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NEW SPACETIME SUPERALGEBRAS
AND THEIR KAC-MOODY EXTENSION

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ABSTRACT

We present new spacetime algebras whose existence is due to special Γ -matrix identities which are also necessary for the existence of super p -branes. They contain a p -th rank antisymmetric tensor, and a $(p-1)$ rank antisymmetric tensor-spinor generator. Furthermore the translations do not commute with the supercharge. In the case of supermembranes, we find a super-Kac-Moody extension of the new spacetime algebra. We give a realization of the algebra in terms of operators in $d=11$ superspace, and find a connection with the $d=11$ supermembrane action which leads to an elegant supergeometric formulation. By double dimensional reduction, we obtain a Kac-Moody algebra for the Type IIA Green-Schwarz superstring. We also discuss the generalization of these results, including the construction of generalized super-Virasoro algebras, for super p -branes.

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It is well known that superstrings admit two different kinds of formulations as far as their supersymmetry properties are concerned. One of them is the Neveu-Schwarz-Ramond (NSR) formulation [1] in which the world-sheet supersymmetry is manifest but spacetime supersymmetry is not. In this formulation, spacetime supersymmetry arises as a consequence of Gliozzi-Scherk-Olive (GSO) projections [2] in the Hilbert space. The field equations of the world-sheet graviton and gravitino are the constraints which obey the super-Virasoro algebra.

In the other formulation, due to Green and Schwarz (GS) [3], the spacetime supersymmetry is manifest but the world-sheet supersymmetry is not. In this case, the theory has an interesting local world-sheet symmetry, known as κ -symmetry, first discovered by Siegel [4,5] for the superparticle, and later generalized by Green and Schwarz [3] for superstrings. In a light-cone gauge, a combination of the residual κ -symmetry and rigid spacetime supersymmetry fuse together to give rise to an $(N=8 \text{ or } 16)$ rigid world-sheet supersymmetry.

The two formulations are essentially equivalent in the light-cone gauge [6,7]. However, for many purposes it is desirable to have a covariant quantization scheme. The covariant quantization of the NSR spinning string is well understood. In many modern treatments of the theory, an increasingly central role is played by the NSR super-Virasoro algebra. In the case of GS superstring, one of the important questions which naturally arises is to find the analog of the NSR super-Virasoro algebra.

Siegel has argued that [8,9] a generalized super-Virasoro algebra for the GS string can be arrived at in two steps: 1. Find a set of operators ("supercovariant derivatives") satisfying a generalized Kac-Moody algebra. 2. Find a suitable set of constraints quadratic in these operators satisfying a generalized Virasoro algebra ("Sugawara construction"). Siegel [8,9] achieved this construction for superparticles and superstrings. Two salient features of his results are: (a) All constraints are first class. In particular, among the constraints is the one corresponding to the κ -symmetry. (b) The supercovariant derivative corresponding to supertranslations does not commute with the bosonic translations.

The feature (b) above is rather interesting: Recently, Green [10] pointed out that the global limit of the Siegel algebra is a new (differing from the super-Poincaré) superalgebra in which the bosonic translations do not commute with supersymmetry. This new superalgebra can be naturally described by an extended superspace which contains new fermionic coordinates. An attractive feature of this superalgebra is that its associated super-Killing form is nondegenerate. This fact has been exploited by Green [10] to find a $(2+1)$ -dimensional Chern-Simons formulation of the GS superstring.

In this note, generalizing Green's result, we find new spacetime superalgebras in all dimensions where super p -branes exist [11-13], i.e. in addition to the superstring case ($p=1$; $d=3,4,6,10$), we find that new spacetime superalgebras also exist for the supermembranes ($p=2$; $d=4,5,7,11$), and the other super p -branes: ($p=3$; $d=6,8$), ($p=4$; $d=9$) and ($p=5$;

$d=10$). In particular, we shall see that (a) the new superalgebras contain a p -th rank antisymmetric tensor, and a $(p-1)$ -rank antisymmetric tensor-spinor generator, (b) the bosonic translations do not commute with supersymmetry, and (c) the existence of the algebras rely on Γ -matrix identities which are precisely the ones needed for the existence of the super p -branes.

In this paper, we shall focus our attention on the most interesting case of the eleven dimensional supermembrane, and construct for that case a generalized Kac-Moody superalgebra. We shall also find the realization of the algebra in terms of operators in $d=11$ superspace. Here, a surprising connection with the $d=11$ supermembrane action emerges in that the same super 3-form which plays a central role in the construction of the κ -invariant supermembrane action [12] also arises in the superspace realization of the Kac-Moody generators. In fact, this leads to a remarkably simple and elegant supergeometric formulation of the full Kac-Moody algebra.

We shall also suggest a way to construct the constraints (including those of reparametrizations and κ -symmetry) which are quadratic in the Kac-Moody generators, but we shall not compute their full algebra. In the next section we describe briefly the results of Siegel [8] and Green [10] for the GS superstring, pertaining to the new spacetime algebra, its Kac-Moody extension and the generalized super-Virasoro algebra. After that we present our results for supermembranes, and Type IIA superstrings. Generalizations to other super p -branes are also discussed.

The Green-Schwarz Superstring

To describe the classical mechanics of the GS superstring, Siegel introduces the operators P_μ and D_α , corresponding to bosonic and fermionic translations in $d=3,4,6$ or 10 superspace respectively, and a new operator Ω_α , which obeys the following Kac-Moody superalgebra [8]

$$\begin{aligned} \{D_\alpha(\sigma), D_\beta(\sigma')\} &= \Gamma_{\alpha\beta}^\mu P_\mu(\sigma) \delta(\sigma - \sigma'), \\ [D_\alpha(\sigma), P_\mu(\sigma')] &= \Gamma_{\mu\alpha\beta} \Omega^\beta(\sigma) \delta(\sigma - \sigma'), \\ \{D_\alpha(\sigma), \Omega^\beta(\sigma')\} &= i\delta_\alpha^\beta \partial_1 \delta(\sigma - \sigma'), \\ [P_\mu(\sigma), P_\nu(\sigma')] &= i\eta_{\mu\nu} \partial_1 \delta(\sigma - \sigma'), \end{aligned} \quad (1)$$

with other commutators and anticommutators vanishing, and $\partial_1 \equiv \partial/\partial\sigma$.

One can check that the Jacobi identities are satisfied by (1), provided that the Γ -matrix identity $\Gamma_{\mu\alpha\beta} \Gamma_{\gamma\delta}^\mu = 0$ holds. This is possible only in $d=3,4,6$ and 10. The algebra (1) can be realized in terms of the following operators [8,9]:

$$\begin{aligned} P_\mu(\sigma) &= \frac{1}{\sqrt{2}} \left(i \frac{\delta}{\delta X^\mu} + \partial_1 X^\mu \right) - i\bar{\theta} \Gamma_\mu \partial_1 \theta, \\ D_\alpha(\sigma) &= \frac{\delta}{\delta\theta^\alpha} + \frac{1}{\sqrt{2}} \Gamma_{\alpha\beta}^\mu \theta^\beta \left(i \frac{\delta}{\delta X^\mu} + \partial_1 X^\mu \right) - \frac{i}{2} \Gamma_{\alpha\beta}^\mu \theta^\beta \bar{\theta} \Gamma^\mu \partial_1 \theta, \\ \Omega^\alpha(\sigma) &= i\partial_1 \theta^\alpha, \end{aligned} \quad (2)$$

where $\bar{\theta} \Gamma^\mu \dots \theta \equiv \theta^\alpha (\Gamma^\mu \dots)^\beta_\alpha \theta_\beta$. These operators are *invariant* under translations and supersymmetry which are generated by [8,9]

$$\begin{aligned} P_\mu &= \int d\sigma \frac{i}{\sqrt{2}} \frac{\delta}{\delta X^\mu} \\ Q_\alpha &= \int d\sigma \left[\frac{\delta}{\delta\theta^\alpha} - \frac{1}{\sqrt{2}} \Gamma_{\alpha\beta}^\mu \theta^\beta \left(i \frac{\delta}{\delta X^\mu} + \partial_1 X^\mu \right) + \frac{i}{6} \Gamma_{\alpha\beta}^\mu \theta^\beta \bar{\theta} \Gamma_\mu \partial_1 \theta \right]. \end{aligned} \quad (3)$$

These generators obey the usual super-Poincaré algebra. However, Green [10] has recently observed that, one can write down a new spacetime superalgebra with the following nonvanishing commutators and anticommutators

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \Gamma_{\alpha\beta}^\mu P_\mu, \\ [Q_\alpha, P_\mu] &= \Gamma_{\mu\alpha\beta} K^\beta, \end{aligned} \quad (4)$$

where K_α is a new fermionic generator. This algebra exists only in $d=3,4,6$ and 10. Note that we can include the Lorentz generators, $M_{\mu\nu}$, as well. We would then add the usual commutators: $[M, M] \sim M$; $[M, P] \sim P$; $[M, Q] \sim Q$; $[M, \Omega] \sim \Omega$ and $[P, P] = 0$.

The algebra (4) can be obtained as the global limit of (1). The global generators P, Q and K can be realized as follows [10]:

$$\begin{aligned} P_\mu &= \int d\sigma \left(-i \frac{\delta}{\delta X^\mu} - i\bar{\theta} \Gamma_\mu \frac{\delta}{\delta\theta} \right), \\ Q_\alpha &= \int d\sigma \left[\frac{\delta}{\delta\theta^\alpha} - X^\mu \left(\Gamma_\mu \frac{\delta}{\delta\theta} \right)_\alpha + i(\Gamma^\mu \theta)_\alpha \frac{\delta}{\delta X^\mu} + \frac{1}{3} i(\Gamma^\mu \theta)_\alpha \bar{\theta} \Gamma_\mu \frac{\delta}{\delta\theta} \right], \\ K_\alpha &= \int d\sigma \frac{\delta}{\delta\theta^\alpha}, \end{aligned} \quad (5)$$

where X^μ, θ^α and ϕ^α are the coordinates of an extended superspace. The algebra (4) has a nonvanishing Killing form. This fact has been used by Green [10] to construct a $d=3$ Chern-Simons gauge theory formulation of the GS superstring. In the quantization of the theory on certain Riemann surfaces, one may have to make the identification $\frac{\delta}{\delta\theta} = \partial_1 \theta$, in which case the algebra of the generators (5) is the usual super-Poincaré algebra [10].

To construct a generalized Virasoro algebra for the GS superstring, Siegel introduces the following generators [8]

$$\begin{aligned} \mathcal{A} &= \frac{1}{2} P^2 + \Omega^\alpha D_\alpha, \\ \mathcal{B}^\alpha &= \Gamma_{\mu\beta}^\alpha P^\mu D_\beta, \\ \mathcal{C}_{\alpha\beta} &= D_\alpha D_\beta, \\ \mathcal{D}_\mu &= i\Gamma_{\mu\alpha\beta} D_\alpha \partial_1 D_\beta. \end{aligned} \quad (6)$$

The generators $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ obey the sought generalized Virasoro algebra [8]. In particular, the generator $\mathcal{A}$ obeys the usual Virasoro algebra:

$$[\mathcal{A}(\sigma), \mathcal{A}(\sigma')] = i\partial_1 \delta(\sigma - \sigma') [\mathcal{A}(\sigma) + \mathcal{A}(\sigma')]. \quad (7)$$

The remaining part of the algebra is very complicated. It has been studied in detail also by Romans [14]. Schematically, it is given by

$$\begin{aligned}
[A, B] &\sim \delta' B, & [A, C] &\sim \delta' C, & [A, D] &\sim \delta' D, \\
\{B, B\} &\sim \delta' C + \delta(PA + D + \Omega B), & [B, C] &= \delta(DA + \Omega C), \\
\{B, D\} &= \delta(DA + \Omega C) + \delta(D'A + \Omega C' + \Omega D), \\
[C, C] &= \delta PC, & [C, D] &= \delta' PC + \delta(D'B + PC'), \\
\{D, D\} &= \delta'DB + \delta(PD + D'B + P'C) + \delta(P'D + D'B' + P'C'),
\end{aligned} \tag{8}$$

where $\delta \equiv \delta(\sigma - \sigma')$ and $\delta' \equiv \partial_1 \delta(\sigma - \sigma')$. This algebra is nonlinear: There are several terms on the RHS of the commutators which are products of a Kac-Moody generator with a "Virasoro" generator. The algebra (8) provides a starting point for a BRST quantization of the theory. See [8] for the use of the constraints (6) in constructing a covariant, generalized GS superstring action which, in particular, contains a gauge field corresponding to the κ -symmetry, generated by B_α , and has only first class constraints in its Hamiltonian formulation. This is in contrast to the usual GS superstring action, which does not contain the gauge field for the κ -symmetry and has both first and second class constraints in its Hamiltonian formulation. We refer the reader to [8] and [14] for further details. We shall now generalize (4) for supermembranes and other super p-branes.

The New Spacetime Superalgebras

The existence of the algebra (4) is due to the existence of the Γ -matrix identity $\Gamma_{\mu(\alpha\beta}\Gamma_{\nu)}^{\mu} = 0$, as can be seen by considering the $\{Q, \{Q, Q\}\}$ Jacobi identity. This Γ -matrix identity is also crucial for the existence of a κ -symmetric GS superstring in $d=3, 4, 6$ and 10. In search of algebras similar to (4), we recall that there exist similar Γ -matrix identities which are crucial for κ -symmetric supermembranes and higher super-extended objects. These identities are [11-13][15]

$$\begin{aligned}
\Gamma_{\mu(\alpha\beta}\Gamma_{\nu_1\nu_2\dots\nu_{p-1}}^{\mu_1\dots\mu_{p-1}} &= 0, & \mu, \nu &= 0, \dots, d-1, \\
(p=1; d=3, 4, 6, 10), & & (p=2; d=4, 5, 7, 11), \\
(p=3; d=6, 8), & & (p=4; d=9), & (p=5; d=10).
\end{aligned} \tag{9}$$

Inspired by these identities, to generalize (4) we introduce a p -th rank antisymmetric generator, $A_{\mu_1\dots\mu_p}$ and a $(p-1)$ rank antisymmetric tensor-spinor generator $K_{\alpha}^{\mu_1\dots\mu_{p-1}}$, and find the following new spacetime algebras:

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= \Gamma_{\alpha\beta}^\mu P_\mu + \Gamma_{\alpha\beta}^{\mu_1\dots\mu_p} A_{\mu_1\dots\mu_p}, \\
[P_\mu, Q^\alpha] &= \Gamma_{\mu\alpha}^{\alpha\beta} K_{\beta}^{\mu_1\dots\mu_{p-1}}, \\
[A_{\mu_1\dots\mu_p}, Q^\alpha] &= \Gamma_{[\mu_1}^{\alpha\beta} K_{\mu_2\dots\mu_p]\beta},
\end{aligned} \tag{10}$$

with all the other commutators and anticommutators vanishing. The Lorentz generators, $M_{\mu\nu}$, can be included, with the obvious commutators: $[M, M] \sim M; [M, P] \sim P; [M, A] \sim A; [M, Q] \sim Q; [M, \Omega] \sim \Omega$ and $[P, P] = 0$. The dimensions and the reality properties of the spinor Q^α is spelled out in detail case by case in [16]. The properties of K_α then follow trivially. One can easily check that the Jacobi identities are indeed satisfied, and in particular the $\{Q, \{Q, Q\}\}$ Jacobi identity does require the Γ -matrix identities (9). We expect the algebra (10) to have a number of interesting consequences which we hope to report on in the future. In the rest of this paper, we shall present our results on the generalization of (1), (2) and (3) for the case of most interest, namely the eleven dimensional supermembrane, and the generalization of (6) for super p-branes in general.

A Generalized Kac-Moody Superalgebra in Eleven Dimensions

In search of the generalization of (1) for the supermembrane, we observe that for $p=1$, the combination $(P_\mu + A_\mu)$ occurs on the RHS of the $\{Q, Q\}$ anticommutator (10). Thus, it is this combination which must be identified with the generator P_μ occurring in (4). Examining (2), we see that the realization of the new P_μ and A^μ should be of the form: $P_\mu \sim \frac{1}{2} \bar{X}^\nu \partial_\nu + \dots$ and $A^\mu \sim \partial_1 X^\mu + \dots$. This suggests for the membrane case realizations of the form: $P_\mu \sim \frac{1}{2} \bar{X}^\nu \partial_\nu + \dots$, and $A^\mu \sim \epsilon^{\mu\nu} \partial_\nu X^\mu \partial_1 X^\nu + \dots$. Further inspection of (2) also suggests that we take $D_\alpha \sim \frac{1}{2} \bar{X}^\nu \partial_\nu + i(\Gamma^\mu \theta)_\alpha \delta \bar{X}^\mu \dots$ and $\Omega^{\mu\nu} \sim \epsilon^{\mu\nu} \partial_\mu X^\nu \partial_1 \theta^\mu + \dots$. These considerations lead to an ansatz for the analog of (1) for the supermembrane. The requirement of satisfying the Jacobi identities leads to the introduction of a new kind of generator, $\Sigma^{\alpha\beta} \sim \epsilon^{\alpha\beta} \partial_\alpha \theta^\mu \partial_\beta \theta^\mu$, which is intrinsically membrany without a string analog. Checking all the Jacobi identities we have arrived at the following supermembrane Kac-Moody algebra:

$$\begin{aligned}
\{D_\alpha(\sigma), D_\beta(\sigma')\} &= (2\Gamma_{\alpha\beta}^\mu P_\mu - \frac{1}{4} \Gamma_{\alpha\beta}^{\mu\nu} A_{\mu\nu}) \delta(\sigma - \sigma'), \\
[D_\alpha(\sigma), P^\mu(\sigma')] &= \frac{1}{2} \Gamma_{\alpha\beta}^{\mu\nu} \Omega_{\nu}^{\beta} \delta(\sigma - \sigma'), \\
\{P^\mu(\sigma), P^\nu(\sigma')\} &= \frac{1}{4} \Gamma_{\alpha\beta}^{\mu\nu} \Sigma^{\alpha\beta} \delta(\sigma - \sigma'), \\
[D_\alpha(\sigma), A^{\mu\nu}(\sigma')] &= -4i\Gamma_{\alpha\beta}^{\mu\nu} \Omega^{\alpha\beta} \delta(\sigma - \sigma'), \\
[P_\mu(\sigma), A^{\mu\nu}(\sigma')] &= -2i\delta_{\mu}^{\nu} \Gamma_{\alpha\beta}^{\alpha\gamma} \Sigma^{\alpha\beta} \delta(\sigma - \sigma') + 2\delta_{\mu}^{\nu} \Gamma_{\alpha\beta}^{\alpha\gamma} \epsilon^{\alpha\beta} \partial_\gamma \delta(\sigma - \sigma'), \\
\{D_\alpha(\sigma), \Omega^{\mu\nu}(\sigma')\} &= i(\Gamma_{\alpha}^{\mu} \delta_{\nu}^{\nu} + 2\Gamma_{\alpha\sigma}^{\mu} \delta_{\nu}^{\sigma}) \Sigma^{\mu\nu} \delta(\sigma - \sigma') - \delta_{\alpha}^{\beta} \Gamma_{\mu}^{\mu} \epsilon^{\alpha\beta} \partial_\beta \delta(\sigma - \sigma'), \\
[P_\mu(\sigma), \Omega^{\mu\nu}(\sigma')] &= \delta_{\mu}^{\nu} \Gamma_{\alpha}^{\alpha} \epsilon^{\alpha\beta} \partial_\beta \delta(\sigma - \sigma'), \\
[D_\gamma(\sigma), \Sigma^{\alpha\beta}(\sigma')] &= 2\delta_{\gamma}^{\alpha} \Gamma_{\mu}^{\mu} \epsilon^{\alpha\beta} \partial_\beta \delta(\sigma - \sigma'),
\end{aligned} \tag{11}$$

with all the other commutators and anticommutators vanishing, and where

$$\begin{aligned}
\Pi_{\alpha}^{\mu} &= \partial_\alpha X^\mu + i\theta^\mu \Gamma_{\alpha\beta}^{\mu} \partial_\beta \theta^\beta, \\
\Pi_{\alpha}^{\sigma} &= \partial_\alpha \theta^\sigma,
\end{aligned} \tag{12}$$

and σ labels the coordinates of a two dimensional parameter space, which is locally a 2-plane. (More generally, we use σ to label the spatial coordinates of a p-brane, and $\delta(\sigma - \sigma')$ for a p-dimensional delta function). The antisymmetrizations are with unit strength, e.g. $\Gamma^{[p]\Omega^p} = \frac{1}{p!}(\Gamma^p \Omega^p - \Gamma^p \Omega^p)$. All the generators (and fields) on the RHS of (11) are σ -dependent. The objects Π_α^μ and Π_α^ν are invariant under rigid supersymmetry transformations whose generators are given below (see (15)). The algebra (11) correctly truncates to the superstring case by double dimensional reduction (see (24)). Unlike in the superstring case, however, here we find explicit X - and θ -dependent terms in the algebra. All such terms appear through Π_α^μ and Π_α^ν . Of course, we can view these as additional generators in the algebra. In that case, the full algebra is still closed without introduction of any new generators, and the structure constants are field independent. The only nonvanishing new (anti) commutators are

$$\begin{aligned} [P_\mu(\sigma), \Pi_\alpha^\mu(\sigma')] &= -\delta_\alpha^\mu \partial_\sigma \delta(\sigma - \sigma'), \\ [D_\alpha(\sigma), \Pi_\alpha^\mu(\sigma')] &= 2i\Gamma_\alpha^\mu \Pi_\alpha^\beta \delta(\sigma - \sigma'), \\ [D_\alpha(\sigma), \Pi_\alpha^\nu(\sigma')] &= -\delta_\alpha^\nu \partial_\sigma \delta(\sigma - \sigma'). \end{aligned} \quad (13)$$

In checking the Jacobi identities, we have used the Γ -matrix identity (9) for $p = 2$. We have also used the fact that $\{D_\alpha, \theta^\beta\} \sim \delta_\alpha^\beta$, $[D_\alpha, X^\mu] \sim (\Gamma^\mu \theta)_\alpha$, $[P_\mu, X^\nu] \sim \delta_\mu^\nu$, and that all the other (anti) commutators of X^μ and θ^μ with the generators vanish. We have found the following representation for all the generators which produces precisely the algebra (11):

$$\begin{aligned} A^{\mu\nu} &= \epsilon^{ab} \Pi_\alpha^\mu \Pi_\beta^\nu, \\ \Omega^{\mu\alpha} &= \epsilon^{ab} \Pi_\alpha^\mu \Pi_\beta^\alpha, \\ \Sigma^{\alpha\beta} &= \epsilon^{ab} \Pi_\alpha^\mu \Pi_\beta^\mu, \\ P_\mu &= \frac{\delta}{\delta\theta^\mu} + \frac{1}{4} \epsilon^{ab} (\partial_b X^\nu - \frac{i}{2} \bar{\theta} \Gamma^\nu \partial_b \theta) \bar{\theta} \Gamma_\mu \partial_a \theta, \\ D_\alpha &= \frac{\delta}{\delta\theta^\alpha} + i\Gamma_\alpha^\mu \frac{\delta}{\delta X^\mu} + \frac{1}{8} \epsilon^{ab} \Gamma_{\mu\nu\alpha\beta} \theta^\beta (\partial_b X^\mu \partial_a \theta - \partial_a X^\mu \partial_b \theta) \\ &\quad - i\partial_b X^\mu \bar{\theta} \Gamma^\nu \partial_a \theta - \frac{1}{3} \bar{\theta} \Gamma^\mu \partial_b \theta \bar{\theta} \Gamma^\nu \partial_a \theta \\ &\quad - \frac{i}{8} \epsilon^{ab} \Gamma_{\alpha\beta}^{\mu\nu} \theta^\beta (\partial_b X^\mu - \frac{i}{3} \bar{\theta} \Gamma^\mu \partial_b \theta) \bar{\theta} \Gamma_\alpha \partial_a \theta \end{aligned} \quad (14)$$

Clearly, $A^{\mu\nu}$, $\Omega^{\mu\alpha}$ and $\Sigma^{\alpha\beta}$ are proportional to different projections of $\epsilon^{ab} \Pi_\alpha^\mu \Pi_\beta^\nu$ where $A = (\mu, \alpha)$. In fact, all the Kac-Moody generators (14) are invariant under the rigid supersymmetry transformations generated by

$$\begin{aligned} P_\mu &= \int d\sigma d\rho \frac{\delta}{\delta X^\mu}, \\ q_\alpha &= \int d\sigma d\rho \left[\frac{\delta}{\delta\theta^\alpha} - i\Gamma_\alpha^\mu \theta^\beta \frac{\delta}{\delta X^\mu} + \frac{1}{8} \epsilon^{ab} \Gamma_{\mu\nu\alpha\beta} \theta^\beta (-\partial_b X^\mu \partial_a \theta \right. \\ &\quad \left. + \frac{i}{3} \partial_b X^\mu \bar{\theta} \Gamma^\nu \partial_a \theta + \frac{1}{15} \bar{\theta} \Gamma^\mu \partial_b \theta \bar{\theta} \Gamma^\nu \partial_a \theta) \right. \\ &\quad \left. + \frac{i}{24} \epsilon^{ab} \Gamma_{\alpha\beta}^{\mu\nu} \theta^\beta (\partial_b X^\mu - \frac{i}{5} \bar{\theta} \Gamma^\mu \partial_b \theta) \bar{\theta} \Gamma_\alpha \partial_a \theta \right]. \end{aligned} \quad (15)$$

(Compare with ref. [18]). Note that $\{q_\alpha, q_\beta\} = -2i\Gamma_{\alpha\beta}^\mu P_\mu$.

Although it is not clear what the analog of the Green's representation (5) is in this case, we do have for supermembranes the analog of the new spacetime algebra (4) as a global limit of the generalized Kac-Moody algebra (11), which is given by:

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \Gamma_{\alpha\beta}^\mu P_\mu + \Gamma_{\alpha\beta}^{\mu\nu} A_{\mu\nu}, \\ [P_\mu, Q^\sigma] &= \Gamma_{\mu\nu}^{\alpha\beta} K_{\nu}^\sigma, \\ [A_{\mu\nu}, Q^\sigma] &= \Gamma_{[\mu}^{\alpha\beta} K_{\nu]}^\sigma, \end{aligned} \quad (16)$$

with other commutators and anticommutators vanishing.

A Supergeometric Formulation of the Membrane Kac-Moody Algebra

Since the Kac-Moody generators (14) are invariant under the rigid supersymmetry transformations, we expect that they admit a supergeometric formulation. In fact, it is evident that the generators $A^{\mu\nu}$, $\Omega^{\mu\alpha}$ and $\Sigma^{\alpha\beta}$ can be combined into a single generator Σ^{AB} given by

$$\Sigma^{AB} = \epsilon^{ab} \Pi_\alpha^A \Pi_\beta^B, \quad (17)$$

where

$$\Pi_\alpha^A = \partial_a Z^M E_M^A. \quad (18)$$

Here, $Z^M = (X^\mu, \theta^\mu)$ are the superspace coordinates and E_M^A is the supervielbein given by

$$\begin{aligned} E_M^A &= (\delta_\mu^A, -i\Gamma_{\alpha\beta}^A \theta^\beta), \\ E_M^{\alpha} &= (0, \delta_\beta^\alpha). \end{aligned} \quad (19)$$

Note that $M = (\mu, \alpha)$ are the world indices while $A = (\underline{a}, \underline{\alpha})$ are the flat tangent space indices.

We now turn our attention to the generators P^μ and D_α . A close examination of the expressions given for these generators in (14) suggests a relation with the super 3-form B_{ABC} which plays a central role in the construction of the supermembrane action [12]. In fact, using the expressions for B_{ABC} given in [17], we find that P_μ and D_α combine into a single object D_A which has a remarkably simple and elegant form given by

$$D_A = E_A^M \frac{\delta}{\delta Z^M} + \frac{3i}{4} \epsilon^{ab} \Pi_\alpha^B \Pi_\beta^C B_{CBA}. \quad (20)$$

We now see that, the generators (D_A, Σ^{AB}) form a complete set whose algebra summarizes the result (11). We find that the algebra (11) can be written as follows:

$$\begin{aligned} [D_A(\sigma), D_B(\sigma')] &= (-T_{AB}^C D_C + \frac{3i}{4} \Sigma^{DC} H_{CDA B}) \delta(\sigma - \sigma'), \\ [D_C(\sigma), \Sigma^{AB}(\sigma')] &= (-\Sigma^{EF} T_{FE}^A \delta_C^B + 2\Sigma^{E[A} T_{EC}^{B]} \delta(\sigma - \sigma') \\ &\quad + 2\epsilon^{ab} \delta_C^A \Pi_\alpha^B \partial_b \delta(\sigma - \sigma')), \end{aligned} \quad (21)$$

$$[\Sigma^{AB}(\sigma), \Sigma^{CD}(\sigma')] = 0,$$

where the square brackets denote graded commutators, and the nonvanishing components of the torsion tensor T_{AB}^C and the field strength H_{ABCD} for the B-tensor are given by [18]

$$\begin{aligned} T_{\alpha\beta}^\mu &= -2i\Gamma_{\alpha\beta}^\mu, \\ H_{\mu\nu\alpha\beta} &= \frac{-i}{3}\Gamma_{\mu\nu\alpha\beta}. \end{aligned} \quad (22)$$

(Due to the delta functions δ_μ^μ and δ_α^α we do not distinguish the world and tangent space indices in flat superspace. Our superspace conventions are those of [18]). Treating Π_α^A as a generator, we must then supplement (21) with the following graded commutators

$$\begin{aligned} [D_A(\sigma), \Pi_\alpha^B(\sigma')] &= -\Pi_\alpha^C T_{CA}^B \delta(\sigma - \sigma') - \delta_A^B \partial_\alpha \delta(\sigma - \sigma'), \\ [\Sigma^{AB}(\sigma), \Pi_\alpha^C(\sigma')] &= 0, \\ [\Pi_\alpha^A(\sigma), \Pi_\beta^B(\sigma')] &= 0. \end{aligned} \quad (23)$$

An important advantage of the above supergeometric formulation of the Kac-Moody algebra is that it allows a natural generalization to higher super extended objects, and perhaps more importantly to curved superspace [19]. We expect that the closure of the Kac-Moody algebra in curved superspace requires constraints on the torsion and the H-tensor, equivalent to supergravity field equations. This is remarkable in that no reference to any action nor κ -symmetry (neither at the classical nor at the quantum level) is made in deriving the supergravity field equations.

A Kac-Moody Algebra for the Type IIA Superstring

One can use our result (11) to also obtain a generalized Kac-Moody algebra for the Type IIA superstring in ten dimensions. To this end, we can set $A_{\mu\nu} = \Omega^{\mu\nu} = 0$ for $\mu = 0, \dots, 10$, $p^{10} = 0$, $X^{10} = \rho$, the second spatial coordinate on the membrane, and take all the generators to be independent of X^{10} and ρ . (For an analogous procedure to obtain the Type IIA superstring from the d=11 supermembrane, see [20]). This procedure, which is a consistent truncation, yields the result:

$$\begin{aligned} \{D_\alpha(\sigma), D_\beta(\sigma')\} &= [2i\Gamma_{\alpha\beta}^\mu P_\mu - \frac{1}{2}(\Gamma^\mu \Gamma^{11})_{\alpha\beta} A_\mu] \delta(\sigma - \sigma') \\ [D_\alpha(\sigma), P_\mu(\sigma')] &= -\frac{1}{2}(\Gamma^\mu \Gamma^{11})_{\alpha\beta} \Omega^\beta \delta(\sigma - \sigma'), \\ [D_\alpha(\sigma), A_\mu(\sigma')] &= 2i\Gamma_{\mu\alpha\beta} \Omega^\beta \delta(\sigma - \sigma'), \\ [P_\mu(\sigma), A^\mu(\sigma')] &= -\delta_\mu^\nu \partial_\nu \delta(\sigma - \sigma'), \\ [D_\alpha(\sigma), \Omega^\beta(\sigma')] &= -\delta_\alpha^\beta \partial_\beta \delta(\sigma - \sigma'), \end{aligned} \quad (24)$$

with all the other commutators and anticommutators vanishing. Here we have used the

notation: $A^{\mu 10} \equiv A^\mu$, $\Omega^{\alpha 10} \equiv \Omega^\alpha$. Similarly, from (14) we obtain the representation:

$$\begin{aligned} A^\mu &= \partial_1 X^\mu - i\bar{\theta} \Gamma^\mu \partial_1 \theta, \\ \Omega^\alpha &= \partial_1 \theta^\alpha, \\ P_\mu &= \frac{\delta}{\delta X^\mu} + \frac{1}{4} \bar{\theta} \Gamma_\mu \Gamma_{11} \partial_1 \theta, \\ D_\alpha &= \frac{\delta}{\delta \theta^\alpha} + i\Gamma_{\alpha\beta}^\mu \theta^\beta \frac{\delta}{\delta X^\mu} \\ &\quad - \frac{1}{4} (\Gamma_\mu \Gamma_{11})_{\alpha\beta} \theta^\beta (\partial_1 X^\mu - \frac{i}{2} \bar{\theta} \Gamma^\mu \partial_1 \theta) + \frac{i}{8} \Gamma_{\alpha\beta}^\mu \theta^\beta \bar{\theta} \Gamma_{11} \partial_1 \theta. \end{aligned} \quad (25)$$

(Compare with (2)). It would be interesting to see whether (24) and (25) might lead to new insights in the string field theory of closed Type IIA superstrings.

In a supergeometric formulation, we have $\Sigma^A = (A^\mu, \Omega^\alpha)$ and $D_A = (P_\mu, D_\alpha)$ which have the representations

$$\begin{aligned} \Sigma^A &= \Pi_1^A, \\ D_A &= E_A^M \frac{\delta}{\delta Z^M} + \frac{3i}{2} \Pi_1^C B_{CA}, \end{aligned} \quad (26)$$

obeying the algebra

$$\begin{aligned} [D_A(\sigma), D_B(\sigma')] &= (-T_{AB}^C D_C + \frac{3i}{2} \Sigma^C H_{CAB}) \delta(\sigma - \sigma'), \\ [D_A(\sigma), \Sigma^B(\sigma')] &= -\Sigma^C T_{CA}^B \delta(\sigma - \sigma') - \delta_A^B \partial_1 \delta(\sigma - \sigma'), \\ [\Sigma^A(\sigma), \Sigma^B(\sigma')] &= 0. \end{aligned} \quad (27)$$

The components of the super 2-form B_{CA} can be directly read from (25).

Towards Generalized Super-Virasoro Algebras for Super p-branes

The super p-brane actions [11-13] have the local κ -symmetry and the world-volume reparametrization symmetry. In particular, the latter are generated by first class constraints which arise as the field equations of the world-volume metric. Let us denote the constraints which generate the time and space reparametrizations by $T(\sigma)$ and $T_a(\sigma)$, $a = 1, \dots, p$, respectively. For simplicity, let us focus our attention on the bosonic p-branes described by the usual Dirac-Nambu-Goto action. In that case, the time reparametrizations are generated by the constraint $T = \frac{1}{2} p^\mu P_\mu + \frac{1}{2} h$ where $h_{ab} = \partial_a X^\mu \partial_b X_\mu$ and $h = \det h_{ab}$, while the space reparametrizations are generated by the constraint $T_a = \partial_a X^\mu P_\mu$. Using the representation $A^{\mu 1} \dots \mu_p = \epsilon^{\mu 1} \dots \mu_p \partial_{\alpha_1} X^{\mu_1} \dots \partial_{\alpha_p} X^{\mu_p}$, we note that

$$h = \frac{1}{p!} A^{\mu_1 \dots \mu_p} A_{\mu_1 \dots \mu_p} \quad (28)$$

Thus, we can express the reparametrization constraints as follows

$$\begin{aligned} T &= \frac{1}{2} P^\mu P_\mu + \frac{1}{2 \times p!} A^{\mu_1 \dots \mu_p} A_{\mu_1 \dots \mu_p}, \\ T_a &= \partial_a X^\mu P_\mu. \end{aligned} \quad (29)$$

Using the commutator

$$[P_\mu(\sigma), A^{\mu_1 \dots \mu_p}(\sigma')] = i p \delta_{\mu}^{\mu_1} \partial_{\sigma} X^{\mu_2} \dots \partial_{\sigma'} X^{\mu_p} \epsilon^{\mu_1 \mu_2 \dots \mu_p} \partial_{\sigma} \delta(\sigma - \sigma'), \quad (30)$$

we find that the constraints (29) obey the following algebra:

$$\begin{aligned} [T(\sigma), T(\sigma')] &= i \partial_\sigma \delta(\sigma - \sigma') [h_{ab} T_b(\sigma) + \sigma \leftrightarrow \sigma'], \\ [T(\sigma), T_a(\sigma')] &= i \partial_\sigma \delta(\sigma - \sigma') [T(\sigma) + T(\sigma')], \\ [T_a(\sigma), T_b(\sigma')] &= i \partial_\sigma \delta(\sigma - \sigma') T_b(\sigma) + i \delta(\sigma - \sigma') \partial_\sigma T_a(\sigma'), \end{aligned} \quad (31)$$

In the case of the minimal super p-brane theories [11-13], the reparametrization algebra has the same form as above [18], with $h_{ab} = \Pi_a^\mu \Pi_{\mu b}$. Of course, in that case the constraints T and T_a have a much more complicated form. The ones given in [18] are appropriate for the "minimal" supermembrane theory of [12]. One should really determine them for a modified "new" supermembrane action which would be based on the Kac-Moody superalgebra (11). Of course, there will be other generators besides T and T_a , which would be the analogs of the generators B, C and $\mathcal{D}$ found by Siegel [8] for the GS superstring (see (6)). As a generalization of (6), we propose the following generators for super p-branes:

$$\begin{aligned} T &= \frac{1}{2} P^\mu P_\mu + c_1 A^{\mu_1 \dots \mu_p} A_{\mu_1 \dots \mu_p} + i c_2 \Omega_{\mu_1 \dots \mu_{p-1}}^{\mu} (\Gamma^{\mu_1 \dots \mu_{p-1}})^\beta D_\beta, \\ T_a &= \Pi_a^\mu P_\mu + c_3 \Pi_a^\mu D_\mu, \\ \mathcal{B}^\alpha &= (\Gamma_{\mu}^{\alpha\beta} P^\mu + i c_4 \Gamma_{\mu_1 \dots \mu_p}^{\alpha\beta} A^{\mu_1 \dots \mu_p}) D_\beta, \\ C_{\alpha\beta} &= D_\alpha D_\beta, \\ &\vdots \end{aligned} \quad (32)$$

where $c_1, \dots, c_4$ are constant coefficients to be determined by closure. Using the realization (14) in the above expressions for the case of supermembranes (i.e. $p=2$), we see that one indeed obtains structures similar to the ones given in [18] for the "minimal" supermembrane theory. However, whether the above generators, with an inclusion of a suitable set, actually do yield the (anti) commutators of a generalized super-Virasoro algebra for certain modified super p-brane actions remains to be seen. If they indeed do so, then the underlying generalized super-Virasoro algebras (consisting of only first class constraints) would provide a convenient starting point for their BRST quantization.

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NOTE ADDED

Recently [21] C.N. Pope and one of the authors (E.S.) have shown that the double dimensional reduction of the generators T, T_a and $\mathcal{B}^\alpha$ indeed gives correctly some of the Virasoro generators of the Type IIA Green-Schwarz superstring in ten dimensions. In fact, by comparing the coefficients, one finds that $c_1 = \frac{1}{16}, c_2 = \frac{1}{4}, c_3 = 2, c_4 = 4$. The closure of the Type IIA Virasoro algebra furthermore requires the inclusion of an infinite set of generators $\mathcal{D}_{\alpha\beta}^{(n)} = D_{(\alpha} \delta_{\beta)}^{(n)} D_\gamma, n = 0, 1, \dots, \infty$. The supermembrane analog of these generators which have been suggested in [21], for super p-branes reads

$$\mathcal{D}_{\alpha\beta}^{(n)} = D_{(\alpha} \Delta_{\beta)}^{(n)} D_\gamma, \quad n = 0, 1, \dots, \infty$$

where

$$\Delta_{\alpha}^{(n)} \equiv (\Gamma_{\mu_1 \dots \mu_{p-1}}^{\alpha\beta_1 \dots \beta_{p-1}} \Pi_{\beta_1}^{\mu_1} \dots \Pi_{\beta_{p-1}}^{\mu_{p-1}})^\beta \partial_\beta.$$

In [21], other generators for the supermembrane case are also suggested. These vanish upon dimensional reduction to ten dimensions. Their analogs for super p-branes are:

$$\begin{aligned} \mathcal{E}^\alpha &= \Gamma_{\mu_1 \dots \mu_{p-1}}^{\alpha\beta} \Gamma_{\mu_1 \dots \mu_{p-1}}^{\mu_1 \dots \mu_{p-1}} \Sigma^{\mu_1 \dots \mu_{p-1}} D_\beta, \\ \mathcal{F} &= \Gamma_{\alpha\beta}^{\mu} \Sigma^{\alpha\beta} P_\mu, \\ \mathcal{G} &= \Gamma_{\alpha\beta}^{\mu_1 \dots \mu_p} \Sigma^{\alpha\beta} A_{\mu_1 \dots \mu_p}. \end{aligned}$$

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