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Stability of Charged Boson Stars

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ABSTRACT

I study the dynamical stability of spherically symmetric gravitational equilibria of scalars coupled to a U(1) gauge field. I derive the pulsation equation, which governs the time evolution of the infinitesimal radial oscillations as well as a variational principle for its eigenvalues. The eigenvalues are real also in the case where the equilibrium configurations have nodes. The equilibrium configurations with a central density bigger than ρ_{crit} , corresponding to the critical mass, are dynamically unstable.

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The more recent developments in particle physics and cosmology suggest that scalar fields may have played an important role in the evolution of the early universe, for instance in primordial phase transitions and that they may make up part of the missing dark matter. Models for galaxy formation using cold dark matter and the inflationary scenario suggest that the ratio of baryonic (luminous) matter to dark matter can be of the order of 10 %. These facts naturally raise the question whether cold gravitational equilibrium configurations of massive scalar fields - Bose stars - may exist and whether such configurations are dynamically stable. Spherically symmetric equilibrium configurations of scalar fields have been found by solving the coupled Einstein-Klein-Gordon equations without [1-4] or with a quartic self-interaction [5]. More recently Jetzer and van der Bij [6] have analysed the static spherically symmetric solutions for a system of complex scalar fields coupled to a U(1) gauge field, allowing also for the possibility of a quartic self-interaction. In all these models if one plots the mass for equilibrium configurations of Bose stars against their central density, one finds a behaviour very similar to neutron stars. The mass quickly rises to a maximum for $\rho = \rho_{crit}$, drops a little, oscillates and approaches an asymptotic value at large central densities. A similar behaviour is found for the particle number. The main effect of having scalar fields coupled to a U(1) gauge field is to increase the maximum mass with increasing gauge coupling constant e . At the same time the binding energy per particle decreases. Static solutions only exist for $e^2/4\pi < Gm^2 = e_{crit}^2$.

In this paper I address the problem of the dynamical stability of spherically symmetric gravitational equilibria of scalars coupled to a U(1) gauge field in the framework of general relativity. I analyse the time evolution of infinitesimal radial oscillations, which conserve the total number of particles, following the method developed by Chandrasekhar [7], and which has already been applied with some modifications to study the stability in the cases without an additional U(1) gauge field [8]. I find an eigenvalue equation, called also pulsation equation, which determines the normal modes of the radial oscillations and a variational principle for determining the eigenvalues. As in the cases without charge [4], the particle number and mass have their extrema, in particular their maximum, at the same value of the central density. From this fact it follows, as pointed out by Lee and Pang [9], that the pulsation equation has a zero mode, where M and N have their extrema. Therefore one expects the central density ρ_{crit} , corresponding to the maximum mass, to be the boundary between stable (for ρ smaller than ρ_{crit}) and unstable equilibrium configurations. However in order to establish this fact completely one still needs to analyse the eigenvalues of the pulsation equation in order to see if they are real and that for central densities smaller than ρ_{crit} they are positive. M and N as a function of the central density have other stationary points for ρ bigger than ρ_{crit} . We expect the behaviour of the stability not to change there, a fact which is already suggested by the shape of the particle number versus radius diagram [6], which is bent counterclockwise at the critical points. A rigorous proof, however, can be given using the variational principle; in fact one can easily find an upper bound to the lowest eigenvalue at the stationary points of M and N for ρ bigger than ρ_{crit} . It turns out that the upper bounds are negative, therefore the lowest eigenvalue is negative which establishes that the equilibrium configurations remain unstable for all ρ bigger than ρ_{crit} .

The system we consider is a complex scalar field with a U(1) charge and coupled to gravity, whose action is given by

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G_N} + g^{\mu\nu} (D_\mu \phi)^* (D_\nu \phi) - m^2 |\phi|^2 - \frac{\lambda}{2} |\phi|^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad (1)$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu \phi = \partial_\mu \phi + ie A_\mu \phi$. The action is invariant under a local U(1) gauge transformation, thus the total particle number is conserved. The corresponding current is given by

$$j^\mu = \sqrt{-g} g^{\mu\nu} (ie [\phi^* \partial_\nu \phi - \phi \partial_\nu \phi^*] - 2e^2 A_\nu |\phi|^2). \quad (2)$$

We take all the fields as functions of r and t alone since we consider only spherically symmetric equilibrium configurations. We express the metric in Schwarzschild coordinates

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (3)$$

where ν and λ are functions of r and t only ($g^{00} = -e^{-\nu}$ and $g^{rr} = e^{-\lambda}$). We chose the gauge field A_μ such that we have only electric charges and no magnetic ones, therefore we set $A_\mu = (A_t = A, A_r = C = 0, A_\theta = 0, A_\varphi = 0)$. For our considerations it is more convenient to write the complex scalar field as $\phi = (\phi_1 + i\phi_2)e^{i\omega t}$, where ϕ_1 and ϕ_2 are two real fields. For the equilibrium configuration we set $\phi_1(r, t) = \phi_0(r)$, $\phi_2(r, t) = 0$ and all the other fields and metric functions are time independent. The coupled system of equations, which can be derived from the above action by varying with respect to the various fields, is given by the two Einstein equations

$$\lambda'_0 = \frac{(1 - e^{\lambda_0})}{r} + 8\pi G_N r e^{\lambda_0} \left[((\omega + eA_0)^2 e^{-\nu_0} + m^2) \phi_0^2 + \frac{\lambda}{2} \phi_0^4 + \phi_0^2 e^{-\lambda_0} + \frac{A_0^2}{2} e^{-\nu_0 - \lambda_0} \right], \quad (4)$$

$$\nu'_0 = \frac{(e^{\lambda_0} - 1)}{r} + 8\pi G_N r e^{\lambda_0} \left[((\omega + eA_0)^2 e^{-\nu_0} - m^2) \phi_0^2 - \frac{\lambda}{2} \phi_0^4 + e^{-\lambda_0} \phi_0^2 - \frac{A_0^2}{2} e^{-\nu_0 - \lambda_0} \right], \quad (5)$$

the Maxwell equation

$$A'_0 + \left(\frac{2}{r} - \frac{\nu'_0 + \lambda'_0}{2} \right) A_0 - 2e\phi_0^2 e^{\lambda_0} (\omega + eA_0) = 0, \quad (6)$$

and the scalar wave equation

$$\phi''_0 + \left(\frac{2}{r} + \frac{\nu'_0 - \lambda'_0}{2} \right) \phi'_0 + e^{\lambda_0} ((\omega + eA_0)^2 e^{-\nu_0} - m^2 - \lambda \phi_0^2) \phi_0 = 0. \quad (7)$$

As boundary conditions we impose $\phi_0(0) = \text{const}$, $\phi'_0(0) = 0$ and $\phi_0(\infty) = \phi'_0(\infty) = 0$ in order to have a localized particle distribution. We allow the possibility of ϕ_0 having nodes, giving rise to excited states. Furthermore we demand that the electric field be absent at the origin, giving $A'_0(0) = 0$, and we normalize the potential by $A_0(\infty) = 0$. The condition $e^{\lambda_0(\infty)} = 1$ is imposed to get asymptotically the ordinary Minkowski metric and the condition $e^{\lambda_0(0)} = 1$ is a regularity condition. With these boundary conditions

the equations (4-7) form an eigenvalue equation for ω . For a discussion of the equilibrium solutions see Ref.[6].

The total particle number N_0 , which is related to the total charge by $Q = eN_0$, is given by

$$N_0 = 8\pi \int_0^\infty dr r^2 (\omega + eA_0) \phi_0^2 e^{(\nu_0 - \lambda_0)/2}. \quad (8)$$

Consider now the situation where the equilibrium configuration is perturbed in a way such that the spherical symmetry is still preserved. These perturbations will give rise to motions in the radial direction. The equations governing the small perturbations are obtained by expanding all functions to first order ($\lambda(\tau, t) = \lambda_0(\tau) + \delta\lambda(\tau, t)$, $\nu(\tau, t) = \nu_0(\tau) + \delta\nu(\tau, t)$, $\phi_1(\tau, t) = \phi_0(\tau) + \delta\phi_1(\tau, t)$, $\phi_2(\tau, t) = \delta\phi_2(\tau, t)$, $A(\tau, t) = A_0(\tau) + \delta A(\tau, t)$, $C(\tau, t) = \delta C(\tau, t)$) and by linearizing the equations. Notice that we have to allow δC , which describes the magnetic field, to be different from zero. A_ϕ and A_ν can still be kept zero, since they do not couple linearly to the other fields. We obtain the following set of Einstein equations for the perturbed quantities

$$\begin{aligned} (re^{-\lambda_0} \delta\lambda)' &= 8\pi G_N r^2 [2m^2 \delta\phi_1 \phi_0 + \bar{\lambda} \phi_0^2 \delta\phi_1 - e^{-\nu_0} (\omega + eA_0)^2 \phi_0^2 \delta\nu + e^{-\nu_0} (\omega + eA_0)^2 2\phi_0 \delta\phi_1 \\ &\quad + e^{-\nu_0} \phi_0^2 (\omega + eA_0) 2e\delta A + 2(\omega + eA_0) \phi_0 \delta\phi_2 e^{-\nu_0} + e^{-\lambda_0} 2\phi_0 \delta\phi_1' - \delta\lambda e^{-\lambda_0} \phi_0^2 \\ &\quad + e^{-(\nu_0 + \lambda_0)} A_0' \delta A' - \frac{1}{2} (\delta\nu + \delta\lambda) e^{-(\nu_0 + \lambda_0)} A_0'^2 - e^{-(\nu_0 + \lambda_0)} A_0' \delta C], \end{aligned} \quad (9)$$

$$\begin{aligned} \delta\nu' &= \frac{\delta\lambda}{r} + \delta\lambda\nu_0' + 8\pi G_N e^{\lambda_0} [-2m^2 \phi_0 \delta\phi_1 - \bar{\lambda} \phi_0^2 \delta\phi_1 + e^{-\nu_0} (\omega + eA_0)^2 2\phi_0 \delta\phi_1 \\ &\quad + e^{-\nu_0} \phi_0^2 2e\delta A (\omega + eA_0) - e^{-\nu_0} \delta\nu (\omega + eA_0)^2 \phi_0^2 + 2(\omega + eA_0) \phi_0 \delta\phi_2 e^{-\nu_0} + e^{-\lambda_0} 2\phi_0 \delta\phi_1' \\ &\quad - \delta\lambda e^{-\lambda_0} \phi_0^2 - e^{-(\nu_0 + \lambda_0)} A_0' \delta A' + \frac{1}{2} (\delta\nu + \delta\lambda) e^{-(\nu_0 + \lambda_0)} A_0'^2 + e^{-(\nu_0 + \lambda_0)} A_0' \delta C], \end{aligned} \quad (10)$$

and

$$\delta\dot{\lambda} = 16\pi G_N [\delta\phi_1 \phi_0' + (\omega + eA_0) e\delta C \phi_0^2 - (\omega + eA_0) (\phi_0' \delta\phi_2 - \delta\phi_2' \phi_0)]. \quad (11)$$

The linearized equations for $\delta\phi_1$ and $\delta\phi_2$ are

$$\begin{aligned} \delta\phi_1'' &+ \delta\phi_1' \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{2} \right) - \delta\ddot{\phi}_1 e^{\lambda_0 - \nu_0} + \delta\phi_1 e^{\lambda_0} (e^{-\nu_0} (\omega + eA_0)^2 - m^2 - \frac{\bar{\lambda}}{2} 3\phi_0^2) \\ &\quad + 2e^{\lambda_0 - \nu_0} (\omega + eA_0) \delta\dot{\phi}_2 + \frac{1}{2} \phi_0' (\delta\nu' - \delta\lambda') - \delta\nu \phi_0 e^{\lambda_0 - \nu_0} (\omega + eA_0)^2 \\ &\quad + 2e\phi_0 e^{\lambda_0 - \nu_0} (\omega + eA_0) \delta A + \delta\lambda e^{\lambda_0} (e^{-\nu_0} (\omega + eA_0)^2 - m^2 - \frac{\bar{\lambda}}{2} \phi_0^2) \phi_0 = 0, \end{aligned} \quad (12)$$

and

$$\begin{aligned} \delta\phi_2'' &+ \delta\phi_2' \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{2} \right) - \delta\ddot{\phi}_2 e^{\lambda_0 - \nu_0} + \delta\phi_2 (e^{-\nu_0} (\omega + eA_0)^2 - m^2 - \frac{\bar{\lambda}}{2} \phi_0^2) e^{\lambda_0} \\ &\quad - 2\delta\dot{\phi}_1 e^{\lambda_0 - \nu_0} (\omega + eA_0) + \frac{1}{2} \phi_0 e^{\lambda_0 - \nu_0} (\omega + eA_0) (\delta\nu - \delta\lambda) - e^{\lambda_0 - \nu_0} e\phi_0 \delta A \\ &\quad + e\phi_0 \delta C' + 2e\phi_0' \delta C + e\phi_0 \delta C \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{2} \right) = 0. \end{aligned} \quad (13)$$

The linearized Maxwell equations are

$$\begin{aligned} \delta A'' &+ \left(\frac{2}{r} + \frac{\lambda_0' + \nu_0'}{2} \right) \delta A' - \left(\frac{\delta\lambda' + \delta\nu'}{2} \right) A_0' - 2e^2 e^{\lambda_0} \phi_0^2 \delta A - \left(\frac{\lambda_0' + \nu_0'}{r} - \frac{\lambda_0' + \nu_0'}{2} \right) \delta\dot{C} \\ &\quad - 4ee^{\lambda_0} (\omega + eA_0) \phi_0 \delta\phi_1 - 2ee^{\lambda_0} \phi_0 \delta\phi_2 - 2ee^{\lambda_0} (\omega + eA_0) \phi_0^2 \delta\lambda - \delta\dot{C}' = 0, \end{aligned} \quad (14)$$

and

$$\delta\ddot{C} + 2e^2 e^{\lambda_0} \phi_0^2 \delta C + \left(\frac{\delta\nu' + \delta\lambda'}{2} \right) A_0' + 2ee^{\lambda_0} (\phi_0 \delta\phi_2' - \phi_0' \delta\phi_2) - \delta\dot{A}' = 0. \quad (15)$$

Not all the above equations are independent due to the Bianchi identities; moreover there is also a gauge freedom in the fields A_μ . However it turns out that for our purpose it will not be necessary to choose explicitly a particular gauge. Eq.(11) integrates directly in time if we write $\delta\phi_2(\tau, t) = \phi_0(\tau) \dot{g}(\tau, t)$ and $\delta C(\tau, t) = h(\tau, t)$. We get

$$\delta\lambda = 16\pi G_N r [\delta\phi_1 \phi_0' + (\omega + eA_0) \phi_0^2 (g' + eh)]. \quad (16)$$

As a next step we eliminate the metric functions $\delta\nu$ and $\delta\lambda$ from eqs.(12)-(15). We notice that an expression for $\delta\lambda' - \delta\nu'$ and $\delta\nu' + \delta\lambda'$ can be obtained respectively by subtracting and adding eqs.(9) and (10). We take $\delta\lambda$ from eq.(16). An expression for $\delta\nu$ can be obtained by solving eq.(9) with respect to $\delta\nu$ and by substituting $\delta\lambda$ and $\delta\lambda'$ using eq.(16). The expression thus obtained can further be simplified by adding to it eq.(13), where we have inserted $\delta C = h$, $\delta\phi_2 = \phi_0 \dot{g}$ and integrated once in time. Finally we get for $\delta\nu$

$$\begin{aligned} \delta\nu &= \frac{2\ddot{g}}{\omega + eA_0} - \frac{2e^{\lambda_0 - \lambda_0}}{\omega + eA_0} (g'' + eh') + \frac{e^{\lambda_0 - \lambda_0} (g' + eh)}{\omega + eA_0} \left(-16\pi G_N r \phi_0'^2 - \frac{4}{r} + 2\lambda_0' - \frac{4\phi_0'}{\phi_0} \right) \\ &\quad + \frac{2e\delta A}{\omega + eA_0} + \delta\phi_1 \left(\frac{4}{r} + 16\pi G_N r \phi_0' \right). \end{aligned} \quad (17)$$

Next we differentiate eq.(17) with respect to r and insert the result thus obtained for $\delta\nu'$, as well as $\delta\nu$ itself, into the expression for $\delta\nu' - \delta\lambda'$, which we get by subtracting eq.(9) from eq.(10). Furthermore we make use of eq.(16) to eliminate $\delta\lambda$, $\delta\lambda'$ and we insert an expression for $\dot{g}$, which we get from eq.(13) by integrating once in time. This way we find the following equation for $g'(r, t)$

$$\begin{aligned} g''' &+ eh'' + (g'' + eh') \left(\frac{3}{2} (\nu_0' - \lambda_0') + \frac{2}{r} + \frac{2\phi_0'}{\phi_0} \right) - e^{\lambda_0 - \nu_0} (g' + eh) \\ &\quad - C_2(\tau) (g' + eh) - \delta\phi_1' \frac{2}{\phi_0} (\omega + eA_0) e^{\lambda_0 - \nu_0} - \delta\phi_1 C_3(\tau) \frac{e^{\lambda_0 - \nu_0}}{\phi_0^2} = 0, \end{aligned} \quad (18)$$

with

$$C_2(\tau) = \frac{2(\lambda'_0 - \nu'_0)}{\tau} + \frac{2}{\tau^2} \cdot \frac{\nu'_0 - \lambda'_0}{2} + 2\left(\frac{\phi'_0}{\phi_0}\right)^2 - 2\frac{\phi''_0}{\phi_0} + \frac{(\lambda'_0 - \nu'_0)}{2} \left(\nu'_0 - \lambda'_0 + 4\frac{\phi'_0}{\phi_0}\right) + 2e^2 \nu_0 \phi_0^2 - 8\pi G_N \tau e^{\lambda_0 - \nu_0} (\omega + eA_0)^2 \phi_0^2 (\nu'_0 - \lambda'_0 + \frac{2}{\tau}) + 32\pi G_N \tau (\omega + eA_0) \phi_0^2 e^{\lambda_0 - \nu_0} eA'_0, \quad (19)$$

and

$$C_3(\tau) = -2\phi'_0(\omega + eA_0) - 8\pi G_N \tau (\omega + eA_0) \phi'_0 (\nu'_0 - \lambda'_0 + \frac{2}{\tau}) \phi_0^2 + 2e\phi_0 A'_0 + 16\pi G_N \tau e^{\lambda_0} (\omega + eA_0) (m^2 \phi_0 + \frac{\lambda}{2} \phi_0^3) \phi_0^2 + 16\pi G_N \tau eA'_0 \phi'_0 \phi_0^2. \quad (20)$$

The analogous equation for $\delta\phi_1$ is obtained by inserting into eq.(12) the expression for $\delta\nu' - \delta\lambda'$ obtained by subtracting eq.(9) from eq.(10). Again $\delta\lambda$ and $\delta\nu$ are given by eqs.(16),(17) and $\delta\phi_2 = \phi_0 g$. The resulting equation is

$$\delta\phi_1'' + \delta\phi_1' \left(\frac{2}{\tau} + \frac{\nu'_0 - \lambda'_0}{2} \right) - \delta\ddot{\phi}_1 e^{\lambda_0 - \nu_0} - C_1(\tau) \delta\phi_1 + (g'' + ek')(\omega + eA_0) 2\phi_0 - (g' + eh) C_3(\tau) + (g' + eh)(2\phi'_0(\omega + eA_0) + 2e\phi_0 A'_0 + \phi_0(\omega + eA_0) \left(\frac{4}{\tau} + \nu'_0 - \lambda'_0 \right)) = 0, \quad (21)$$

with

$$C_1(\tau) = e^{\lambda_0} (3e^{-\nu_0} (\omega + eA_0)^2 + m^2 + \frac{\lambda}{2} 3\phi_0^2) - 8\pi G_N \tau \phi_0^2 \left(\frac{2}{\tau} + \nu'_0 - \lambda'_0 \right) + 32\pi G_N \tau e^{\lambda_0} \phi'_0 (m^2 \phi_0 + \frac{\lambda}{2} \phi_0^3). \quad (22)$$

As next step we could also eliminate the metric functions $\delta\nu$ and $\delta\lambda$ from the eqs.(14) and (15) for δA and δC . However we notice that if we write $(g' + eh) = f_2$ (and for notational clarity we set also $\delta\phi_1 = f_1$) we see immediately from the previous eqs.(18) and (21) that they no longer explicitly depend upon h (and thus δC) and δA . Furthermore, for f_1 and f_2 we can suppose a time dependence of the form $e^{i\omega t}$, since all the coefficients in the eqs.(18) and (21) are time-independent. Since we are interested only in the eigenvalues σ , which determine the time evolution and thus the stability behaviour of the system, we do not explicitly need eqs.(14),(15) and therefore do not need to choose a particular gauge condition for the fields δA and δC . This fact greatly simplifies the subsequent analysis of the stability, since, like in the case without a U(1) gauge field [8], we will have to deal with an eigenvalue equation consisting of two coupled equations only.

Before going on with the discussion of eqs.(18) and (21), we will address the question of the particle number conservation. We consider only radial perturbations, which do not change the total particle number N_0 , as given by eq.(8). In fact we can compute δN , and eliminating again the metric functions $\delta\nu$ and $\delta\lambda$, we find that the integrand of δN is a total derivative with respect of r , thus getting

$$\delta N = 8\pi(g' + eh)\phi_0^2 e^{(\nu_0 - \lambda_0)/2} \Big|_0^\infty. \quad (23)$$

Therefore the requirement $\delta N = 0$ translates into conditions on the boundary values for $f_2 = (g' + eh)$. For $r \rightarrow \infty$ we require $(f_2 r^2 \phi_0^2) \rightarrow 0$ ($e^{\lambda_0 - \nu_0} \rightarrow 1$). For $r \rightarrow 0$, since $\phi_0(0) = \text{const}$ and $e^{\lambda_0 - \nu_0} \rightarrow \text{const}$, we require $r^2 f_2 \rightarrow 0$.

In the following we are interested only in the lowest eigenvalue σ_0 , so that we will drop the subscript 0, and f_1, f_2 will denote the radial part from now on. We multiply eq.(18) by $G_2 = r^2 \phi_0^2 e^{3(\nu_0 - \lambda_0)/2}$ and eq.(21) by $G_1 = r^2 e^{(\nu_0 - \lambda_0)/2}$. The eigenvalue equation is then

$$L_{ij} f_j = \sigma^2 M_{ij} f_j \quad i, j = 1, 2, \quad (24)$$

where

$$M_{ij} = e^{\lambda_0 - \nu_0} \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}, \quad (25)$$

$$L_{ij} = \begin{pmatrix} -\frac{\partial}{\partial r} G_1 \frac{\partial}{\partial r} + G_1 C_1(\tau) & -2\frac{\partial}{\partial r} ((\omega + eA_0) G_1 \phi_0) + G_1 C_3(\tau) \\ 2(\omega + eA_0) G_1 \phi_0 \frac{\partial}{\partial r} + G_1 C_3(\tau) & -\frac{\partial}{\partial r} G_2 \frac{\partial}{\partial r} + G_2 C_2(\tau) \end{pmatrix}, \quad (26)$$

and C_1, C_2 and C_3 are defined in eqs.(22),(19) and (20). Eq.(24) is the required "pulsation equation". The appropriate boundary conditions are for $r \rightarrow \infty$: $f_1 \rightarrow 0$, $r^2 \phi_0^2 f_2 \rightarrow 0$ and for $r \rightarrow 0$: $f_1 = \text{const}$, $r^2 f_2 \rightarrow 0$. If ϕ_0 is nodeless it follows immediately that both the operators L_{ij} and M_{ij} are symmetric given the above boundary conditions and therefore the corresponding eigenvalues σ_n^2 are real [8]. Dynamical instability will occur whenever $\sigma_0^2 \leq 0$.

Eq.(24) can also be obtained from the following variational principle

$$\sigma^2 \int_0^\infty \frac{1}{2} e^{\lambda_0 - \nu_0} (G_1 f_1^2 + G_2 f_2^2) dr = \int_0^\infty \left(\frac{1}{2} G_1 f_1^2 + \frac{1}{2} G_1 f_2^2 C_1(\tau) + \frac{1}{2} G_2 f_2^2 + \frac{1}{2} G_2 f_2^2 C_2(\tau) + f_2 f_1' 2\phi_0(\omega + eA_0) G_1 + f_2 f_1 G_1 C_3(\tau) \right) dr. \quad (27)$$

Eigenfunctions related to different eigenvalues satisfy the following orthogonality relation

$$\int_0^\infty \frac{1}{2} e^{\lambda_0 - \nu_0} (G_1 f_{1i} f_{1j} + G_2 f_{2i} f_{2j}) dr = \delta_{ij}. \quad (28)$$

Eq.(27) is a minimal principle; from this it is clear that a sufficient condition for dynamical instability to occur is that the right-hand side of eq.(27) vanishes (or becomes negative) for some chosen pair of trial functions f_1, f_2 which satisfy the above-mentioned boundary conditions.

If ϕ_0 has nodes, thus corresponding to an excited state, it turns out that L_{ij} and M_{ij} are still symmetric and therefore that the eigenvalues are real. To show this one can repeat the same analysis as done in Ref.[10] for the case without a U(1) gauge interaction. The crucial point is that f_1 and f_2 are regular at the points where ϕ_0 has a node. In particular a singular behaviour for f_2 at the node point can again be excluded for the same reason as in Ref.[10]: by inserting an expansion for f_2 around the node point in the pulsation equation it follows that the coefficient multiplying a possible singular term has to vanish.

M and N as functions of ρ , the central density, have also stationary points for ρ bigger than ρ_{crit} and as a consequence of this fact, as shown in Ref.[9], the pulsation equation has a zero mode there. Using the above variational principle ,eq.(27), in the same way as done in Ref.[8], one can easily find there a negative upper bound to the lowest eigenvalue; the zero mode corresponds therefore to a higher mode.

In conclusion, we have established that the eigenvalues of the pulsation equation are real, also in the case of excited states, and that the equilibrium configurations corresponding to a central density bigger than ρ_{crit} are unstable, whereas for a central density smaller than ρ_{crit} we expect them to be stable [9]. It is remarkable that, despite the complexity of the perturbation equations (9)-(15), one can still find a pulsation equation consisting of only two coupled equations as well as a variational principle. It is well possible that the same type of simplification will occur also in the cases where instead of a $U(1)$ charge one has a more complicated gauge group.

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