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Massive Kaluza-Klein Modes and Effective Theories of Superstring Moduli

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ABSTRACT

We discuss the low-energy effective theories that one obtains by compactifying ten-dimensional string theories on Calabi-Yau spaces. In particular, we focus attention on the effect of integrating out the Kaluza-Klein massive modes. We show that provided the four-dimensional massless scalar fields do not have potentials, the structure of their σ -model metric is unchanged by these integrations. This will be the case for type II theories, but not for the heterotic theory when non-perturbative effects are taken into account.

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1 – Introduction

One of the major problems in superstring theory is to determine the effective low-energy action for the light modes of the theory. The most promising theories with chiral fermions and phenomenologically interesting gauge groups are heterotic superstrings compactified so as to preserve $N = 1$ spacetime supersymmetry in four dimensions. Unfortunately, the number of candidate vacua is very large and also the structure of the low-energy interactions would *a priori* appear to be influenced by many higher-order terms in the effective action. Among the massless fields of the effective theory, a particular set represent the moduli of the vacuum solution. By moduli, we mean the undetermined integration constants for the solutions to the effective field equations for the internal metric and the internal components of any other fields of the higher-dimensional theory, in a compactification of the form $(\text{Minkowski})_4 \times \mathcal{K}$. The scalar fields associated with these moduli can conveniently be parametrized by allowing the modulus parameters to become functions of the four-dimensional spacetime coordinates.

In a number of recent papers [1–5], it has been suggested that information on the interactions of these fields may be obtained by using the same internal spaces to compactify type II and heterotic strings. Under certain circumstances, the type II compactifications yield twice the number of spacetime supersymmetries as the heterotic string with the same internal manifold $\mathcal{K}$. It has been argued [4] that in such cases the σ -model metric for those modulus fields common to both theories should be the same in the compactifications of the two theories, and that this metric can be regarded as a “natural” metric on the space of moduli of the internal manifold. Note that the internal manifold in general is not itself a homogeneous space (e.g., the cases of compactification on a Calabi-Yau manifold or $K3 \times T^2$), and the metric on it satisfying the superstring effective field equations may be unknown. Nonetheless, in special cases the space of *moduli* of the internal manifold may turn out to be endowed with a simple homogeneous-space metric. Indeed, this is known to be the case for $K3$ compactifications, where one knows from the mathematical study of its space of moduli that there is a natural, transitive $SO(19, 3)$ action [6]. It is not clear that analogous mathematical statements can be made in general for Calabi-Yau spaces, since the homogeneous structure for $K3$ depends upon features particular to four-dimensional spaces. In the case of the $K3$ compactification, one can also understand the coset-space structure of the σ -model metric from the point of view of $N = 4$, $d = 4$ supergravity couplings, which are so constrained by the $N = 4$ local supersymmetry that the σ -model metric is uniquely determined to have a specific homogeneous form [7]. For Calabi-Yau compactifications, on the other hand, the compactified theory has only $N = 2$, $d = 4$ local supersymmetry and this is not restrictive enough to determine a unique structure for the σ -model metric. Nor does the imposition of Pecci-Quinn-type symmetries [2,5] for the pseudoscalar modulus fields fix the σ -model metric uniquely. Metrics satisfying these requirements are generically inhomogeneous and there is no known reason to expect that the metric for the total σ -model of the

modulus fields should have a coset-space structure.* The σ -model metrics actually occurring in Calabi-Yau compactifications of string or Kaluza-Klein theories form a discrete subclass of those occurring in $N = 2$ supersymmetric and Pecci-Quinn symmetric models in $d = 4$; for each topologically-distinct Calabi-Yau manifold there is a unique low-energy σ -model metric. The problem is that there is as yet no known way to determine these metrics by “macroscopic” low-energy arguments.

In this paper, we shall examine the extent to which the structure of the low-energy theory may be modified by integrating out the heavy modes of the compactified higher-dimensional theory. We shall concentrate in particular on the effects of the Kaluza-Klein massive modes on the σ -model metric for the modulus fields. It is expected that this metric can be identified with the Zamolodchikov metric [8] that occurs in the derivation of the effective action for the β -functions of the massless background fields. The motivation for examining the stability of the σ -model metric against corrections from the massive modes arises from the example of $K3$ compactification discussed above. Since the σ -model metric in this case is uniquely determined by supersymmetry, it evidently must also be stable against corrections due to the massive modes. We shall explain how this happens, despite the fact that the couplings of the modulus fields to the massive Kaluza-Klein modes are not such as to admit a consistent truncation. The key point is that by integrating out the massive Kaluza-Klein modes in a *manifestly supersymmetric* fashion, one can see that the effective action remains unchanged at lowest order in derivatives.

For Calabi-Yau compactifications of type II theories, we shall also show that the σ -model metric for the modulus fields is stable against the effects of integrating out the massive Kaluza-Klein modes. However in this case the argument is more involved than in the $K3$ case, because four-dimensional $N = 2$ supersymmetry is less restrictive than $N = 4$. The essential difference is that the $N = 2$, $d = 4$ supersymmetry allows independent massive Kaluza-Klein supermultiplets with maximum spin 1, as well as supermultiplets with maximum spins $\frac{3}{2}$ and 2. The higher-spin supermultiplets have the same dimensional requirements on their couplings as the massive supermultiplets encountered in the $K3$ compactification (i.e. they are of dimension 5), but the maximum-spin-1 multiplet has the possibility, *a priori*, of a “minimal” (dimension 4) coupling to massless hypermultiplets. ($N = 2$ -supersymmetric and gauge-invariant couplings of a massive vector multiplet to massless vector multiplets will be seen not to be possible.) If these minimal couplings were to occur, they would lead to an instability of the σ -model metric for the hypermultiplet modulus fields against the effects of integrating out the massive Kaluza-Klein modes. However, such couplings will be shown to be ruled out for no-scale compactifications of $d = 10$, $N = 2a$ supergravity by virtue of the factorization of the space of moduli into a direct product of $H^{2,1}$ and $H^{1,1}$ sectors. For

* By restricting attention to a subset of the modulus fields, e.g. the untwisted sector in an orbifold compactification, explicit forms of the σ -model metrics have been worked out by a number of methods [2,3,5]. The coset-space structure of the metrics so found arises for clear-cut reasons which cannot be expected to apply in general for the metrics on the total spaces of moduli for Calabi-Yau compactifications.

no-scale compactifications of $d = 10$, $N = 2b$ supergravity, such couplings will be shown to be ruled out by Peccei-Quinn symmetries for the hypermultiplet modulus fields arising from gauge antisymmetric tensors in $d = 10$.

A crucial feature of the stability argument is that there should be no potential for the modulus fields alone. In compactifications of type-II theories, this can be deduced purely from $N = 2$, $d = 4$ supersymmetry. This continues to be true even in the face of non-perturbative effects due to world-sheet instantons.

In the case of the heterotic string, however, there are potentials involving modulus fields coupled to massless scalars arising from the Yang-Mills sector of the theory. The modulus fields are always singlets under the unbroken Yang-Mills gauge group. Thus, there is always a consistent truncation that can be made to an effective theory containing the modulus fields, in which the charged fields are set to zero. Moreover, although there are no perturbative contributions to potentials for the heterotic theory's modulus fields alone, it has been argued that non-perturbative effects due to world-sheet instantons do contribute to such potentials in some cases [9–11]. The superpotentials giving rise to these potentials would involve massless fields arising from the $d = 10$ Yang-Mills field itself that are also singlets under the unbroken Yang-Mills gauge group. The number of these extra heterotic massless fields is given by $\dim(H^1(\text{End } T))$ [10]. If one were to truncate to the set of modulus fields, which occur also in the type-II theories, the instanton-induced superpotentials would be removed. However, since the $H^1(\text{End } T)$ fields of the heterotic theory are all singlets under the unbroken gauge symmetry, there is no reason why such a truncation should be consistent, unlike the truncation to the uncharged sector discussed above. In fact, explicit instanton calculations indicate that trilinear superpotentials occur, containing one of the $H^1(\text{End } T)$ fields and two $H^{2,1}$ modulus fields, or an $H^{1,1}$ modulus field and an $H^{2,1}$ modulus field [11]. Since the $H^1(\text{End } T)$ fields occur linearly, truncating them out would be inconsistent with their own equations of motion. (In particular, the spinors associated to the $H^{1,1}$ and $H^{2,1}$ moduli would act as sources for these fields.) These non-perturbative effects place limitations on one's ability to learn about the low-energy effective theory of the heterotic string from the corresponding sector of the type-II theories.

One may view the process of compactification from several different points of view. One of these is first to find the effective low energy theory for the string in ten dimensions. This can be done either using the approach of calculating the β -functions for the non-linear σ -model defined by the set of massless background fields, or directly in string theory by computing string scattering amplitudes using the techniques of operator-product expansions. After finding the effective theory of the massless fields in ten dimensions, one can then look for Kaluza-Klein compactifications of this field theory down to four dimensions. In this procedure, one must face the consequences of having towers of massive Kaluza-Klein modes, which need to be integrated out in order to obtain the final effective theory of massless fields in four dimensions.

Another way to view the process of compactification is to consider strings propagating on a background that is a direct product of the σ -model target manifold K of a conformal field theory and a four-dimensional spacetime. The massless background fields in four dimensions include those related to the moduli of the internal manifold, which are promoted to fields by allowing them to depend on the coordinates of four-dimensional spacetime. The effective field equations for the four-dimensional fields can then be obtained in principle from the β -functions of these backgrounds. It should be said that the process of working out β -functions for strings propagating in such a generalized direct-product spacetime has not been explicitly carried out to date. A related way to view the compactification is to consider the string theory formulated directly in four dimensions, but with internal degrees of freedom corresponding to an abstract conformal field theory. The interactions of the massless fields of the effective four-dimensional field theory may then be derived from the string amplitudes. It is not clear whether all abstract conformal field theories relevant to this approach correspond to the internal manifolds that are used in the β -function calculations. In the case of the heterotic string, non-trivial superconformal field theories exist with $(2, q)$ or $(4, q)$ worldsheet supersymmetry, leading to $N = 1$ or $N = 2$ spacetime supersymmetry in $d = 4$. In the language of spacetime compactification, the left-right symmetric $(2, 2)$ or $(4, 4)$ systems are believed to correspond to Calabi-Yau manifolds of complex dimension 3 or 2.

While it is to be hoped that there is an equivalence between the two basic approaches, of obtaining an effective field theory first in ten dimensions and then performing Kaluza-Klein compactification, or of obtaining an effective field theory directly in four dimensions, there is no proof of such an equivalence. In particular, any such equivalence must accommodate the fact that in the Kaluza-Klein compactification, there is a final integration over massive modes coming from the higher-dimensional effective field theory. Since these massive modes will in general couple to currents constructed from the massless fields, integrating out the massive modes will in general change the interactions of the massless fields. The equivalence of the two approaches to string compactification can hold only after integrating out all of the massive Kaluza-Klein modes.

As mentioned above, a crucial feature of the modulus fields is that they do not appear by themselves in potential terms in the effective action. This is easiest to see using the natural parametrization for the modulus fields referred to earlier. A ten-dimensional spacetime that is a direct product of flat four-dimensional spacetime and an internal manifold is required to be an exact solution of the ten-dimensional field equations, irrespective of the values of the modulus parameters. If the action of the 10-dimensional theory has the property that it vanishes when the fields are constrained to satisfy their equations of motion, then the value of the action with the Kaluza-Klein ansatz inserted will remain equal to zero when the modulus fields are taken to be constants. In this case, there can be no potential for the modulus fields alone.

Since the absence of a potential for the moduli will play a major rôle in our discussion, it is worth considering what happens when one includes also the corrections at higher order

in α' to the 10-dimensional effective action. We shall discuss this specifically in the case of (2,2) worldsheet-supersymmetric compactifications, for which the internal space $\mathcal{K}$ is Kähler, and is Ricci flat at lowest order in α' . One can then argue perturbatively that the corrected equations of motion for $\mathcal{K}$ will have solutions with the same number of moduli as the lowest-order solutions. To see this, note that the higher-order corrections to $R_{a\bar{b}} = 0$ are of the cohomologically-trivial form $R_{a\bar{b}} = \partial_a \partial_{\bar{b}} S$, where S is constructed from the Riemann tensor and its covariant derivatives [12]. One can therefore proceed iteratively, starting from a Ricci-flat metric g_0 and then invoking Yau's theorem to solve for a new metric g_1 satisfying $R_{a\bar{b}}(g_1) = \partial_a \partial_{\bar{b}} S(g_0)$, and so on for g_2, g_3 , etc. [13]. The iterative scheme can be applied for each starting point g_0 in the moduli space of Ricci-flat Kähler metrics, thereby yielding a set of exact solutions g_∞ parametrised by the same set of moduli. This argument for the preservation of the number of moduli assumes the convergence of the perturbative expansion (or at least its resummability). Having established the preservation of the number of moduli, we see that all of the modulus fields of the lowest-order theory continue to be modulus fields even in the presence of the perturbative higher-order corrections, and thus still can have no potential. Correspondingly, it can also be verified that the perturbative effective action continues to vanish on shell [14].

2 – Kaluza-Klein Massive Modes

In the Kaluza-Klein procedure [15], one begins with a theory in some higher dimension which admits a *spontaneous compactification* on some internal space $\mathcal{K}$; i.e. there is a solution of the higher-dimensional equations of motion of the form (lower-dimensional spacetime) $\times \mathcal{K}$. One then expands all the higher-dimensional fields in generalized Fourier series, in terms of complete sets of the appropriate tensor or spinor harmonics on $\mathcal{K}$, thereby obtaining in the lower-dimensional spacetime a theory comprising infinite towers of fields. These in general consist of a finite number of massless fields, together with infinite numbers of massive fields. The masses can be found by working just to quadratic order in fields; at this order one simply obtains infinite sets of non-interacting fields in the lower-dimensional effective theory. At this order, one can truncate freely to any desired subset of fields; for example, to the massless sector.

Once the non-linearities of the higher-dimensional theory are taken into account, the situation becomes much more complicated. The nature of the interactions amongst the lower-dimensional fields are now governed by “overlap integrals”, in the internal space $\mathcal{K}$, of products of three or more of the harmonics involved in the generalized Fourier expansions of the higher-dimensional fields. Broadly speaking, there are two radically different kinds of situation that can arise, depending upon the nature of the internal space $\mathcal{K}$. If $\mathcal{K}$ is an homogeneous space, which admits a transitively-acting group of isometries $\mathcal{G}$, then the tensor and spinor harmonics all carry representations of $\mathcal{G}$ and the structure of the overlap integrals is very tightly constrained. In particular, only isolated overlap integrals are non-zero; this is

because, generically, the product of two or more harmonics can be expressed as a sum over only a finite number of harmonics, namely those permitted by the group theory [16]. If, on the other hand, $\mathcal{K}$ is not an homogeneous space — a typical example would be $K3$ [17] or a six-dimensional Calabi-Yau space [18] — then there are no group-theoretical constraints and a product of harmonics generically requires an infinite number of harmonics in an expansion. Thus a compactification on an homogeneous space $\mathcal{K}$ leads to an orderly set of couplings between the massless and massive modes, governed by the group $\mathcal{G}$, while compactification on an inhomogeneous space leads to a messy set of couplings which cannot be understood purely group-theoretically. In fact, in the latter case, there is no alternative but to calculate the metric and the harmonics on $\mathcal{K}$ explicitly and then perform the overlap integrals. If $\mathcal{K}$ is a Ricci-flat Calabi-Yau space, for which even the metric is unknown, the task is clearly beyond practical computability.

One might hope that it would be possible to truncate the full lower-dimensional theory to some finite subset of the fields; in particular, one would like to be able to truncate to just the massless sector. However, except in very exceptional circumstances, this is not possible. This is because, in general, the product of harmonics corresponding to the massless modes involves an expansion that includes harmonics corresponding to massive modes. In other words, there will be couplings of the form HL^2 , HL^3 , etc., where H denotes a massive field and L a massless field in the lower dimension. Thus, the field equation for the massive field will have source terms that involve only massless fields, and so one cannot set the massive field to zero without imposing algebraic constraints on the massless fields that are to be retained [16].

In lieu of truncating the massive modes in the lower-dimensional theory, one can instead adopt the device of integrating them out, so as to obtain an effective lower-dimensional theory solely involving massless fields. At the classical level, which is all that we are concerned with at present, this amounts to solving the field equations for the heavy fields, and substituting the solutions back into the full lower-dimensional effective Lagrangian. If a given heavy field H happens to be consistently truncatable (i.e. if it does not enter linearly in the Lagrangian multiplying powers of the massless fields and their derivatives) then one can simply substitute the solution $H = 0$. If, on the other hand, H is not consistently truncatable, then its solution will take the form of some non-local expression in the massless fields that act as a source for H . Since the general form of the H equation will be

$$\square H + m^2 H = J(L), \quad (2.1)$$

where m is the mass of the field H and $J(L)$ is the source term constructed from the massless fields L , it follows that H can be solved for in an expansion of the form

$$H = \frac{1}{m^2}(1 + m^{-2}\square)^{-1}J = \frac{1}{m^2}(1 - m^{-2}\square + \dots)J. \quad (2.2)$$

Thus to leading order, the effect of integrating out H will be to introduce a term of the form

$$\frac{1}{m^2} J^2 \quad (2.3)$$

in the effective action for the remaining, massless, fields. Since $J(L)$ is quadratic or higher in L , this means that the extra terms in the Lagrangian resulting from integrating out the massive fields will be quartic or higher in massless fields.

Since the details of the interaction terms HL^2 , HL^3 , etc. leading to these modifications of the massless modes' effective action depend on the details of the harmonics on K and their overlap integrals, it follows that, in the case of an internal space like a Calabi-Yau manifold, one has in general essentially no information about the types and magnitudes of the modifications. A case of particular interest in this regard is that of the couplings between massive vector fields and the massless scalar fields in a Calabi-Yau or $K3$ compactification. Let us consider $K3$ for definiteness. In a gravity or supergravity theory admitting compactification on a Ricci-flat $K3$ space, we know that there will be 58 massless scalar fields in the lower dimension, corresponding to the 58 moduli of Ricci-flat $K3$ metrics. Of these, one is just the overall scale of the Ricci-flat metric, while the remaining 57 correspond to the 57 varieties of unit-volume Ricci-flat $K3$ spaces. The tensor harmonic associated with the overall scale is just the metric tensor itself, which of course is covariantly constant. Clearly, products of covariantly-constant harmonics, such as the metric tensor, generate tensors that can be expanded in terms of covariantly-constant harmonics, and so there will always be a consistent truncation to the subset of fields associated with the covariantly-constant harmonics. However, the other 57 massless scalars arise as the coefficients of tensor harmonics that are *not* covariantly constant, and these, when multiplied together, will generate tensors that are expanded as infinite sums of massless *and* massive harmonics. In other words, the 57 scalars ϕ will have couplings in which the massive fields appear *linearly* in the lower-dimensional effective Lagrangian. In particular, one generally expects couplings to massive vectors H^μ of the form

$$H^\mu f(\phi) \partial_\mu \phi. \quad (2.4)$$

By our general arguments above, these will lead, after integrating out the heavy vectors, to extra interactions of the form

$$a(\phi) \partial^\mu \phi \partial_\mu \phi, \quad (2.5)$$

where $a(\phi)$ is some function quadratic and higher in the scalar fields ϕ .

The apparent paradox in the comparison of string-theory effective actions with known supergravity theories is clearest in the case of compactification on $K3$. We know that *before* integrating out the massive fields, the lower-dimensional effective action will involve a sigma-model term for the massless scalars of the form

$$g_{ij}(\phi) \partial^\mu \phi^i \partial_\mu \phi^j. \quad (2.6)$$

Even before integrating out the massive modes, one might expect that some "natural" choice of parametrization would exist in which the σ -model metric $g_{ij}(\phi)$ for the modulus fields would have a simple homogeneous-space form. In fact, we know that the space of metric moduli for the 57 volume-preserving parameters is $SO(19,3)/SO(19,3;\mathbf{Z})$ (note that $SO(19,3)/SO(3) \cong F^{57}$) [6] and that this space admits an $SO(19,3)$ -invariant metric.* However, there is the other "natural" choice of parametrization for the modulus fields in which the free parameters of the classical solution are simply promoted to scalar fields in the lower dimension. Whether these two notions of "natural" parametrization are the same is not obvious. Moreover, after integrating out the massive vectors, we have seen that generically there will be corrections of the form (2.5), which can be expected to be very complicated and effectively incalculable. The apparent paradox arises because in the case of compactification of a supergravity or superstring theory on $K3$, one ends up with an effective action for a lower-dimensional supergravity theory whose interactions are tightly constrained by the extended supersymmetry. For example, in the case of compactification from 11 dimensions to 7 on $K3$, the $N = 1$, $d = 7$ supersymmetry implies [7] that the 57 volume-preserving modulus fields must have the $SO(19,3)$ -invariant metric in the final low-energy theory *after* integrating out the Kaluza-Klein massive modes. Thus it appears that a "miracle" has to occur, and the generic incalculable modifications of the form (2.5) must conspire to cancel out in the final theory.

In fact, supersymmetric theories do sometimes display miracles of the kind that seem to be needed here. A striking example is provided by the compactification of $d = 11$ supergravity on the seven-sphere, to give $N = 8$, $SO(8)$ -gauged supergravity in four dimensions. *A priori*, one might have expected that the full four-dimensional theory with all the massless and massive modes retained would involve couplings linear in massive fields to products of massless fields and their derivatives, since the seven-sphere harmonics corresponding to the massless $N = 8$ supermultiplet can be group-theoretically multiplied to generate harmonics associated with the massive fields. However, it turns out that the specific details of the $d = 11$ theory are such that some of the *a priori* allowed couplings occur with zero coefficient [16,20]. From a purely Kaluza-Klein point of view, these zero coefficients have no very obvious explanation, and seem to be the result of chance cancellations. In fact, however, they can be understood when one looks at the implications of $N = 8$ supersymmetry in the four-dimensional theory, together with the general structure of couplings in Kaluza-Klein theories.

The key point [21] about the towers of lower-dimensional fields in a Kaluza-Klein theory is that they have their origin in *gauge* fields in the higher-dimensional theory. Thus, even

* In a compactification of the type-IIA string on $K3$ to $d = 6$, one obtains 22 additional scalar modulus fields coming from the 2-index antisymmetric tensor in $d = 10$. Together with the $K3$ scaling mode, the total of 80 scalar fields are described by the coset space $SO(20,4)/(SO(20) \times SO(4))$. It has been argued that there is also a discrete factoring by $SO(20,4;\mathbf{Z})$ in this case [19].

though the massive members of the towers correspond to broken gauge symmetries, the forms of their couplings are nevertheless restricted by the higher-dimensional gauge invariances. In particular, if there are couplings linear in a particular massive field H in the lower-dimensional effective Lagrangian, then at lowest order the form of any coupling to a set of massless fields L must be via a current that is quadratic in the field strengths of the massless fields, and that is also conserved by virtue of the lowest-order equations of motion for these massless fields. A typical example is a current that occurs in the Kaluza-Klein compactification of gravity on a sphere, where the massive spin-2 fields $H^{(ij)}_{\mu\nu}$ (carrying the symmetric traceless representation of the isometry group $SO(N)$) couple to the massless spin-1 fields $A^{(ij)}_\mu$ (carrying the adjoint representation of $SO(N)$) through the generalized stress-tensor

$$T^{(ij)}_{\mu\nu} = F^{k(i)}_{\mu\rho} F^{j)k}_{\nu\rho} - \frac{1}{4} \eta_{\mu\nu} F^{k(i)}_{\rho\sigma} F^{j)k}_{\rho\sigma}. \quad (2.7)$$

Although this current carries both spacetime and internal symmetry indices, and thus would seem to be ruled out in an interacting theory by the Coleman-Mandula theorem, there is nothing to prevent its conservation at the level of *linearized* field equations for the massless vector fields, as one may check — conservation of (2.7) follows from $\partial^\mu F^{(ij)}_{\mu\nu} = 0$, subject only to the symmetric structure of (2.7). When one includes the couplings of $H^{(ij)}_{\mu\nu}$ to other massive modes and also includes the higher-order couplings in $A^{(ij)}_\mu$ that covariantize (2.7), conservation in the fully-interacting Kaluza-Klein theory is possible despite the Coleman-Mandula theorem by virtue of the infinite number of modes present in the theory. For our purposes, it is sufficient to note that if the field strengths $F^{(ij)}_{\mu\nu}$ in (2.7) are taken to be the full non-abelian field strengths for $SO(N)$, then (2.7) becomes *covariantly* conserved by virtue of the $A^{(ij)}_\mu$ field equations $\mathcal{D}^\mu F^{(ij)}_{\mu\nu} = 0$. Thus, couplings of massive Kaluza-Klein modes such as $H^{(ij)}_{\mu\nu}$ to massless fields must be such as to preserve both the gauge symmetries of the massless fields and the gauge symmetry of the massive mode $H^{(ij)}_{\mu\nu}$ itself.

If one is dealing with a supersymmetric compactification, then the couplings must respect the relevant supersymmetry. As a consequence, establishing the absence of one such coupling in a supermultiplet can be sufficient to establish the absence of couplings for the entire multiplet. The details of this mechanism are discussed in [21], where it was shown that it is *always* consistent to truncate the infinite towers of Kaluza-Klein states to just those of the massless supergravity multiplet. The seven-sphere compactification of $d = 11$ supergravity is an extreme case in which the massless supermultiplet comprises just one irreducible representation of the supersymmetry algebra with spins running from 0 up to 2.

The proof of the consistency of truncation to the supergravity multiplet given in [21] involved some detailed considerations of the dimensions of the possible interaction terms. Specifically, the proof depended upon the fact that in a theory with at most second-order field equations, the only possible interaction terms would be of too low a dimension to satisfy simultaneously the requirements of supersymmetry, the massless multiplet's gauge invariances and the lowest-order conservation of the currents formed from the massless field

strengths. In the special case of the seven-sphere, there is in fact an even simpler argument for why the truncation is consistent. As we have already noted, the couplings between the lower-dimensional fields are, at lowest order, cubic interactions between one field that appears linearly and two fields that couple to it via a conserved current quadratic in their field strengths. These conserved currents comprise *off-shell* supermultiplets, which are therefore necessarily *long* supermultiplets, since multiplet shortening is always an on-shell phenomenon, whereas here the current multiplets are quadratic products of field-strength multiplets and do not themselves satisfy any field-equation type constraints. In the case of the seven-sphere, the currents therefore would have to form $N = 8$ long multiplets, which have at least maximum spin 4.* Since all of the massive fields in the four-dimensional theory must occur in short multiplets with maximum spin 2, it is therefore not possible for there to be couplings of these fields to the long-multiplet currents.** Thus, there cannot be any couplings such as (2.7) in which any massive field appears linearly, and so the truncation to $N = 8$ supergravity is necessarily consistent.

The results described above relate to the consistency of truncation to the supergravity multiplet. In a compactification on $K3$ or on a Calabi-Yau space, there are also independent massless “matter” supermultiplets with maximum spin 1. In the next section, we shall examine the nature of the possible couplings of these supermultiplets to a single massive supermultiplet. This situation is different from that of the $N = 8$ supergravity compactification in that couplings of the forms (2.4) or (2.7) are not ruled out by supersymmetry and hence would seem bound to occur, given the lack of other restrictions on the harmonic overlap integrals, as discussed earlier. However, we shall show that once one integrates out entire supermultiplets of massive fields, and takes into account the “no-scale” nature of the Kaluza-Klein compactifications, as well as the Peccei-Quinn symmetries specific to the compactified $d = 10$ supergravity theory, the final result is to leave the original form of the sigma-model metric for the scalar fields (2.6) unaltered.

3 – Massive $N = 1$, $d = 4$ supergravity coupling to massless multiplets

In the light of the discussion given in the last section on the general structure of the couplings of massive Kaluza-Klein fields to the massless sector, we now consider the implications of supersymmetry for these couplings and determine the forms of the corrections to the massless fields’ effective action that can arise from integrating out the massive fields. First, we discuss the couplings of a massive $N = 1$, $d = 4$ graviton multiplet in superspace at the linearised level. This will establish the general features of such couplings, and similar results will apply to the couplings of massive extended supersymmetric graviton multiplets.

* Details of the $N = 8$ Poincaré supersymmetry current multiplets formed from field strengths are given in [22].

** Self-couplings of the massless multiplet are not ruled out by this argument because they are not made via currents constructed from field strengths.

Even if we are dealing with a higher-dimensional theory for which there is no off-shell formulation with auxiliary fields, in many cases the compactified theory has a less-extended supersymmetry for which an off-shell formulation is possible. Or, one might just select a portion of the lower-dimensional supersymmetry and provide auxiliary fields for it, leaving the rest non-linearly realized. It is important to consider a linear realization of at least some of the supersymmetry because we need to establish what happens to the residual theory of massless fields when massive multiplets are integrated out supersymmetrically. In order to make it clear what this procedure entails, we need to work in a superspace formulation.

For compactifications on homogeneous spaces with isometries, the massive multiplets will carry representations of the isometry group. For compactifications on Calabi-Yau spaces or on other inhomogeneous manifolds, there will not be such an obvious index labelling of the massive states. For considerations of supersymmetry, however, the types of such indices carried by the massive fields are immaterial. It is only important to note that in the compactified theory, any group structure will occur in a direct product with $N = 1$, $d = 4$ supersymmetry. Thus, in the following we shall treat the coupling of a single massive $N = 1$, $d = 4$ graviton multiplet without loss of generality.

Linearized supergravity is represented in superspace in the minimal formulation by a vector superfield* [23,24] $V_{\alpha\dot{\beta}}$, for which the linearized gauge transformations are

$$\delta V_{\alpha\dot{\beta}} = \bar{D}_{\dot{\beta}} \Lambda_{\alpha} - D_{\alpha} \bar{\Lambda}_{\dot{\beta}}. \quad (3.1)$$

The transformation (3.1) contains both linearized super-Poincaré and super-Weyl gauge transformations. In order to write an action that reduces to pure $N = 1$ massless linearized supergravity, but that is also invariant under the full gauge transformation (3.1), it is necessary to include also a conformal compensating multiplet which is a chiral superfield ϕ . The action for massless linearized supergravity is then

$$I_{\text{kinetic}} = \int d^4x d^4\theta V_{\alpha\dot{\alpha}} E^{\alpha\dot{\alpha}} - 2 \int d^4x d^2\theta R \phi - 2 \int d^4x d^2\bar{\theta} \bar{R} \bar{\phi} + \int d^4x d^4\theta \phi \bar{\phi}. \quad (3.2)$$

The field-strength superfields appearing in (3.2) are

$$E_{\alpha\dot{\beta}} = 2D^2 \bar{D}^2 V_{\alpha\dot{\beta}} + 4i\partial_{\gamma\dot{\gamma}} D^{\gamma} \bar{D}^{\dot{\gamma}} V_{\alpha\dot{\beta}} - 4i\partial_{\gamma}^{\dot{\gamma}} D^{\gamma} \bar{D}^{\dot{\gamma}} V_{\alpha\dot{\beta}} - 8\partial_{\alpha\dot{\beta}} D^{\gamma} \bar{D}^{\dot{\gamma}} V_{\gamma\dot{\gamma}}, \quad (3.3)$$

$$R = -2i\bar{D}^2 \partial^{\alpha\dot{\alpha}} V_{\alpha\dot{\alpha}}. \quad (3.4)$$

Since the gauge transformations (3.1) contain also the linearized super-Weyl transformations, the field strengths (3.3) and (3.4) are not invariant, but transform as

$$\delta E_{\alpha\dot{\beta}} = -2i\partial_{\alpha\dot{\beta}}(U - \bar{U}) \quad (3.5)$$

$$\delta R = -\frac{1}{2}\bar{D}^2 \bar{U} \quad (3.6)$$

* Our conventions are $D_{\alpha} = \partial/\partial\theta^{\alpha}$, $\{D_{\alpha}, \bar{D}_{\dot{\beta}}\} = -2i\partial_{\alpha\dot{\beta}}$, $D^2 = D^{\alpha}D_{\alpha}$, $\bar{D}^2 = \bar{D}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}}$, $V^{\alpha} = \epsilon^{\alpha\dot{\beta}}V_{\dot{\beta}}$, $V_{\dot{\alpha}} = \epsilon_{\alpha\dot{\beta}}V^{\beta}$.

where U is the superfield parameter containing the super-Weyl part of Λ_{α} ,

$$U = \bar{D}^2 D^{\gamma} \Lambda_{\gamma}. \quad (3.7)$$

The kinetic action (3.2) is invariant under the gauge transformations (3.1) provided the conformal compensating multiplet ϕ transforms as

$$\delta\phi = -U. \quad (3.8)$$

In verifying the gauge invariance of the kinetic action (3.2), it is necessary to use the linearized superspace Bianchi identity

$$D^{\alpha} E_{\alpha\dot{\beta}} = \bar{D}_{\dot{\beta}} \bar{R} \quad (3.9)$$

and its complex conjugate.

In order to write a mass term for the graviton multiplet in the minimal formulation,* it is necessary to solve the chirality constraint on the conformal compensator superfield ϕ by introducing a prepotential for ϕ :

$$\phi = \bar{D}^2 X. \quad (3.10)$$

Under the gauge transformation (3.1), X transforms as

$$\delta X = -D^{\gamma} \Lambda_{\gamma}. \quad (3.11)$$

The introduction of X is necessary because the mass term $\int d^4x d^4\theta V_{\alpha\dot{\alpha}} V^{\alpha\dot{\alpha}}$ does not contain the correct Pauli-Fierz mass term for the spin-2 member of the multiplet. Moreover, $\int d^4x d^2\theta \phi^2$ is not of the correct dimension. The correct mass term that does include the Pauli-Fierz mass term for the spin-2 field is

$$\frac{m^2}{2} \int d^4x d^4\theta V_{\alpha\dot{\alpha}} V^{\alpha\dot{\alpha}} - \frac{m^2}{2} \int d^4x d^4\theta X \bar{X}. \quad (3.12)$$

Varying $V_{\alpha\dot{\alpha}}$ and X gives the field equations

$$2E_{\alpha\dot{\alpha}} - 4i\partial_{\alpha\dot{\alpha}}(\phi - \bar{\phi}) + m^2 V_{\alpha\dot{\alpha}} = 0 \quad (3.13)$$

$$-2R + \bar{D}^2 \bar{\phi} - \frac{m^2}{2} \bar{X} = 0. \quad (3.14)$$

In order to see that these correctly describe massive supergravity, apply D^{α} to the $V_{\alpha\dot{\alpha}}$ equation and $\bar{D}_{\dot{\alpha}}$ to the $\bar{X}$ field equation and add to find

$$D^{\alpha} V_{\alpha\dot{\alpha}} - \frac{1}{2} \bar{D}_{\dot{\alpha}} X = 0. \quad (3.15)$$

* Massive supergravity in the new minimal formulation is given in [25].

From this equation and its conjugate, one finds

$$-2i\partial^\alpha\bar{\partial}^\beta V_{\alpha\beta} = \frac{1}{2}(\bar{\phi} - \phi), \quad (3.16)$$

which can be inserted into the definition of $\bar{R}$ to give

$$\bar{R} = \frac{1}{2}D^2\bar{\phi}. \quad (3.17)$$

Thus, from the field equation (3.14), it follows that

$$X = 0, \quad (3.18)$$

so from (3.15) we have finally

$$D^\alpha V_{\alpha\dot{\alpha}} = 0, \quad (3.19)$$

which is the necessary constraint for the massive graviton multiplet [24].

Now we must examine the coupling of the massive graviton multiplet to massless matter. As explained in section 2, although the gauge invariances of supergravity are broken by the mass term, consistency of coupling for the massive spin- $\frac{3}{2}$ and spin-2 fields requires that their gauge symmetries be broken *only* by the mass term. For the massive graviton multiplet, the constraint (3.19) must still hold even in the coupled theory. This constraint may be obtained more manifestly than in the above discussion by including a Stueckelberg compensator supermultiplet for the gauge symmetries broken by the mass term (3.12). As in the purely bosonic case [26], the field equations for the Stueckelberg multiplet must give directly the constraint (3.19). In order for this to happen, the Stueckelberg multiplet must couple *only* to the massive supergravity prepotentials $V_{\alpha\dot{\alpha}}$ and X in the compensated generalization of the mass term (3.12). All other couplings of the massive graviton multiplet to other matter must be via gauge-invariant couplings. This requires that the coupling terms linear in the massive graviton multiplet must be to a generalized “supercurrent” $J_{\alpha\dot{\alpha}}$ with a conservation condition

$$D^\alpha J_{\alpha\dot{\alpha}} = \bar{D}_{\dot{\alpha}}\bar{S} \quad (3.20)$$

that must hold in lowest order, subject to the equations of motion of the massless supermultiplets from which $J_{\alpha\dot{\alpha}}$ and S are constructed. The generalized supercurrent may carry isometry-group indices in the case of compactification on an homogeneous space, even though this would normally be contrary to the Coleman-Mandula theorem. As discussed in section 2, the exception is allowed in Kaluza-Klein theories because of the infinite number of massive fields that are present, so that the Coleman-Mandula theorem does not apply.

The only objects built from massless multiplets with the correct dimensions for Kaluza-Klein couplings that can satisfy the conservation condition (3.20) are direct generalizations of the actual supercurrents to which the massless supergravity multiplet itself couples. For

example, for a Yang-Mills supermultiplet with field strength W_α^A , where A is some group index, the generalized supercurrent

$$J_{\alpha\dot{\alpha}}^{(AB)} = W_\alpha^A \bar{W}_{\dot{\alpha}}^B \quad (3.21)$$

will still satisfy the necessary conservation law in lowest order provided a *symmetric* combination is taken in (3.21). In fact, for generalized supercurrents like (3.21) carrying gauge-group indices for the gauge symmetries of the massless multiplets, one must require that even beyond the lowest order of coupling, a *gauge-covariant* conservation condition must hold: $D^\alpha J_{\alpha\dot{\alpha}} = \bar{D}_{\dot{\alpha}}\bar{S}$. The generalized supercurrent (3.21) contains spin-2 terms of the form of the bosonic generalized stress-tensor (2.7).

For the moduli of the Calabi-Yau spaces with which we are mainly concerned, the generalized supercurrent will have to have a structure derived from that of the true supercurrent

$$J_{\alpha\dot{\beta}} = g_{\alpha\dot{\beta}} D_\alpha \phi^a \bar{D}_{\dot{\beta}} \bar{\phi}^b + 2iK_{\dot{\beta}}^b \partial_{\alpha\dot{\beta}} \bar{\phi}^b - 2iK_{\alpha}^a \partial_{\alpha\dot{\beta}} \phi^a, \quad (3.22)$$

which satisfies (3.20) with

$$\bar{S} = -\frac{1}{2}D^2 K, \quad (3.23)$$

where K is the Kähler potential for the σ -model metric of the modulus fields' kinetic term (K is also the superspace Lagrangian for the corresponding supermultiplets). The generalized supercurrent may have the a and b indices opened up, provided symmetry is maintained between the separate pairs of chiral superfields and their conjugates ($\phi^a, \bar{\phi}^{\dot{a}}$), which is necessary for (3.20) to hold at the quadratic level in chiral multiplets. For chiral multiplets such as those carrying charges under E_6 , for which there can be a superpotential $\int d^4x d^2\theta f(\phi)$, the “trace” part of the supercurrent (3.22) becomes

$$\bar{S} = -\frac{1}{2}D^2 K + 3\bar{f}. \quad (3.24)$$

Equations (3.22, 3.24) indicate the structure of the chiral fields' generalized supercurrent. The actual generalized supercurrent may have indices opened up similarly to the Maxwell generalized supercurrent (3.21), provided that symmetric combinations of the pairs of chiral superfields and their conjugates are taken so as to ensure that (3.20) holds in lowest order.

Suppressing any such possible extra index structure on the generalized supercurrent and the corresponding extra indices on the massive graviton multiplet, the allowed couplings linear in the massive multiplet to the massless fields take the form

$$\int d^4x d^4\theta V^{\alpha\dot{\alpha}} J_{\alpha\dot{\alpha}} - \int d^4x d^2\theta \phi S - \int d^4x d^2\bar{\theta} \bar{\phi} \bar{S}. \quad (3.25)$$

The last two terms can be rewritten as $-\int d^4x d^2\theta (XS + \bar{X}\bar{S})$, and one may check that such couplings will be invariant under (3.1, 3.11) provided $J_{\alpha\dot{\alpha}}$ and S satisfy (3.20).

It is now a trivial matter to integrate out the massive graviton multiplet ($V_{\alpha\dot{\alpha}}, X$). Keeping only the lowest-order terms in the resulting corrections to the massless multiplets' effective action is equivalent to dropping the kinetic action (3.2), but keeping the mass term (3.12), so that to this order the massive graviton multiplet is effectively a set of auxiliary superfields, generating the massless-multiplet effective-action terms

$$\Delta I_{\text{eff}} = -\frac{1}{2m^2} \int d^4x d^4\theta J_{\alpha\dot{\alpha}} J^{\alpha\dot{\alpha}} + \frac{2}{m^2} \int d^4x d^4\theta \bar{S} S. \quad (3.26)$$

If $f = 0$, then ΔI_{eff} is a pure higher-derivative term, containing, e.g., vector-field terms $\sim \frac{1}{m^2} \int (F_{\mu\nu})^4$, and their supersymmetric partners. If $f \neq 0$, then (3.26) will indeed contain corrections to the kinetic terms of the light fields. Thus, the absence of a superpotential is crucial for the integrations over massive multiplets not to disturb the structure of the lowest-order action for the light fields. This is the case for the effective field theories of type II strings compactified on $K3$ or on Calabi-Yau spaces. For the heterotic string compactified on these spaces, the absence of superpotentials is obtained only in perturbation theory and only when one sets all the charged superfields to zero, i.e. to their vacuum values.

Note that although the pure higher-derivative nature of the correction (3.26) to the massless fields' effective action is manifest for $f = 0$ in a superspace formulation, it becomes hidden if one works without auxiliary fields. For example, in the form in which supersymmetric Kaluza-Klein theories originally arise from the compactification of a higher-dimensional theory for which there is no off-shell formulation, *minimal* gauge-field couplings of massive vector fields to the massless fields occur, and these do produce changes in the lowest-order terms in the effective action when the massive vectors are integrated out.

The resolution of this apparent discrepancy is easy to see if one considers integrating out first the auxiliary fields in the massive graviton multiplet, and then subsequently the physical vector field of the multiplet. Because the auxiliary field b^μ of the graviton multiplet is of dimension two, the current coupling (3.25) contains a minimal coupling of b^μ to currents like $\phi_{\text{matter}} \vec{\partial}_\mu \phi_{\text{matter}}$. The mass term (3.12) also contains an off-diagonal mass term for the physical vector C_μ of the graviton multiplet. In integrating out the auxiliary field, the term quadratic in this field in I_{kinetic} must be used. Thus, one has a system comprising the physical vector and the auxiliary vector of the form

$$\frac{1}{2} b_\mu b^\mu + m^2 b_\mu C^\mu + b^\mu \phi_{\text{matter}} \vec{\partial}_\mu \phi_{\text{matter}}. \quad (3.27)$$

If one integrates out *only* the auxiliary field b_μ , then the result is

$$-\frac{1}{2} (m^2 C_\mu + \phi_{\text{matter}} \vec{\partial}_\mu \phi_{\text{matter}})^2. \quad (3.28)$$

Thus, a correction to the ϕ_{matter} kinetic term has been generated as well as a minimal coupling between the physical vector mC_μ and the ϕ_{matter} current (note that the physical vector C_μ must be scaled by a factor of m to have dimension one).

Now, while it is true that subsequent integration over the physical vector C_μ will make further changes to the ϕ_{matter} kinetic action beyond (3.28), the effect of these subsequent changes will be just to remove the term $(\phi_{\text{matter}} \vec{\partial}_\mu \phi_{\text{matter}})^2$ that was introduced by having previously integrated out b_μ . All of the terms involving ϕ_{matter} in (3.27) may be absorbed into a shift in C_μ , so that in the end the massive vector and the auxiliary vector conspire to make no overall changes in the lowest-order effective action for ϕ_{matter} . Of course, the usual Kaluza-Klein action without auxiliary fields corresponds to a situation like that of (3.28), and one does not immediately see that the final integration over C_μ actually simplifies the resulting effective action for ϕ_{matter} , removing at the end all corrections in lowest order to the massless fields' effective action.

4 – Couplings of extended massive supermultiplets to the massless sector

In the last section, we have examined in detail how the constraints arising from $N = 1$, $d = 4$ supersymmetry and the gauge invariances associated with the massive fields determine the allowable forms of the lowest-order couplings linear in the fields of the massive graviton supermultiplet to the massless fields. When one has massive multiplets of N -extended supersymmetry, the above discussion naturally generalizes. For example, when $d = 11$ supergravity is compactified on $K3 \times T^3$ to four dimensions, one ends up with $N = 4$, $d = 4$ unbroken supersymmetry in the compactified theory. In this case, there is only one type of long supermultiplet that can contain the massive fields — the superspin-0 massive multiplet, which contains 1 graviton, 8 gravitinos, 15 vectors, 12 two-forms, 48 spinors and 42 scalars. All other $N = 4$, $d = 4$ long massive multiplets contain spins higher than 2, and thus do not occur in a Kaluza-Klein theory. In addition to the $N = 4$ superspin-0 massive supermultiplets in $d = 4$, there are also short massive $N = 4$ supermultiplets carrying central charges arising from compactification on the T^3 part of the internal manifold. However, since the massless fields carry zero central charge, there are no currents through which they can couple to the fields of the centrally-charged short massive multiplets. The lowest-order couplings of the superspin-0 supermultiplet to massless fields must be via $N = 4$ generalized supercurrents $J_{ijkl}^{(AB)} = W_{ij}^{(A} W_{kl}^{B)} - (1/24) \epsilon_{ijkl} W_{rs}^{(A} W_{tu}^{B)} \epsilon^{rstu}$ analogous to (3.21) [27,22]. The effect of integrating out the massive Kaluza-Klein modes will be an $N = 4$ supersymmetric generalization of (3.26). It is clear from the discussion of section 3 and from dimensional considerations that integrating out the massive supermultiplets will not produce any corrections in lowest order to the effective action for the massless fields. In arriving at this conclusion, it is important to note that the massless matter multiplets are $N = 4$ supermultiplets without gauge self-coupling. The absence of gauge self-coupling for these maximum spin-1 multiplets follows from the fact that the modulus fields* cannot have a potential,

* The total of 131 scalar and pseudoscalar modulus fields of the $K3 \times T^3$ compactification are distributed in $N = 4$ supermultiplets as follows [17]. The graviton supermultiplet contains the metric volume modulus field and an antisymmetric-tensor modulus field. Then there are 19 Maxwell multiplets, each containing 3

which would have been implied by $N = 4$ supersymmetry had the multiplets possessed any gauge self-interactions.

Instead of the above compactification to four dimensions, one could equally well look at a compactification to $d = 7$ on $K3$ alone. In this case, the massive supermultiplets each contain 1 graviton, 2 gravitinos, 1 three-form, 3 two-forms, 6 vectors, 6 spinors and 7 scalars. The massless fields reside in the $N = 1$, $d = 7$ graviton supermultiplet plus 19 Maxwell supermultiplets; the 57 scalar fields in the latter comprise all of the volume-preserving metric modulus fields for $K3$ (in the $d = 7$ theory, there are no modulus fields coming from the antisymmetric tensors). As in the four-dimensional case discussed above, supersymmetrically integrating out the massive Kaluza-Klein modes produces no net changes in the lowest-order effective action for the massless modes. In this case, the constraint of locally supersymmetric coupling to the $N = 1$, $d = 7$ supergravity multiplet *uniquely* determines the form of the lowest-order Lagrangian containing all terms with up to two derivatives. The result [7] for the volume-preserving scalar modulus fields' kinetic term is a σ -model metric for the homogeneous space $\frac{SO(19,3)}{SO(19) \times SO(3)}$. ** Thus, the "miracle" referred to in section 2 has in fact occurred. The massive Kaluza-Klein modes produce no overall change in the lowest-order kinetic term for the moduli, and the result is a remarkably simple homogeneous-space metric.

We now return to the question of whether it is possible to choose a "natural" parametrization of the Kaluza-Klein ansatz for a $K3$ compactification in which the scalar fields' kinetic term has a simple homogeneous-space metric even before integrating out the massive modes. (We remind the reader that it is only in the case of $K3$ compactifications, as opposed to Calabi-Yau compactifications, that one can expect the σ -model metric for the full set of modulus fields to be homogeneous.) From the above discussion plus that of section 3, it is clear that at an intermediate stage in the Kaluza-Klein analysis, with the auxiliary fields eliminated but with the physical vector fields of the massive multiplets retained, the σ -model metric for the massless scalars would be intractably complicated. At such a stage, the metric would contain many corrections, such as those of equation (3.28), arising from the auxiliary fields that would have been eliminated from the start. Thus, in a supergravity theory without auxiliary fields, one would be mistaken to expect that a simple homogeneous-space metric for the modulus fields could be obtained before integrating out the massive fields simply by promoting the modulus parameters to modulus fields. However, in a manifestly supersymmetric treatment including auxiliary fields for the massive supermultiplets, the massive metric modulus fields (from the $K3$ moduli) and 3 antisymmetric-tensor modulus fields. Finally, there are 3 linear multiplets, each containing 2 metric modulus fields (from the T^2 moduli), 3 antisymmetric-tensor modulus fields and one spin-zero degree of freedom represented by a gauge antisymmetric-tensor field (which we do not count as a modulus field).

** Note, however, that the supergravity couplings give information only about the local structure of this space. The information about the discrete factoring by $SO(19, 3; \mathbb{Z})$ comes from the detailed mathematical study of the modulus space of $K3$ [6].

multiplets may simply be dropped as far as the lowest-order effective action for the massless fields is concerned. If one simply drops the linearly-transforming massive multiplets and then integrates out the auxiliary fields for the massless matter and supergravity multiplets, then the modulus fields will have a kinetic term containing the simple homogeneous-space metric.

The compactifications of $d = 11$ supergravity on $K3$ or on $K3 \times T^3$ are maximal examples of the type of Kaluza-Klein compactification that we have been considering in this paper. It is not possible to have any higher extended supersymmetry in the lower dimension and still have couplings linear in massive fields to the massless fields of the compactified theory. This is because any higher supersymmetry in a compactified Kaluza-Klein theory (of maximum spin 2) would require that all the massive multiplets be "short" multiplets — either having central charges or being short $OSp(N|4)$ multiplets (in a compactification with a cosmological constant in $d = 4$). Such short massive multiplets cannot couple supersymmetrically to the long multiplets bilinear in massless field strengths that are the only ones potentially available as supercurrents. The case of the S^7 compactification of $d = 11$ supergravity illustrates the point [16,20,21], for in this $N = 8$ supersymmetric compactification one may consistently truncate to the massless sector. As a result, integrating out all the massive modes produces no changes to the massless effective action in any order of tree-level perturbation theory.* Only compactifications preserving $N \leq 4$ supersymmetry in $d = 4$ (or the equivalent extension in higher dimensions) allow couplings linear in massive fields. As we have seen in the case with $N = 4$, $d = 4$ supersymmetry, the massive multiplets are irreducible supermultiplets with spins running from 0 up to and including 2.

Now consider the non-maximal compactifications preserving $N < 4$ supersymmetry in $d = 4$. One approach that can sometimes be used is to consider embedding a non-maximal compactification into a maximal one. For example, suppose one wished to consider the compactification of $N = 1$, $d = 10$ supergravity on $K3 \times T^2$ to $d = 4$ (of course, at the quantum level, this theory is anomalous, so this example is purely classical). This would yield an $N = 2$ supersymmetric theory in $d = 4$ (or equivalently an $N = 1$ theory in $d = 6$). This compactification can be analyzed by considering how it may be embedded into the corresponding compactification of the $N = 2a$, $d = 10$ supergravity theory. The $N = 1$ and $N = 2a$ theories both admit compactification on the internal space $K3 \times T^2$. The compactification of the $N = 2a$ theory is essentially the same as that of $d = 11$ supergravity on $K3 \times T^3$ which we discussed earlier — the important point for our purposes is that the resulting compactified theory in $d = 4$ has $N = 4$ supersymmetry, so the massive Kaluza-Klein multiplets necessarily have spins running up to 2. Thus, only one type of massive supermultiplet need be considered, and the considerations of section 3 may be applied directly, as we have already seen.

* Of course, if one includes loop diagrams, the massive modes will contribute corrections to the effective action for the massless modes.

Now consider the truncation from the $N = 2a$ theory to the $N = 1$, $d = 10$ theory, but consider this truncation as it takes place in the $N = 4$, $d = 4$ compactified theory to an $N = 2$, $d = 4$ compactified theory. The only massive fields of the $N = 4$ theory that could conceivably upset the stability of the modulus fields' kinetic term are vector fields with off-diagonal mass terms, as we saw in section 3. These fields acquire mass terms only after integrating out the auxiliary vectors with which they are paired. In the process of integrating out the auxiliary vectors, minimal couplings can appear between the physical vectors and the massless scalars, as we saw in (3.28). These minimal couplings can arise only after integrating out the auxiliary fields, because only the auxiliary fields initially have the correct dimension for such couplings. After integrating out the auxiliary fields, factors of the mass m can arise so as to make minimal couplings to the physical vectors dimensionally consistent.

The only danger to the stability of the modulus fields' effective action against lowest-order corrections from the massive fields would arise if, in the truncation from $N = 4$ supersymmetry, vector fields and their associated auxiliary vectors were not treated as pairs, either truncated or retained together. If such fields are always treated as a pair, the stability property of the massless sector of the $N = 4$ theory will be "inherited" by the $N = 2$ theory. Although the details of this truncation of the compactified $N = 4$ theory to the $N = 2$ theory have not been worked out, the paired retention or truncation of physical and auxiliary vector fields would seem inevitable.

This argument for the stability of the $N = 2$, $d = 4$ supersymmetric compactification of $N = 1$ supergravity in $d = 10$ suggests the conjecture that a similar "inheritance" of stability of the massless sector against lowest-order corrections due to the massive fields may also take place if one truncates all the way down to pure Einstein gravity in $d = 10$. Again, the dangerous terms for the stability of the kinetic action for the modulus fields all arise from massive-multiplet auxiliary-physical vector pairs in the $N = 2$ compactified theory. In a truncation to $N = 0$ supersymmetry in $d = 4$ (the compactified pure $d = 10$ Einstein gravity), it would seem likely that pairs of physical vectors and their associated auxiliary vectors must be discarded together, i.e. that in the final truncated $N = 0$ theory prior to integrating over the massive modes, there will remain none of the physical vectors needing auxiliary fields in order to have mass terms. In that case, the truncation from $N = 2$ to $N = 0$ supersymmetry should work in a manner similar to that of the truncation from $N = 4$ to $N = 2$: the stability of the kinetic terms for the modulus fields in the truncated theory against lowest-order effects from the massive modes would be inherited from its ability to be embedded in a theory with extended supersymmetry.

The above argument applies only in cases where there is an available higher-dimensional theory with a larger supersymmetry than that of the theory in question, so that information can be derived from the possibility of embedding into the larger theory. Compactifications on Calabi-Yau spaces of $d = 10$, $N = 2a$ or $N = 2b$ supergravity to $d = 4$ do not fall into this class, and require a second kind of argument. This second argument also applies

to the compactifications considered above of the $N = 1$, $d = 10$ theory. The argument generalizes the analysis of section 3 and depends both on local supersymmetry and on the antisymmetric-tensor gauge invariances of the higher-dimensional theory.

A compactified theory with $N = 2$, $d = 4$ supersymmetry can have massive supermultiplets with maximum spin $2, \frac{3}{2}$ or 1. The maximum spin-2 massive multiplets can be treated in direct analogy to the discussion of section 3, making the necessary changes for massive $N = 2$ supergravity similarly to the case for the $N = 4$ compactification discussed above. The resulting couplings will contain as a subset the massive $N = 1$ supergravity couplings discussed in section 3, plus new couplings for the extra fields of the $N = 2$ massive graviton multiplet. The latter must be dimensionally consistent with the massive $N = 1$ supergravity multiplet's couplings, and thus can contribute only higher-derivative corrections to the massless sector's effective action (again, as in section 3, this argument is crucially dependent upon the absence of potential terms for the massless fields). Similarly, the $N = 2$ multiplets with maximum spin $\frac{3}{2}$ must couple to the massless fields in a way that preserves the massless fields' gauge invariances and also yields a generalized supersymmetry current for the massive spin- $\frac{3}{2}$ field itself. All of these couplings must therefore be dimensionally consistent with those discussed in section 3, and thus can only contribute higher-derivative corrections to the massless sector's effective action.

The $N = 2$ maximum spin-1 massive supermultiplets (vector supermultiplets) pose a new problem. Supersymmetry and conservation of the gauge invariances of the massless fields and of the massive spin-1 fields are not by themselves enough to rule out minimal-type couplings for these supermultiplets. If present, these would destroy the stability of the massless sector against the effects of integrating out the massive modes. Let us recall how a massive $N = 2$ vector multiplet may be formulated in superspace. In terms of off-shell superfields, it is possible to view this massive multiplet as a compensated vector gauge multiplet in a fashion analogous to that explained in section 3 for the $N = 1$ massive graviton multiplet. This formulation requires a compensator multiplet whose propagating fields are all either spin-0 or spin- $\frac{1}{2}$. Restricting attention to multiplets with finite numbers of component fields, the only available compensator is the "relaxed hypermultiplet" [28], described in $N = 2$ superspace by three real* superfields L^{ij} , L^{ijk} and V ($i, j = 1, 2$) satisfying the off-shell constraints

$$D_\alpha^i L^{jk} = D_{\alpha t} L^{ijk} \quad (4.1)$$

$$D_\alpha^i L^{jktm} = 0 \quad (4.2)$$

$$D_{\alpha\beta} V = [D_\alpha^i, \bar{D}_{\beta i}] = 0, \quad (4.3)$$

where $D_{\alpha\beta} = \frac{1}{2} D_{\alpha i} D_{\beta}^i$. The resulting massive $N = 2$ vector multiplet is then a fairly complicated construction.

* By real we mean $L^{ij} = \epsilon^{ik} \epsilon^{jl} \bar{L}_{kl}$, etc.

A simpler way to describe the massive $N = 2$ vector multiplet avoids the use of a compensating multiplet, using instead the gauge-mixing mechanism, in which a massive spin-1 field is described by a coupled system consisting of a gauge vector and a gauge antisymmetric tensor. For the spin-1 sector in such a formulation, the Lagrangian is $\int F_{\mu\nu} F^{\mu\nu} + m \int A_\mu H^\mu + \int H_\mu H^\mu$, where $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ and $H_\mu = \epsilon_{\mu\nu\rho\sigma} \partial^\nu A^{\rho\sigma}$; one may verify that this describes pure massive spin-1. Coupling to other matter is then effected through gauge-invariant couplings of the A_μ field. For the full $N = 2$ massive vector supermultiplet, one has the superspace Lagrangian [22]

$$I = \int d^4x d^2\theta d^2\bar{\theta} d^4\theta G_{ij}(C) G_{kl}(C) + m \int d^4x d^4\theta d^4\bar{\theta} G_{ij}(C) V^{ij} + \left[\int d^4x d^4\theta W^2(V) + \text{c.c.} \right], \quad (4.4)$$

where the chiral prepotential C has the field-strength superfield $G_{ij} = D_{ij}C + \bar{D}_{ij}\bar{C}$ ($D_{ij} \equiv (1/2)D_{\alpha i}D_{\alpha j}$) and the real prepotential V_{ij} has the chiral field-strength superfield $W = i\bar{D}^4 D^{ij}V_{ij}$. Coupling to other fields may then be effected by $N = 2$ supersymmetric and gauge-invariant minimal-type couplings

$$\int d^4x d^4\theta d^4\bar{\theta} V^{ij} J_{ij}, \quad (4.5)$$

where the real current superfield J_{ij} must satisfy the conservation conditions

$$D_{\alpha(i} J_{k)\alpha} = \bar{D}_{\alpha(i} J_{k)\alpha} = 0, \quad (4.6)$$

subject to the equations of motion for the massless fields.

As discussed in sections 2 and 3 of this paper, couplings of massive Kaluza-Klein fields to the massless sector must preserve the gauge invariances of the massless fields as well as those of the massive gauge fields. This means that at lowest order the currents involved in such couplings must be built from field strengths of the massless gauge fields, just as we saw with the generalized stress tensor (2.7). Now, the massless sector of an $N = 2$ supersymmetric compactified Kaluza-Klein theory is described by hypermultiplets, vector multiplets and the massless graviton multiplet. Owing to the requirement that the current superfield J_{ij} be constructed from field-strength superfields, no such couplings are possible for the massless vector and graviton multiplets. In fact, the simplest real superfields X_{ij} that can be built from the field strengths of massless $N = 2$ vector or graviton multiplets have component currents with spins running up to 2 (and these also do not satisfy (4.6)).

For the hypermultiplets, on the other hand, it is possible to build a current superfield J_{ij} satisfying (4.6). For hypermultiplets described off-shell by the relaxed hypermultiplet defined in (4.1–4.3) (in which case the four scalar fields in each hypermultiplet split up into a real 3 and 1 of the $N = 2$ supersymmetry automorphism group $SU(2)$), the current superfield takes the form

$$J_{ij}^{[AB]} = L_{[i} L_{j]}^{[A} B]} - \frac{2}{3} L_{ij}^{[A} V^{B]}, \quad (4.7)$$

which satisfies (4.6) subject to the free hypermultiplet equations of motion [28]

$$L_{ij;kl} = 0; \quad D_{\alpha j} L^{ij} = D_{\alpha}{}^i V. \quad (4.8)$$

For interacting hypermultiplets of this type, similar currents can be constructed. (A and B indicate indices for the massive multiplets' gauge symmetries; as in section 3, we need not be concerned with the details of these indices except for the indicated antisymmetrization, which is necessary in this case owing to the reality of the hypermultiplet representation (4.1–4.3).)

For hypermultiplets in which the scalar fields carry the complex 2 representation of the automorphism symmetry $SU(2)$, there is no off-shell formulation with a finite number of component fields. Such multiplets can be represented in harmonic superspace [29], which gives an infinite number of component fields. However, for our present purposes it is not necessary to go into all the details of the off-shell formulation, since we just need to know if there is a current superfield satisfying (4.6) subject to the massless multiplets' equations of motion. For free hypermultiplets ϕ^A , satisfying the field equations

$$D_{\alpha(i} \phi^A_{j)} = \bar{D}_{\alpha(i} \phi^A_{j)} = 0, \quad (4.9)$$

the current superfield is of the form [22]

$$J_{ij}^{[AB]} = \bar{\phi}^{[A}_{(i} \phi^{B]}_{j)}. \quad (4.10)$$

Note that the current for this complex representation of the hypermultiplet does not require that the A and B indices be antisymmetrized. For interacting hypermultiplets, similar currents can be constructed.

Thus, supersymmetry and the gauge invariances of the massive and massless fields are not by themselves sufficient to rule out couplings in $N = 2$ supersymmetric Calabi-Yau compactifications that could destroy the stability of the massless sector against the effects of integrating out massive Kaluza-Klein modes. For couplings of the form (4.8) or (4.10), integrating out a massive multiplet could be effected in lowest order as before by dropping the kinetic terms in the action (4.4), keeping just the off-diagonal mass term

$$m \int d^4x d^4\theta d^4\bar{\theta} G_{ij}^{[AB]}(C) V^{[AB]ij} \quad (4.11)$$

together with the coupling

$$\int d^4x d^4\theta d^4\bar{\theta} V^{[AB]ij} J_{ij}^{[AB]}. \quad (4.12)$$

Integrating out $V_{ij}^{[AB]}$ then would just impose the relation

$$G_{ij}^{[AB]}(C) = \frac{1}{m} J_{ij}^{[AB]}. \quad (4.13)$$

Note that, owing to the off-diagonal form of the mass term (4.11), integrating out $V_{ij}^{[AB]}$ would not introduce any potential-type terms in the effective action for the massless fields. The lowest-order corrections to the massless fields' effective action would come from substituting (4.12) into the kinetic terms for the massive vector multiplets. From (4.4), we see that this would imply corrections of the form

$$\Delta \mathcal{L}_{\text{eff}} = \frac{1}{m^2} \int d^4x d^2\theta (\bar{y}^{\dot{2}} \bar{\theta}^{\dot{2}k\ell}) J_{ij}^{[AB]} J_{k\ell}^{[AB]}, \quad (4.14)$$

which would contribute incalculable corrections to the σ -model metric for the hypermultiplet modulus fields without causing any inadmissible potential to appear for these fields.

Couplings of the form (4.5, 4.12) could be expected to occur in general Kaluza-Klein compactifications preserving $N = 2, d = 4$ supersymmetry, and so uncontrollable corrections of the form (4.14) could in general be expected when the massive vector multiplets are integrated out. In the case of no-scale compactifications of pure $d = 10$ supergravity theories, however, the situation is rescued by the fact that the scale of the m^{-2} factor in (4.14) itself must be set by the vacuum expectation value of some field. This scale-setting field, or “breathing mode” is itself one of the modulus fields, so the structure of the correction (4.14) would have to accommodate an extra superfield containing the breathing mode. The reasons why such couplings are ruled out in the end involve specific details of the supergravity theory being compactified. We first consider the type IIA theory.

In the type IIA theory, compactification on a Calabi-Yau space $\mathcal{K}$ yields $(h^{2,1} + 1)$ massless hypermultiplets containing modulus fields, where $h^{2,1}$ denotes the number of harmonic $(2, 1)$ -forms on $\mathcal{K}$, i.e. $h^{2,1} = \dim[H^{2,1}(\mathcal{K})]$. One hypermultiplet (referred to as the “universal” hypermultiplet in ref. [2]) contains the massless dilaton and a massless spin-0 degree of freedom corresponding to the dualized surviving $d = 4$ part of the 2-form gauge field $A_{\mu\nu}$. The remaining $h^{2,1}$ hypermultiplets are associated to the $(2, 1)$ -forms on $\mathcal{K}$; each contains two modulus fields coming from the metric $g_{\mu\nu}$ and two from the 3-form gauge field $A_{\mu\nu\rho}$. In the following, we will not need to consider in detail whether the hypermultiplets are represented in the form (4.1-4.3, 4.8) or in the form (4.9). However, we shall have to consider the implications of replacing the factor of m^{-2} in (4.14) by factors of the breathing mode more carefully.

The breathing mode is just one of $h^{1,1}$ metric modulus fields associated to the $(1, 1)$ -forms on $\mathcal{K}$ ($h^{1,1} = \dim[H^{1,1}(\mathcal{K})]$). Together with equal numbers of modulus fields coming from the 2-form gauge field $A_{\mu\nu}$ and of massless spin-1 fields coming from the 3-form gauge field $A_{\mu\nu\rho}$, these constitute the propagating bosonic components of $h^{1,1}$ vector multiplets. Thus, a correction such as (4.14) to the hypermultiplet sector of the effective action would necessarily entail a mixing of vector multiplets and hypermultiplets upon replacement of the factors of m^{-2} by factors of the superfield containing the breathing mode. This would contribute off-diagonal terms between $H^{1,1}$ and $H^{2,1}$ modulus fields to the σ -model metric.

But this cannot occur, since the total σ -model for the modulus fields must factorise into separate sectors for the $H^{1,1}$ and $H^{2,1}$ moduli, with a direct product metric.

There are several ways to see that the total modulus space is a product space [30, 1, 2, 31, 5]. It follows from the fact that the $H^{1,1}$ and $H^{2,1}$ modulus fields belong to distinct types of $N = 2$ supermultiplets (vector multiplets and hypermultiplets), and there are no canonical-dimension (i.e. with two derivatives) $N = 2$ supersymmetric kinetic terms that mix these two types of multiplet. It turns out that the total σ -model target space for massless scalars in an $N = 2$ supergravity coupling must be a product of a Kähler and a quaternionic Kähler manifold [1, 2], the first for the scalars transforming in vector multiplets and the second for the hypermultiplet scalars. As discussed in ref. [2], there is a map that projects the quaternionic Kähler manifold onto a submanifold that is a second Kähler manifold. Both of these Kähler manifolds are of a restricted type, in which the Kähler potential is determined by a holomorphic function of the scalar fields. In ref. [31], the theory of complex manifolds has been used to show that such a product structure for the space of moduli is mathematically natural. More recently, a string-based discussion [5] has used the relation of the four-point function of the modulus fields to the full curvature tensor for the space of moduli to show that the holonomy group of the space of moduli has a product structure. It then follows that the metric on this space has a product structure. As a result of this direct-product structure, the type IIA low-energy effective theory of modulus fields is stable, both perturbatively and non-perturbatively, against corrections from the massive Kaluza-Klein modes in no-scale compactifications.

In the case of the type IIB theory, the rôles of the $H^{1,1}$ and $H^{2,1}$ modulus fields are reversed [2, 3, 32]. Accordingly, there are now $h^{2,1}$ vector multiplets associated to the elements of $H^{2,1}(\mathcal{K})$, each containing two metric modulus fields and a massless vector coming from the 4-form self-dual gauge field $A_{\mu\nu\rho\sigma}$ of the type IIB theory. The $H^{1,1}(\mathcal{K})$ modulus fields now transform in $h^{1,1}$ hypermultiplets, each containing one metric modulus field, two modulus fields from the complex 2-form gauge field $A_{\mu\nu}$ and one real spin-0 degree of freedom coming from $A_{\mu\nu\rho\sigma}$ after dualization (with two indices projected into $d = 4$ and the other two internal, giving an antisymmetric tensor in $d = 4$). There is also the one “universal” hypermultiplet, containing two spin-zero degrees of freedom from the surviving $d = 4$ parts of the complex $A_{\mu\nu}$ gauge field plus the two scalar degrees of freedom of the complex Φ field of the type IIB theory.

Since the hypermultiplets now contain $H^{1,1}$ modulus fields, there would be no inconsistency with the direct-product structure of the low-energy σ -model metric if the factors of m^{-2} were to be replaced by factors of the breathing-mode field. We shall not enter into a discussion of how such expressions might be made $N = 2$ supersymmetric. The reason that couplings such as (4.12) and the corrections (4.14) following from them are ruled out in the type IIB theory is that the two modulus fields in each of the $H^{1,1}$ hypermultiplets that arise from the 2-form gauge field $A_{\mu\nu}$ have an inhomogeneous “shift” symmetry of Peccei-Quinn type arising from the original $d = 10$ gauge symmetry of $A_{\mu\nu}$. Any $N = 2$

supersymmetric expression involving $H^{1,1}$ hypermultiplets that reduces to (4.14) upon letting the breathing-mode hypermultiplet assume a constant value would necessarily contain terms with undifferentiated Peccei-Quinn scalars. Consequently, such corrections, and similarly the couplings generalizing (4.12) leading to them, are ruled out by the Peccei-Quinn symmetries. As a result, the Peccei-Quinn symmetries ensure that the low-energy effective theory of the modulus fields of the type IIB theory is also stable against corrections from the massive Kaluza-Klein modes in no-scale compactifications.

The above argument for the type-IIB theory holds in perturbation theory, where the Peccei-Quinn symmetry is unbroken. On the other hand, one expects the Peccei-Quinn symmetry to be violated by non-perturbative effects such as genus-0 world-sheet instantons, which are known to modify the perturbative result for the σ -model metric of the $H^{1,1}$ modulus fields [32]. Thus, at the non-perturbative level, it is not possible to establish the stability of the type-IIB low-energy σ -model against massive-mode corrections.

5 – Conclusions

In the preceding sections, we have focused primarily on the pure supergravity sector of the higher-dimensional field theory or superstring theory. When one restricts attention to this sector alone, all the lower-dimensional massless scalars are modulus fields, and hence have no potential in compactifications of the form $(\text{Minkowski})_4 \times \mathcal{K}$. As discussed above, one way to see this is to note that the action of the higher-dimensional pure supergravity theory vanishes on shell. In the case of type-II theories, the $d = 10$ effective field theory consists of an $N = 2$ pure supergravity sector only, and so the analysis of sections 3 and 4 may be applied directly.

In the case of the heterotic superstring, the situation is different, because this theory contains a super-Yang-Mills sector in addition to the $d = 10$ supergravity sector. In the presence of non-vanishing field strengths for the super-Yang-Mills sector, the effective action no longer vanishes on shell. Correspondingly, there is now a potential in the compactified theory for the massless scalar fields. As a result, the stability of the low-energy effective theory against corrections arising from massive Kaluza-Klein modes as discussed in the preceding sections will be lost. In perturbation theory, however, the potential for the massless scalar fields that causes the trouble has a particular structure that nonetheless allows us to retain some of the conclusions of the preceding sections. There are no terms in the perturbative contributions to the potential that involve the modulus fields alone, because the heterotic theory's perturbative effective action does vanish on shell when one sets to zero all of the massless scalar fields from the Yang-Mills sector (i.e. the massless fields that are charged under the unbroken Yang-Mills gauge group and the $H^1(\text{End } T)$ singlets).

Accordingly, one can only attempt to deduce the structure of the low-energy effective field theory of the heterotic string from the more tightly constrained low-energy $N = 2$

supersymmetric compactified type-II theories to the extent of the modulus sector alone.* As noted in the introduction, the heterotic theory possesses massless gauge-singlet fields that do not occur in the type-II theories. In order to compare the low-energy effective actions, one would have to truncate out these extra fields. (We shall return to the question of whether this truncation is consistent.)

Owing to the presence of perturbative superpotentials in the heterotic theory, describing interactions of the modulus fields and the $H^1(\text{End } T)$ fields with the charged fields, and among the charged fields themselves, there will be massive-mode corrections that are constrained only by the surviving $N = 1$, $d = 4$ supersymmetry of the compactified heterotic theory. Moreover, these corrections to the σ -model metric of the massless fields may be expected to arise in every order in α' .

We shall see that non-perturbative effects imply even greater limitations on one's ability to deduce the low-energy effective theory of the modulus fields for the heterotic string from the corresponding sector of the type-II theories. In the following, we shall discuss first the perturbative and then the non-perturbative corrections to the effective theories of the modulus fields.

Perturbative contributions

As we saw in section 1, the simplest way to see that there are no perturbative contributions to the potential for the metric modulus fields is to observe that the world-sheet $(2,2)$ -supersymmetric field equation for the internal Kähler metric has the cohomologically-trivial form

$$R_{a\bar{b}} = \partial_a \partial_{\bar{b}} S, \quad (5.1)$$

and then to use Yau's theorem iteratively to show that the number of free parameters is preserved order by order in α' . Thus, massless fields in the effective field theory that derive from the metric moduli of the lowest-order internal equation $R_{a\bar{b}} = 0$ continue to be associated to moduli, i.e. to free parameters, of the higher-order equation (5.1), and consequently they can have no potential.

Equation (5.1) is written in a holomorphic coordinate system corresponding to a specific choice of complex structure for the Kähler manifold $\mathcal{K}$. Given this choice of complex structure, Yau's theorem tells us that the number of free parameters arising in the solution to (5.1) is given by $h^{1,1}$. Furthermore, the dimension of the space of complex structures for a

* To see that the possibility of such a connection is non-trivial, one should bear in mind the rôle of the heterotic Yang-Mills fields, which are absent in the type II case, in compactifying the heterotic string on a Calabi-Yau space [18]. Although an $N = 1$ low-energy theory can always be obtained by truncation from an $N = 2$ theory, there is no *a priori* reason to suppose that the resulting σ -model is the one appearing in the heterotic case.

Kähler manifold of complex dimension 3, i.e. the number of free parameters in the solution to the Nijenhuis equation

$$N^k_{ij} \equiv J^k_i(\partial_t J^k_j - \partial_j J^k_t) - J^k_j(\partial_t J^k_i - \partial_i J^k_t) = 0, \quad (5.2)$$

is given by $2h^{2,1}$ [33]. Thus, the total number of metric moduli for such manifolds is given by $h^{1,1} + 2h^{2,1}$.*

The essential point that underlies the argument for the absence of perturbative potentials for the $h^{1,1} + 2h^{2,1}$ metric modulus fields is the fact that the perturbative counterterms for a (2,2)-worldsheet-supersymmetric σ -model must have the form (5.1). Equivalently, since (2,2)-worldsheet supersymmetry leads to $N = 2$ spacetime supersymmetry in a compactified type-II superstring theory, one can consider the possible presence of potentials from the viewpoint of $N = 2$, $d = 4$ spacetime supersymmetry. For the type-IIA superstring, the $H^{1,1}$ moduli give rise to $N = 2$ vector multiplets, while the $H^{2,1}$ moduli give rise to hypermultiplets. For the type-IIB superstring, the rôles of the two kinds of moduli are reversed. In either case, from the $N = 2$ spacetime-supersymmetric viewpoint, a potential for the modulus fields could arise only through some kind of gauge coupling of the vector supermultiplets, either non-abelian self-couplings or minimal gauge couplings of the vector supermultiplets to the hypermultiplets. In fact, the vector multiplets arising from Calabi-Yau compactifications are necessarily abelian, and there are no direct interactions between the two kinds of supermultiplets (they do, of course, interact via $N = 2$ supergravity), and thus a potential cannot arise. The abelian nature of the multiplets follows from the differential-geometric fact that real cohomology is abelian, and the absence of abelian minimal coupling follows from the fact that there are no non-zero overlap integrals that mix $H^{1,1}$ and $H^{2,1}$ harmonics on a Calabi-Yau space.

* It is not *a priori* obvious that all of the metric deformations satisfying the internal field equations preserve the Kähler condition. For example, at lowest order the question is whether all Ricci-flat deformations of a given Ricci-flat Kähler metric preserve the Kähler condition. In this case, one can easily see that the total number of Ricci-flat metric deformations (i.e. the number of zero modes of the Lichnerowicz operator) is also equal to $h^{1,1} + 2h^{2,1}$. Thus the dimension of the space of moduli for Ricci-flat metrics is the same as the dimension of the space of moduli for Ricci-flat Kähler metrics, and so the connected components of the two spaces of moduli must coincide. This leaves open the question of whether there might be non-Kähler Ricci-flat metrics on the Calabi-Yau manifold that are not connected continuously to the Kähler ones. Interestingly enough, there is a much more elegant argument that can be given in the case of manifolds of real dimension $4n$ rather than $4n + 2$. If such a manifold admits a Ricci-flat Kähler metric, then one knows that there are two covariantly-constant spinors of the same chirality, and hence that the index of the Dirac operator is either $+2$ or -2 (since Lichnerowicz's theorem implies that any Dirac zero mode is covariantly constant if the Ricci scalar vanishes). Thus there will also necessarily be two covariantly-constant spinors of the same chirality for any other Ricci-flat metric on the same manifold, and hence any Ricci-flat metric on the manifold must be Kähler, whether or not it is continuously connected to the original Ricci-flat Kähler metric. This global topological argument fails in $4n + 2$ dimensions because in that case the two covariantly-constant spinors have opposite chiralities and so the Dirac index is zero, and hence one gains no information about the existence of covariantly-constant spinors for other Ricci-flat metrics.

Non-perturbative contributions

Although $N = 2$ spacetime supersymmetry implies that there are no non-perturbative corrections to the potentials for the type-II theories, there can be non-perturbative corrections to the σ -model metric for the modulus fields. In order to derive the stability of the low-energy effective theory of modulus fields against corrections from the massive modes, an important assumption was made in section 4, namely that the Peccei-Quinn symmetries of the $H^{1,1}$ pseudoscalar modulus fields remain unbroken. Under these symmetries, originating as we have seen from the gauge invariances of the antisymmetric tensor fields in $d = 10$, the pseudoscalar modulus fields undergo constant shifts, and so this invariance puts further strong restrictions on the low-energy effective theory beyond those implied by $N = 2$ spacetime supersymmetry [2].

For definiteness, let us consider the compactification of the type IIA superstring. In this case, the vector multiplets contain the $H^{1,1}$ modulus fields. $N = 2$ local supersymmetry then implies that the σ -model metric for the $H^{1,1}$ modulus fields in these vector multiplets must be Kähler, with a Kähler potential $K = -\log Y$ that is determined [3,31] by a single holomorphic function [34,35] $F(\phi^a)$:

$$Y = F + \bar{F} - \frac{1}{2}(F_a - \bar{F}_a)(\phi^a - \bar{\phi}^a), \quad (5.3)$$

where the real and imaginary parts of ϕ^a correspond to the pseudoscalar and scalar $H^{1,1}$ moduli and $a = 1, \dots, h^{1,1}$.

The requirement of a Peccei-Quinn symmetry $\delta\phi^a = C^a$, with C^a real, imposes a strong constraint on the form of $F(\phi^a)$, as can be seen from the fact that the $N = 2$ supersymmetric action contains axion couplings like

$$\text{Im}(F_{,abc}(\phi))F_{\mu\nu}^b\tilde{F}^{\mu\nu c}, \quad (5.4)$$

where $F_{\mu\nu}^b$ are the field strengths of the spin-1 members of the vector supermultiplets containing the ϕ^b . Thus, we see that Peccei-Quinn invariance requires that $F(\phi)$ take the form

$$F(\phi) = id_{abc}\phi^a\phi^b\phi^c, \quad (5.5)$$

for which the axion coupling (5.4) has the Peccei-Quinn invariant form

$$d_{abc}b^aF_{\mu\nu}^b\tilde{F}^{\mu\nu c} \quad (5.6)$$

with $\phi^a = b^a + is^a$, where b^a and s^a are respectively the pseudoscalar and scalar modulus fields in the vector supermultiplet. The coefficients d_{abc} are given by the intersection coefficients [36]

$$d_{abc} = \int \omega_a \wedge \omega_b \wedge \omega_c \quad (5.7)$$

in some suitable basis ω_a of the harmonic (1,1) forms on the compactifying space $\mathcal{K}$.

The Peccei-Quinn symmetry remains valid in all orders of σ -model perturbation theory, but it is expected to be violated [32] by non-perturbative effects such as world-sheet instantons. These effects therefore will change the function $F(\phi)$ but do not alter its holomorphic character, which is purely a consequence of $N = 2$ spacetime supersymmetry. From (5.3), it follows that the violation of the Peccei-Quinn symmetry is also responsible for non-perturbative corrections to the Kähler metric $G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K$ on the space of $H^{1,1}$ moduli. As we have seen in section 4, the violation of the Peccei-Quinn symmetry by world-sheet instanton effects also makes it impossible to extend our argument for the perturbative stability of the type IIB low-energy effective theory to the non-perturbative level. This raises the possibility that non-perturbative effects may in fact destroy the stability of the type IIB theory.

As for the $H^{2,1}$ modulus fields, which correspond to deformations of the complex structure and which are the bosonic components of hypermultiplets in the type IIA theory, the σ -model metric is quaternionic Kähler when these hypermultiplets are coupled to $N = 2$ supergravity [37]. Using the fact that the correspondence of hypermultiplets and vector multiplets to $H^{2,1}$ and $H^{1,1}$ moduli is interchanged when one goes from the type IIA to the type IIB theory, it has been argued [31, 4] that the quaternionic Kähler metric must have a very special form that is determined by a second holomorphic function $\mathcal{F}(\varphi^A)$ of the $h^{2,1}$ complex scalar modulus fields φ^A . Explicit string calculations and non-renormalization arguments [32] indicate that this function does not receive either perturbative or non-perturbative σ -model corrections. Here one again uses the argument that the σ -model corrections depend on the $H^{1,1}$ modulus R that at lowest order is the radial or volume-scaling mode of the internal space. On the other hand, as discussed in section 4, the total modulus space is a product space $[30, 1, 2, 31, 5] \mathcal{M}_{(1,1)} \times \mathcal{M}_{(2,1)}$. Consequently, the metric on $\mathcal{M}_{(2,1)}$ cannot acquire σ -model corrections, and is thus determined entirely by the lowest-order results.

The situation is different for the compactified heterotic string. As we have seen above, the Yang-Mills sector in this theory does give rise to massless scalars with non-vanishing superpotentials in lowest order of perturbation theory and these superpotentials also involve the modulus fields. Moreover, it has been argued that there are non-perturbative contributions to the low-energy effective theory due to world-sheet instantons that give superpotentials involving only the modulus fields and the $H^1(\text{End } T)$ fields [11]. (We note, however, that the existence of such superpotentials would seem to disagree with the general analysis of ref. [5].) These superpotentials always contain at least one of the $H^1(\text{End } T)$ fields of the heterotic theory, and thus would vanish if those fields were set to zero. However, such a truncation would not be consistent with the equations of motion for the $H^1(\text{End } T)$ fields. A typical trilinear term in the superpotential has one of the forms [11]

$$\begin{aligned} & \Phi_{(2,1)} \Phi_{(1,1)} \Phi_{H^1(\text{End } T)} \\ & \Phi_{(2,1)} \Phi_{(2,1)} \Phi_{H^1(\text{End } T)}. \end{aligned} \quad (5.8)$$

Thus, although one may formally recover the correspondence between the type-II and heterotic low-energy theories by such a truncation, this is not necessarily indicative of a true correspondence in the dynamics of the two types of theory once non-perturbative effects are taken into account.

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