\lambda)-2\lambda}}{E(z, P_-)^{j+s(\lambda)}} \frac{\theta(z + (j - s(\lambda))P_+ - (j + s(\lambda))P_- + (1 - 2\lambda)\Delta)}{\theta(z - gP_+ + \Delta)^{1-2\lambda}}. \quad (2.6)$$

where Δ is the Riemann class. The θ - function in the denominator insures the correct singularity in P_+ , while the θ - function in the numerator guarantees the single-valuedness of $f_j^{(\lambda,l)}(z)$. Since the latter has g zeroes, it follows that the degree of $f_j^{(\lambda,l)}(z)$ is precisely $2\lambda(g - 1)$.

Using the definition of the σ - differential (see Appendix C) and inserting the theta characteristics, eq.(2.6) is generalized to the λ - differential with $[\frac{\delta}{e}]$ - structure, i.e.

$$f_j^{(\lambda, l)}(z) = \frac{\theta_{[\epsilon]}^{[\delta]}(z + (j - s(\lambda))P_+ - (j + s(\lambda))P_- + (1 - 2\lambda)\Delta)}{E(z, P_+)^{-j+s(\lambda)} E(z, P_-)^{j+s(\lambda)}} \sigma(z)^{1-2\lambda}, \quad (2.7)$$

where $j \in \mathbb{Z} + s(\lambda) + \lambda(l-1)$, $l \in \mathbb{Z}^+$ (*). For $\delta, \epsilon = 0$ eq.(2.7) differs from eq.(2.6) by the constant term $s(P_+, \dots, P_+)$ defined in eq.(C.8). When $\lambda = \frac{m}{n}$ then $\delta_i, \epsilon_i \in \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$, so that there are n^{2g} " $[\epsilon]$ - structures", n for each one of the a_i and b_i homology cycles. However, for the time being, we will leave the " $[\epsilon]$ - structures" undetermined. The dual of $f_j^{(\lambda, l)}(z)$ is

$$f_{(1-\lambda, l)}^j(z) = \frac{N(\lambda, l, j) \theta_{[\epsilon]}^{[-\delta]}(z - (j - s(\lambda) + 1)P_+ + (j + s(\lambda) - 1)P_- + (2\lambda - 1)\Delta)}{E(z, P_+)^{j-s(\lambda)+1} E(z, P_-)^{-j-s(\lambda)+1} \sigma(z)^{2\lambda-1}}, \quad (2.8)$$

where $N(\lambda, l, j)$ is a suitable normalization constant fixed by the condition (2.5), which, for the sake of brevity, we do not write down explicitly.

The multivaluedness of $f_{(1-\lambda, l)}^j(z)$ is given by (2.2) with λ replaced by $1 - \lambda$, so that the integrand in eq.(2.5) is a well defined 1-differential.

To conclude this section we give the expression of the covariant delta function for λ - differentials in the sector l :

$$\Delta^{(l)}(z, w) = \sum_j f_j^{(\lambda, l)}(z) f_{(1-\lambda, l)}^j(w), \quad (2.9)$$

that is, if $g(z)$ is a smooth λ - differential with the multivaluedness given in eq.(2.2), we have

$$g(z) = \oint_{C^*} \Delta^{(l)}(z, w) g(w). \quad (2.10)$$

3. THE "FAT" b - c SYSTEM IN THE OPERATOR FORMALISM

We start with the chiral anticommuting b - c system. It is a first order system whose action is

$$S = \int_{\Sigma} b \bar{\partial} c, \quad (3.1)$$

(*) Of course when $\lambda = \frac{m}{n}$, with m and n relatively prime, $l = 1, \dots, n$ represent the only distinct possibilities.

the fields b and c being λ and $1 - \lambda$ differentials, respectively. The anticommutation relations are

$$\{b^{i,l}, c_j^l\} = \delta_j^i, \quad \{b^{i,l}, b^{j,l}\} = 0 = \{c_i^l, c_j^l\}, \quad (3.2)$$

where $j \in \mathbb{Z} + s(\lambda) + \lambda(l - 1)$, $l \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}$. In the operator formalism these fields are expanded in a basis of λ -differentials in such a way that the equations of motion $\bar{\partial}b = 0$ and $\bar{\partial}c = 0$ are satisfied everywhere except possibly at $P_{\pm}$. We use the differentials defined in section 2 as bases to expand the b and c fields

$$b^{(l)}(z) = \sum_j b^{j,l} f_j^{(\lambda,l)}(z), \quad c^{(l)}(z) = \sum_j c_j^l f_{(1-\lambda,l)}^j(z). \quad (3.3)$$

For any $\lambda \in \mathbb{R}$ we define the vacuum $|0\rangle_l$ associated to Σ , following ref.[10] (for $\lambda = 0, 1$ or $g = 1$ there are some modifications, see Appendix A), by requiring that

$$\begin{aligned} b^{j,l}|0\rangle_l &= {}_l\langle 0|c_j^l = 0, & \text{for } j \leq s(\lambda) + \mu(l) - 1; \\ c_j^l|0\rangle_l &= {}_l\langle 0|b^{j,l} = 0, & \text{for } j \geq s(\lambda) + \mu(l), \end{aligned} \quad (3.4)$$

$${}_l\langle 0|0\rangle_l = 1,$$

$$\mu(l) = \overline{\lambda(l - 1)},$$

where the bar denotes the non integral part of $\lambda(l - 1)$. Notice that the requirement ${}_l\langle 0|0\rangle_l = 1$ is consistent with the algebra (3.2).

Since they are going to play a very important role in the following, let us digress a bit on the properties of the vacua we have just defined. They are different from the vacua one usually meets in the literature [1]. First of all both $|0\rangle_l$ and ${}_l\langle 0|$ depend on the moduli of Σ . The second remark is that the vacuum ${}_l\langle 0|$ accounts for the insertion of the total charge $Q(\lambda)$, which is entirely concentrated in P_- . The subscript l means a further insertion of " λ -spin fields" $S^{-,l}(P_-)$ and $S^{+,l}(P_+)$ in ${}_l\langle 0|$ and $|0\rangle_l$, respectively. In particular as we will show below we have

$$|0\rangle_l = S^{+,l}(P_+)|0\rangle_1, \quad {}_l\langle 0| = \frac{{}_1\langle 0|S^{-,l}(P_-)}{{}_1\langle 0|S^{-,l}(P_-)S^{+,l}(P_+)|0\rangle_1},$$

where $S^{-,1}(P_-)$ and $S^{+,1}(P_+)$ are the identity operators. We recall that for $Q(\lambda) \in Z^+$ the higher genus generalization of the standard (i.e. without zero mode insertion) vacuum are defined by the requirement that $b^{(1)}(z)|0\rangle_\Sigma$, $c^{(1)}(z)|0\rangle_\Sigma$ and $\Sigma < 0|b^{(1)}(z)$, $\Sigma < 0|c^{(1)}(z)$ be holomorphic in P_+ and P_- respectively. This condition gives $|0\rangle_\Sigma = |0\rangle_1$ and

$$\Sigma < 0|c_j^1 = 0, \quad \text{for } j \leq -s(\lambda);$$

$$\Sigma < 0|b^{j,1} = 0, \quad \text{for } j \geq 1 - s(\lambda).$$

If $Q(\lambda) \notin Z^+$, it is not possible to define a bra vacuum such that the exponents of z_- in the expansion of $\Sigma < 0|b^{(1)}(z)$, $\Sigma < 0|c^{(1)}(z)$ be integral. The solution of this problem is to use a modified basis, where an "amount $Q(\lambda)$ of singularity" in P_- of the differential $f_j^{(\lambda,1)}(z_\pm)$ ($f_{(1-\lambda,1)}^j(z_\pm)$) is shifted to other points. As we will show, this corresponds to a shift in the location of the charge $Q(\lambda)$. Moreover, using this modified basis in the expansion of the fields $b^{(1)}(z)$ and $c^{(1)}(w)$, the vacuum $1 < 0|$ is defined by the holomorphicity condition on $1 < 0|b^{(1)}(z)$, $1 < 0|c^{(1)}(z)$ in P_- . An analogous argument holds for the vacua in an arbitrary sector l . The bases representing λ -spin field insertions at arbitrary points will be discussed later.

After this digression concerning the vacua (3.3), let us compute the following propagator

$$S^{(l)}(z, w) \equiv 1 < 0|\mathcal{R}(b^{(l)}(z)c^{(l)}(w))|0\rangle_l = \begin{cases} 1 < 0|b^{(l)}(z)c^{(l)}(w)|0\rangle_l, & \text{if } \tau_z > \tau_w; \\ -1 < 0|c^{(l)}(w)b^{(l)}(z)|0\rangle_l, & \text{if } \tau_w > \tau_z. \end{cases} \quad (3.5)$$

Inserting the expansions (3.2) into eq.(3.5) we obtain

$$S^{(l)}(z, w) = \begin{cases} \sum_{j \leq s(\lambda) + \mu(l) - 1} f_j^{(\lambda,l)}(z) f_{(1-\lambda,l)}^j(w), & \text{if } \tau_z > \tau_w; \\ -\sum_{j \geq s(\lambda) + \mu(l)} f_j^{(\lambda,l)}(z) f_{(1-\lambda,l)}^j(w), & \text{if } \tau_w > \tau_z. \end{cases} \quad (3.6)$$

To evaluate $S^{(l)}(z, w)$ one looks at the behaviour of the right hand side of eq.(3.6) in a neighborhood of $P_\pm$ and in the limit $z \rightarrow w$. A careful analysis similar to the one carried

out in [6] shows that

$$S^{(l)}(z, w) = \frac{1}{E(z, w)} \left(\frac{E(z, P_-)}{E(w, P_-)} \right)^{Q(\lambda) - \mu(l)} \left(\frac{E(z, P_+)}{E(w, P_+)} \right)^{\mu(l)} \cdot \left(\frac{\sigma(z)}{\sigma(w)} \right)^{2\lambda - 1} \frac{\theta[\epsilon]_\epsilon(z - w + (Q(\lambda) - \mu(l))P_- + \mu(l)P_+ - (2\lambda - 1)\Delta)}{\theta[\epsilon]_\epsilon((Q(\lambda) - \mu(l))P_- + \mu(l)P_+ - (2\lambda - 1)\Delta)}. \quad (3.7)$$

That equations (3.6) and (3.7) coincide, can be seen also in another way: one considers the propagator $S^{(l)}(z, w)$ in (3.7) as λ - differential in z and expands it in the basis $f_j^{(\lambda)}(z)$

$$S^{(l)}(z, w) = \sum_j a^j(w) f_j^{(\lambda)}(z), \quad (3.8)$$

where

$$a^j(w) = \frac{1}{2\pi i} \oint_{C_r} S^{(l)}(z, w) f_{(1-\lambda)}^j(z), \quad (3.9)$$

It is easy to verify that

$$a^j(w) = \begin{cases} f_{(1-\lambda)}^j(w), & \text{if } j \leq s(\lambda) + \mu(l), \\ 0, & \text{if } j \geq s(\lambda) + \mu(l) + 1, \end{cases} \quad \tau_z > \tau_w; \quad (3.10)$$

$$a^j(w) = \begin{cases} 0, & \text{if } j \leq s(\lambda) + \mu(l), \\ -f_{(1-\lambda)}^j(w), & \text{if } j \geq s(\lambda) + \mu(l) + 1, \end{cases} \quad \tau_w > \tau_z.$$

Notice also that $S^{(l)}(z, w)$ can be seen as the generalization of the Szegő kernel to λ - and $(1 - \lambda)$ - differentials in the w and z variables respectively.

Let us come now to the main point of this section (and a crucial point of our construction), i.e. generalizing the bases (2.6, 2.7) in such a way as to record in the bases themselves the insertion of suitable fields (V-fields) which, as in ref.[4], will be eventually assimilated to vertex operators of the bosonized formulation. The goal will be achieved with the introduction of the bases (3.18) and (3.19). In the following we would like to motivate the introduction of these bases by discussing several intermediate steps that lead to these equations, bearing in mind that the basic idea is simply to shift (inside the bases)

the singularities corresponding to the charge $Q(\lambda)$ and the spin-fields, from P_+ and P_- to generic points on Σ . The first step in this direction consists, following the procedure of ref.[4] for the genus zero case, in locating the zero modes (whose total number is $Q(\lambda)$) outside the point P_- (zero modes exist only when $Q(\lambda) \in \mathbb{Z}^+$ in the sector $l = 1$; however their presence is reflected in any sector l). This corresponds to expanding the b field in terms of [10]

$$\tilde{f}_j^{(\lambda, l)}(z) = \frac{\prod_{i=1}^{Q(\lambda)} E(z, P_i) \theta_{\epsilon}^{[\delta]}(z + u)}{E(z, P_-)^{j-s(\lambda)+1} E(z, P_+)^{-j+s(\lambda)} \sigma(z)^{1-2\lambda}}, \quad (3.11)$$

$$u = (j - s(\lambda))P_+ - (j - s(\lambda) + 1)P_- + \sum_{i=1}^{Q(\lambda)} P_i + (1 - 2\lambda)\Delta,$$

where $j \in \mathbb{Z} + s(\lambda) + \lambda(l - 1)$, $l \in \mathbb{Z}^{(*)}$ and $Q(\lambda) \in \mathbb{Z}^+$. The dual basis (up to a normalization) is

$$\tilde{f}_{(1-\lambda, l)}^j(z) = \frac{\theta_{\epsilon}^{[-\delta]}(z - (j - s(\lambda) + 1)P_+ + (j - s(\lambda))P_- - \sum_{i=1}^{Q(\lambda)} P_i + (2\lambda - 1)\Delta)}{E(z, P_+)^{j-s(\lambda)+1} E(z, P_-)^{-j+s(\lambda)} \sigma(z)^{2\lambda-1} \prod_{i=1}^{Q(\lambda)} E(z, P_i)}. \quad (3.12)$$

The propagator with the insertion of zero-modes at the points $P_1, \dots, P_{Q(\lambda)}$ is

$$S^{(l)}(z, w) = \frac{1}{E(z, w)} \left(\frac{E(z, P_+)E(w, P_-)}{E(w, P_+)E(z, P_-)} \right)^{\mu(l)} \left(\prod_{i=1}^{Q(\lambda)} \frac{E(z, P_i)}{E(w, P_i)} \right) \cdot \left(\frac{\sigma(z)}{\sigma(w)} \right)^{2\lambda-1} \frac{\theta_{\epsilon}^{[\delta]}(z - w - \mu(l)P_- + \mu(l)P_+ + \sum_{i=1}^{Q(\lambda)} P_i - (2\lambda - 1)\Delta)}{\theta_{\epsilon}^{[\delta]}(-\mu(l)P_- + \mu(l)P_+ + \sum_{i=1}^{Q(\lambda)} P_i - (2\lambda - 1)\Delta)}. \quad (3.13)$$

When $\lambda \in \mathbb{R}$ is generic we cannot do the same as above, i.e. insert vertex fields with charge ± 1 . However we can, for example, insert a vertex field at the point $P \in \Sigma$ that absorbs the entire charge $Q(\lambda)$. This can be done substituting in eqs.(3.11-12) the terms $\prod_{i=1}^{Q(\lambda)} E(z, P_i)$ and $\sum_{i=1}^{Q(\lambda)} P_i$ with $E(z, P)^{Q(\lambda)}$ and $Q(\lambda)P$, respectively. The propagator becomes

$$S^{(l)}(z, w) = \frac{1}{E(z, w)} \left(\frac{E(z, P_+)E(w, P_-)}{E(w, P_+)E(z, P_-)} \right)^{\mu(l)} \left(\frac{E(z, P)}{E(w, P)} \right)^{Q(\lambda)}.$$

(*) The range of distinct values of l is determined by the value of λ (see comments after eq.(2.1)).

$$\left(\frac{\sigma(z)}{\sigma(w)}\right)^{2\lambda-1} \frac{\theta[\delta]_\epsilon(z-w-\mu(l)P_- + \mu(l)P_+ + Q(\lambda)P - (2\lambda-1)\Delta)}{\theta[\delta]_\epsilon(-\mu(l)P_- + \mu(l)P_+ + Q(\lambda)P - (2\lambda-1)\Delta)}. \quad (3.14)$$

Notice that $S^{(l)}(z, w)$ can be seen as the propagator in the sector $l = 1$ with the insertion of λ -spin fields at the points $P_\pm$:

$$S^{(l)}(z, w) = \frac{1 \langle 0 | R(S^{-,l}(P_-) b^1(z) c^1(w) S^{+,l}(P_+)) | 0 \rangle_1}{1 \langle 0 | R(S^{-,l}(P_-) S^{+,l}(P_+)) | 0 \rangle_1} \quad (3.15)$$

where $S^{\pm,1}$ are identity operators. The factors $\left(\frac{E(z, P_+)}{E(w, P_+)}\right)^{\mu(l)}$ and $\left(\frac{E(z, P_-)}{E(w, P_-)}\right)^{-\mu(l)}$ in eqs.(3.13-14) are the effects of the λ -spin field insertions $S^{+,l}(P_+)$ and $S^{-,l}(P_-)$.

The remarks just made, concerning eq. (3.15), suggests that also the spin-fields can be moved away from P_+ and P_- . Let us consider a simple example. We want to find the analog of eqs.(3.11-12) with the insertion of the λ - spin fields at arbitrary points Q_1 and Q_2 . The bases fit for that are for $Q(\lambda) \in \mathbb{Z}^+$ (for $\lambda \in \mathbb{R}$ the modification is analogous)

$$\hat{f}_j^{(\lambda, l)}(z) = \left(\frac{E(z, Q_1)}{E(z, Q_2)}\right)^{\mu(l)} \frac{\prod_{i=1}^{Q(\lambda)} E(z, P_i) \sigma(z)^{2\lambda-1} \theta[\delta]_\epsilon(z+u)}{E(z, P_+)^{-j+s(\lambda)+\mu(l)} E(z, P_-)^{j-s(\lambda)+1-\mu(l)}}, \quad (3.16)$$

where $u = (j-s(\lambda)-\mu(l))(P_+-P_-)-P_-+\mu(l)(Q_1-Q_2)+(1-2\lambda)\Delta$, $j \in \mathbb{Z}+s(\lambda)+\lambda(l-1)$. For the sake of conciseness we do not write down explicitly the dual basis. We remark that the numbers $j-s(\lambda)-\mu(l)$ are l -independent and integral. So the conditions (3.4) define a unique (l -independent) vacuum.

Using such modified bases we find a propagator $\hat{S}^{(\lambda, l)}$, which is equal to (3.13) with $P_+(P_-)$ replaced by $Q_1(Q_2)$. As a consequence this propagator is independent of the points $P_\pm$ where the vacua are defined.

As a final natural extension of the procedure outlined above, we introduce now "fat" $b-c$ systems, referred to as $B-C$ systems. The idea is to reshuffle the location of the λ -spin fields and of the total charge, and to represent them as insertions of V -fields at the points P_i . As we have shown in the $g=0$ case ^[4] the fields $V^i(P_i)$ can be regarded as the fermionized counterparts of (chiral) insertions of vertex operators : $e^{i\alpha_i \phi(P_i)}$: (for suitable α_i). In other words

$$S(z, w) = \langle 0 | \mathcal{R}((B(z)C(w))) | 0 \rangle = \frac{\langle \prod_k V^k(P_k) b(z) c(w) \rangle}{\langle \prod_k V^k(P_k) \rangle}. \quad (3.17)$$

The last expression refers to the standard formalism.

In order to implement the idea just outlined, we have to use modified bases

$$g_j^{(\lambda)}(z) = \frac{\prod_{i=1}^n E(z, P_i)^{\tilde{\alpha}_i} \theta_{[\epsilon]}^{[\delta]}(z+u)}{E(z, P_-)^{j-s(\lambda)+1} E(z, P_+)^{-j+s(\lambda)} \sigma(z)^{1-2\lambda}}, \quad (3.18)$$

where $u = (j - s(\lambda))P_+ - (j - s(\lambda) + 1)P_- + \sum_{i=1}^n \tilde{\alpha}_i P_i + (1 - 2\lambda)\Delta$, $j \in \mathbb{Z} + s(\lambda)$. We expand

$$B(z) = \sum_j B^j g_j^{(\lambda)}(z).$$

The dual bases are

$$g_{(1-\lambda)}^j(z) = \frac{\theta_{[-\epsilon]}^{[-\delta]}(z + (-j + s(\lambda) - 1)P_+ - (-j + s(\lambda))P_- - \sum_{i=1}^n \tilde{\alpha}_i P_i + (2\lambda - 1)\Delta)}{E(z, P_+)^{j-s(\lambda)+1} E(z, P_-)^{-j+s(\lambda)} \prod_{i=1}^n E(z, P_i)^{\tilde{\alpha}_i} \sigma(z)^{2\lambda-1}}, \quad (3.19)$$

and

$$C(z) = \sum_j C_j g_{(1-\lambda)}^j(z).$$

The vacuum $|0\rangle$ in eq.(3.17) is defined by the analog of eq.(3.4) and since in eqs.(3.18-19) $j \in \mathbb{Z} + s(\lambda)$ it corresponds to $|0\rangle_1$. The requirement that $g_j^{(\lambda)}$ be of weight λ in z gives the constraint

$$\sum_{i=1}^n \tilde{\alpha}_i = (2\lambda - 1)(g - 1). \quad (3.20)$$

From this equation we can see the topological origin of the constraint over the total charge of the V -fields.

The propagator of the $B - C$ system is

$$S(z, w) \equiv \langle B(z) C(w) \rangle = \frac{1}{E(z, w)} \prod_{i=1}^n \left(\frac{E(z, P_i)}{E(w, P_i)} \right)^{\tilde{\alpha}_i} \left(\frac{\sigma(z)}{\sigma(w)} \right)^{2\lambda-1} \frac{\theta_{[\epsilon]}^{[\delta]}(z - w + \sum_{i=1}^n \tilde{\alpha}_i P_i - (2\lambda - 1)\Delta)}{\theta_{[\epsilon]}^{[\delta]}(\sum_{i=1}^n \tilde{\alpha}_i P_i - (2\lambda - 1)\Delta)}. \quad (3.21)$$

4. SPIN - FIELD CORRELATION FUNCTIONS

As an introduction to the calculation of V -field correlation functions in the next section, here we show how one can compute correlation functions of spin fields in a straightforward way. In particular we test the formalism introduced in the previous section by recovering the 2-point spin-field insertions in the $\lambda = \frac{1}{2}$ and in the commuting $\lambda = \frac{3}{2}$ cases [11]. We then generalize these calculations to rational λ . We recall that λ -spin fields are a particular case of V -fields. We start with the $\lambda = \frac{1}{2}$ case. Let $S^\pm(P_\pm)$ be the spin-field connecting the Neveu-Schwarz with the Ramond vacuum^(*):

$$S^{(2)}(z, w) = {}_2\langle 0 | \mathcal{R}(b(z)c(w)) | 0 \rangle_2 = \frac{\langle S^-(P_-)b(z)c(w)S^+(P_+) \rangle}{\langle S^-(P_-)S^+(P_+) \rangle}. \quad (4.1)$$

Once the propagator $S^{(2)}(z, w)$ is known, the correlation function $\langle S^-(P_-)S^+(P_+) \rangle$ can be computed with a procedure similar to the one introduced by Dixon et al. in [12].

The normal ordered stress energy tensor $T(z)$ of a $b - c$ system of weight λ is

$$T(z) = (1 - \lambda)\partial b c - \lambda b \partial c. \quad (4.2)$$

By means of the operator product expansion

$$\langle T(z)V^1(P_1)\dots V^n(P_n) \rangle = \sum_i \left(\frac{h_i}{(z - P_i)^2} + \frac{\partial P_i}{(z - P_i)} \right) \langle V^1(P_1)\dots V^n(P_n) \rangle \quad (4.3)$$

we obtain a set of first order differential equations which can be solved to get the correlator $\langle V^1(P_1)\dots V^n(P_n) \rangle$.

For $\lambda = \frac{1}{2}$ we have the following correlation function for the stress-energy tensor in presence of spin-field insertions:

$$\frac{\langle S^-(P_-)T(z)S^+(P_+) \rangle}{\langle S^-(P_-)S^+(P_+) \rangle} = \lim_{z \rightarrow w} \frac{1}{2}(\partial_z - \partial_w) \left(S^{(2)}(z, w) - \frac{1}{(z - w)} \right) \quad (4.4)$$

with

$$S^{(2)}(z, w) = \frac{1}{E(z, w)} \left(\frac{E(z, P_+)E(w, P_-)}{E(w, P_+)E(z, P_-)} \right)^{\frac{1}{2}} \frac{\theta[\frac{\delta}{2}](z - w + \frac{1}{2}(P_+ - P_-))}{\theta[\frac{\delta}{2}](\frac{1}{2}(P_+ - P_-))} \quad (4.5)$$

(*) For brevity here we do not consider odd spin structures.

We obtain

$$\langle S^-(P_-)S^+(P_+) \rangle = K_{\delta,\epsilon} E(P_-, P_+)^{-\frac{1}{4}} \theta[\epsilon] \left(\frac{1}{2}(P_+ - P_-) \right) \quad (4.6)$$

where $K_{\delta,\epsilon}$ is an integration constant which carries a dependence on the spin structure. The conformal weight of the spin-fields $S^\pm(P_\pm)$ is $\frac{1}{8}$ as can be seen in two independent ways: either by looking at the residue in the leading singularity in (4.4) or directly from the geometrical weight in eq. (4.6).

Before addressing the commuting $\lambda = \frac{3}{2}$ case, we consider the anticommuting $\lambda = \frac{3}{2}$ system. In order for the propagator to be non vanishing we must put $2g - 2$ insertions of b zero-modes at the points P_i . Therefore the propagator for the $\lambda = \frac{3}{2}$ case is

$$\tilde{S}^{(2)}(z, w) = {}_2\langle 0 | \mathcal{R}(b(z)c(w)) | 0 \rangle_2 = \frac{\langle S^-(P_-)b(z)c(w) \prod_{i=1}^{2g-2} b(z_i) S^+(P_+) \rangle}{\langle S^-(P_-) \prod_{i=1}^{2g-2} b(z_i) S^+(P_+) \rangle} \quad (4.7)$$

where

$$\tilde{S}^{(2)}(z, w) = \frac{1}{E(z, w)} \left(\frac{E(z, P_+)E(w, P_-)}{E(w, P_+)E(z, P_-)} \right)^{\frac{1}{2}} \prod_{i=1}^{2g-2} \frac{E(z, P_i)}{E(w, P_i)} \left(\frac{\sigma(z)}{\sigma(w)} \right)^2 \frac{\theta[\epsilon](z - w + u)}{\theta[\epsilon](u)} \quad (4.8)$$

with $u = \frac{1}{2}(P_+ - P_-) + \sum_{i=1}^{2g-2} P_i - 2\Delta$.

The same procedure outlined above allows us to compute the correlation function with zero-modes and spin-field insertions:

$$\begin{aligned} \langle S^-(P_-)b(P_1)\dots b(P_{2g-2})S^+(P_+) \rangle &= K_{\delta,\epsilon} \frac{\sigma(P_+)}{\sigma(P_-)} \prod_i \sigma(P_i)^2 E(P_-, P_+)^{-\frac{1}{4}} \\ &\cdot \prod_{j,k} E(P_-, P_j)^{-\frac{1}{2}} E(P_k, P_+)^{\frac{1}{2}} \prod_{l \leq m} E(P_l, P_m) \theta[\epsilon] \left(\frac{1}{2}(P_+ - P_-) + \sum_i P_i - 2\Delta \right). \end{aligned} \quad (4.9)$$

In P_-, P_+ the conformal weight is respectively $-\frac{3}{8}$ and $\frac{5}{8}$. The $2g - 2$ points P_i represent the zero-mode insertions, as can be seen by noting that the conformal weight at P_i is precisely $\frac{3}{2}$.

Let us now discuss the commuting case. The quantization of a generic commuting $\beta - \gamma$ system of weight λ and $1 - \lambda$ respectively is achieved by imposing the commutation relations

$$[\beta^{i,l}, \gamma_j^l] = \delta_j^i, \quad [\beta^{i,l}, \beta^{j,l}] = 0 = [\gamma_i^l, \gamma_j^l]. \quad (4.10)$$

The bra and ket vacua of the bosonic system are assumed to satisfy the relation (3.4) as for the anticommuting case.

Once a specific choice of the zero-mode insertions is made, the bosonic propagator

$$\bar{S}^{(l)}(z, w) \equiv {}_l \langle 0 | \mathcal{R}(\beta^{(l)}(z) \gamma^{(l)}(w)) | 0 \rangle_l,$$

coincides with the fermionic one

$$S^{(l)}(z, w) \equiv {}_l \langle 0 | \mathcal{R}(b^{(l)}(z) c^{(l)}(w)) | 0 \rangle_l.$$

The regularized bosonic stress energy tensor is

$$T(z) = - \lim_{z \rightarrow w} ((1 - \lambda) \partial_z - \lambda \partial_w) \left(\beta(z) \gamma(w) - \frac{1}{z - w} \right). \quad (4.11)$$

For $\lambda = \frac{3}{2}$ a possible expression for the bosonic propagator is the one given in eq. (4.8). The points P_i represent the insertion of vertex operators $V_1(P_i)$ which carry a charge 1. These vertex operators can no longer (as in the fermionic case) be considered as zero-mode insertions because their conformal weight, as shown below, is $-\frac{3}{2}$ (*). Starting from the correlator of the stress-energy tensor with vertex operators V_1 and spin-field insertions

$$\frac{\langle S^-(P_-) \left(\prod_{i=1}^{2g-2} V_1(P_i) \right) T(z) S^+(P_+) \rangle}{\langle S^-(P_-) \left(\prod_{i=1}^{2g-2} V_1(P_i) \right) S^+(P_+) \rangle},$$

we get the correlation function:

$$\begin{aligned} \langle S^-(P_-) V_1(P_1) \dots V_1(P_{2g-2}) S^+(P_+) \rangle &= K_{\delta, \epsilon} \frac{\sigma(P_-)}{\sigma(P_+)} \prod_i \sigma(P_i)^{-2} E(P_-, P_+)^{\frac{1}{4}}. \\ &\cdot \prod_{j,k} E(P_-, P_j)^{\frac{1}{2}} E(P_k, P_+)^{-\frac{1}{2}} \prod_{l < m} E(P_l, P_m)^{-1} \frac{1}{\theta[\delta] \left(\frac{1}{2}(P_+ - P_-) + \sum_i P_i - 2\Delta \right)}. \end{aligned} \quad (4.12)$$

In P_-, P_+ the conformal weight is respectively $\frac{3}{8}$ and $-\frac{5}{8}$. In order to compare our result with Atick-Sen's [11] we notice that they absorbed the extra charge by inserting, instead

(*) In the path integral approach these vertex operators are represented by a delta function $\delta(\beta)$.

of $2g - 2$ vertex operators with charge 1, $g - 1$ vertex operators V_2 with charge 2 (which turn out to have conformal weight -4)^(*). In our formalism their result can be recovered starting from the propagator

$$\tilde{S}^{(2)}(z, w) = \frac{1}{E(z, w)} \left(\frac{E(z, P_+)E(w, P_-)}{E(w, P_+)E(z, P_-)} \right)^{\frac{1}{2}} \prod_{i=1}^{g-1} \left(\frac{E(z, P_i)}{E(w, P_i)} \right)^2 \left(\frac{\sigma(z)}{\sigma(w)} \right)^2 \frac{\theta[\epsilon](z - w + u)}{\theta[\epsilon](u)} \quad (4.13)$$

with $u = \frac{1}{2}(P_+ - P_-) + 2\sum_{i=1}^{g-1} P_i - 2\Delta$. With this choice the correlator is

$$\begin{aligned} < S^-(P_-) V_2(P_1) \dots V_2(P_{g-1}) S^+(P_+) > = K_{\delta, \epsilon} \frac{\sigma(P_-)}{\sigma(P_+)} \prod_i \sigma(P_i)^{-4} E(P_-, P_+)^{\frac{1}{4}} \cdot \\ \cdot \prod_{j, k} E(P_-, P_j) E(P_k, P_+)^{-1} \prod_{l < m} E(P_l, P_m)^{-4} \frac{1}{\theta[\epsilon](\frac{1}{2}(P_+ - P_-) + 2\sum_i P_i - 2\Delta)}. \end{aligned} \quad (4.14)$$

This result completely agrees with Atick and Sen's. From the discussion carried out in section 3, it is clear that our approach allows us to compute the correlation function for an arbitrary number of spin-field insertions and, moreover, that the vertex operators, which must be inserted in order to fulfill the condition (3.20) on the total charge, can be placed in general position and assumed to have the most general charge. The corresponding formulas can be obtained by specializing the expressions given in the next section for real λ , eq.(5.2).

In this section however we limit ourselves to give some more examples with rational λ . The general expression for the spin-field correlation functions in the case of anticommuting $b - c$ systems of half-integer weight $\frac{N}{2}$, N odd, is:

(*) The relation between the conformal weight and the charge in the case of a fermionic $b - c$ system of weight λ is shown in the next section and it is given by eq. (5.2). We remark here that since the stress-energy tensor for the bosonic system differs by a minus sign from the fermionic one, in the bosonic case the relation between conformal weight and charge is just the opposite: $h_{\tilde{\alpha}_i} = -\frac{1}{2}\tilde{\alpha}_i^2 + \frac{1}{2}(1 - 2\lambda)\tilde{\alpha}_i$. An immediate consequence of this fact is that only in a certain range of values for λ is it possible to introduce real charge operators which reproduce the $(\lambda, 1 - \lambda)$ $\beta - \gamma$ system.

$$\begin{aligned}
& \langle S^+(P_1) \dots S^+(P_q) b(Q_1) \dots b(Q_r) c(R_1) \dots c(R_s) S^-(T_1) \dots S^-(T_t) \rangle = \\
& = K_{\delta, \epsilon} \prod_i \sigma(P_i)^{\frac{1}{2}(N-1)} \prod_j \sigma(Q_j)^{N-1} \prod_k \sigma(R_k)^{1-N} \prod_l \sigma(T_l)^{\frac{1}{2}(1-N)} \cdot \\
& \cdot \prod_{i < j} E(P_i, P_j)^{\frac{1}{4}} \prod_{k < l} E(Q_k, Q_l) \prod_{m < n} E(R_m, R_n) \prod_{p < q} E(T_p, T_q)^{\frac{1}{4}} \cdot \\
& \cdot \prod_{i, j, k, l} E(P_i, Q_j)^{\frac{1}{2}} E(P_i, R_k)^{-\frac{1}{2}} E(P_i, T_l)^{-\frac{1}{4}} E(Q_j, R_k)^{-1} E(Q_j, T_l)^{-\frac{1}{2}} E(R_k, T_l)^{\frac{1}{2}} \cdot \\
& \cdot \theta[\epsilon] \left(\frac{1}{2} \left(\sum_i P_i - \sum_l T_l \right) + \sum_j Q_j - \sum_k R_k - (N-1)\Delta \right). \tag{4.15}
\end{aligned}$$

The charge conservation condition requires the constraint

$$\frac{1}{2}(q-t) + r - s = (N-1)(g-1)$$

to be satisfied, otherwise the correlation function vanishes. The spin-fields $S^\pm$ have charge $\pm \frac{1}{2}$ and their conformal weight is $\frac{2N-1}{8}$ for S^+ and $\frac{3-2N}{8}$ for S^- .

Likewise the general spin-field correlation function for a commuting $\beta - \gamma$ system with $\lambda = \frac{N}{2}$ is:

$$\begin{aligned}
& \langle S^+(P_1) \dots S^+(P_q) V(Q) S^-(T_1) \dots S^-(T_s) \rangle = \\
& = K_{\delta, \epsilon} \prod_i \sigma(P_i)^{-\frac{1}{2}(N-1)} \sigma(Q)^{\gamma(1-N)} \prod_j \sigma(T_j)^{\frac{1}{2}(N-1)} \cdot \\
& \cdot \prod_{i < j} E(P_i, P_j)^{-\frac{1}{4}} \prod_{k < l} E(T_k, T_l)^{-\frac{1}{4}} \prod_m E(P_m, Q)^{-\frac{1}{2}\gamma} \prod_n E(T_n, Q)^{\frac{1}{2}\gamma} \prod_{t, u} E(P_t, T_u)^{\frac{1}{4}} \cdot \\
& \cdot \left[\theta[\epsilon] \left(\frac{1}{2} \left(\sum_i P_i - \sum_l T_l \right) + \gamma Q - (N-1)\Delta \right) \right]^{-1}. \tag{4.16}
\end{aligned}$$

The spin-fields $S^\pm$ have charge $\pm \frac{1}{2}$ and conformal weight $\frac{1-2N}{8}$ for S^+ and $\frac{2N-3}{8}$ for S^- . The extra charge has been absorbed by inserting at the point Q a single operator V, whose charge γ is given by

$$\gamma = \frac{2(N-1)(g-1) + s - q}{2}. \tag{4.17}$$

Finally we consider the case of a fermionic $b - c$ system of rational weight $\lambda = \frac{N}{M}$, with N, M relatively prime integers. The λ -spin fields with charge $\pm k, k = \frac{1}{M}, \dots, \frac{M-1}{M}$ are denoted by $S_k^\pm$. These fields, of course, live on a suitable covering of the Riemann surface

Σ . The correlation function with the insertion of $N_{\pm k}$ λ -spin fields $S_k^{\pm}$ at the points $P_{\pm k,i}$, where $i_k = 1, \dots, N_{\pm k}$, is given by:

$$\begin{aligned}
& \langle S_1^+(P_{1,1}) \dots S_1^+(P_{1,N_1}) S_2^+(P_{2,1}) \dots S_{M-1}^+(P_{M-1,N_{M-1}}) b(Q_1) \dots b(Q_r) \cdot \\
& \cdot c(R_1) \dots c(R_s) S_{M-1}^-(P_{M-1,1}) \dots S_1^-(P_{-1,N_{-1}}) \rangle = \\
& = K_{\delta,\epsilon} \left[\prod_k \prod_{l=1, \dots, N_{\pm k}} \sigma(P_{\pm k,l})^{\pm(2N-M)\frac{k}{M^2}} \right] \prod_i \sigma(Q_i)^{\frac{2N-M}{M}} \prod_j \sigma(R_j)^{\frac{M-2N}{M}} \cdot \\
& \cdot \left[\prod_{k,i} \prod_{k',j} E(P_{\pm k,i}, P_{\mp k',j})^{-\frac{kk'}{2M^2}} \right] \prod_{k,l,m} E(P_{\pm k,l}, Q_m)^{\pm \frac{k}{M}} \prod_{k,n,q} E(P_{\pm k,n}, R_q)^{\mp \frac{k}{M}} \cdot \\
& \cdot \prod_{(k,i) \neq (k',j)} E(P_{\pm k,i}, P_{\pm k',j})^{\frac{kk'}{2M^2}} \prod_{l < m} E(Q_l, Q_m) \prod_{n < p} E(R_n, R_p) \cdot \\
& \cdot \theta[\epsilon] \left(\pm \sum_{k,i} \frac{k}{M} P_{\pm k,i} + \sum_j Q_j - \sum_l R_l - \frac{(2N-M)}{M} \Delta \right). \tag{4.18}
\end{aligned}$$

The correlator is non-vanishing only if the constraint

$$\frac{2N-M}{M}(g-1) = r-s + \sum_{k=1}^{M-1} \left(\frac{k}{M} N_{+k} - \frac{k}{M} N_{-k} \right) \tag{4.19}$$

is satisfied. The conformal weight of $S_k^{\pm k}$ is given by

$$\frac{k^2 \mp (M-2N)k}{2M^2}. \tag{4.20}$$

The correlation functions calculated so far are in general non-single-valued as the points, where the insertions occur, are shifted by a homology cycle. So the above formulas are to be considered as starting points where single-valuedness and modular invariance are still to be implemented.

5. VERTEX INSERTION CORRELATION FUNCTIONS FOR MINIMAL MODELS

In this section we return to the general case. We consider the “fat” $b-c$ systems defined in section 3 (eqs. (3.17)-(3.21)) and apply to them the method applied in the last section to a few simple cases. We will use

$$\frac{\langle T(z) V^1(P_1) \dots V^n(P_n) \rangle}{\langle V^1(P_1) \dots V^n(P_n) \rangle} = \lim_{z \rightarrow w} ((1-\lambda)\partial_z - \lambda\partial_w) \left(S(z, w) - \frac{1}{z-w} \right) \tag{5.1}$$

where $S(z, w)$ is given by eq. (3.21).

Analyzing the residues of the leading and subleading singularities at P_i , and comparing them with the OPE of $T(z)V^i(P_i)$ we are able to identify the weights h_i of the fields V^i and to extract differential equations which allow us to determine the form of $\langle \prod_i V^i(P_i) \rangle$. Specifically we obtain the relation which determines the conformal weight in terms of the charge $\tilde{\alpha}_i$:

$$h_i \equiv h_{\tilde{\alpha}_i} = \frac{1}{2}\tilde{\alpha}_i^2 - \frac{1}{2}(1 - 2\lambda)\tilde{\alpha}_i, \quad (5.2)$$

and

$$\partial_{P_i} \ln \langle \prod_k V^k(P_k) \rangle = (2\lambda - 1)\tilde{\alpha}_i \frac{\sigma'(P_i)}{\sigma(P_i)} + \tilde{\alpha}_i \frac{\theta'[\frac{\delta}{\epsilon}](v)}{\theta[\frac{\delta}{\epsilon}](v)} + \sum_{i \neq j} \tilde{\alpha}_i \tilde{\alpha}_j \frac{E'(P_i, P_j)}{E(P_i, P_j)}, \quad (5.3)$$

where

$$v = \sum_i \tilde{\alpha}_i P_i - (2\lambda - 1)\Delta. \quad (5.4)$$

Integrating (5.3) we obtain

$$\langle \prod_k V^k(P_k) \rangle = K_{\delta, \epsilon} \prod_k \sigma(P_k)^{\tilde{\alpha}_k(2\lambda-1)} \prod_{i < j} E(P_i, P_j)^{\tilde{\alpha}_i \tilde{\alpha}_j} \theta[\frac{\delta}{\epsilon}](v) \quad (5.5)$$

where $K_{\delta, \epsilon}$ is an integration constant.

As for eq.(5.2) we remark that the weight of V^i can be obtained directly from eq.(5.5) by calculating the conformal weight of the RHS at the points P_i and applying the constraint (3.20).

In general the correlation functions given by eq.(5.5) are not single-valued on Σ . They have cuts with endpoints P_i as expected. But they, in general, pick up a phase and the theta characteristics are shifted when P_i winds around a homology cycle. Precisely, when $P_i \rightarrow P_i + na + mb$, eq.(5.5) goes over to

$$K_{\delta, \epsilon} \prod_k \sigma(P_k)^{\tilde{\alpha}_k(2\lambda-1)} \prod_{i < j} E(P_i, P_j)^{\tilde{\alpha}_i \tilde{\alpha}_j} e^{-2\pi i \tilde{\alpha}_i^2 mn - 2\pi i \tilde{\alpha}_i m \epsilon} \theta[\frac{\delta + \tilde{\alpha}_i m}{\epsilon + \tilde{\alpha}_i n}](v). \quad (5.6)$$

The phase and the shifts can be rational or irrational depending on the values taken by $\tilde{\alpha}_i, \epsilon$ and δ . In order to carry on the discussion of this very important point we need to be more specific, so let us refer as in [4] to the minimal models.

Let us add the information that $B - C$ systems represent minimal models. It has been shown indeed in ref.[4] that in genus 0 “fat” $b - c$ systems constitute a dual (fermionized) version of the Coulomb gas approach, so that by the use of $B - C$ systems we can represent primary fields and reconstruct their correlation functions. The correspondence between the $b - c$ system formalism and Dotsenko-Fateev’s Coulomb gas approach is established by means of the identifications

$$\tilde{\alpha}_i = \sqrt{2}\alpha_i, \quad \tilde{\alpha}_0 \equiv \frac{1}{2}(1 - 2\lambda) = \sqrt{2}\alpha_0, \quad (5.7)$$

and the value of λ suitable for a given minimal model is obtained by equating the central charges

$$c(\lambda) = -12\lambda^2 + 12\lambda - 2 = 1 - 24\alpha_0^2 \equiv 1 - \frac{6(p - q)^2}{pq}.$$

Here α_0 and α_i are the same symbols appearing in [3]. So for the minimal model identified by the integers p and q we have

$$\lambda = \frac{1}{2} + \frac{p - q}{\sqrt{2pq}}. \quad (5.8)$$

Therefore in general λ is irrational. Incidentally we remark that exchanging p and q turns λ into $1 - \lambda$. This is the way the electromagnetic duality of the bosonic formulation is recovered in our formalism. We remember that the constraint (3.20) on the bases implies that the V-fields correlation functions are non-vanishing only if the total charge $\sum_i \tilde{\alpha}_i$ satisfies the relation:

$$\sum_i \tilde{\alpha}_i = -2\tilde{\alpha}_0(g - 1). \quad (5.9)$$

We can now introduce screening charges, i.e. weight one V-fields that by eq.(5.2) have charge

$$\tilde{\alpha}_{\pm} = \tilde{\alpha}_0 \pm \sqrt{\tilde{\alpha}_0^2 + 2}. \quad (5.10)$$

Eqs.(5.9) and (5.10) imply charge quantization, the same as in the case $g = 0$:

$$\tilde{\alpha}_{r,s} = \frac{1}{2}(1 - r)\tilde{\alpha}_+ + \frac{1}{2}(1 - s)\tilde{\alpha}_- = (1 - r)\sqrt{\frac{q}{2p}} - (1 - s)\sqrt{\frac{p}{2q}}, \quad (5.11)$$

where $1 \leq r \leq p - 1, 1 \leq s \leq q - 1$. We remark that in higher genus, w.r.t. the genus 0 case, an extra charge $2\tilde{\alpha}_0 g$ must be reabsorbed (we can think of $2\tilde{\alpha}_0(g - 1)$ as a “bra vacuum

charge"). In general this can easily be done by inserting in the correlation function an appropriate number of screening charges and relative contour integrations; this is possible thanks to the fact that for the $\tilde{\alpha}'_i$ s given by eqs.(5.7-9) the following relation holds

$$(p-1)\tilde{\alpha}_+ + (q-1)\tilde{\alpha}_- = -2\tilde{\alpha}_0 = \sqrt{2}\frac{p-q}{\sqrt{pq}}, \quad (5.12)$$

with the parametrization of $\tilde{\alpha}_i$ and λ given by eqs. (5.6 - 8).

For the sake of simplicity, from now on, we will be dealing only with unitary minimal models. The relevant formulas are obtained from the previous ones by the substitutions $q \rightarrow p$ and $p \rightarrow p+1$.

Let us now return to eq. (5.5) with the parametrization of $\tilde{\alpha}_i$ and λ given by eqs. (5.11) and (5.8), respectively, and the above substitutions. Looking at eq.(5.6) we see that the shifts of the theta-characteristics are in general irrational. This fact is not surprising since it is inherited from the bases (3.18) we started from: they are well-behaved as far as the z -dependence is concerned, but not with respect to the P_i 's (well-behaved meaning that the relevant shifts along homology cycles involve rational phases and shifts). Now, if phases and shifts in (5.6) were rational (as they are in some particular cases), we could take the attitude of the previous section and consider eq.(5.5) as our basic result, from which single-valued quantities can be calculated by taking linear combinations of the RHS of (5.5) with suitable coefficients $K_{\delta,\epsilon}$. Since this is not the case, we take another attitude (see however the remark at the end of this section): we change the bases (3.18). In order to construct the new bases we argue as follows.

When a P_i , i.e. a point where a vertex is inserted, winds around a homology cycle, in general the theta-characteristics change and a phase appears. The phase and the theta-characteristic shifts must correspond to the behaviour of the vertex insertions at the points P_i . We describe the latter in the following way: we say that there exists a finite covering Σ' of the Riemann surface Σ , such that on Σ' the correlation functions $\langle \prod_k V^k(P_k) \rangle$ are single-valued (up to a possible phase depending only on the θ -characteristics). We will see that this simple statement will allow us not only to characterize the covering, but also to determine the θ -characteristics δ and ϵ .

To see this let us modify the bases (3.18) as follows

$$g_j^{(\lambda)}[\epsilon]^\delta(z, P_1, \dots, P_n) = \frac{E(z, P_+)^{j-s(\lambda)}}{E(z, P_-)^{j-s(\lambda)+1}} \prod_{i=1}^n \left(\frac{E(P_i, P_+)^{j-s(\lambda)}}{E(P_i, P_-)^{j-s(\lambda)+1}} \right)^{\tilde{\alpha}_i} \prod_{i=1}^n E(z, P_i)^{\tilde{\alpha}_i} \cdot \prod_{i \leq j}^n E(P_i, P_j)^{\tilde{\alpha}_i \tilde{\alpha}_j} \sigma(z)^{2\lambda-1} \prod_{i=1}^n \sigma(P_i)^{\tilde{\alpha}_i (2\lambda-1)} \theta[\epsilon]^\delta(cz + cv|d\Omega), \quad c, d \in \mathbb{R}, \quad (5.13)$$

where $u = (j - s(\lambda))P_+ - (j - s(\lambda) + 1)P_- + \sum_{i=1}^n \tilde{\alpha}_i P_i + (1 - 2\lambda)\Delta$, $j \in \mathbb{Z} + s(\lambda)$, and c, d are numerical constants to be determined. We recover the bases (3.18) by setting $c = d = 1$, up to a normalization. This normalization is actually very important ^[13] since it allows us to put the conformal weights $h_{\tilde{\alpha}_i} = \frac{1}{2}\tilde{\alpha}_i^2 - \frac{1}{2}(1 - 2\lambda)\tilde{\alpha}_i$ at the points P_i . We point out that this normalization is not *ad hoc*: it is possible only if condition (3.20) is satisfied, being thus another indication of the geometrical consistency of our method.

Our aim now is to determine c and d in such a way that the bases be single-valued (up to theta-characteristic dependent phases) when one of the P_i winds around a homology cycle of the covering Σ' . This will allow us to determine also δ and ϵ . The detailed derivation is contained in Appendix B. Here we write down the result.

Using the freedom in the choice of theta-function basis, pointed out in Appendix B, we can write the new bases (5.13) in the form:

$$g_j^{(\lambda)}[\epsilon]^\delta(z, P_1, \dots, P_n) = \frac{E(z, P_+)^{j-s(\lambda)}}{E(z, P_-)^{j-s(\lambda)+1}} \prod_{i=1}^n \left(\frac{E(P_i, P_+)^{j-s(\lambda)}}{E(P_i, P_-)^{j-s(\lambda)+1}} \right)^{\tilde{\alpha}_i} \prod_{i=1}^n E(z, P_i)^{\tilde{\alpha}_i} \cdot \prod_{i \leq j}^n E(P_i, P_j)^{\tilde{\alpha}_i \tilde{\alpha}_j} \sigma(z)^{2\lambda-1} \prod_{i=1}^n \sigma(P_i)^{\tilde{\alpha}_i (2\lambda-1)} \theta\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix} \middle| \frac{u}{2p(p+1)}\right] (\sqrt{2p(p+1)}(z+v)|2p(p+1)\Omega) \quad (5.14).$$

where $u = 0, 1, \dots, 2p(p+1) - 1$. These bases are characterized by the fact that when any P_i winds $2p(p+1)$ times around any homology cycle of Σ , they remain unchanged. This is not true for the z -dependence: the effect of a z -shift along the homology cycles implies an irrational shift both for the δ and ϵ characteristics. Evidently the z -dependence can be understood only in terms of an infinite covering of Σ . The propagator of the new $b - c$

system is given by

$$\mathcal{S}(z, w) = \frac{1}{E(z, w)} \prod_{i=1}^n \left(\frac{E(z, P_i)}{E(w, P_i)} \right)^{\tilde{\alpha}_i} \left(\frac{\sigma(z)}{\sigma(w)} \right)^{2\lambda-1} \cdot \frac{\theta\left[\frac{u}{2p(p+1)}\right](\sqrt{2p(p+1)}(z - w + \sum_{i=1}^n \tilde{\alpha}_i P_i - (2\lambda - 1)\Delta) | 2p(p+1)\Omega)}{\theta\left[\frac{u}{2p(p+1)}\right](\sqrt{2p(p+1)}(\sum_{i=1}^n \tilde{\alpha}_i P_i - (2\lambda - 1)\Delta) | 2p(p+1)\Omega)}. \quad (5.15)$$

Redoing now the calculations from (5.1) to (5.5) with these new bases we find

$$\langle \prod_k V^k(P_k) \rangle = K_\delta \prod_k \sigma(P_k)^{\tilde{\alpha}_k(2\lambda-1)} \prod_{i < j} E(P_i, P_j)^{\tilde{\alpha}_i \tilde{\alpha}_j} \theta\left[\frac{\delta}{0}\right](y | 2p(p+1)\Omega), \quad (5.16)$$

where v is given by:

$$y = \sqrt{2p(p+1)}(\sum_j \tilde{\alpha}_j P_j + (1 - 2\lambda)\Delta),$$

and the characteristic δ is of the form $\frac{u}{2p(p+1)}$ where $u = 0, \dots, 2p(p+1) - 1$.

A few comments are in order. The result (5.16) is now single-valued when any P_i winds $n = 2p(p+1)$ times around any homology cycle. We have already noticed that this in general cannot be true for any integer n for the formula (5.5). However it is likely that one can recover eq.(5.16) starting from the formula (5.5), by means of an averaging procedure over the theta characteristics similar to the one outlined in the third article of ref. [7].

Eq.(5.16) is our final result in this section. A similar result was obtained via the bosonic approach in ref. [14] (see also [15]). It can be taken as a starting point for computing conformal blocks of minimal models in a generic Riemann surface.

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APPENDIX A

In this Appendix we discuss some particular cases in which bases defined in Section 2 need to be modified. Due to the Riemann - Roch theorem, eq.(2.1) does not work in the

following cases (this follows also from the application of the Riemann vanishing theorem to the explicit form of eq.(2.1) given in eq.(2.7))

$$\begin{aligned} &\forall g \text{ for } |j| \leq \frac{g}{2} \text{ and } \lambda = 0, 1; \\ &\forall g \text{ for } |j| = \frac{1}{2} \text{ and } \lambda = \frac{1}{2}, l = 1 \text{ with odd spin structure}; \\ &g = 1 \text{ for } |j| = \frac{1}{2} \text{ and } \lambda \in \mathbb{Z}; \\ &g = 1 \text{ for } |j| = \frac{1}{2} \text{ and } \lambda \in \mathbb{Z} + \frac{1}{2}, l = 1. \end{aligned}$$

Then we define

$$f_j^{(0)}(z_{\pm}) = a_j^{(0)\pm} z_{\pm}^{\pm j - \frac{g}{2} - \frac{1 \mp 1}{2}} (1 + O(z_{\pm})), \quad -\frac{g}{2} \leq j \leq \frac{g}{2} - 1, \quad (\text{A.1})$$

and $f_{\frac{g}{2}}^{(0)}(z) = 1$. For $\lambda = 1$ we put

$$f_j^{(1)}(z_{\pm}) = a_j^{(1)\pm} z_{\pm}^{\pm j + \frac{g}{2} - \frac{1 \pm 1}{2}} (1 + O(z_{\pm})) dz_{\pm}, \quad -\frac{g}{2} + 1 \leq j \leq \frac{g}{2}, \quad (\text{A.2})$$

and choose $f_{-\frac{g}{2}}^{(1)}(z)$ to be the third kind abelian differential with simple poles at $P_{\pm}$ with residue ± 1 , normalized in such a way that its periods be purely imaginary, i.e. $\text{Re } \oint_{a_i} f_{-\frac{g}{2}}^{(1)}(z) = \text{Re } \oint_{b_i} f_{-\frac{g}{2}}^{(1)}(z) = 0$. For $g = 1$ we define

$$f_j^{(\lambda)}(z) = f_j^{(0)}(z) (f_{\frac{1}{2}}^{(1)}(z))^{\lambda}, \quad \lambda \in \mathbb{Z}, \quad (\text{A.3})$$

$$f_j^{(\lambda, 1)}(z) = f_j^{(0)}(z) (f_{\frac{1}{2}}^{(1)}(z))^{\lambda}, \quad \lambda \in \mathbb{Z} + \frac{1}{2}, \quad (\text{A.4})$$

where the spin structure of $(f_{\frac{1}{2}}^{(1)}(z))^{\frac{1}{2}}$ is chosen to be odd. Finally, for $\lambda = \frac{1}{2}, l = 1$ and odd spin structure:

$$f_{-\frac{1}{2}}^{(\frac{1}{2}, 1)}(z_{\pm}) = a_{-\frac{1}{2}}^{(\frac{1}{2})\pm} (1) z_{\pm}^{-1} (1 + O(z_{\pm})) (dz_{\pm})^{\frac{1}{2}}, \quad (\text{A.5})$$

$$f_{\frac{1}{2}}^{(\frac{1}{2}, 1)}(z_{\pm}) = a_{\frac{1}{2}}^{(\frac{1}{2})\pm} (1) (1 + O(z_{\pm})) (dz_{\pm})^{\frac{1}{2}}. \quad (\text{A.6})$$

The multivaluedness of $f_{(1-\lambda, 1)}^j(z)$ is given by (2.2) with λ replaced by $1 - \lambda$, so that the integrand in eq.(2.5) is a well defined 1 - differential.

The explicit form of the differentials in eqs.(A.1 - 2) is

$$f_j^{(0)}(z) = \frac{\theta(z + (j - \frac{g}{2})P_+ + R - (j + \frac{g}{2} + 1)P_+ + \Delta)}{E(z, R)^{-1} E(z, P_-)^{j + \frac{g}{2} + 1} E(z, P_+)^{-j + \frac{g}{2}} \sigma(z)}, \quad -\frac{g}{2} \leq j \leq \frac{g}{2} - 1, \quad (\text{A.7})$$

$$f_j^{(1)}(z) = \frac{\theta(z + (j + \frac{g}{2} - 1)P_+ - R - (j - \frac{g}{2})P_- - \Delta)}{E(z, R)E(z, P_-)^{j - \frac{g}{2}}E(z, P_+)^{-j - \frac{g}{2} + 1}\sigma(z)^{-1}}, \quad -\frac{g}{2} + 1 \leq j \leq \frac{g}{2}, \quad (A.8)$$

where the point $R \in \Sigma$ is arbitrary; its presence in (A.7) reflects the fact that the meromorphic function defined in (A.7) is unique up to addition of a constant. In eq.(A.8) the pole in R is cancelled by the zero of the θ - function; indeed, being $\theta(-z) = \theta(z)$, from the Riemann vanishing theorem we have for $-\frac{g}{2} + 1 \leq j \leq \frac{g}{2}$

$$\theta((j + \frac{g}{2} - 1)P_+ - (j - \frac{g}{2})P_- - \Delta) = 0.$$

Note that this result allows us to identify the differential $f_{\frac{1}{2}}^{(1)}(z)$ in eq.(A.3) with $\sigma(z)^2$.

For $j = -\frac{g}{2}$ the 1 - differential is

$$f_{-\frac{g}{2}}^{(1)}(z) = d \ln \frac{E(z, P_+)}{E(z, P_-)} - 2\pi i \sum_{j,k=1}^g \text{Im} \left(\int_{P_-}^{P_+} \omega_j \right) (\text{Im} \Omega)_{jk}^{-1} \omega_k(z). \quad (A.9)$$

Finally the conditions (A.5 - 6) give

$$f_{-\frac{1}{2}}^{(\frac{1}{2}, 1)}(z) = \frac{E(z, S)}{E(z, P_+)E(z, P_-)} \theta_{[\epsilon]}^{[\delta]}(z + S - P_+ - P_-), \quad (A.10)$$

$$f_{\frac{1}{2}}^{(\frac{1}{2}, 1)}(z) = \frac{\theta_{[\epsilon]}^{[\delta]}(z - P_-)}{E(z, P_-)}, \quad (A.11)$$

where the θ - characteristics are odd. The presence in eq.(A.10) of the arbitrary point $S \in \Sigma$ is due to the fact that eq.(A.5) does not fix uniquely $f_{-\frac{1}{2}}^{(\frac{1}{2}, 1)}(z)$. Note that $f_{\frac{1}{2}}^{(\frac{1}{2}, 1)}(z)$, as defined in (A.11), is equal to the expression (2.7) with $j = \frac{1}{2}$, $\lambda = \frac{1}{2}$, $l = 1$. This means that in the framework of θ - functions theory the modification (A.6) to eq.(2.1) is automatically taken into account. Moreover since $P_- \in \text{div} \theta_{[\epsilon]}^{[\delta]}(z - P_-)$, $f_{\frac{1}{2}}^{(\frac{1}{2}, 1)}(z)$ does not have poles, it is the zero - mode associated to the odd spin structure $[\epsilon]^{[\delta]}$. In particular using the definition of the prime form, we have

$$f_{\frac{1}{2}}^{(\frac{1}{2}, 1)}(z) = h(P_-)h(z), \quad (A.12)$$

so that we can identify $f_{\frac{1}{2}}^{(\frac{1}{2}, 1)}(z)$ with the $\frac{1}{2}$ - differential which appears in eq.(A.4).

APPENDIX B

Our aim in this Appendix is to determine the values of c , d , δ and ϵ in eq.(5.13). To this end we will use the formula

$$\theta_{[\epsilon]}^{(\delta)}(cz + cn + cm\Omega|d\Omega) = e^{-(1/d)(\pi icm\Omega cm + 2\pi icm(cz + \epsilon + cn))} \theta_{[\epsilon + cn]}^{(\delta + cm/d)}(cz|d\Omega), \quad (B.1)$$

which holds for any b, c, δ and ϵ . As a consequence of this the change of the basis under a shift along the homology cycles is

$$g_j^{(\lambda)}[\epsilon]^{(\delta)}(z, P_1, \dots, P_k + na + mb, \dots, P_n) = e^{-2\pi im(\tilde{\alpha}_k^2 n + \tilde{\alpha}_k \epsilon/c)} g_j^{(\lambda)}[\epsilon + c\tilde{\alpha}_k n]^{(\delta + \tilde{\alpha}_k m/c)}(z, P_1, \dots, P_k, \dots, P_n), \quad (B.2)$$

where we have put $d = c^2$ in order to eliminate the dependence on z , P_+ , P_- , P_i , Ω and Δ . We notice that, had we started from an even more general expression of the bases, in which z, P_+, P_-, P_i, Ω and Δ all had a different coefficient in front of them in the argument of the θ function, the requirement of the phase being independent on z, P_+, P_-, P_i, Ω and Δ in eq.(5.13), would have implied the equality of the coefficients in front of z, P_+, P_-, P_i and Δ , and therefore would have entailed the same conclusion.

Now we impose the new bases (5.13) to live on the covering Σ' , as far as the points P_i are concerned (while we ignore the z -dependence). In other words we assume that there exists an integer N such that when any P_i winds N times around an a or b cycle, the bases return to the initial form (up to a phase). From eq.(B.2) we have

$$g_j^{(\lambda)}[\epsilon]^{(\delta)}(z, P_1, \dots, P_k + Na, \dots, P_n) = g_j^{(\lambda)}[\epsilon + c\tilde{\alpha}_k N]^{(\delta)}(z, P_1, \dots, P_k, \dots, P_n), \quad (B.3)$$

and

$$g_j^{(\lambda)}[\epsilon]^{(\delta)}(z, P_1, \dots, P_k + Nb, \dots, P_n) = e^{-2\pi i N(\tilde{\alpha}_k \epsilon/c)} g_j^{(\lambda)}[\epsilon + \tilde{\alpha}_k N/c]^{(\delta + \tilde{\alpha}_k N/c)}(z, P_1, \dots, P_k, \dots, P_n), \quad (B.4)$$

Therefore, first of all, $c\tilde{\alpha}_k N$ and $\tilde{\alpha}_k N/c$ must be integers. Now due to eq.(5.11) with $q \rightarrow p$ and $p \rightarrow p + 1$, we can write

$$\tilde{\alpha}_k = \frac{n_k}{\sqrt{2p(p+1)}},$$

where n_k is an integer. Therefore we have

$$\frac{cNn_k}{\sqrt{2p(p+1)}} = m_k \in \mathbb{Z}, \quad \frac{Nn_k}{c\sqrt{2p(p+1)}} = l_k \in \mathbb{Z}. \quad (B.5)$$

It follows that

$$m_k = an_k, \quad l_k = bn_k, \quad (B.6)$$

and since in any given minimal unitary model n_k always takes on the value 1, a and b must be integers. Therefore we can rewrite eq.(B.7) as follows

$$cN = a\sqrt{2p(p+1)} \quad \frac{N}{c} = b\sqrt{2p(p+1)}, \quad (B.8)$$

from which we have

$$N^2 = ab \, 2p(p+1) \quad c^2 = \frac{a}{b}. \quad (B.9)$$

Now we choose $N = 2p(p+1)$ (we will discuss later all the other possible choices). Then

$$ab = 2p(p+1), \quad c = \frac{\sqrt{2p(p+1)}}{b}. \quad (B.10)$$

The theta-characteristics δ and ϵ can be deduced from the same formulas (B.3-4). Under the shift $P_k \rightarrow P_k + a$ the characteristic ϵ becomes $\epsilon + c\tilde{\alpha}_k$. So ϵ must be an integer times $c\tilde{\alpha}_k = n_k/b$. Similarly, as a consequence of a shift of a b -cycle, we deduce that δ must be an integer times $\tilde{\alpha}_k/c = \frac{bn_k}{2p(p+1)}$. Finally we arrive at the bases (5.13) where the theta function is

$$\theta_{[t/b]^{bu/2p(p+1)}} \left(\frac{\sqrt{2p(p+1)}}{b} (z+v) \middle| \frac{2p(p+1)}{b^2} \Omega \right) \quad (B.11)$$

Where $t, u \in \mathbb{Z}$. Recalling now that b is an integer we remark that an equivalent basis of b -th order theta-functions ^[9] is given by eq. (B.11) with $b = 1$. We will use this freedom in the choice of a theta-function basis to write our new bases (5.13) in the form (5.14).

Let us recall now the choice we did after eq.(B.9) for N . We could have chosen also $N = 2p(p+1)s$ where s is an integer. In this case $ab = 2p(p+1)s$ and $c = \sqrt{2p(p+1)s}/b$; but this would have entailed a simple redefinition of the basis of theta functions of the type considered above. Similarly we could have chosen $N = p(p+1)$, so that $ab = p(p+1)/2$ etc.. But again this corresponds to a redefinition of the basis of the theta functions, as above. Other choices for N are possible for some particular values of p , but these cases too

can be treated in the same way. So, in conclusion, there is no loss of generality in choosing the bases (5.14).

APPENDIX C

Here we recall some facts about theta functions theory [9]. The θ - function with characteristic $[\epsilon]^\delta$ is defined by

$$\theta[\epsilon]^\delta(z) = \sum_{n \in \mathbb{Z}^g} e^{\pi i(n+\delta)\Omega(n+\delta) + 2\pi i(n+\delta)(z+\epsilon)} = e^{\pi i\delta\Omega\delta + 2\pi i\delta(z+\epsilon)} \theta(z + \epsilon + \Omega\delta), \quad (C.1)$$

$$\theta(z) = \theta[0]^\delta(z), \quad z \in \mathbb{C}^g, \quad \delta, \epsilon \in \mathbb{R}^g,$$

where $\Omega_{i,j} \equiv \oint_{b_i} \omega_j$, $\Omega_{i,j} = \Omega_{j,i}$, $\text{Im}(\Omega) > 0$. The holomorphic differentials ω_i , $i = 1, \dots, g$ are normalized in such a way that $\oint_{a_i} \omega_j = \delta_j^i$, a_i, b_i being the homology cycle basis.

When $\delta_i, \epsilon_i \in \{0, \frac{1}{2}\}$, the θ - function is even or odd depending on the parity of $4\delta\epsilon$. The θ - function is multivalued under a lattice shift in the z - variable:

$$\theta[\epsilon]^\delta(z + n + \Omega m) = e^{-\pi i m \Omega m - 2\pi i m z + 2\pi i(\delta n - \epsilon m)} \theta[\epsilon]^\delta(z). \quad (C.2)$$

The prime form is defined by ($I(z)$ is the Jacobi map)

$$E(z, w) = \frac{\theta[\epsilon]^\delta(I(z) - I(w))}{h(z)h(w)} = -E(w, z), \quad z, w \in \Sigma, \quad (C.3)$$

it is a holomorphic (multivalued) $(-\frac{1}{2}, -\frac{1}{2})$ - differential with a simple zero in $z = w$:

$$E(z, w) \sim z - w, \text{ as } z \rightarrow w. \quad (C.4)$$

$h(z)$ is the square root of $\sum_{i=1}^g \omega_i(z) \partial_{u_i} \theta[\epsilon]^\delta(u)|_{u_i=0}$, it is the holomorphic $\frac{1}{2}$ - differential with non singular (i.e. $\partial_{u_i} \theta[\epsilon]^\delta(u)|_{u_i=0} \neq 0$) odd spin structure $[\epsilon]^\delta$. Notice that $E(z, w)$ does not depend on the particular choice of $[\epsilon]^\delta$. The prime form has the following multivaluedness around the b 's homology cycles:

$$E(z + na + mb, w) = e^{-\pi i m \Omega m - 2\pi i m(I(z) - I(w))} E(z, w). \quad (C.5)$$

The σ - differential is defined by

$$\sigma(z) = e^{-\sum_{j=1}^g \oint_{a_j} \omega_j(w) \ln E(w, z)} \quad (C.6)$$

It is a $\frac{g}{2}$ -differential without zeroes and poles defined on a covering of Σ . It has the following multivaluedness

$$\sigma(z + na + mb) = e^{\pi i(g-1)m\Omega m - 2\pi i m(I(\Delta) - (g-1)I(z))} \sigma(z). \quad (C.7)$$

A useful relation involving $\sigma(z)$ is

$$\theta(I(z) - I(P_1 + \dots + P_g) + I(\Delta)) = s(P_1, \dots, P_g) \sigma(z) E(z, P_1) \dots E(z, P_g), \quad (C.8)$$

where $s(P_1, \dots, P_g)$ is a holomorphic section of a line bundle of degree $g - 1$ in each variable.

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