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REPRESENTATIONS OF THE QUANTUM GROUP $GL_q(2)$
FOR VALUES OF q ON THE UNIT CIRCLE

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ABSTRACT

We construct finite dimensional matrix representations of the quantum group $GL_q(2, \mathbb{C})$ for values of the deformation parameter $q = \exp 2\pi i \lambda$, $\lambda \in \mathbb{R}$. When $\lambda = 1/N$, N being a prime integer, using the representations of the Heisenberg group algebra we construct N -dimensional matrix representations of $GL_q(2, \mathbb{C})$ which can be used as building blocks for representations with q being a rational number. For the case where λ is a real irrational number we provide some examples of the generators of $GL_q(2, \mathbb{C})$.

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In the study of classical and quantum integrable models on a lattice or in the continuum, a universal mathematical structure has emerged through the Yang-Baxter or factorization equations [1-3]. Recently the same structure has appeared in the studies of conformal field theories [4,5] and in the 2+1-dimensional Chern-Simons topological field theories [5].

Since some years ago, the inverse problem of determining new integrable systems by studying solutions of the Yang-Baxter equations has been formulated by Faddeev et al. [6]. There is already a great deal of literature on the subject, but we want to mention important progress in the representation theory of these equations over the recent years [7-12]. To formulate these equations, let H be the Hilbert space of states of a quantum mechanical system L , let T be the monodromy (or scattering) matrix and

$$T_1 = T \otimes I \quad (1)$$

$$T_2 = I \otimes T, \quad (2)$$

the left and right tensor products of T with the identity operator on H . The operators T_1 and T_2 act on the tensor product space $H^{\otimes 2}$. What has been found in all integrable systems in 1+1 dimensions (for higher dimensions see recent work [13]) is that there is a c-number matrix acting on $H^{\otimes 2}$, R such that (quantum group equation)

$$R T_1 T_2 = T_2 T_1 R. \quad (3)$$

The integrability condition of this equation can be formulated (Yang-Baxter equation)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \quad (4)$$

where, if

$$R = \sum_i a_i \otimes b_i; \quad (5)$$

with a_i, b_i c-number matrices acting on H , then R_{12}, R_{13}, R_{23} act on $H^{\otimes 3}$, with

$$R_{12} = \sum_i a_i \otimes b_i \otimes I \quad (6a)$$

$$R_{13} = \sum_i a_i \otimes I \otimes b_i; \quad (6b)$$

$$R_{23} = \sum_i I \otimes a_i \otimes b_i. \quad (6c)$$

For every classical Lie algebra there is a standard way to construct a matrix R satisfying the Yang-Baxter equation [14,8,7,10,5], and then the relations (3) provide equations for the operator matrix T .

We shall deal only with the case of $SL(2)$, when

$$R = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}, \quad q \in \mathbb{C} \quad (7)$$

and if we write T as a 2×2 matrix

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (8)$$

then we obtain from Eq. (3) the following relations:

$$ab = qba \quad cd = qdc \quad (9a-b)$$

$$ac = qca \quad bc = cb \quad (9c-d)$$

$$bd = qdb \quad ad - da = (q - q^{-1})bc. \quad (9e-f)$$

In this short letter, we want to present matrix representations of Eqs. (9a-f) which define the elements of the quantum group in the case $q = e^{2\pi i \lambda}$, λ being a rational number. The importance of the above structure stems from the fact that if we define the so-called comultiplication operation on T , which satisfies relations (9a-f)

$$\Delta(T_{ij}) = \sum_k T_{ik} \otimes T_{kj}, \quad i, j, k = 1, 2. \quad (10)$$

the new matrix also satisfies Eqs. (9a-f). So there must be some building blocks which satisfy Eqs. (9a-f) and from which all others can be found by multiplication [7].

One can verify that the quantum determinant

$$D_q(T) = ad - qbc \quad (11)$$

commutes with all elements a, b, c, d , and if $D_q \neq 0$ it is possible to define the inverse quantum matrix

$$S(T) = T^{-1} = \frac{1}{D_q} \begin{bmatrix} d & -qb \\ -qc & a \end{bmatrix} \quad (12)$$

with elements satisfying relations (9a-f) where q is replaced by q^{-1} . Also T^n satisfies the same relations with q replaced by q^n [15,16]. All matrices T which satisfy relations (9a-f) with fixed q and with non-vanishing quantum determinant constitute the $GL_q(2, C)$ quantum group. One can define in an obvious way $SL_q(2, C)$ or $SU_q(2, C)$.

After these preliminaries we go over our construction. In a recent work we constructed irreducible unitary representations of the Heisenberg-Weyl group for a quantum-mechanical particle living on the N -vertices of a canonical polygon of the plane [17]. The exponential of the momentum operator is the matrix ($N = \text{prime}$)

$$h = \begin{bmatrix} 0 & 1 & \dots & \dots \\ \vdots & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ 1 & \dots & \dots & 0 \end{bmatrix} \quad (13)$$

and the exponential of the position operator is

$$g = \begin{bmatrix} 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^{N-1} & \omega^{N-2} & \omega^{N-3} & \dots & 1 \end{bmatrix} \quad (14)$$

Then the representation of the Heisenberg group divided by its centre is provided by the N^2 matrices:

$$J_{\vec{n}} = \omega^{\frac{n_1 n_2}{2}} g^{n_1} h^{n_2}, \quad 0 \leq n_1, n_2 \leq N-1, \text{ integers.} \quad (15)$$

The Heisenberg relation

$$hg = \omega g h, \quad (16)$$

implies that the matrices $J_{\vec{n}}$ provide a projective representation of the translation group of the lattice, $\Lambda = \mathbb{Z}_N \times \mathbb{Z}_N$:

$$J_{\vec{n}} J_{\vec{m}} = \omega^{\frac{\vec{n} \cdot \vec{m}}{2}} J_{\vec{n} + \vec{m}}, \quad \forall \vec{n}, \vec{m} \in \Lambda. \quad (17)$$

Since $J_{\vec{n}}$ are N^2-1 linearly independent matrices which are unitary and traceless, they may be used to define the fundamental representation of $SU(N)$ [19,20]. We are going to use the following basic relation

$$J_{\vec{n}} J_{\vec{m}} = \omega^{\frac{\vec{n} \cdot \vec{m}}{2}} J_{\vec{n}} J_{\vec{m}}, \quad \forall \vec{n}, \vec{m} \in \Lambda, \quad (18)$$

which is implied by (17), in order to construct representations of the quantum group $GL_q(2, C)$ by $N \times N$ matrices, with $q = \omega$.

We start by noticing that we may construct quantum matrices of $GL_q(2, C)$ by choosing the elements a, b, c satisfying the relations

$$ab = qba, \quad ac = qca \quad (19a-b)$$

$$bc = cb. \quad (19c)$$

If a is an invertible operator, then using the fact that the quantum determinant commutes with a, b, c , we may choose d as:

$$d = q a^{-1} bc + \lambda a^{-1}. \quad (20)$$

We shall choose λ to be a non-zero complex number although it may be an invertible operator which commutes with a, b, c . It easily follows that the commutation relations of d with a, b, c :

$$ad - da = (q - q^{-1})bc, \quad bd = qdb \quad (21a-b)$$

$$cd = qcd, \quad (21c)$$

are satisfied by the "solution" (20). The above construction shows that the six relations (19a-c) and (21a-c) between the four elements a, b, c, d are not independent

and the problem of determining the quantum group $GL_q(2, C)$ is reduced to the set of equations (19a-c) (if a is invertible). We now choose q to be a primitive root of unity

$$q = w = e^{2\pi i/N}, \quad N \text{ prime odd integer.} \quad (22)$$

The case $q = w^k$, $k = 0, 1, \dots, N-1$ is easily treated in the same way as for $k = 1$. We start by choosing the invertible element a to be one of the matrices $J_{\vec{m}}^{\pm}$, $\vec{m} \in \Lambda$ [relation (15)], $\vec{m} \neq (0, 0)$. When N is a prime integer there is a modular group, $SL(2, \mathbb{Z}_N)$, acting transitively modulo N on the lattice Λ . In Ref. [17] we show that each $\vec{m} \in \Lambda$, $\vec{m} \neq 0$ can be brought by an element $A \in SL(2, \mathbb{Z}_N)$ to the point $\vec{m}_0 = (0, 1)$, and this induces the metaplectic group of $N \times N$ matrices $\mu(A)$ such that

$$\mu(A) J_{\vec{m}_0}^{\pm} \mu(A) = J_{A\vec{m}_0}^{\pm} = J_{\vec{m}}^{\pm}. \quad (23)$$

So when N is a prime, we may choose

$$a = J_{\vec{m}_0}^{\pm} = b, \quad (24)$$

relation (13).

The next step is to find matrices $N \times N$, b and c , satisfying relations (19a-c). Suppose that we choose b or c to be also one of the matrices $J_{\vec{m}}^{\pm}$; then we must have

$$ab = wba, \quad b \equiv J_{\vec{m}_b}, \quad (25)$$

but this implies, from relation (18), that

$$\vec{m}_b \times \vec{m}_0 = 1 \bmod N \quad (26)$$

so

$$\vec{m}_b = (1, n), \quad n = 0, 1, \dots, N-1. \quad (27)$$

The same is true for c :

$$\vec{m}_c = (1, m), \quad m = 0, 1, \dots, N-1. \quad (28)$$

Since the relations (19a-b) are linear in b and c , any linear combination

$$b = g \sum_{m=0}^{N-1} b_m h^m, \quad c = g \sum_{m=0}^{N-1} c_m h^m, \quad b_m, c_m \in \mathbb{C}, \quad (29)$$

will satisfy relations (19a-b). The last relation

$$bc = cb \quad (30)$$

implies that

$$BgC = CgB \quad (31)$$

where

$$B = \sum_{m=0}^{N-1} b_m h^m, \quad C = \sum_{m=0}^{N-1} c_m h^m. \quad (32)$$

In the following we shall determine B, C using properties of the eigenvalues and eigenvectors of the matrix h . In Ref. [17] we discuss the finite Fourier transform matrix U , which has the property:

$$hU = Uq, \quad U_{mn} = \frac{1}{\sqrt{N}} w^{m \cdot n}, \quad m, n = 0, 1, \dots, N-1. \quad (33)$$

Equation (33) corresponds to the passage from the position to the momentum space description and includes all the eigenvectors and eigenvalues of h : ($k = 0, 1, \dots, N-1$)

$$hX_k = \lambda_k X_k \quad (34a)$$

$$(X_k)_j = \frac{1}{\sqrt{N}} w^{jk}, \quad \lambda_k = w^k, \quad j = 0, 1, \dots, N-1. \quad (34b)$$

It easily follows that:

$$gX_k = X_{k+1}, \quad k = 0, 1, \dots, N-1, \quad X_N = X_0. \quad (35)$$

We apply Eq. (31) to the complete and orthonormal set of vectors X_k and we obtain the relations

$$\lambda_{k+1}(B) \lambda_k(C) = \lambda_k(B) \lambda_{k+1}(C), \quad k=0, 1, \dots, N-1. \quad (36)$$

For the eigenvalues of B and C

$$\lambda_k(B) = \sum_{m=0}^{N-1} b_m \omega^{k \cdot m}$$

$$\lambda_k(C) = \sum_{n=0}^{N-1} c_n \omega^{k \cdot n}, \quad k=0, 1, \dots, N-1. \quad (37)$$

First we find "elementary" solutions of Eq. (36) and then we form linear combinations compatible with (36). Consider the matrices

$$e_i = \frac{1}{N} \sum_{m=0}^{N-1} \omega^{(N-i)m} \cdot h^m, \quad i=0, 1, \dots, N-1. \quad (38)$$

These matrices have eigenvalues

$$\lambda_k(e_i) = \delta_{i,k}, \quad i, k=0, 1, \dots, N-1. \quad (39)$$

It is obvious that if i, j are such that

$$|i-j| \geq 2 \bmod N \quad (40)$$

then the matrices e_i and e_j satisfy

$$\lambda_{k+1}(e_i) \lambda_k(e_j) = \lambda_k(e_i) \lambda_{k+1}(e_j). \quad (41)$$

Indeed, the matrices e_i form a projector system in the N -dimensional e^N space

$$e_i e_j = \delta_{ij} e_i, \quad \sum_{i=0}^{N-1} e_i = I \quad (42)$$

and

$$g e_i = e_{i+1} g, \quad i=0, 1, \dots, N-1. \quad (43)$$

The forms of the matrices $b(i)$ and $c(i)$ which correspond to $B = e_1$, $C = e_j$, $|i-j| \geq 2 \bmod N$ and satisfy relation (30) are

$$b_{jk}(i) = \frac{1}{N} \omega^{(N-i)(k-j)+j} \quad (44)$$

Using $b(i)$'s we can form the general solution of Eq. (30):

$$b(\lambda_1, \dots, \lambda_p) = \lambda_1 (b(\omega_1 + b(i) + \dots + b(\omega_1)) + \lambda_2 (b(\omega_2 + 2) + \dots + b(\omega_2) + \dots + \lambda_p (b(\omega_p + 2) + \dots + b(\omega_p)) \quad (45)$$

$$c(\lambda'_1, \dots, \lambda'_p) = b(\lambda'_1, \dots, \lambda'_p) \quad (46)$$

with

$$\lambda'_1 \lambda_p = \lambda'_p \lambda_1, \quad \lambda_i, \lambda'_i \in \mathbb{C}, \quad i=1, \dots, p. \quad (47)$$

We consider now the case $q = \omega^k$, $k = 2, 3, \dots, N-1$. This is similar to the above case, when we chose as a the element

$$a = h^k. \quad (48)$$

When $N = p^n$, where p is a prime, the same construction holds because $SU(2, 2, p)$ acts transitively on the lattice $\mathbb{Z}_p \times \mathbb{Z}_p$. The case N any positive integer can be constructed using comultiplication and the relation

$$\mathbb{Z}_{N_1} \otimes \mathbb{Z}_{N_2} \cong \mathbb{Z}_{\text{LCM}(N_1, N_2)}, \quad (49)$$

where $\text{LCM}(N_1, N_2)$ is the least common multiple of N_1, N_2 .

If λ is an irrational real number we may, roughly speaking, take the limit of the construction:

$$p_N = \frac{k_N}{N}, \quad k_N \in \mathbb{Z}_N, \quad p_N \xrightarrow{N \rightarrow \infty} \lambda, \quad k_N \xrightarrow{N \rightarrow \infty} \lambda N \quad (50)$$

More care is needed, though, because the matrices h and g are replaced by infinite dimensional matrices [17]:

$$(H)_{ij} = \delta_{i,j+1}, \quad (G)_{kj} = e^{2\pi i k \lambda} \delta_{k,j} \quad (51)$$

and the generators J_n^+ by

$$J_n^+ = e^{in_1 n_2} \otimes H^{n_2} \quad n_1, n_2 \in \mathbb{Z} \quad (52)$$

Again we may choose

$$a = H. \quad (53)$$

The case $b \neq c$ will be presented elsewhere. Below we give an example:

$$b = c = J_{(1,n)} \quad , \quad n \in \mathbb{Z} \quad , \quad q = e^{2\pi i \lambda} \quad (54)$$

for $q = e^{2\pi i \lambda}$, and d is given by the relation (20):

$$d = q a^{-1} b^2 + \lambda a^{-1} \quad (55)$$

where $\lambda \in \mathbb{C}$.

Representation of the quantum groups and quantum algebras for $q = \text{root of unity}$ have been studied in Ref. [22] and for $SU_q(2)$ in Ref. [23]. Finally we would like to mention an interesting work by A. Belavin [24]*, where the matrices $g_{n_1 n_2}^{\lambda}$ have been used as a lattice Lorentz symmetry of the Yang-Baxter equation for a system of N kinds of particles, in order to determine solutions of these equations. Last of all we would like to mention that in Ref. [10] the passage from the q -deformed classical Lie algebras to the corresponding quantum groups has been constructed.

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