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Topological Super YM and Super Donaldson Polynomials

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Abstract

After commenting on the recently given superspace formulation of topological super YM in four dimensions, we discuss the space-time projection of this theory. A geometric derivation of the BRS algebra is presented and the supersymmetric generalization of the Donaldson polynomials is determined.

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1 Introduction

Some time ago, a manifestly supersymmetric formulation of $N=1$ and $N=2$ topological super YM theories (SYM) in four dimensions was presented [1]. These theories were constructed by considering a BRS invariant gauge-fixing of the supersymmetric generalization of $\int \text{Tr}(F \wedge F)$. The general approach is based on the discussion of Witten's topological YM [2]-[5] given by the authors of reference [6].

In the present work, we look for the supersymmetric generalization of the Donaldson invariants [7] which correspond to the observables of topological YM [2]. Superspace expressions for these quantities may eventually be found by applying algebraic techniques [8]. However, the results appear to be prohibitively complex and to depend exponentially on the gauge superfield (very much like the superchiral anomaly [8]). For these reasons, we rather look for minimal component field expressions and, in detail, we will proceed as follows.

In the next section, we recall the construction of topological SYM in superspace and we make some comments on the equations of motion. Thereafter, we work out the BRS component field algebra in the WZ-gauge by determining various compensating transformations. The final space-time algebra can also be obtained in a more geometric way by covariant projection from a superspace differential algebra which we will introduce in section 4. As a direct application of these results, the supersymmetric extension of the Donaldson polynomials is determined. To conclude, the superspace approach to the Donaldson invariants will be commented upon.

For simplicity, we only discuss the $N=1$ theory (for which we use the superspace conventions of Wess and Bagger [9]). Since the gauge-fixing procedure involves instanton configurations, consistency requires us to work in Euclidean superspace, where the spinor indices α and $\dot{\alpha}$ represent unrelated $SU(2)$ -indices, and where the structure group is to be complexified (see [1] and references therein).

2 Topological SYM in superspace

In this section, we discuss the equations of motion for topological SYM and we show that the complete quantum action contains, in particular, the Lagrangian for ordinary SYM.

In the antichiral representation, the action for ordinary SYM is given by

$$S_{YM} = \int d^4x \left[\int d^2\bar{\theta} \text{Tr} W^\alpha W_\alpha + \int d^2\bar{\theta} \text{Tr} \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right], \quad (1)$$

$W_\alpha \equiv e^V [-\bar{D}^2(e^{-V} D_\alpha e^V)] e^{-V}$ and $\bar{W}_{\dot{\alpha}} \equiv D^2(e^V \bar{D}_{\dot{\alpha}} e^{-V})$ denoting the field strengths of the gauge superfield V [9]. The equations of motion following from (1) read

$$\nabla^\alpha W_\alpha = -\bar{\nabla}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}, \quad (2)$$

where $\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}$ represent the covariant derivatives of the antichiral representation:

$$\begin{aligned} \nabla_\alpha &= D_\alpha \\ \bar{\nabla}_{\dot{\alpha}} &= e^V \bar{D}_{\dot{\alpha}} e^{-V} \end{aligned}$$

On the other hand, the solution of the constrained Bianchi identities in superspace leads to

$$\nabla^\alpha W_\alpha = \bar{\nabla}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}, \quad (3)$$

a relation which holds in any representation (real, chiral and antichiral) of SYM. Therefore, the equations of motion for the gauge field V can be written as

$$\nabla^\alpha W_\alpha = 0 = \bar{\nabla}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}. \quad (4)$$

By contrast, the action for topological SYM [1] involves a different relative sign,

$$S_{inv} = \int d^4x \left[\int d^2\bar{\theta} \text{Tr} W^\alpha W_\alpha - \int d^2\bar{\theta} \text{Tr} \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right], \quad (5)$$

so that the associated equation of motion for V is given by (3) instead of (2). As part of the superspace Bianchi identities, this equation does not contain any dynamical information. This fact becomes even more apparent in the WZ-supergauge where

$$S_{inv} \propto \int d^4x \left\{ \frac{1}{8} \epsilon_{mnpq} \text{Tr}(F^{mn} F^{pq}) + \frac{1}{2} \partial_m \text{Tr}(\lambda \sigma^m \bar{\lambda}) \right\}, \quad (6)$$

the equations of motion for the gauge field reducing to the ordinary Bianchi identity $\epsilon^{mnq} D_n F_m = 0$.

The topological action (5) is not only invariant under gauge transformations, but under arbitrary variations of the gauge field V ($\delta V = \Psi$). The gauge-fixing of this symmetry is realized by localizing on the self-dual gauge fields ($\bar{W}_{\dot{\alpha}} = 0$). To be more concrete, one introduces a Lagrange multiplier $\bar{b}_{\dot{\alpha}}$ and an anti-ghost field $\bar{\psi}_{\dot{\alpha}}$ and implements the gauge condition by adding the following action to (5):

$$\begin{aligned} \hat{S}^{(1)} &= s \int d^4x \int d^2\bar{\theta} \text{Tr} \bar{\psi}_{\dot{\alpha}} (\bar{b}^{\dot{\alpha}} - \bar{W}^{\dot{\alpha}}) \\ &= \int d^4x \int d^2\bar{\theta} \text{Tr} \bar{b}_{\dot{\alpha}} (\bar{b}^{\dot{\alpha}} - \bar{W}^{\dot{\alpha}}) + \text{ghost terms} \end{aligned} \quad (7)$$

Here, s denotes the BRS differential defined by

$$\begin{aligned} sV &= \Psi - (c - \bar{c}) - \frac{1}{2} [V, c + \bar{c}] - \frac{1}{12} [V, [V, c - \bar{c}]] - \dots \\ s\Psi &= -\frac{1}{2} [\Psi, c + \bar{c}] - \frac{1}{12} \{ [\Psi, [V, c - \bar{c}]] + [V, [\Psi, c - \bar{c}]] \} - \dots \\ &\quad + (\Phi - \bar{\Phi}) + \frac{1}{2} [V, \Phi + \bar{\Phi}] + \frac{1}{12} [V, [\Phi - \bar{\Phi}]] + \dots \\ sc &= -\frac{1}{2} [c, c] + \Phi \quad s\bar{c} = -\frac{1}{2} [\bar{c}, \bar{c}] + \bar{\Phi} \\ s\bar{\psi}_{\dot{\alpha}} &= \bar{b}_{\dot{\alpha}} - [\bar{\psi}_{\dot{\alpha}}, \bar{c}] \\ s\bar{b}_{\dot{\alpha}} &= -[\bar{b}_{\dot{\alpha}}, \bar{c}] + [\bar{\psi}_{\dot{\alpha}}, \bar{\Phi}] \\ s\bar{\Phi} &= -[\bar{\Phi}, c] \quad s\bar{\Phi} = -[\bar{\Phi}, \bar{c}] \end{aligned} \quad (8)$$

The chiral (antichiral) superfields $c, \bar{\Phi}$ ($\bar{c}, \bar{\Phi}$) parametrize gauge transformations and a residual invariance of $S_{inv} + \hat{S}^{(1)}$, respectively [1].

In the antichiral basis, the superfield $\bar{W}^\alpha$ is antichiral ($0 = \nabla_\beta \bar{W}^\alpha = D_\beta \bar{W}^\alpha$), henceforth the multiplier $\bar{b}^\alpha$ also has this property. Now, the functional (7) represents an integral over antichiral superspace (in which antichiral fields are unconstrained) and, therefore, we can freely vary (7) with respect to $\bar{b}^\alpha$. By reinserting the equation of motion of $\bar{b}^\alpha$ into (7), we find that

$$\hat{S}^{(1)} = -\frac{1}{4} \int d^4x \int d^2\bar{\theta} \text{Tr} \bar{W}_\alpha \bar{W}^\alpha + \text{ghost terms}.$$

Since $-1/4 \int d^4x \int d^2\bar{\theta} \text{Tr} \bar{W}_\alpha \bar{W}^\alpha = -1/8 (S_M - S_{inv})$, it is clear that the quantum action $S_{inv} + \hat{S}^{(1)}$ includes the Lagrangians of both ordinary and topological SYM.

The Φ -invariance of $S_{inv} + \hat{S}^{(1)}$ can be fixed by choosing an appropriate gauge-fixing function and introducing additional fields. However, the latter do not occur in the Donaldson invariants and therefore we will not discuss them here (see reference [1] for the determination of the complete quantum action).

3 Symmetry algebra and WZ-gauge

In this section, we use a step-by-step procedure to determine the minimal space-time algebra associated to the superspace BRS algebra (8).

The components of the vector multiplet V may be defined by projection using the invariant derivatives D_α and $\bar{D}_{\dot{\alpha}}$:

$$V| \equiv C, \quad D_\alpha V| \equiv i\chi_\alpha, \quad \bar{D}^2 V| \equiv -2i(M + iN) \\ [D_\alpha, \bar{D}_{\dot{\alpha}}]V| \equiv -2\sigma_{\alpha\dot{\alpha}}^m A_m, \quad \bar{D}^2 D_\alpha V| \equiv 4i\lambda_\alpha, \quad D^\alpha \bar{D}^2 D_\alpha V| \equiv 8D. \quad (9)$$

As usual, the vertical bar denotes the projection to the $\theta = 0 = \bar{\theta}$ component of the corresponding superfield. The SUSY transformations of the space-time fields (9) follow by applying the rules

$$\delta_\zeta(F|) = (\delta_\zeta F)| \\ \delta_\zeta F = (\zeta^\alpha D_\alpha + \bar{\zeta}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}) F \quad (10)$$

to the superfields $F = V, D_\alpha V, \dots$. In the WZ-gauge ($0 = C = \chi_\alpha = M + iN$), these variations have to be accompanied by an infinitesimal gauge transformation δ_N^{WZ} which is such that [10]

$$(\delta_\zeta + \delta_N^{WZ})(C, \chi_\alpha, M + iN) = 0.$$

The latter are parametrized by a chiral superfield Λ with components

$$\Lambda| = 0, \quad D_\alpha \Lambda| = -i(\sigma^\mu \bar{\zeta})_\alpha A_\mu, \quad \bar{D}^2 \Lambda| = 4(\bar{\zeta}\bar{\lambda}). \quad (11)$$

The field content of the superfield Ψ is formally identical to the one of V ,

$$\Psi \sim (B, \varphi_\alpha, P + iQ, \psi_m, \eta_\alpha, d)$$

and the SUSY variations of these space-time fields also follow from the definitions (10). However, in the WZ-gauge ($0 = B = \varphi_\alpha = P + iQ$), we not only have to take into account

the gauge transformations (11), but also compensating Φ -transformations δ_Φ^{WZ} which are such that

$$(\delta_\zeta + \delta_\Lambda^{WZ} + \delta_\Phi^{WZ})(B, \varphi_\alpha, P + iQ) = 0.$$

(The infinitesimal Φ -transformation of the superfield Ψ was given in eq.(8).) This last condition fixes the components of Φ to be given by

$$\Phi| = 0, \quad D_\alpha \Phi| = -i(\sigma^\mu \bar{\zeta})_\alpha \psi_m, \quad \bar{D}^2 \Phi| = 4(\bar{\zeta}\bar{\eta}). \quad (12)$$

The residual gauge freedom for the WZ-gauge fields (A_m, λ_α, D) are ordinary gauge transformations parametrized by a Lie algebra-valued field $\omega(x)$. The non-vanishing components (ψ_m, η_α, d) of Ψ vary, in addition, under ordinary Φ -transformations parametrized by a space-time field $\varphi(x)$:

$$\delta_\Phi \psi_m = -D_m \varphi, \quad \delta_\Phi \eta_\alpha = -\frac{i}{2} [\lambda_\alpha, \varphi], \quad \delta_\Phi d = -\frac{i}{2} [D, \varphi].$$

The algebra of the "SUSY variations in the WZ-gauge" closes up to translations, ordinary gauge and ordinary Φ -transformations. The results are best summarized in differential algebra form by turning all symmetry parameters into ghost fields. The differentials of the latter can be read off from the commutation relations of the infinitesimal symmetry transformations [11]. Altogether, we find

$$sA_m = \psi_m + D_m \omega + \zeta \sigma_m \bar{\lambda} + \bar{\zeta} \bar{\sigma}_m \lambda + a\partial A_m \\ s\lambda_\alpha = \eta_\alpha - \frac{i}{2} \{\lambda_\alpha, \omega\} - i(\sigma^{\mu\nu} \zeta)_\alpha F_{\mu\nu} - i\zeta_\alpha D + a\partial \lambda_\alpha \\ sD = d + \frac{i}{2} [D, \omega] - \zeta \sigma^m (D_m \bar{\lambda}) + \bar{\zeta} \bar{\sigma}^m (D_m \lambda) + a\partial D \\ s\psi_m = -D_m \varphi - \frac{i}{2} \{\psi_m, \omega\} - \zeta \sigma_m \bar{\eta} - \bar{\zeta} \bar{\sigma}_m \eta + a\partial \psi_m \\ s\eta_\alpha = -\frac{i}{2} [\lambda_\alpha, \varphi] + \frac{i}{2} [\eta_\alpha, \omega] + i(\sigma^{\mu\nu} \zeta)_\alpha D_{[\mu} \psi_{\nu]} + i\zeta_\alpha d + a\partial \eta_\alpha \\ sd = -\frac{i}{2} [D, \varphi] - \frac{i}{2} \{d, \omega\} + \zeta \sigma^m (\bar{D}_m \bar{\eta}) - \bar{\zeta} \bar{\sigma}^m (D_m \eta) + a\partial d \\ s\omega = \varphi - \frac{i}{4} \{\omega, \omega\} - 2i(\zeta \sigma^m \bar{\zeta}) A_m + a\partial \omega \\ s\varphi = \frac{i}{2} [\varphi, \omega] + 2i(\zeta \sigma^m \bar{\zeta}) \psi_m + a\partial \varphi \\ sa^m = 2i(\zeta \sigma^m \bar{\zeta}) \\ s\zeta^\alpha = 0 = s\bar{\zeta}_{\dot{\alpha}}. \quad (13)$$

Here, the vector field $a\partial = a^m \partial_m$ generates translations and the covariant derivatives $D_m, \bar{D}_m$ are defined by

$$D_m \omega = \partial_m \omega + \frac{i}{2} [A_m, \omega] \quad (\text{similarly for } \lambda_\alpha, \eta_\alpha, \psi_m, \dots) \\ \bar{D}_m \eta_\alpha = D_m \eta_\alpha + \frac{i}{2} \{\psi_m, \lambda_\alpha\} \quad (\text{similarly for } \bar{\eta}^{\dot{\alpha}}).$$

The field strengths of A_m and ψ_m are given by

$$\begin{aligned} F_{mn} &= \partial_m A_n - \partial_n A_m + \frac{i}{2} [A_m, A_n] \\ \mathcal{D}_{[m} \psi_{n]} &= \mathcal{D}_m \psi_n - \mathcal{D}_n \psi_m \end{aligned}$$

For completeness, we also quote the s -variations of $\tilde{\lambda}^{\dot{\alpha}}$ and $\tilde{\eta}^{\dot{\alpha}}$:

$$\begin{aligned} s\tilde{\lambda}^{\dot{\alpha}} &= \tilde{\eta}^{\dot{\alpha}} - \frac{i}{2} \{ \tilde{\lambda}^{\dot{\alpha}}, \omega \} - i(\tilde{\sigma}^{mn}\tilde{\zeta})^{\dot{\alpha}} F_{mn} + i\tilde{\zeta}^{\dot{\alpha}} D + \alpha\partial\tilde{\lambda}^{\dot{\alpha}} \\ s\tilde{\eta}^{\dot{\alpha}} &= -\frac{i}{2} \{ \tilde{\lambda}^{\dot{\alpha}}, \varphi \} + \frac{i}{2} [\tilde{\eta}^{\dot{\alpha}}, \omega] + i(\tilde{\sigma}^{mn}\tilde{\zeta})^{\dot{\alpha}} \mathcal{D}_{[m}\psi_{n]} - i\tilde{\zeta}^{\dot{\alpha}} d + \alpha\partial\tilde{\eta}^{\dot{\alpha}} \end{aligned}$$

In equation (13), the operator s acts from the left as an antiderivation of degree 1 and the grading of any field is determined by adding its ghost number to its Grassmann parity, e.g. $\deg a^m = 1 = \deg \lambda_a$ and $\deg \zeta_a = 2 = \deg \eta_a$. With these definitions, the s -variations (13) have been verified to be nilpotent. They contain, in particular, the BRS algebra for topological YM [2] [6] and they generate the super Donaldson polynomials by application to the topological Lagrangian (6) (section 5). In the appendix, we have shown that the cohomology of the s -operator defined by (13) contains, at most, polynomials in the constant parameters a^m, ζ^a and $\tilde{\zeta}_a$.

4 Geometric approach to the BRS component field algebra

The derivation of the space-time algebra (13) relies on different supergauge-fixings and it is desirable to have a more geometric approach for which the variations (13) directly follow from superspace expressions without a particular gauge-fixing. We will now present such an approach along the lines of our previous treatment of ordinary SYM [11], to which we refer for further technical details.

The basic superspace algebra is defined by

$$\begin{aligned} sA &= \Psi - \nabla\Omega - L_{\Xi}A \\ s\Psi &= -\nabla\Phi - [\Psi, \Omega] - L_{\Xi}\Psi \\ s\Omega &= \Phi - \frac{1}{2}[\Omega, \Omega] - L_{\Xi}\Omega \\ s\Phi &= -[\Phi, \Omega] - L_{\Xi}\Phi \\ s\Xi &= -\frac{1}{2}[\Xi, \Xi] \end{aligned} \quad (14)$$

Here, $A = E^A A_A$ denotes the connection super 1-form and $\Psi = E^A \Psi_A$ an arbitrary variation of A . The Lie algebra-valued superfields Ω and Φ parametrize gauge and Φ -transformations, respectively. The operator $L_{\Xi} = i\mathcal{d} - d\mathcal{L}_{\Xi}$ represents the Lie derivative with respect to the ghost vector field $\Xi = \Xi^M \partial_M$ which generates translations and SUSY-transformations in superspace ($L_{\Xi} E^A = 0$):

$$(\Xi^M) = (a^m + i\theta\sigma^m\tilde{\zeta} - i\zeta^m\bar{\theta}, \zeta^{\mu}, \tilde{\zeta}_{\mu})$$

The variations of the field strengths $\mathbf{F} = dA + A \wedge A$ and $\nabla\Psi = d\Psi + [\Psi, A]$ immediately follow from (14) and $sd + ds = 0$:

$$\begin{aligned} s\mathbf{F} &= -\nabla\Psi - [\mathbf{F}, \Omega] - L_{\Xi}\mathbf{F} \\ s(\nabla\Psi) &= -[\mathbf{F}, \Phi] - [\nabla\Psi, \Omega] - L_{\Xi}(\nabla\Psi) \end{aligned} \quad (15)$$

By the change of variables $\tilde{\Omega} = \Omega - i_{\Xi}A$, $\tilde{\Phi} = \Phi - i_{\Xi}\Psi$, the previous differential algebra becomes

$$\begin{aligned} sA &= \Psi - \nabla\tilde{\Omega} - i_{\Xi}\mathbf{F} \\ s\mathbf{F} &= -\nabla\Psi - [\mathbf{F}, \tilde{\Omega}] - L_{\Xi}\mathbf{F} \\ s\Psi &= -\nabla\tilde{\Phi} - [\Psi, \tilde{\Omega}] - i_{\Xi}\nabla\Psi \\ s(\nabla\Psi) &= -[\mathbf{F}, \tilde{\Phi}] - [\nabla\Psi, \tilde{\Omega}] - L_{\Xi}(\nabla\Psi) - [\mathbf{F}, i_{\Xi}\Psi] \\ s\tilde{\Omega} &= \tilde{\Phi} - \frac{1}{2}[\tilde{\Omega}, \tilde{\Omega}] + \frac{1}{2}i_{\Xi}\mathbf{F} \\ s\tilde{\Phi} &= -[\tilde{\Phi}, \tilde{\Omega}] + \frac{1}{2}i_{\Xi}\nabla\Psi \\ s\Xi &= -\frac{1}{2}[\Xi, \Xi] \end{aligned} \quad (16)$$

where $L_{\Xi} = i_{\Xi}\nabla - \nabla i_{\Xi}$ denotes the gauge covariant Lie derivative. Finally, the s -invariance of the rigid vielbein E^A allows to determine the variations of the components of all superforms:

$$\begin{aligned} sA_A &= \Psi_A - \nabla_A\tilde{\Omega} - \Xi^B F_{BA} \\ sF_{BA} &= -\Psi_{BA} - [\mathbf{F}_{BA}, \tilde{\Omega}] - \Xi^C \nabla_C F_{BA} \\ s\Psi_A &= -\nabla_A\tilde{\Phi} - [\Psi_A, \tilde{\Omega}] - \Xi^B \Psi_{BA} \\ s\Psi_{BA} &= -[\mathbf{F}_{BA}, \tilde{\Phi}] - [\Psi_{BA}, \tilde{\Omega}] - \Xi^C \nabla_C \Psi_{BA} \\ s\tilde{\Omega} &= \tilde{\Phi} - \frac{1}{2}[\tilde{\Omega}, \tilde{\Omega}] + \frac{1}{2}\Xi^A \Xi^B F_{BA} \\ s\tilde{\Phi} &= -[\tilde{\Phi}, \tilde{\Omega}] + \frac{1}{2}\Xi^A \Xi^B \Psi_{BA} \\ s\Xi^M &= -\Xi^N (\partial_N \Xi^M) \end{aligned} \quad (17)$$

Here,

$$\begin{aligned} \Psi_{BA} &\equiv (\nabla\Psi)_{BA} = \nabla_{[B}\Psi_{A]} - T_{BA}{}^C \Psi_C \\ \tilde{\nabla}_C \Psi_{BA} &\equiv \nabla_C \Psi_{BA} - [\Psi_C, \mathbf{F}_{BA}] \end{aligned} \quad (18)$$

where $T_{BA}{}^C$ denotes the components of the rigid superspace torsion. We note that the covariant derivative $\tilde{\nabla}_C$ bears some similarity with the so-called ‘‘supercovariant derivatives’’ of component field supergravity [9].

In order to obtain a sensible theory upon projection to space-time, we impose the standard constraints on the curvature $\mathbf{F}$ of the connection A , i.e. $\mathbf{F}_{\alpha\beta} = 0$ where $\alpha = \alpha$ or $\alpha = \dot{\alpha}$. Consistency then requires us to impose the same constraints on the field strength

5 (Super) Donaldson polynomials

Let us first discuss topological YM [2]. The simplest way to derive the Donaldson invariants in this framework is the following [3].

First, introduce differentials forms in 4-space:

$$A \equiv A_m dx^m, \quad F \equiv dA + A \wedge A = \frac{1}{2} F_{mn} dx^m \wedge dx^n \quad (24)$$

$$\psi \equiv \psi_m dx^m, \quad D\psi = d\psi + [A, \psi] = \frac{1}{2} (D_{[m} \psi_{n]}) dx^m \wedge dx^n.$$

For each variable, add the form degree to the ghost number. Then, define generalized differentials and gauge fields by [12]

$$\check{d} = d + s, \quad \check{A} = A + \omega, \quad \check{F} = \check{d}\check{A} + \check{A} \wedge \check{A}. \quad (25)$$

By construction, the field $\check{A}$ satisfies the Bianchi identity $0 = \check{D}\check{F} = \check{d}\check{F} + [\check{A}, \check{F}]$. Now, postulate the generalized "Russian formula" (or "horizontal condition")

$$\check{F} = F + \psi + \varphi, \quad (26)$$

where the right-hand side contains all fields of total degree 2. Equation (26) describes, respectively at ghost number 0, 1 and 2, the definition of F , the s -variation of the classical variable A and the s -variation of the Faddeev-Popov ghost ω . Substitution of (26) into the Bianchi identity

$$0 = \check{D}\check{F} = (d + s)\check{F} + [\check{A} + \omega, \check{F}] \quad (27)$$

reproduces, respectively at ghost number 0, 1, 2 and 3, the usual Bianchi identity and the s -variations of F , ψ and φ .

Finally, expansion of $\check{W} \equiv -1/2 \text{Tr} \check{F} \wedge \check{F}$ with respect to the ghost number,

$$\check{W} = W_4^0 + W_3^1 + W_2^2 + W_1^3 + W_0^4,$$

(where W_p^s has ghost number g and form degree p) yields the Donaldson polynomials:

$$\begin{aligned} W_4^0 &= -\frac{1}{2} \text{Tr} [F \wedge F] \\ W_3^1 &= -\text{Tr} [\psi \wedge F] \\ W_2^2 &= -\text{Tr} [\varphi F + \frac{1}{2} \psi \wedge \psi] \\ W_1^3 &= -\text{Tr} [\varphi \psi] \\ W_0^4 &= -\frac{1}{2} \text{Tr} [\varphi \varphi]. \end{aligned} \quad (28)$$

From

$$0 = -\frac{1}{2} \text{Tr} \check{D}(\check{F} \wedge \check{F}) = \check{d}\check{W} = (d + s)(W_4^0 + W_3^1 + W_2^2 + W_1^3 + W_0^4),$$

of $\Psi : \Psi_{ab} = 0$. As a consequence, the Bianchi identity for $\nabla\Psi$ (i.e. $\nabla\nabla\Psi = -[\mathbf{F}, \Psi]$) is not satisfied anymore by construction; rather it becomes a system of differential equations for the non-constrained components of $\nabla\Psi$:

$$\sum_{(C,B,A)} \left\{ \hat{\nabla}_C \Psi_{BA} - T_{CB}^D \Psi_{DA} \right\} = 0 \quad (19)$$

(the sum running over all graded cyclic permutations of C, B and A). The solution of equations (19) is very similar to the solution of the Bianchi identities for ordinary SYM [9]. The result depends on two Lie algebra-valued superfields $V_\alpha, \bar{V}^\alpha$ which are analogous to the SYM-field strengths W_α and $\bar{W}^\alpha$:

$$\begin{aligned} \Psi_{\alpha\alpha} &= i(\sigma_\alpha)_{\alpha\beta} \bar{V}^\beta \\ \Psi_{\alpha\dot{a}} &= iV^\beta (\sigma_\alpha)_{\beta\dot{a}} \\ \Psi_{\dot{a}\dot{b}} &= -\frac{1}{2} \left\{ (\nabla\sigma_{\dot{a}\dot{b}} V - [\Psi, \sigma_{\dot{a}\dot{b}} W]) - (\bar{\nabla}\bar{\sigma}_{\dot{a}\dot{b}} \bar{V}) - [\bar{\Psi}, \bar{\sigma}_{\dot{a}\dot{b}} \bar{W}] \right\} \end{aligned} \quad (20)$$

with

$$\begin{aligned} \hat{\nabla}_\alpha V^\beta &= 0 = \hat{\nabla}_\alpha \bar{V}^\beta \\ \hat{\nabla}^\alpha V_\alpha &= \hat{\nabla}_\alpha \bar{V}^\alpha. \end{aligned} \quad (21)$$

Thanks to the complete covariance of our approach, the equations for $\mathbf{F}_{BA}$ and Ψ_{BA} remain valid for the (super-) covariant derivatives of these fields, e.g.

$$s(\nabla^\alpha W_\alpha) = \hat{\nabla}^\alpha V_\alpha - [\nabla^\alpha W_\alpha, \hat{\Omega}] - \Xi^C \nabla_C (\nabla^\alpha W_\alpha). \quad (22)$$

Now, we are ready to project the superspace BRS algebra (17) to space-time. For this purpose, we consider the gauge real representation of SYM and introduce x -space fields by

$$\begin{aligned} A_\alpha &\equiv -\frac{i}{2} A_\alpha & \Psi_\alpha &\equiv -\frac{i}{2} \psi_\alpha \\ \mathbf{F}_{\dot{a}\dot{b}} &\equiv -\frac{i}{2} F_{\dot{a}\dot{b}} & \Psi_{\dot{a}\dot{b}} &\equiv -\frac{i}{2} D_{[\dot{a}} \psi_{\dot{b}]} \\ W_\alpha &\equiv -\frac{i}{2} \lambda_\alpha & V_\alpha &\equiv -\frac{i}{2} \eta_\alpha \\ \nabla^\alpha W_\alpha &\equiv -D & \hat{\nabla}^\alpha V_\alpha &\equiv -d \\ \hat{\Omega} &\equiv \frac{i}{2} \tilde{\omega} & \hat{\Phi} &\equiv \frac{i}{2} \tilde{\varphi}. \end{aligned} \quad (23)$$

The variations of the space-time fields (23) then follow from eqs.(17) and $s(Q) = (sQ)|$ for $Q = A_\alpha, W_\alpha, \dots$. In conclusion, the differential algebra (13) is recovered¹ after performing the change of variables $\omega = \tilde{\omega} - a^m A_m, \varphi = \tilde{\varphi} - a^m \psi_m$. Obviously, the derivation presented in this section elucidates the origin of the "supercovariant derivatives" $\hat{D}_m \eta_\alpha$ occurring in eqs.(13).

¹Up to signs due to the different grading of the superspace BRS algebra [11]

one gets the descent equations

$$\begin{aligned}
dW_4^0 &= 0 \\
sW_4^0 + dW_3^1 &= 0 \\
sW_3^1 + dW_2^2 &= 0 \\
sW_2^2 + dW_1^3 &= 0 \\
sW_1^3 + dW_0^4 &= 0 \\
sW_0^4 &= 0
\end{aligned} \tag{29}$$

which state that the polynomials W_p^g are s -invariant up to d -exact terms. Consequently, by Stoke's theorem, W_p^g is s -trivial when integrated over a p -cycle γ_p :

$$s \int_{\gamma_p} W_p^g = 0.$$

The local functionals $\int_{\gamma_p} W_p^g$ are the observables of topological YM [2]. They are non-trivial as elements of the so-called basic or equivariant cohomology [6]. This means that they are gauge and s -invariant as well as independent of ω and that they cannot be written as s -variations of a local functional which is gauge invariant and ω -independent. For instance [5], $sW_0^4 = 0$ and

$$W_0^4 = s \left[-\frac{1}{2} \text{Tr}(\varphi\omega - \frac{1}{3}\omega^3) \right].$$

Space-time transformations generated by a vector field $\xi = \xi^m \partial_m$ can easily be included in the presented framework by modifying condition (26) as in gravitational theories [13]:

$$\tilde{F} = e^{\xi_i} (F + \psi + \varphi). \tag{30}$$

Here, $\tilde{F}$ is defined as in eq.(25), but, for convenience, we now write $\tilde{\omega}$ for ω . The r.h.s. of the last equation reads

$$(1 + i_\xi + \frac{1}{2} i_\xi i_\xi) (F + \psi + \varphi) = F + \psi + \varphi + i_\xi F + i_\xi \psi + \frac{1}{2} i_\xi i_\xi F,$$

i_ξ denoting the interior multiplication by the ghost vector field ξ . Expansion of (30) and of the Bianchi identity (27) with respect to the ghost number leads to the differential algebra

$$\begin{aligned}
sA &= \psi - \mathcal{D}\omega + L_\xi A \\
s\psi &= -\mathcal{D}\varphi + [\psi, \omega] + L_\xi \psi \\
s\omega &= \varphi - \frac{1}{2} [\omega, \omega] + L_\xi \omega \\
s\varphi &= [\varphi, \omega] + L_\xi \varphi \\
s\xi &= \frac{1}{2} [\xi, \xi],
\end{aligned} \tag{31}$$

where $\omega = \tilde{\omega} - i_\xi A$ and where all commutators are assumed to be graded. Similarly, expansion of $-1/2 \text{Tr} \tilde{F} \wedge \tilde{F}$,

$$-\frac{1}{2} \text{Tr} \tilde{F} \wedge \tilde{F} = W_4^0(\xi) + W_3^1(\xi) + W_2^2(\xi) + W_1^3(\xi) + W_0^4(\xi),$$

yields the polynomials

$$\begin{aligned}
W_4^0(\xi) &= W_4^0 \\
W_3^1(\xi) &= W_3^1 + i_\xi W_4^0 \\
W_2^2(\xi) &= W_2^2 + i_\xi W_3^1 + \frac{1}{2} i_\xi i_\xi W_4^0 \\
W_1^3(\xi) &= W_1^3 + i_\xi W_2^2 + \frac{1}{2} i_\xi i_\xi W_3^1 + \frac{1}{6} i_\xi i_\xi i_\xi W_4^0 \\
W_0^4(\xi) &= W_0^4 + i_\xi W_1^3 + \frac{1}{2} i_\xi i_\xi W_2^2 + \frac{1}{6} i_\xi i_\xi i_\xi W_3^1 + \frac{1}{24} i_\xi i_\xi i_\xi i_\xi W_4^0,
\end{aligned} \tag{32}$$

the W_p^g being given by eq.(28). By construction,

$$0 = d \left[-\frac{1}{2} \text{Tr} \tilde{F} \wedge \tilde{F} \right] = (d + s) [W_4^0(\xi) + W_3^1(\xi) + W_2^2(\xi) + W_1^3(\xi) + W_0^4(\xi)];$$

henceforth, the polynomials (32) satisfy the descent equations (29) where the s -operations are now defined by eqs.(31).

After this digression on Donaldson polynomials, we now proceed to discuss their supersymmetric generalization. The polynomial with ghost number zero is given by the topological Lagrangian (6):

$$\begin{aligned}
W_4^0 &= -\frac{1}{2} \text{Tr} \{ F \wedge F \} + \frac{1}{2} d^* \left[dx^m \text{Tr}(\lambda \sigma_m \bar{\lambda}) \right] \\
&= \left[\text{Tr} \left(\frac{1}{8} \epsilon^{mnpq} F_{mn} F_{pq} \right) + \frac{1}{2} \partial_m \text{Tr}(\lambda \sigma^m \bar{\lambda}) \right] dx^1 \wedge \dots \wedge dx^4.
\end{aligned} \tag{33}$$

Here, F is the curvature 2-form introduced in eq.(24) and the star represents the Hodge operator. Obviously, $dW_4^0 = 0$ and by successive application of the differential algebra (13), one obtains the Donaldson superpolynomials. Their dependence on the translation parameter a^m is described by (32) with $\xi = a^m \partial_m$. The results are best expressed in terms of the variable

$$P \equiv P_m dx^m \equiv (\psi_m + \zeta \sigma_m \bar{\lambda} + \bar{\zeta} \sigma_m \lambda) dx^m$$

which is part of the s -variation of A .

To summarize, the super Donaldson polynomials are given by eq.(32) with $\xi \equiv a^m \partial_m$ and W_p^g defined by (33) and by

$$\begin{aligned}
W_4^0 &= -\frac{1}{2} \text{Tr} \{ \varphi \varphi \} \\
W_3^1 &= -\text{Tr} \{ \varphi P \} = -\text{Tr} \{ \varphi P_m \} dx^m \\
W_2^2 &= -\text{Tr} \{ \varphi F + \frac{1}{2} P \wedge P \} = -\frac{1}{2} \text{Tr} \{ \varphi F_{mn} - P_m P_n \} dx^m \wedge dx^n \\
W_1^3 &= -\frac{1}{4} \text{Tr} \left[(\psi_m + P_m) F_{pq} - \frac{i}{2} (\zeta \sigma_m \bar{\lambda} - \bar{\zeta} \sigma_m \lambda) \epsilon_{pqrs} F^{rs} \right. \\
&\quad \left. + \frac{1}{3} \{ (-\eta \sigma^n \bar{\lambda} + \bar{\eta} \sigma^n \lambda) + i D(\zeta \sigma^n \bar{\lambda} + \bar{\zeta} \sigma^n \lambda) \} \epsilon_{nmpq} \right] dx^m \wedge dx^p \wedge dx^q.
\end{aligned} \tag{34}$$

By construction, they satisfy the descent equations (29) with s defined by (13). Obviously, they are independent of ω and generalize the polynomials (28).

In deriving (34), we used the fact that the exterior derivative acts on p -forms with odd ghost number according to

$$d \left(\omega_{i_1 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r} \right) = - \left(\partial_j \omega_{i_1 \dots i_r} \right) dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r} .$$

6 Concluding comments

If we do not include diffeomorphisms in the superspace BRS algebra (14), it formally coincides with the one of topological YM [6]. Consequently, one can obtain superpolynomials satisfying the descent equations (29) by replacing the forms in the ordinary Donaldson polynomials (28) by superforms. However, it is impossible to construct BRS invariant quantities from these objects (as proposed in reference [14]), since general superforms cannot be integrated over supermanifolds (see [15]).

Moreover, if one wants to maintain a connection with SYM theories, one is obliged to implement the constraints and the solution of the superspace Bianchi identities. For the super 4-form $\text{Tr}(\mathbf{F} \wedge \mathbf{F})$ this implies [1] :

$$\text{Tr} \mathbf{F} \wedge \mathbf{F} = E^a E^b \left\{ (d\bar{\theta}) \bar{\sigma}_{ab} (d\bar{\theta}) \text{Tr} W^a W_a - (d\theta) \sigma_{ab} (d\theta) \text{Tr} \bar{W}_a \bar{W}^a \right\} + \text{other terms} . \quad (35)$$

Obviously, this polynomial contains more terms than the original action (5) from which the whole theory was constructed by a gauge-fixing procedure.

A possible way out of these problems and towards a superspace realization of Donaldson invariants might be to extract the relevant components of the super 4-form

$$w_4^0 \equiv -\frac{1}{2} \text{Tr} \{ \mathbf{F} \wedge \mathbf{F} \} \equiv \frac{1}{4!} E^A E^B E^C E^D (w_4^0)_{DCBA} ,$$

for instance

$$\text{Tr} \bar{W}_a \bar{W}^a = \frac{1}{2} (\epsilon \sigma^{ba})^{\gamma\delta} (w_4^0)_{\gamma\delta ba} , \quad (36)$$

and to study the decomposed descent equations [8],

$$\begin{aligned} 0 &= s w_4^0 + d w_3^1 \\ &= \frac{1}{4!} E^A E^B E^C E^D \left[s (w_4^0)_{DCBA} + 4 D_D (w_3^1)_{CBA} - 6 T_{DC}{}^E (w_3^1)_{EBA} \right] \end{aligned}$$

while taking into account the properties of the rigid superspace torsion $T_{DC}{}^E$. Although very useful for the determination of the superchiral anomaly [8], these techniques (as well as the results to which they lead) are quite complex and the completion of this program is certainly beyond the scope of the present work.

7 Appendix: Cohomological aspects

The projection from superfields to WZ-gauge component fields modifies the geometric structure of the theory. This fact has important consequences for the cohomology of the corresponding BRS operators. For example, in the superspace formulation of SYM, there is no supersymmetry anomaly [16], whereas such anomalies generally occur in the WZ-gauge description (see [11] and references therein). In the present case, the cohomology of the BRS operator given by (8) is trivial [1], but this does not need to be true for the BRS differential defining the component field algebra (13). To clarify this situation, we now apply standard methods [17] [16] [18].

Let $P(f; a^m, \zeta^a, \bar{\zeta}_a)$ be a polynomial in the constant parameters $a^m, \zeta^a, \bar{\zeta}_a$ and a local functional of the basic fields :

$$f \in \left\{ A_m, \lambda^a, \bar{\lambda}_a, D, \psi_m, \eta^a, \bar{\eta}_a, d, \omega, \varphi \right\} .$$

On the space of these functionals, the BRS operator acts by the functional derivative

$$\mathcal{S} = \sum_f \int d^4x (sf)(x) \frac{\delta}{\delta f(x)} + 2i (\zeta^m \bar{\zeta}) \frac{\partial}{\partial a^m} , \quad (37)$$

where the s -variations $sf(x)$ are defined by eq.(13). The nilpotency of s implies the one of $\mathcal{S}$.

On the same space, we now introduce a number operator involving all fields:

$$\mathcal{N} = \sum_f \int d^4x f(x) \frac{\delta}{\delta f(x)} + a^m \frac{\partial}{\partial a^m} + \zeta^a \frac{\partial}{\partial \zeta^a} + \bar{\zeta}_a \frac{\partial}{\partial \bar{\zeta}_a} . \quad (38)$$

This operator admits eigenvalues 0, 1, 2, ... and induces a decomposition of $\mathcal{S}$,

$$\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(1)} + \mathcal{S}^{(2)}$$

such that $[\mathcal{N}, \mathcal{S}^{(n)}] = n \mathcal{S}^{(n)}$. Explicitly,

$$\begin{aligned} \mathcal{S}^{(0)} A_m &= \psi_m + \partial_m \omega & \mathcal{S}^{(0)} \psi_m &= -\partial_m \varphi \\ \mathcal{S}^{(0)} \lambda_a &= \eta_a & \mathcal{S}^{(0)} \eta_a &= 0 \\ \mathcal{S}^{(0)} D &= d & \mathcal{S}^{(0)} d &= 0 \\ \mathcal{S}^{(0)} \omega &= \varphi & \mathcal{S}^{(0)} \varphi &= 0 \end{aligned} \quad (39)$$

and

$$0 = \mathcal{S}^{(0)} a^m = \mathcal{S}^{(0)} \zeta^a = \mathcal{S}^{(0)} \bar{\zeta}_a . \quad (40)$$

The nilpotency of $\mathcal{S}$ implies the one of $\mathcal{S}^{(0)}$ (the check being trivial). Introducing a new variable, $\chi_m \equiv \psi_m + \partial_m \omega$, the differential algebra (39) can be cast into the form

$$\begin{aligned} \mathcal{S}^{(0)} A_m &= \chi_m & \mathcal{S}^{(0)} \chi_m &= 0 \\ \mathcal{S}^{(0)} \lambda_a &= \eta_a & \mathcal{S}^{(0)} \eta_a &= 0 \\ \mathcal{S}^{(0)} D &= d & \mathcal{S}^{(0)} d &= 0 \\ \mathcal{S}^{(0)} \omega &= \varphi & \mathcal{S}^{(0)} \varphi &= 0 . \end{aligned}$$

This is a combination of contractible differential algebras [19], i.e. algebras of the form $sa = b$, $sb = 0$. Their cohomology is trivial². Therefore, the cohomology algebra $H(S^{(0)})$ of $S^{(0)}$ is determined by equation (40) which implies that it contains arbitrary polynomials of a^m , ζ^α and $\tilde{\zeta}_\alpha$.

According to the general theory, the cohomology of the complete operator S is isomorphic to a subspace of $S^{(0)}$,

$$H(S) \subset H(S^{(0)}) \quad ,$$

henceforth, $H(S)$ involves, at most, polynomials of the translation and SUSY parameters.

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²The term "contractible" is motivated by the following analogy. If a smooth manifold is smoothly contractible to a point, its de Rham cohomology is that of a point, i.e. trivial [20].

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