**FUSION AND BRAIDING IN W-ALGEBRA
EXTENDED CONFORMAL THEORIES (II)**
Generalization to chiral screened vertex operators
labelled by arbitrary Young tableaux

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ABSTRACT

In a previous work, we defined the chiral screened vertex operators of W-algebra extended conformal theories by fusion of elementary ones. After reviewing how to obtain the braid group representation matrices realizing the exchange algebra for those chiral vertex operators corresponding to the symmetric tensor representations of A_n , we generalize our results to chiral screened vertex operators associated with arbitrary A_n representations. The fused braiding matrices for antisymmetric tensor screened vertex operators are computed explicitly and shown to have a very simple form. Closure of the exchange algebra in the general case is proved using the relation with the Boltzmann weights of the A_n face models. Since, in the unitary case, the W-algebras are realized as cosets $\hat{g}_k \oplus \hat{g}_1 / \hat{g}_{k+1}$, the present results can also be reinterpreted in terms of fusion of braiding matrices of the $\hat{g}$ WZW models. As an example, the simplest W-algebra extended theory, the 3-state Potts model, is discussed in some detail.

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1. Introduction

Among the conformal field theories the rational ones are of particular interest. They are characterized by having only a finite number of primary fields. The statement of being primary (highest weight) is made with respect to the chiral symmetry algebra $\mathcal{A}$ of the theory, where $\mathcal{A}$ contains at least the Virasoro algebra. It is clear that if $\mathcal{A}'$ is an extension of $\mathcal{A}$, i.e. $\mathcal{A} \subset \mathcal{A}'$, there are less primary fields with respect to $\mathcal{A}'$ than with respect to $\mathcal{A}$. In particular, a conformal field theory might be rational with respect to the extended algebra $\mathcal{A}'$ while it is not with respect to $\mathcal{A}$. Hence it is important to find the maximally extended chiral algebra of a given conformal field theory.

In this connection, during the last two years, a lot of study has been devoted to the so-called W -algebras¹⁻¹²⁾. They are extensions of the Virasoro algebra by Virasoro-primary fields of integer dimensions greater than two. The operator product algebra of these fields in general has a non-linear structure, i.e. the OPE algebra closes only in the enveloping algebra.*

A W -algebra is associated with a finite-dimensional Lie algebra g . Series of unitary representation have been constructed⁴⁾ (at least for simply-laced g) by cosets from the affine algebra $\hat{g}: \hat{g}_k \oplus \hat{g}_1/\hat{g}_{k+1}$. The dimensions of the fields generating the W -algebra are in one-to-one correspondence with the orders of the Casimir operators of g . An alternative approach to W -algebras starts from the classical Toda field theory⁵⁻⁸⁾, associated with g (not with $\hat{g}$ [†]). Its classical solutions are built up from a certain set of fundamental functions (classical analogs of screened vertex operators) that satisfy an ordinary linear differential equation. The coefficient functions of this differential equation give, upon quantization, the W -algebra generators. This Toda field theory approach leads naturally to a multicomponent Feigin-Fuchs-type construction of W -algebras. The latter

* Hence, the W -algebras are also called non-linearly extended Virasoro algebras.

† The Toda field theories associated with $\hat{g}$ describe deformations of the corresponding conformal field theory by the relevant operator, which is the vertex operator associated with the extra null root in $\hat{g}$.

was also elaborated independently by Zamolodchikov and Fateev²⁾, and Fateev and Lykhanov³⁾. The structure of the above-mentioned differential equation was also used by other authors⁹⁾ to explore the classical W -algebras.

Fundamental properties of a conformal field theory are its modular invariance and its braiding and fusion properties¹³⁻¹⁸⁾. The latter two describe $s-u$ and $s-t$ channel duality. They are represented on the conformal blocks by (constant) matrices. Consistency demands that these matrices satisfy polynomial equations:¹⁷⁾ the hexagon (Yang-Baxter equation) and pentagon identities. The fusion and braiding matrices are linearly related. On the other hand, one knows from the work of Verlinde¹⁹⁾ that the matrix implementing the S -modular transformation is related to the fusion properties of the theory (see e.g. ref.¹⁸⁾ for the precise relation with the braiding or fusion matrices). It was shown¹⁷⁾ that one-loop modular invariance and consistent tree-level (genus-zero) fusion and braiding implies modular invariance on any genus Riemann surfaces.

In a recent paper¹⁾, henceforth referred to as I, the braiding and fusion properties of the ($g = A_n$) W -algebras were determined. We started with a Feigin-Fuchs type construction. "Higher" screened vertex operators were obtained by fusion of certain "fundamental" screened vertex operators. This allowed us to obtain the braiding matrices for the former by a fusion procedure from the braiding matrices of the latter, which were already known^{7,8)}. The braiding matrices were shown to factorize (up to a trivial factor) into a pair of braiding matrices r and r' . r , resp. r' , gives the braiding for the α_- , resp. α_+ , part of the screened vertex operators. ($\alpha_{\pm}$ are the two screening charges.) Comparing with the coset approach, $\hat{g}_k \oplus \hat{g}_1/\hat{g}_{k+1}$ ^{*}, r and $(r')^{-1}$ are associated with the braiding for the $\hat{g}_k$ and $\hat{g}_{k+1}$ WZW models¹⁾.

It was shown in I that the braiding matrices r (resp. r') coincide, up to

* Note that $\alpha_{\pm}$ and k are related by $\alpha_- = -\sqrt{\frac{m}{m+1}}$, $\alpha_+ = \sqrt{\frac{m+1}{m}}$, $m = k + h_c$, where h_c is the dual Coxeter number of g .

trivial factors, with the critical limit[†] of the Boltzmann weights of the A_n face models constructed by Jimbo et al.²⁰⁻²². Knot polynomials associated with these braiding matrices obtained from the Boltzmann weights were studied in ref.²³.

The "fundamental" screened vertex operators correspond to the fundamental vector representation of A_n . The "higher" screened vertex operators constructed in I correspond to symmetric tensor representations of A_n , i.e. to Young tableaux with one row. The aim of the present paper is to show how the fusion procedure can be generalized to obtain screened vertex operators (and their braiding matrices) corresponding to arbitrary Young tableaux, and to antisymmetric tensors in particular.

The plan of this paper is the following. In sections 2 and 3, we briefly recall some basic properties of W -algebras and review the results of I. In section 4, we discuss the rank two antisymmetric tensor case in some detail. We show that the corresponding screened vertex operators can be obtained by antisymmetric fusion of the fundamental (vector representation) vertex operators. Again, this induces a fusion relation for the braiding matrices. In section 5, we discuss the generalization to screened vertex operators associated with arbitrary A_n representations, and compute explicitly the matrix elements for the braiding of a rank k ($< n$) antisymmetric screened vertex operator with the fundamental screened vertex operators. The closure of the braiding operation in general is shown to follow from the relation with the Boltzmann weights of the A_n face models of Jimbo et al.²² In section 6, we discuss an explicit example: the 3-state Potts model with $c = \frac{4}{5}$, which is the simplest W -algebra extended theory.

† One also has to take the spectral parameter to infinity.

2. The W -algebra extended conformal theories

The W -algebra : As already mentioned, in W -algebra extended conformal theories, the algebraic structure is enlarged beyond the Virasoro algebra. In addition to the energy-momentum tensor $T(z) \equiv W^{(2)}(z)$ which is a primary field of weight 2 (modulo the conformal anomaly), there are conformal fields $W^{(k)}(z)$ of weight $k \in \mathbb{Z}$. The values of k are in one-to-one correspondence with the orders of the Casimir operators of the associated finite dimensional Lie algebra g , e.g. $k = 2, 3, \dots, n+1$ for A_n . The $W^{(k)}$ form a closed operator product algebra (W -algebra) in the enveloping algebra:^{*}

$$W^{(k)}(z)W^{(k')}(z') = \frac{c_{kk'}}{(z-z')^{k+k'}} + \sum_{l=2}^{k+k'-1} \frac{c_{kk'}^{k+l-1}}{(z-z')^{k+k'-l}} P_l(W^{(j)}(z)) \quad (2.1)$$

where $P_l(W^{(j)})$ is some polynomial of the $\partial^r W^{(j)}$ such that $\sum j + \sum r = l$. Although the $W^{(k)}$ do not form a Lie algebra, the algebraic structure (2.1) is all one needs to define generalized Verma modules⁷⁾. The representation theory of the W -algebras is much less well known than the one of the standard Virasoro algebra. However, using a Feigin-Fuchs-type realization, it was possible to obtain the spectrum of highest weights^{2-3,7)} in general. For the unitary $c < n = \text{rank } g$ series (and for simply-laced g) these results are confirmed by the GKO construction⁴⁾ based on $\hat{g}_k \oplus \hat{g}_1/\hat{g}_{k+1}$. There exist also some results^{10,11)} concerning the zeros of the generalized Kac determinant.

Here, we will use the Feigin-Fuchs-type representation²⁵⁾ of these extended algebras^{2-3,5-8)}. We will only present the case of simply-laced g (see refs.⁶⁻⁸⁾ for the non-simply-laced case). This Feigin-Fuchs-type construction uses n free

* Although at present there exists no proof that this is true for any n and any k, k' , there is a lot of evidence from sample calculations^{7,8)} that it is. However, recently closure in the enveloping algebra has been proven to hold in the $n \rightarrow \infty$ limit²⁴⁾.

bosonic fields ϕ_i with a background charge α_0 placed at infinity:

$$\langle \phi_i(z)\phi_j(z') \rangle = -\delta_{ij} \ln(z - z') \quad (2.2)$$

The ϕ_i are standard Fubini-Veneziano fields. Normal-ordered chiral vertex operators are defined by

$$V_\alpha(z) = : e^{i\alpha\phi(z)} : \equiv T_\alpha z^{\alpha_0} \exp \left(\alpha \cdot \sum_{n=1}^{\infty} a_{-n} \frac{z^n}{n} \right) \exp \left(-\alpha \cdot \sum_{n=1}^{\infty} a_n \frac{z^{-n}}{n} \right) \quad (2.3)$$

where $[a_n^i, a_m^j] = \delta^{ij} n \delta_{n,-m}$ and $[a_n^i, T_\alpha] = \delta_{n,0} \alpha^i T_\alpha$.

One introduces the simple roots $e_i, i = 1, \dots, n$, of g (normalized as $e_i \cdot e_i = 2$), the n fundamental weights λ_i satisfying

$$\lambda_i \cdot e_j = \delta_{ij} \quad (2.4)$$

and the weights h_μ of the fundamental vector representation ($h_1 = \lambda_1$). For $g = A_n$, e.g., one has $\mu = 1, \dots, n+1$, $\sum_{\mu} h_\mu = 0$, $h_\mu \cdot h_\nu = \delta_{\mu\nu} - \frac{1}{n+1}$, $e_i = h_i - h_{i+1}$, $\lambda_i = \sum_{\mu=1}^i h_\mu$ and $h_\mu = \lambda_\mu - \lambda_{\mu-1}$ ($\lambda_0 = \lambda_{n+1} = 0$). The Weyl vector ρ is defined as half the sum of the positive roots and equals

$$\rho = \sum_{i=1}^n \lambda_i \quad (2.5)$$

The energy-momentum tensor then reads

$$T(z) = -\frac{1}{2} \partial \phi(z) \cdot \partial \phi(z) + 2i\alpha_0 \rho \cdot \partial^2 \phi(z) \quad (2.6)$$

where the background charge α_0 is related to the Virasoro central charge by

$$c = n - 48c_0^2 \rho^2 = n(1 - h_c(h_c + 1)4\alpha_0^2) \quad (2.7)$$

h_c being the (dual) Coxeter number of g ($h_c = n + 1$ for A_n). In a similar way, the other fields $W^{(k)}$ are of the form $W^{(k)} \simeq \sum \prod \partial^{r_i} \phi$ with $\sum r_i = k$ and $r_i \geq 1$. Their exact expression^{3,6-7)} is not relevant to us here.

The vertex operators (2.3) are (chiral) primary fields of conformal dimension

$$\Delta(\alpha) = \frac{1}{2} \alpha \cdot (\alpha - 4\alpha_0 \rho) = \frac{1}{2} (\alpha - 2\alpha_0 \rho)^2 + \frac{c-n}{24} \quad (2.8)$$

Screened vertex operators are built up from the screening operators

$$V_{\alpha_\pm}^i(z) = V_{\alpha_\pm e_i}(z) \equiv : e^{i\alpha_\pm e_i \cdot \phi(z)} : \quad (2.9)$$

which have unit conformal dimensions provided

$$\alpha_\pm^2 - 2\alpha_0 \alpha_\pm - 1 = 0 \quad (2.10)$$

The spectrum of highest weights : As usual in Feigin-Fuchs-type constructions²⁶⁾, the spectrum of highest weights of the theory is determined by the screening charges $\alpha_\pm$. It is found to be given by^{3,7)} ($m = (m_1, \dots, m_n)$)

$$\Delta_{mm'} = \Delta(\alpha_{mm'}), \quad \alpha_{mm'} = - \sum_{i=1}^n ((m_i - 1)\alpha_- + (m'_i - 1)\alpha_+) \lambda_i \quad (2.11)$$

In refs.^{7,8)} it was argued that the rational theories are precisely such that the highest weights are rational. This requirement, which is equivalent to having rational $\alpha_\pm^2$, then gives the generalized BPZ series of minimal models:

$$\alpha_-^2 = \frac{r}{s} \Leftrightarrow c = n \left(1 - \frac{(r-s)^2 h_c(h_c + 1)}{rs} \right) \quad (2.12)$$

For $s = r + 1$, $r = 1, 2, \dots$, one gets the unitary theories also obtained by coset constructions based on $(A_n^{(1)})_k \oplus (A_n^{(1)})_1 / (A_n^{(1)})_{k+1}$ ⁴⁾. Let us quote from ref.⁴⁾ that in these coset constructions the highest weights $\Delta(\alpha)$ of the Virasoro algebra (eqs.(2.8,11)) are obtained as

$$\Delta(\alpha = -\lambda\alpha_- - \lambda'\alpha_+) = \frac{\lambda' \cdot (\lambda' + 2\rho)}{2(k + h_c)} + \frac{\epsilon' \cdot (\epsilon' + 2\rho)}{2(1 + h_c)} - \frac{\lambda \cdot (\lambda + 2\rho)}{2(k + h_c + 1)} \bmod \mathbf{Z} \quad (2.13)$$

$\lambda^{(\prime)} = \sum_{i=1}^n (m_i^{(\prime)} - 1) \lambda_i$ being a highest weight of A_n .

The Hilbert space of the extended conformal theories is a direct sum of irreducible representation spaces for a pair of (left and right) W -algebras: $\mathcal{H} = \bigoplus_L \mathcal{H}_L \otimes \mathcal{H}_L$ where L labels the irreducible highest weight representations. Generally, these correspond to a fixed eigenvalue of the zero-mode a_0 . The highest weight states are obtained by applying the chiral primary vertex operators (2.3) to the $sl(2, \mathbb{C})$ invariant vacuum state $|0\rangle$ (satisfying $a_n^\dagger |0\rangle = 0, \forall n \geq 0, \forall i$): $|(l')\rangle = \lim_{z \rightarrow 0} V_{a_{l'}}(z)|0\rangle$ which is a Fock space vacuum state with a_0 eigenvalue $\alpha_{l'}$. Hence the index L corresponds to (l') . For rational theories, L only takes on a finite number of values.

As usual, the primary fields of the extended conformal theory decompose into holomorphic and antiholomorphic parts:

$$\Phi_{\Delta_{mm'}} = \sum_{L, K} d_{(mm')L}^K V_{(mm')L}^K(z) \otimes V_{(mm')L}^K(\bar{z}) \quad (2.14)$$

$V_{(mm')L}^K$ has conformal dimension $\Delta_{mm'}$ and maps $\mathcal{H}_L$ to $\mathcal{H}_K$.

The screened vertex operators : The chiral primary fields are screened vertex operators $V_{(mm')L}^K$ built up from (unscreened) vertex operators (2.3) with $\alpha = \alpha_{mm'}$ and the screening operators which are integrals over closed contours of the vertex operators $V_{\alpha_{\pm}}^i$ with unit conformal dimensions (see (2.9,10)). Hence, the screened vertex operators have the same dimensions as the unscreened ones, $\Delta = \Delta(\alpha_{mm'})$. However, the screening operators $V_{\alpha_{\pm}}^i$ shift the eigenvalues of the zero-mode a_0 by $\alpha_{\pm} e_i$.

The integration contours for the screened vertex operators

$$\begin{aligned} & V_{mm'}^{i_1 \dots i_r j_1 \dots j_{r'}}(z) \\ &= \int V_{\alpha_{mm'}}(z) V_{\alpha_-}^{i_1}(u_1) \dots V_{\alpha_-}^{i_r}(u_r) V_{\alpha_+}^{j_1}(v_1) \dots V_{\alpha_+}^{j_{r'}}(v_{r'}) \prod_{a=1}^r \prod_{b=1}^{r'} du_a dv_b \end{aligned} \quad (2.15)$$

have to be chosen carefully. For the unextended conformal theories ($\simeq \mathcal{A}_1$) this is discussed in detail in refs. ^{16,26}. Since the integrand is many-valued, it must be

defined by analytic continuation from some reference configuration of the integration variables. The latter is chosen to have all u_a, v_b ordered on the positive real axis (cf ref. ¹⁶) for $\mathcal{A}_1$ as $0 < v_{r'} < \dots < v_1 < u_r < \dots < u_1 < z$, see fig. 1.

The screened vertex operators (2.15) should be thought of as being inserted in correlation functions $< 0|V(z_N)V(z_{N-1})\dots V(z_2)V(z_1)|0\rangle$. This of course shows that the integration contours in fig. 1 *cannot* be deformed arbitrarily. In particular, if one considers a screened vertex operator acting on $\mathcal{H}_L$, one should imagine a $V_{a_{l'}}(z_1)$ inserted at $z_1 = 0$ in fig. 1. The contours over which vertex operators V^i and V^j with $e_i \cdot e_j = 0$ are integrated can be interchanged without altering the value of (2.15)¹. The same is true for $V_{\alpha_+}^i V_{\alpha_-}^j$ and also for $V_{\alpha_-}^i V_{\alpha_+}^j$ if $e_i \cdot e_j = -1$ ¹, and hence for any $V_{\alpha_-}^i V_{\alpha_+}^j$ (since g is simply-laced). In the other cases, interchanging the contours is non-trivial and requires their decomposition into pieces with appropriate phase factors for each piece. For the unextended conformal theories this has been discussed in detail by Dotsenko and Fateev ²⁶ and Felder, Froehlich and Keller ¹⁶.

So far the discussion was for arbitrary simply-laced g . Now let us restrict to the A_n theories. We define the "fundamental" chiral (screened) vertex operators associated with the fundamental vector representation of A_n :

$$\begin{aligned} \mathcal{V}_{\mu}^{-}(z) &= e^{i\theta_L^{12\dots\mu-1}} \int V_{\alpha_{(21\dots1)(1\dots1)}}(z) V_{\alpha_-}^1(u_1) V_{\alpha_-}^2(u_2) \dots V_{\alpha_-}^{\mu-1}(u_{\mu-1}) du_{\mu-1} \dots du_1 \\ \mathcal{V}_{\mu}^{+}(z) &= e^{i\bar{\theta}_L^{12\dots\mu-1}} \int V_{\alpha_{(1\dots1)(21\dots1)}}(z) V_{\alpha_+}^1(v_1) V_{\alpha_+}^2(v_2) \dots V_{\alpha_+}^{\mu-1}(v_{\mu-1}) dv_{\mu-1} \dots dv_1 \end{aligned} \quad (2.16)$$

These operators shift the a_0 eigenvalue by $-\alpha_{\pm}(\lambda_1 - e_1 - \dots e_{\mu-1}) = -\alpha_{\pm} h_{\mu}$. Recall that h_{μ} is the μ^{th} weight of the vector representation of A_n . $\theta(\bar{\theta})$ is given below.

In order to indicate clearly between which Hilbert spaces the screened vertex operators are acting, it is convenient to write

$$V_{ML}^K(z) \equiv V_{(mm')\chi(l'')}^{(kk')} (z) = e^{i\theta^{i_1 \dots i_r j_1 \dots j_{r'}}} V_{mm'}^{i_1 \dots i_r j_1 \dots j_{r'}}(z) : \mathcal{H}_{(l'')} \rightarrow \mathcal{H}_{(kk')} \quad (2.17)$$

The phase factors θ which ensure that the fusion of screened vertex operators (2.17) will look as simple as possible (cf. eqs. (2.24,25) of I) are

$$\begin{aligned} \theta_{\alpha_L}^{i_1 \dots i_r j_1 \dots j_{r'}} &= -\pi (\alpha_- (e_{i_1} + \dots + e_{i_r}) + \alpha_+ (e_{j_1} + \dots + e_{j_{r'}})) \\ &\quad \cdot (\tfrac{1}{2} \alpha_- (e_{i_1} + \dots + e_{i_r}) + \tfrac{1}{2} \alpha_+ (e_{j_1} + \dots + e_{j_{r'}}) + \alpha_L) \end{aligned} \quad (2.18)$$

In particular, for the operators $\mathcal{V}_\mu^-$ one has simply $\theta_L^{12 \dots \mu-1} \equiv \theta_{\alpha_L}^{12 \dots \mu-1} = -\pi \sum_{i=1}^{\mu-1} (l_i - 1) + \pi \alpha_-^2 (-1 + \sum_{i=1}^{\mu-1} (l_i - 1))$, and similar for $\bar{\theta}$.

3. Braiding and fusion for chiral screened vertex operators associated with the symmetric tensor representation of A_n

The braiding matrix : The braiding matrix R is defined by^{*}

$$V_{MD}^C(z) V_{NA}^D(w) = \sum_B R(C, M, N, A)_{DB} V_{NB}^C(w) V_{MA}^D(z) \quad (3.1)$$

C and A label the final and initial Hilbert spaces, D and B the intermediate ones on the right-hand and left-hand sides, respectively. M and N label the A_n representation with which the V 's are associated. The operator product on the right-hand side is well defined for $|w| > |z|$. The left-hand side must be understood as the analytic continuation from the region $|w| < |z|$ along a path which we choose such that z circumvents w counterclockwise. In the notation of ref.¹⁸⁾ this

braiding matrix is denoted by $R(C, M, N, A)_{DB} = B_{DB} \begin{bmatrix} M & N \\ C & A \end{bmatrix}$.

It is sometimes convenient to represent the R -matrix graphically as

$$R(C, M, N, A)_{DB} \equiv \begin{array}{c} A \\ \boxed{(M, N)} \\ D \end{array} \begin{array}{c} B \\ C \end{array} \quad (3.2)$$

This notation makes it clear that one goes from $\mathcal{H}_A$ (with a_0 eigenvalue $\alpha_{aa'}$) to $\mathcal{H}_C$ (with a_0 eigenvalue $\alpha_{cc'}$) via $\mathcal{H}_B$ ($a_0 = \alpha_{bb'}$) on the r.h.s. or via $\mathcal{H}_D$ ($a_0 = \alpha_{dd'}$) on

* We use the same notation as in ref.¹⁶⁾.

the l.h.s. With the correspondence to the Boltzmann weights $W_{MN} \begin{pmatrix} A & B \\ D & C \end{pmatrix}$ of Jimbo et al²⁰⁻²²⁾ in mind (see next section), we call (3.1,2) the IRF (interaction round a face) representation of the R -matrix.

There exists also a dual representation of the R -matrix, which we call the vertex representation. Here the word "vertex" does not refer to the chiral vertices of the conformal theory, but to the geometric object which is dual to the plaquette (face). Instead of labelling R by the Hilbert space indices A, B, C, D , in the dual vertex representation R is labelled by the *differences* of the a_0 eigenvalues between these Hilbert spaces, i.e. by the shifts of a_0 realized by the chiral vertex operators V_{NA}^D , etc. Call $\alpha = \alpha_{bb'} - \alpha_{aa'}$, $\beta = \alpha_{cc'} - \alpha_{bb'}$, $\gamma = \alpha_{cc'} - \alpha_{dd'}$, $\delta = \alpha_{dd'} - \alpha_{aa'}$ ($\alpha + \beta = \gamma + \delta$). Then, replacing A, B, C, D by $\alpha, \beta, \gamma, \delta$ and A we write[†]

$$V_{M\gamma}(z) V_{N\delta}(w) \Big|_{\mathcal{H}_A} = \sum_{\substack{\alpha, \beta \\ \alpha + \beta = \gamma + \delta}} R_{(MN)\gamma\delta}^{\beta\alpha}(A) V_{N\beta}(w) V_{M\alpha}(z) \Big|_{\mathcal{H}_A} \quad (3.3)$$

This is the vertex representation of the R -matrix. It can be denoted graphically as

$$R_{(MN)\gamma\delta}^{\beta\alpha} \equiv \begin{array}{c} A \\ \delta \end{array} \begin{array}{c} \boxed{(MN)} \\ \gamma \end{array} \begin{array}{c} \alpha \\ \beta \end{array} \begin{array}{c} (M, \alpha) \\ (N, \beta) \end{array} \begin{array}{c} (M, \gamma) \\ (N, \delta) \end{array} \quad (3.4)$$

It is clear from the associativity of the operator product algebra of three V_{NA}^B 's that the braiding matrices satisfy a cubic relation, the Yang-Baxter equation (YBE) or Star-Triangle relation (STR):

$$\begin{aligned} &\sum_G R(G, K, N, A)_{FB} R(D, K, M, F)_{EG} R(D, M, N, B)_{GC} \\ &= \sum_G R(E, M, N, A)_{FG} R(C, K, M, A)_{GB} R(D, K, N, G)_{EC} \end{aligned} \quad (3.5)$$

[†] $R(A)$ means $R(\alpha_{aa'})$, where the argument A should not be confused with a spectral parameter (cf. eq. (3.29) below), and of course $V_{N\beta} \Big|_{\mathcal{H}_A} \equiv V_{NA}^{A+\beta}$, etc.

This equation can be equally well formulated in the vertex representation.

The fundamental braiding matrix : We will now recall^{7,1)} the braiding matrix for the fundamental chiral screened vertex operators $\mathcal{V}_\mu^\pm$. The basic identity used for determining the braiding matrices is (with the same convention for the analytic continuation as for (3.1)):

$$V_\alpha(z)V_\beta(w) = e^{i\pi\alpha\cdot\beta} V_\beta(w)V_\alpha(z) \quad (3.6)$$

for (unscreened) vertex operators (2.3).

First of all, the braiding of $\mathcal{V}_\mu^+$ with $\mathcal{V}_\nu^-$ is trivial. Indeed, as discussed above, all integration contours of the $V_{\alpha+}^i$ can be interchanged with those of the $V_{\alpha-}^j$. In the same way, one can pass a $V_{\alpha+}$ ($V_{\alpha-}$) contour across $V_{\alpha(2,1,\dots,1)(1,\dots,1)}(w)$ ($V_{\alpha(1,\dots,1)(2,1,\dots,1)}(z)$). Hence the braiding of $\mathcal{V}_\mu^+$ with $\mathcal{V}_\nu^-$ is only determined by the braiding of $V_{\alpha(1,\dots,1)(2,1,\dots,1)}(z)$ with $V_{\alpha(2,1,\dots,1)(1,\dots,1)}(w)$ which gives a simple phase factor $e^{i\pi\lambda_1^2\alpha_-\alpha_+}$:

$$\mathcal{V}_\mu^\pm(z)\mathcal{V}_\nu^\mp(w) = e^{-i\pi\lambda_1^2} \mathcal{V}_\nu^\mp(w)\mathcal{V}_\mu^\pm(z), \quad \lambda_1^2 = \frac{n}{n+1} \quad (3.7)$$

Next we give the braiding matrix of $\mathcal{V}_\mu^-$ ($\mathcal{V}_\mu^+$) with $\mathcal{V}_\nu^-$ ($\mathcal{V}_\nu^+$). In this case, the value of a' (a) remains fixed and we will not write it explicitly. The R -matrix will then be denoted simply by $r(c, m, n, a)_{db}$ (resp. $r'(c', m', n', a')_{d'b'}$).^{*} In general we write

$$\begin{aligned} & V_{(m,(1,\dots,1))(da')}^{(ca')} (z) V_{(n,(1,\dots,1))(aa')}^{(da')} (w) \\ &= \sum_b \tau(c, m, n, a)_{db} V_{(n,(1,\dots,1))(ba')}^{(ca')} (w) V_{(m,(1,\dots,1))(aa')}^{(da')} (z) \end{aligned} \quad (3.8)$$

or more explicitly (using the vertex representation of the r -matrix)

$$\mathcal{V}_\mu^-(z)\mathcal{V}_\nu^-(z) \Big|_{\mathcal{H}_{(aa')}} = \sum_{\substack{\kappa, \rho \\ h_\kappa + h_\rho = h_\mu + h_\nu}} r_{\mu\nu}^{\kappa\rho}(a) \mathcal{V}_\kappa^-(w)\mathcal{V}_\rho^-(z) \Big|_{\mathcal{H}_{(aa')}} \quad (3.9)$$

^{*} A priori, r could depend also on a' . However, it will turn out that this is not the case.

$$r_{\mu\nu}^{\kappa\rho}(a) \equiv r_{(\lambda_1, \lambda_1)}^{\kappa\rho}(a) = r(a - \alpha_- h_\mu - \alpha_- h_\nu, \lambda_1, \lambda_1, a)_{a - \alpha_- h_\nu, a - \alpha_- h_\rho} = \nu \begin{array}{|c|} \hline \kappa \\ \hline \mu \\ \hline \end{array} \quad (3.10)$$

We will concentrate on the $\mathcal{V}^-$, the equations for the $\mathcal{V}^+$ being completely analogous. The braiding matrix of $\mathcal{V}_1^-(z)$ with $\mathcal{V}_1^-(w)$ is trivial since there are no screening charges and one has (see (3.6)): $r_{11}^{11}(t) = e^{i\pi\lambda_1^2\alpha_-^2} = q^{\frac{1}{2}\lambda_1^2}$ where we introduced

$$q = e^{2i\pi\alpha_-^2}, \quad q' = e^{2i\pi\alpha_+^2} \quad (3.11)$$

The braiding of $\mathcal{V}_1^-$ with $\mathcal{V}_2^-$ is more complicated, since the latter now contains one screening operator. The computation of this braiding matrix involves the decomposition of the integration contour into two pieces, one from w to z and one from z to w . For each of the two pieces, we move the reference points for the analytic continuations to a given standard position, which results in multiplying by certain phase factors. Doing the same manipulations for $\mathcal{V}_1^-(z)\mathcal{V}_2^-(w)$, $\mathcal{V}_2^-(z)\mathcal{V}_1^-(w)$, $\mathcal{V}_1^-(w)\mathcal{V}_2^-(z)$ and $\mathcal{V}_2^-(w)\mathcal{V}_1^-(z)$ allows us to express all four products in terms of two integrals multiplied by certain phase factors. Solving for $\mathcal{V}_1^-(z)\mathcal{V}_2^-(w)$ and $\mathcal{V}_2^-(z)\mathcal{V}_1^-(w)$ in terms of $\mathcal{V}_1^-(w)\mathcal{V}_2^-(z)$ and $\mathcal{V}_2^-(w)\mathcal{V}_1^-(z)$ gives the braiding matrix elements r_{21}^{12} , r_{21}^{21} , r_{12}^{21} and r_{12}^{12} . The details of this calculation are a step-by-step repetition of those for the non-extended theories¹⁶⁾, only the phase factors are modified.

Computing the braiding of $\mathcal{V}_\mu^-$ with $\mathcal{V}_\nu^-$ for $\mu, \nu \neq 1, 2$ using the same technique is in general rather complicated, since one has to decompose more than one integration contour.

However, it was shown in ref.⁷⁾ that there exists a permutation symmetry among the $\mathcal{V}_\mu^-$. Indeed, there exists a canonical Fock space transformation $\mathcal{P}_{\mu\nu}$ mapping all oscillators a_n^i to a different set of oscillators $(a^{(\mu\nu)})_n^i$ such that

1. the new set still satisfies $[(a^{(\mu\nu)})_n^i, (a^{(\mu\nu)})_m^j] = \delta^{ij} n \delta_{n, -m}$ and

2. in terms of the $(a^{(\mu\nu)})_{\mu\nu}^j$ the $\mathcal{V}_\mu^-$ is written as an *unscreened* vertex operator and $\mathcal{V}_\nu^-$ contains only one screening operator.

From a Lie algebra representation theoretic point of view, $\mathcal{P}_{\mu\nu}$ permutes the weights h_κ of the vector representation of A_n such that $h_1 \rightarrow h_\mu, h_2 \rightarrow h_\nu$. This allows the deduction⁷⁾ of all $r_{\mu\nu}^{\kappa\rho}$ from $r_{11}^{11}, r_{12}^{12}, r_{21}^{21}, r_{12}^{21}, r_{12}^{12}$. In particular, $r_{\mu\mu}^{\mu\mu} = r_{11}^{11}$ for all μ , which is readily verified for $\mu = 2$ by a direct computation (decomposing the two screening contours) in the same way as above. The result is

$$\begin{aligned} r_{\mu\mu}^{\mu\mu}(l) &\equiv r(l - 2\alpha_- h_\mu, \lambda_1, l)_{l-\alpha_- h_\mu, l-\alpha_- h_\mu} = q^{\frac{1}{2}\lambda_1^2} \\ r_{\mu\nu}^{\mu\nu}(l) &\equiv r(l - \alpha_- h_\mu - \alpha_- h_\nu, \lambda_1, l)_{l-\alpha_- h_\mu, l-\alpha_- h_\nu} = -q^{-\frac{1}{2}(1-\lambda_1^2) - \frac{1}{2}l_{\mu\nu}} \frac{[1]}{[l_{\mu\nu}]} \quad (\mu \neq \nu) \\ r_{\mu\nu}^{\nu\mu}(l) &\equiv r(l - \alpha_- h_\mu - \alpha_- h_\nu, \lambda_1, l)_{l-\alpha_- h_\nu, l-\alpha_- h_\mu} = q^{-\frac{1}{2}(1-\lambda_1^2) - \frac{1}{2}l_{\mu\nu}} \frac{[l_{\mu\nu} + 1]}{[l_{\mu\nu}]} \quad (\mu \neq \nu) \end{aligned} \quad (3.12)$$

where

$$\lambda_1^2 = \frac{n}{n+1} \quad \text{and} \quad l_{\mu\nu} = \sum_{i=\nu}^{\mu-1} l_i \quad \text{if } \mu > \nu, \quad l_{\mu\nu} = -l_{\nu\mu} \quad \text{if } \nu > \mu \quad (3.13)$$

and where we introduced the following notation, borrowed from quantum groups:

$$[a] = \frac{q^{\frac{1}{2}a} - q^{-\frac{1}{2}a}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \quad (3.14)$$

In the Appendix, we collect a certain number of algebraic relations between products of the r -matrix elements (other than the YBE) which will be useful in the sequel.

It is obvious that all equations (3.8-14) are equally valid for the $\mathcal{V}^+$, provided one replaces $q \rightarrow q', l_i \rightarrow l'_i$.

The braiding matrix (3.12) was first obtained in ref.⁷⁾, although in a slightly different basis, using the fact that the four-point functions $\langle 0 | V_{\alpha_{\kappa'}} \mathcal{V}_1^\pm(z) \mathcal{V}_2^\pm(w) V_{\alpha_{\mu'}} | 0 \rangle$, etc. are hypergeometric functions.

In I, the braiding matrices (3.12) have been related to those of the A_n WZW theories, in the cases when the W-algebra is given by the coset $(A_n^{(1)})_k \oplus (A_n^{(1)})_1 / (A_n^{(1)})_{k+1}$. We found the following conjectured²⁷⁾ relation to be satisfied:

$$R_{G/H} = R_G (R_H)^{-1} \quad (3.15)$$

Symmetric fusion of the chiral screened vertex operators : In I we defined the fusion of the chiral (screened) vertex operators $\mathcal{V}_\mu$ associated with the vector representation of A_n (highest weight: λ_1). This yields chiral screened vertex operators $\mathcal{V}_{\mu_1 \mu_2 \dots \mu_N}$ associated with the rank N symmetric tensor representation of A_n (highest weight: $N\lambda_1$). For the A_1 case, this had been done in ref.¹⁶⁾

In general, when fusing two screened vertex operators of the type (2.15) it is not enough to multiply with $(\zeta - z)^{-\alpha_{nn'} \cdot \alpha_{mm'}}$ in order to suppress the short distance singularity. One also has to change the reference points for the analytic continuations to the standard configuration for the fused operator. This results in an extra phase factor. The V_{ML}^K are defined in (2.17) including the $\theta_{\alpha_L}^{i_1 \dots i_{j'}}$ so as to precisely cancel this extra phase factor. This is discussed in detail in ref.¹⁾

The result is that

the fusion of two arbitrary screened vertex operators

$$V_{(n+m, n'+m')A}^C(z) = \lim_{\zeta \rightarrow z} (\zeta - z)^{-\alpha_{nn'} \cdot \alpha_{mm'}} V_{(nn')B}^C(\zeta) V_{(mm')A}^B(z) \quad (3.16)$$

*does not contain any left-over phase factor.**

In I it was shown that if we define (screened) vertex operators recursively by

$$\mathcal{V}_{\mu\mu_1 \dots \mu_N}^-(z) = \lim_{\zeta \rightarrow z} (\zeta - z)^{-N\lambda_1^2 \alpha_z^2} \mathcal{V}_\mu^-(\zeta) \mathcal{V}_{\mu_1 \dots \mu_N}^-(z) \quad (3.17)$$

* In the general case (3.16), however, one has to make some convention for the ordering of the screening operators, as was done for the $\mathcal{V}_\mu$ and as will be done implicitly in the following.

they are completely symmetric in all indices:

$$\mathcal{V}_{\mu_1 \dots \mu_N}^- = \mathcal{V}_{(\mu_1 \dots \mu_N)}^- \quad (3.18)$$

This symmetry guarantees that it does not matter whether one first fuses $\mathcal{V}_\mu$ with $\mathcal{V}_\nu$ and then the resulting operator with $\mathcal{V}_\rho$ or whether one does it in a different order. The result only depends on *which* operators have been fused.

The preceding discussion for the $\mathcal{V}^-$ is equally valid for the fusion of operators $\mathcal{V}^+$. If we fuse a $\mathcal{V}_\mu^-$ with a $\mathcal{V}_\nu^+$, again all phase factors cancel: $\mathcal{V}_{\mu-\nu}^+(z) = \lim_{\zeta \rightarrow z} (\zeta - z)^{\lambda_1^2} \mathcal{V}_\mu^-(\zeta) \mathcal{V}_\nu^+(\zeta)$. Using the R -matrix (3.7) a simple calculation shows that again $\mathcal{V}_{\mu-\nu}^+ = \mathcal{V}_{\nu+\mu}^-$. Hence a general operator obtained by fusion in an arbitrary order of N operators $\mathcal{V}^-$ and N' operators $\mathcal{V}^+$ is

$$\mathcal{V}_{\mu_1 \dots \mu_N \nu_1 \dots \nu_{N'}}^+ = \lim_{\zeta \rightarrow z} (\zeta - z)^{N' \lambda_1^2} \mathcal{V}_{\mu_1 \dots \mu_N}^- (\zeta) \mathcal{V}_{\nu_1 \dots \nu_{N'}}^+ (\zeta) \quad (3.19)$$

and is associated with a pair of rank N and rank N' symmetric tensor representations of A_n .

Finally, the fusion of any two operators of the type (3.19) is very simple:

$$\begin{aligned} & \mathcal{V}_{\mu_1 \dots \mu_N \rho_1 \dots \rho_{N'}}^+ \dots \nu_1^+ \sigma_1^+ \dots \sigma_{M'}^+ (z) \\ &= \lim_{\zeta \rightarrow z} (\zeta - z)^{(N' M' + N' M - N M \alpha_2^2 - N' M' \alpha_1^2) \lambda_1^2} \mathcal{V}_{\mu_1 \dots \mu_N \nu_1 \dots \nu_{N'}}^+ (\zeta) \mathcal{V}_{\rho_1 \dots \rho_{N'} \sigma_1^+ \dots \sigma_{M'}^+}^+ (z) \end{aligned} \quad (3.20)$$

Symmetric fusion of the braiding matrices : As was shown in I, the fusion procedure of the previous subsection for the chiral vertex operators implies a fusion procedure for the corresponding R -matrices.

Consider first the braiding of $\mathcal{V}_\mu^-(z)$ with $\mathcal{V}_{\mu_1 \dots \mu_M}^-(w)$. Writing the product of the two operators as

$$\mathcal{V}_\mu^-(z) \mathcal{V}_{\mu_1 \dots \mu_M}^-(w) = \lim_{\zeta \rightarrow w} (\zeta - w)^{-(M-1) \lambda_1^2} \alpha_2^2 \mathcal{V}_\mu^-(z) \mathcal{V}_\mu^-(\zeta) \mathcal{V}_{\mu_1 \dots \mu_{M-1} \mu_{M+1} \dots \mu_M}^-(w) \quad (3.21)$$

and braiding $\mathcal{V}(z)$ to the right we obtain the braiding matrix of $\mathcal{V}_\mu^-$ with $\mathcal{V}_{\mu_1 \dots \mu_M}^-$ as a product of the elementary braiding matrix $r_{\mu \mu_i}^{\nu \nu_i}$ and of the braiding matrix of

$\mathcal{V}_\mu^-$ with $\mathcal{V}_{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_M}^-$. This yields a recurrence relation (fusion rule) for the r -matrices, most neatly written in the IRF representation:

$$r(c, \lambda_1, M \lambda_1, a)_{db} = \sum_{\tilde{b}} r(c, \lambda_1, \lambda_1, \tilde{a})_{d\tilde{b}} r(\tilde{b}, \lambda_1, (M-1) \lambda_1, a)_{\tilde{a}b} \quad (3.22)$$

Any $\tilde{a}$ compatible with the fusion (3.17) can be chosen. $\tilde{b}$ can take at most two values (cf. (3.12)). An analogous relation holds for the r' -matrices.

It is useful to rewrite the recursion relation (3.22) using the diagrammatic notation (3.2):

$$\begin{array}{c} \boxed{(1, M)} \\ a \quad b \\ c \quad d \end{array} = \sum_{\tilde{b}} \begin{array}{c} \boxed{(1, M-1)} \\ a \quad b \\ \tilde{a} \quad \tilde{b} \end{array} \times \begin{array}{c} \boxed{(1, 1)} \\ \tilde{a} \quad \tilde{b} \\ d \quad c \end{array} \quad (3.23)$$

In the same way one finds

$$\begin{array}{c} \boxed{(N, 1)} \\ a \quad b \\ c \quad d \end{array} = \sum_{\tilde{a}} \begin{array}{c} \boxed{(N-1, 1)} \\ a \quad b \\ \tilde{a} \quad \tilde{d} \end{array} \times \begin{array}{c} \boxed{(1, 1)} \\ \tilde{a} \quad \tilde{c} \\ \tilde{c} \quad c \end{array} \quad (3.24)$$

One sees that $r(c, \lambda_1, M \lambda_1, a)_{db}$ is obtained by multiplying M elementary braiding matrices $r(\cdot, \lambda_1, \lambda_1, \cdot)$ "vertically", whereas $r(c, N \lambda_1, \lambda_1, a)_{db}$ is obtained by multiplying N elementary $r(\cdot, \lambda_1, \lambda_1, \cdot)$ "horizontally".

Next, one obtains the general recursion formula for $r(c, N \lambda_1, M \lambda_1, a)_{db}$ using the fusion relations as before:

$$r(c, (N+1) \lambda_1, M \lambda_1, a)_{db} = \sum_{\tilde{a}} r(c, \lambda_1, M \lambda_1, \tilde{a})_{d\tilde{b}} r(\tilde{c}, N \lambda_1, M \lambda_1, a)_{\tilde{a}b} \quad (3.25)$$

$$r(c, N \lambda_1, (M+1) \lambda_1, a)_{db} = \sum_{\tilde{b}} r(c, N \lambda_1, \lambda_1, \tilde{a})_{d\tilde{b}} r(\tilde{b}, N \lambda_1, M \lambda_1, a)_{\tilde{a}b} \quad (3.26)$$

The diagrammatic representation of these equations is analogous to (3.23,24) (which are just the special cases $M=1$ and $N=1$). Again, the same relations are true for the r' .

Finally, in I, we gave the braiding of general operators $\mathcal{V}_{\mu_1 \dots \mu_N}^{-\nu_1^+ \dots \nu_N^+}$ (cf. (3.19)). The same argument as the one that led to eq. (3.7) shows that the braiding of $\mathcal{V}_{\mu_1 \dots \mu_N}^{-}$ with $\mathcal{V}_{\rho_1 \dots \rho_M}^{+}$ simply gives a phase factor:

$$R(C, (N\lambda_1, 0), (0, M\lambda_1), A)_{DB} = R(C, (0, M\lambda_1), (N\lambda_1, 0), A)_{DB} = e^{-i\pi N M \lambda_1^2} \quad (3.27)$$

Then, using (3.19), we obtained the braiding matrix of the general operators

$$\mathcal{V}_{\mu_1 \dots \mu_N}^{-\nu_1^+ \dots \nu_N^+} \times R(C, (N\lambda_1, N'\lambda_1), (M\lambda_1, M'\lambda_1), A)_{DB} = e^{-i\pi(NM' + N'M)\lambda_1^2} r(c, N\lambda_1, M\lambda_1, a)_{db} r'(c', N'\lambda_1, M'\lambda_1, a')_{d'b'} \quad (3.28)$$

This equation, together with eqs. (3.23-24) and (3.12) gives the braiding matrices for any two chiral screened vertex operators (and hence also for the conformal blocks), each of them being labelled by a pair of symmetric tensor representations (rank N and N') of A_n . The complete R -matrix (3.28) factorizes - up to a trivial phase factor independent of A, B, C, D - into the product of two r (r')-matrices.

The case of screened vertex operators associated with general A_n -representations will be treated in the next two sections.

Relation with the Boltzmann weights of the A_n face models :

We have shown in I that the braiding matrices r (resp. r') given above coincide, up to a trivial factor, with the critical limit of the Boltzmann weights of the A_n face models of Jimbo et al. ²⁰⁻²². This was shown in two steps: first we related the fundamental braiding matrices $r_{\mu\nu}^{\rho\sigma}$ (eq.(3.12)) to the Boltzmann weights of the A_n face models associated with the vector representation, as constructed by Jimbo et al. ^{20,21}. Then we pointed out that the fusion procedure of ref. ²² for the Boltzmann weights is the same as the one for the braiding matrices r (r') given above.

The A_n face models are two-dimensional solvable lattice models with fluctuation variables a, b, c, d interacting by four around a plaquette (face). The corresponding Boltzmann weights are given in terms of elliptic theta functions of nome p and depend on a spectral parameter u . In the critical limit, $p \rightarrow 0$, the elliptic functions become trigonometric, as are the braiding matrices. The precise relation is (see I for further details):

$$r(c, N\lambda_1, M\lambda_1, a)_{db} = q^{-\frac{1}{2}NM(1-\lambda_1^2)} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{NM} \times \lim_{u \rightarrow -i\infty} \lim_{p \rightarrow 0} q^{-\frac{1}{2}NMu} W_{NM} \left(\begin{array}{cc} a & b \\ d & c \end{array} \middle| u \right) \quad (3.29)$$

4. Screened vertex operators associated with the antisymmetric tensor representation of A_n

In the previous section we have defined symmetric tensor vertex operators $\mathcal{V}_{\mu_1 \dots \mu_N}^{-\nu_1^+ \dots \nu_N^+}$ by (symmetric) fusion of the fundamental (vector) vertex operators $\mathcal{V}_{\mu}^{-}$ and $\mathcal{V}_{\nu}^{+}$. Let us stress that these symmetric tensor vertex operators are again screened vertex operators (cf. (2.15,17))

$$\mathcal{V}_{\mu_1 \dots \mu_N}^{-\nu_1^+ \dots \nu_N^+}(z) = e^{i\theta} \int V_{\alpha}(z) V_{\alpha_1}^{i_1}(u_1) \dots V_{\alpha_r}^{i_r}(u_r) V_{\alpha_+}^{j_1}(v_1) \dots V_{\alpha_+}^{j_{r'}}(v_{r'}) \prod_{a=1}^r \prod_{b=1}^{r'} du_a dv_b \quad (4.1)$$

with

$$\alpha = -(\alpha_- N + \alpha_+ N') \lambda_1 \quad (4.2)$$

and the screening operators are those contained in the $\mathcal{V}_{\mu_i}^{-}$, $\mathcal{V}_{\nu_j}^{+}$ which were fused to obtain $\mathcal{V}_{\mu_1 \dots \mu_N}^{-\nu_1^+ \dots \nu_N^+}$. The ordering of the screening operators is determined by the fusion procedure. The phase factors θ are given by eq. (2.18).

In this section we will define screened vertex operators corresponding to the antisymmetric tensor representation of A_n and study their braiding. They will have $\alpha = -\alpha_+ \lambda_2$. We will show how they are obtained by antisymmetric fusion of the vector representation vertex operators $\mathcal{V}_\mu^\pm$. The generalization to arbitrary A_n -representations will be discussed in the next section.

As an example, consider first

$$e^{i\theta_{\alpha_L}^{2\alpha_-}} \int V_{\alpha(z)} V_{\alpha_-}^2(u) du \Big|_{\mathcal{H}_L} \equiv \hat{V}^{13}(z) \quad , \quad \alpha_{(i)} = -\alpha_- \lambda_i \quad (4.3)$$

In order to compute its braiding with $\mathcal{V}_1^-(w) \equiv V_{\alpha_1}(w)$, we decompose the u -integration contour into two pieces, as was discussed in the previous section, below eq. (3.11). A priori, the braiding could give rise to a linear combination of $\mathcal{V}_1^-(w) \hat{V}^{13}(z)$ and $\left(\int V_{\alpha_1}(w) V_{\alpha_-}^2(u) du \right) V_{\alpha_2}(z)$. However, $\lambda_1 - e_2$ is not a weight of the A_n vector representation, and hence the second possibility is expected to be absent. Indeed, the braiding simply is

$$\hat{V}^{13}(z) \mathcal{V}_1^-(w) = q^{\frac{1}{2}\lambda_1 \cdot \lambda_2} \mathcal{V}_1^-(w) \hat{V}^{13}(z) \equiv q^{\lambda_1^2 - \frac{1}{2}} \mathcal{V}_1^-(w) \hat{V}^{13}(z) \quad (4.4)$$

This result can also be understood if we think of $\hat{V}^{13}(z)$ as being obtained by antisymmetric fusion of $\mathcal{V}_1^-$ and $\mathcal{V}_3^-$:

$$\hat{V}^{13}(z) \Big|_{\mathcal{H}_L} \simeq \text{const}(L) \lim_{\zeta \rightarrow z} (\mathcal{V}_1^-(\zeta) \mathcal{V}_3^-(z) - \mathcal{V}_3^-(\zeta) \mathcal{V}_1^-(z)) \quad (4.5)$$

where $\text{const}(L)$ and x are appropriate constants determined below. Using this relation, we obtain the matrix element for the braiding of $\hat{V}^{13}(z)$ with $\mathcal{V}_1^-(w)$ from the antisymmetric fusion of the elementary braiding matrix elements (3.12). One has:

$$\begin{aligned} \mathcal{V}_1^-(\zeta) \mathcal{V}_3^-(z) \mathcal{V}_1^-(w) &= \frac{q^{\lambda_1^2 - \frac{1}{2}}}{[l_{31} + 1]} \left(-\frac{[1]^2}{[l_{31}]} \mathcal{V}_1^-(w) \mathcal{V}_3^-(\zeta) \mathcal{V}_1^-(z) \right. \\ &\quad \left. - q^{-\frac{1}{2}(l_{31}+1)} [1] \mathcal{V}_3^-(w) \mathcal{V}_1^-(\zeta) \mathcal{V}_1^-(z) \right. \\ &\quad \left. + \frac{[l_{31} + 1]^2}{[l_{31}]} \mathcal{V}_1^-(w) \mathcal{V}_1^-(\zeta) \mathcal{V}_3^-(z) \right) \end{aligned} \quad (4.6a)$$

$$\begin{aligned} \mathcal{V}_3^-(\zeta) \mathcal{V}_1^-(z) \mathcal{V}_1^-(w) &= \frac{q^{\lambda_1^2 - \frac{1}{2}}}{[l_{31} + 1]} \left([l_{31} + 2] \mathcal{V}_1^-(w) \mathcal{V}_3^-(\zeta) \mathcal{V}_1^-(z) \right. \\ &\quad \left. - q^{-\frac{1}{2}(l_{31}+1)} [1] \mathcal{V}_3^-(w) \mathcal{V}_1^-(\zeta) \mathcal{V}_1^-(z) \right) \end{aligned} \quad (4.6b)$$

and combining both equations one has, using elementary trigonometric identities:

$$\begin{aligned} &(\mathcal{V}_1^-(\zeta) \mathcal{V}_3^-(z) - \mathcal{V}_3^-(\zeta) \mathcal{V}_1^-(z)) \mathcal{V}_1^-(w) \\ &= q^{\lambda_1^2 - \frac{1}{2}} \frac{[l_{31} + 1]}{[l_{31}]} \mathcal{V}_1^-(w) (\mathcal{V}_1^-(\zeta) \mathcal{V}_3^-(z) - \mathcal{V}_3^-(\zeta) \mathcal{V}_1^-(z)) \end{aligned} \quad (4.7)$$

First, note that the unwanted terms $\mathcal{V}_3^-(w) \mathcal{V}_1^-(\zeta) \mathcal{V}_1^-(z)$ in (4.6) have cancelled. Second, comparing (4.7) with (4.4), we see that the braiding gives the same result provided, in (4.5), we take

$$\text{const}(L) = \text{const} \times \frac{1}{[l_{31}]} \quad (4.8)$$

We will now show that equation (4.5) indeed holds, i.e. that the antisymmetric fusion of two operators $\mathcal{V}_\mu^-, \mathcal{V}_\nu^-$ yields a screened vertex operator $\int V_{\alpha_2} V_{\alpha_-}^{i_1} \dots V_{\alpha_-}^{i_r}$ associated with the rank 2 antisymmetric tensor representation of A_n .

Before embarking on the quantum computation, it is useful to pause and recall some facts about the classical solutions of the Toda field equations^{5,6}. The general solution for the n Toda fields $\varphi_i(x^+, x^-)$ obeying $\partial_+ \partial_- \varphi_i = e^{\sum_j K_{ij} \varphi_j}$ (where $K_{ij} = 2\delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1}$ is the Cartan matrix of A_n) is

$$e^{-\varphi_i(x^+, x^-)} = \text{const} \sum_{\mu_1, \dots, \mu_i} \chi_{\mu_1 \dots \mu_i}(x^+) \psi_{\mu_1 \dots \mu_i}(x^-) \quad , \quad \mu_j = 1, \dots, n+1$$

$$\psi_{\mu_1 \dots \mu_i} = \psi_{[\mu_1} \partial \psi_{\mu_2} \partial^2 \psi_{\mu_3} \dots \partial^{i-1} \psi_{\mu_i]} \quad (4.9)$$

Here [...] denotes antisymmetrisation of the indices. The $n+1$ functions ψ_μ are the classical analogues of the screened vertex operators associated with the vector

representation (highest weight λ_1):

$$\begin{aligned}\psi_1(x) &= \exp\left(\frac{i}{2\alpha_0}\lambda_1 \cdot \phi(x)\right) \\ \psi_i(x) &= \psi_1(x) \int_{x_1}^x dx_1 f_1(x_1) \int_{x_1}^{x_2} dx_2 f_2(x_2) \dots \int_{x_{i-1}}^{x_i} dx_i f_i(x_i) \dots \int_{x_{i-1}}^{x_i} dx_{i-1} f_{i-1}(x_{i-1}) \\ f_j(x) &= \exp\left(-\frac{i}{2\alpha_0}e_j \cdot \phi(x)\right)\end{aligned}\quad (4.10)$$

Here, $\phi(x)$ is an arbitrary (c-number) function. At this classical level it is very easy to check⁶⁾ that

$$\psi_{12\dots j} = \exp\left(\frac{i}{2\alpha_0}\lambda_j \cdot \phi(x)\right) \quad (4.11)$$

and that $\psi_{\mu_1\dots\mu_j}$ is built up from $\psi_{12\dots j}$ times classical analogues of screening operators (as in (4.10)). For example:

$$\psi_{12} \equiv \psi_1 \partial \psi_2 - \psi_2 \partial \psi_1 = (\psi_1)^2 f_1 = e^{\frac{i}{2\alpha_0}(2\lambda_1 - e_1) \cdot \phi(x)} = e^{\frac{i}{2\alpha_0}\lambda_2 \cdot \phi(x)},$$

$$\psi_{13} \equiv \psi_1 \partial \left(\psi_1 \int_{f_1} \int_{f_2} \right) - \left(\psi_1 \int_{f_1} \int_{f_2} \right) \partial \psi_1 = \psi_1^2 f_1 \int_{f_2} = e^{\frac{i}{2\alpha_0}\lambda_2 \cdot \phi} \int_{f_2} \quad (4.12)$$

This little classical computation should be compared with the quantum identity (4.3,5) we wish to prove.

To establish the quantum analogues of (4.12) is less easy, but the classical case gives us a hint how to proceed. Consider the antisymmetric combination $\mathcal{V}_1^-(\zeta)\mathcal{V}_2^-(z) - \mathcal{V}_2^-(\zeta)\mathcal{V}_1^-(z)$, for $|\zeta| > |z|$, acting on $\mathcal{H}_L$. Using (2.18) and (3.6) one finds

$$\mathcal{V}_1^-(\zeta)\mathcal{V}_2^-(z) - \mathcal{V}_2^-(\zeta)\mathcal{V}_1^-(z) = (-)^{l'_1-1} q^{\frac{1}{2}l_1-1} \left(\int_{\Gamma_1} - \int_{\Gamma_2} \right) V_{\alpha_{(1)}}(\zeta) V_{\alpha_{(1)}}(z) \mathcal{V}_{\alpha_-}^1(u) du \quad (4.13)$$

where the integration contours Γ_1 and Γ_2 (as well as $\Gamma_3, \Gamma_4, \Gamma_5$ used next) are shown in fig. 2. The dashed lines indicate how the integrand is continued away

from the real axis. Next, one decomposes Γ_2 into three pieces:

$$\Gamma_2 = \Gamma_1 - \Gamma_3 + \Gamma_4 \quad (4.14)$$

The integral over Γ_1 cancels in (4.13), and one further has

$$\begin{aligned}\int_{\Gamma_3} \dots &= q^{\frac{1}{2}} \int_{\Gamma_3} \dots \\ \int_{\Gamma_4} \dots &= q^{\frac{1}{2}-l_1} \int_{\Gamma_3} \dots\end{aligned}\quad (4.15)$$

Combining these equations yields

$$\mathcal{V}_1^-(\zeta)\mathcal{V}_2^-(z) - \mathcal{V}_2^-(\zeta)\mathcal{V}_1^-(z) = (-)^{l'_1+1} (1 - q^{-1}) [l_1] \int_z^\zeta V_{\alpha_{(1)}}(\zeta) \mathcal{V}_{\alpha_-}^1(u) V_{\alpha_{(1)}}(z) du \quad (4.16)$$

We evaluate the OPE of the vertex operators in the integrand and keep only the term of leading order as $\zeta \rightarrow z$. We get $(2\alpha_{(1)} + \alpha_- e_1 = \alpha_{(2)})$:

$$\begin{aligned}(-)^{l'_1+1} (1 - q^{-1}) [l_1] : V_{\alpha_{(1)}}(z) \mathcal{V}_{\alpha_-}^1(z) V_{\alpha_{(1)}}(z) : &= (\zeta - z)^{\lambda_1^2 \alpha_-^2} \\ &\times \int_z^\zeta du (\zeta - u)^{-\alpha_-^2} (u - z)^{-\alpha_-^2}\end{aligned}$$

$$= (-)^{l'_1+1} (1 - q^{-1}) \frac{(\Gamma(1 - \alpha_-^2))^2}{\Gamma(2 - 2\alpha_-^2)} [l_1] (\zeta - z)^{\lambda_1^2 \alpha_-^2 + 1 - 2\alpha_-^2} V_{\alpha_{(2)}}(z) \quad (4.17)$$

and hence

$$\begin{aligned}V_{\alpha_{(2)}}(z) &= \frac{const (-)^{l'_1+1}}{[l_1]} \lim_{\zeta \rightarrow z} (\zeta - z)^{2\alpha_-^2 - \lambda_1^2 \alpha_-^2 - 1} (\mathcal{V}_1^-(\zeta) \mathcal{V}_2^-(z) - \mathcal{V}_2^-(\zeta) \mathcal{V}_1^-(z)) \\ const &= \frac{\Gamma(2 - 2\alpha_-^2)}{(1 - q^{-1}) (\Gamma(1 - \alpha_-^2))^2}\end{aligned}\quad (4.18)$$

Before taking $\zeta \rightarrow z$, one could have derived with respect to z , as was done in the classical case, see eqs. (4.9,12). Here, however, this is not necessary to

obtain antisymmetrisation, and at the present stage it seems to be a matter of taste whether one includes the derivation or not. In eq.(4.18), it only changes the constant and the power of $(\zeta - z)$ by 1.

In a similar way, one can consider $\mathcal{V}_1^-(\zeta)\mathcal{V}_\mu^-(z) - \mathcal{V}_\mu^-(\zeta)\mathcal{V}_1^-(z)$, $|\zeta| > |z|$. Again, decomposing the $\mu - 1$ integration contours of $\mathcal{V}_\mu^-(\zeta)\mathcal{V}_1^-(z)$ into $2(\mu - 1) + 1$ pieces, one of them cancels $\mathcal{V}_1^-(\zeta)\mathcal{V}_\mu^-(z)$. The surviving $2(\mu - 1)$ integrals each contain a factor of

$$\int_z^\zeta V_{\alpha_{(1)}}(\zeta) V_{\alpha_{(1)}}^i(u) V_{\alpha_{(1)}}(z) du \sim (\zeta - z)^{\lambda_1^2 \alpha_-^2 + 1 + 2\alpha_- \epsilon, \alpha_{(1)}} = (\zeta - z)^{\lambda_1^2 \alpha_-^2 + 1 - 2\alpha_-^2 \epsilon_{11}} \quad (4.19)$$

For real background charge $\alpha_0 (\neq 0)$, we have $\alpha_-^2 > 0$, and all $2(\mu - 2)$ integrals with $i \neq 1$ are subleading as $\zeta \rightarrow z$. Only the two integrals with $i = 1$ survive, and hence

$$\begin{aligned} & \lim_{\zeta \rightarrow z} (\zeta - z)^{2\alpha_-^2 - \lambda_1^2 \alpha_-^2 - 1} (\mathcal{V}_1^-(\zeta)\mathcal{V}_\mu^-(z) - \mathcal{V}_\mu^-(\zeta)\mathcal{V}_1^-(z)) \\ &= \text{const}(L) \int V_{\alpha_{(2)}}(z) V_{\alpha_-}^2(u_1) \dots V_{\alpha_-}^{\mu-1}(u_{\mu-2}) du_2 \dots du_{\mu-2} \end{aligned} \quad (4.20)$$

Finally, it is only a little bit more complicated to show that

$$\begin{aligned} & \lim_{\zeta \rightarrow z} (\zeta - z)^{2\alpha_-^2 - \lambda_1^2 \alpha_-^2 - 1} (\mathcal{V}_\mu^-(\zeta)\mathcal{V}_\nu^-(z) - \mathcal{V}_\nu^-(\zeta)\mathcal{V}_\mu^-(z)) \\ &= \text{const}(L) \int V_{\alpha_{(2)}}(z) V_{\alpha_-}^2(u_1) \dots du_2 \dots du_{\mu+\nu-2} \end{aligned} \quad (4.21)$$

where the screened vertex operator on the right-hand side contains those screening operators contained in $\mathcal{V}_\mu^-$ and $\mathcal{V}_\nu^-$ except for the one $V_{\alpha_-}^1$ that has been fused into the $V_{\alpha_{(2)}}(z)$. Of course, similar considerations apply to the $\mathcal{V}_\mu^+$, and also to higher rank antisymmetric tensors.

Whereas for the symmetric tensors it was obvious that the braiding of $\mathcal{V}_{\mu_1 \dots \mu_N}^-(z)$ with $\mathcal{V}_{\nu_1 \dots \nu_M}^-(w)$ closes again on linear combinations of $\mathcal{V}_{\rho_1 \dots \rho_M}^-(w)\mathcal{V}_{\sigma_1 \dots \sigma_N}^-(z)$ (for fixed N, M), the analogous property must be explicitly checked for the antisymmetric tensor representations. Consider the braiding of $(\mathcal{V}_\mu^-(\zeta)\mathcal{V}_\nu^-(z) - \mathcal{V}_\nu^-(\zeta)\mathcal{V}_\mu^-(z))$ with $\mathcal{V}_\rho^-(w)$. First, we take μ, ν and ρ all different. Braiding $\mathcal{V}(w)$ to the left, gives two factors of the elementary r -matrix (3.12):

$$\begin{aligned} & (\mathcal{V}_\mu^-(\zeta)\mathcal{V}_\nu^-(z) - \mathcal{V}_\nu^-(\zeta)\mathcal{V}_\mu^-(z)) \mathcal{V}_\rho^-(w) \\ &= (r_{\nu\rho}^{\rho\mu\mu\nu} - r_{\mu\rho}^{\mu\rho\mu\nu}) \mathcal{V}_\mu^-(w)\mathcal{V}_\nu^-(\zeta)\mathcal{V}_\rho^-(z) + r_{\nu\rho}^{\rho\nu\mu\mu} \mathcal{V}_\mu^-(w)\mathcal{V}_\rho^-(\zeta)\mathcal{V}_\nu^-(z) \\ &+ (r_{\nu\rho}^{\nu\rho\mu\mu} - r_{\mu\rho}^{\mu\rho\nu\mu}) \mathcal{V}_\nu^-(w)\mathcal{V}_\mu^-(\zeta)\mathcal{V}_\rho^-(z) - r_{\mu\rho}^{\rho\mu\nu\nu} \mathcal{V}_\nu^-(w)\mathcal{V}_\rho^-(\zeta)\mathcal{V}_\mu^-(z) \\ &+ r_{\nu\rho}^{\rho\nu\mu\mu} \mathcal{V}_\rho^-(w) (\mathcal{V}_\mu^-(\zeta)\mathcal{V}_\nu^-(z) - \mathcal{V}_\nu^-(\zeta)\mathcal{V}_\mu^-(z)) \end{aligned} \quad (4.22)$$

Since μ, ν, ρ are all different, the arguments of $r(i)$ are all the same and have been dropped. Only the last term gives an explicitly antisymmetric combination of $\mathcal{V}(\zeta)$ and $\mathcal{V}(z)$. However, using the identity (A.2) of the appendix, one obtains

$$\begin{aligned} V^{\mu\nu}(\zeta, z) \mathcal{V}_\rho^-(w) &= r_{\mu\rho}^{\mu\rho\nu\nu} \mathcal{V}_\mu^-(w) V^{\rho\nu}(\zeta, z) + r_{\nu\rho}^{\nu\rho\mu\mu} \mathcal{V}_\nu^-(w) V^{\mu\rho}(\zeta, z) \\ &+ r_{\nu\rho}^{\rho\nu\mu\mu} \mathcal{V}_\rho^-(w) V^{\mu\nu}(\zeta, z) \end{aligned} \quad (4.23)$$

where we abbreviated

$$V^{\mu\nu}(\zeta, z) = (\mathcal{V}_\mu^-(\zeta)\mathcal{V}_\nu^-(z) - \mathcal{V}_\nu^-(\zeta)\mathcal{V}_\mu^-(z)) \quad (4.24)$$

Now suppose that ρ coincides with one of the indices μ, ν , say $\rho = \nu$. Then of course $\rho \neq \mu$. Proceeding in the same way as before, but now using eqs. (A.5,6), one obtains

$$V^{\mu\rho}(\zeta, z) \mathcal{V}_\rho^-(w) = r_{\rho\rho}^{\rho\rho\mu\mu} \mathcal{V}_\rho^-(w) V^{\mu\rho}(\zeta, z) \quad (4.25)$$

Upon taking the limit $\zeta \rightarrow z$, equations (4.23,25) show that the braiding of the antisymmetric tensor vertex operators with the fundamental (vector) vertex operators closes on themselves. A similar computation shows that an analogous result holds for the braiding of the antisymmetric tensor vertex operators with themselves. Of course, all equations of this section (and of the next one) remain valid upon exchanging α_- with α_+ .

5. The braiding of screened vertex operators associated with arbitrary Young tableaux of A_n

Higher-rank antisymmetric tensor screened vertex operators : In the previous section we have shown that the antisymmetric fusion of the fundamental (vector) screened vertex operators $\mathcal{V}_\mu^-, \mathcal{V}_\nu^-$ gives a screened vertex operator associated with the rank 2 antisymmetric tensor representation (highest weight λ_2): $\int V_{\alpha(2)} V_{\alpha_-}^{i_1} \dots V_{\alpha_-}^{i_r}$. In the same way, one can show the following higher-rank generalization of (4.21):

$$V^{\mu_1 \dots \mu_k}(z_1, \dots, z_k) \equiv \mathcal{V}_{\mu_1}^-(z_1) \mathcal{V}_{\mu_2}^-(z_2) \dots \mathcal{V}_{\mu_k}^-(z_k) \sim \int V_{\alpha(k)}(z) V_{\alpha_-}^{i_1} \dots V_{\alpha_-}^{i_r} \quad (5.1)$$

as $z_1, \dots, z_k \rightarrow z$

This allows us to express the braiding of the operators on the r.h.s. by sums over products of the elementary braiding matrix elements (3.12) for the $\mathcal{V}_{\mu_i}^-$.

We will show that the braiding of the operator (5.1) with $\mathcal{V}_\rho^-(w)$ closes on operators of the same type:

PROPOSITION 1 : For $\rho \neq \mu_1, \dots, \mu_k$ one has

$$V^{\mu_1 \dots \mu_k}(z_1, \dots, z_k) \mathcal{V}_\rho^-(w) = \sum_{i=1}^k r_{\mu_i \rho}^{\mu_i \rho} \prod_{j \neq i}^k r_{\mu_j \rho}^{\rho \mu_j} \mathcal{V}_{\mu_i}^-(w) V^{\mu_1 \dots \mu_{i-1} \rho \mu_{i+1} \dots \mu_k}(z_1, \dots, z_k) + \prod_{i=1}^k r_{\mu_i \rho}^{\rho \mu_i} \mathcal{V}_\rho^-(w) V^{\mu_1 \dots \mu_k}(z_1, \dots, z_k) \quad (5.2)$$

Proof : For $k = 1$, eq. (5.2) is just the braiding of $\mathcal{V}_\mu^-(z)$ with $\mathcal{V}_\rho^-(w)$, cf. eq. (3.9). For $k = 2$, it reduces to eq. (4.23). Suppose now that eq. (5.2) holds for k . Then one has for $k + 1$:

$$V^{\mu_1 \dots \mu_k \mu_{k+1}}(z_1, \dots, z_k, z_{k+1}) = \sum_{i=1}^{k+1} (-)^{k+1-i} V^{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_k}(z_1, \dots, z_k) \mathcal{V}_{\mu_i}^-(z_{k+1}) \quad (5.3)$$

Braiding with $\mathcal{V}_\rho^-(w)$ yields

$$\begin{aligned} & \sum_{i=1}^{k+1} (-)^{k+1-i} \left(r_{\mu_i \rho}^{\mu_i \rho} \sum_{j \neq i}^{k+1} r_{\mu_j \rho}^{\rho \mu_j} \prod_{l \neq i, j}^{k+1} r_{\mu_l \mu_i}^{\mu_l \mu_i} \mathcal{V}_{\mu_j}^-(w) V^{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_{k+1}}(z_1, \dots, z_k) \mathcal{V}_\rho^-(z_{k+1}) \right. \\ & \quad + r_{\mu_i \rho}^{\mu_i \rho} \prod_{j \neq i}^{k+1} r_{\mu_j \mu_i}^{\mu_j \mu_i} \mathcal{V}_{\mu_i}^-(w) V^{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_{k+1}}(z_1, \dots, z_k) \mathcal{V}_\rho^-(z_{k+1}) \\ & \quad + \sum_{j \neq i}^{k+1} r_{\mu_j \rho}^{\rho \mu_j} \prod_{l \neq i, j}^{k+1} r_{\mu_l \rho}^{\rho \mu_l} \mathcal{V}_{\mu_j}^-(w) V^{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_{k+1}}(z_1, \dots, z_k) \mathcal{V}_{\mu_i}^-(z_{k+1}) \\ & \quad \left. + \prod_{j=1}^{k+1} r_{\mu_j \rho}^{\rho \mu_j} \mathcal{V}_\rho^-(w) V^{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_{k+1}}(z_1, \dots, z_k) \mathcal{V}_{\mu_i}^-(z_{k+1}) \right) \\ & \equiv \text{I} + \text{II} + \text{III} + \text{IV} \quad (5.4) \end{aligned}$$

(Here an index $(\mu_i)_j$ or $(\rho)_j$ means that μ_i or ρ replaces μ_j .) On the other hand, we want to obtain

$$\begin{aligned} & \sum_{j=1}^{k+1} r_{\mu_j \rho}^{\mu_j \rho} \prod_{l \neq j}^{k+1} r_{\mu_l \rho}^{\rho \mu_l} \mathcal{V}_{\mu_j}^-(w) V^{\mu_1 \dots \mu_{j-1} \rho \mu_{j+1} \dots \mu_{k+1}}(z_1, \dots, z_k) + \text{IV} \\ & = \sum_{j=1}^{k+1} r_{\mu_j \rho}^{\mu_j \rho} \prod_{l \neq j}^{k+1} r_{\mu_l \rho}^{\rho \mu_l} \mathcal{V}_{\mu_j}^-(w) V^{\mu_1 \dots \mu_{j-1} \mu_{j+1} \dots \mu_{k+1}}(z_1, \dots, z_k) \mathcal{V}_\rho^-(z_{k+1}) (-)^{k+1-j} \\ & \quad + \text{III} + \text{IV} \end{aligned} \quad (5.5)$$

Expressions (5.4) and (5.5) coincide, and the proposition is proved, provided the following identity between r -matrix elements holds:

LEMMA 1 : One has (for fixed i)

$$r_{\mu_i \rho}^{\mu_i \rho} \left(\prod_{l \neq i}^k r_{\mu_l \mu_i}^{\mu_l \mu_i} - \prod_{l \neq i}^k r_{\mu_l \rho}^{\rho \mu_l} \right) = \sum_{j \neq i}^k r_{\mu_j \rho}^{\mu_j \rho} r_{\mu_i \mu_j}^{\rho \mu_i} \prod_{l \neq i, j}^k r_{\mu_l \mu_j}^{\mu_l \mu_j} \quad (5.6)$$

Proof : For $k = 1$, this equation is trivial, and for $k = 2$, it reduces to the identity (A.2). For general k , using (A.2) one can factorize $r_{\mu_i \rho}^{\mu_i \rho}$ and (5.6) is

equivalent to

$$\sum_{j \neq i}^k r_{\mu_j \mu_i}^{\mu_i \mu_j} \prod_{l \neq i, j}^k r_{\mu_l \mu_j}^{\mu_j \mu_l} - \prod_{l \neq i}^k r_{\mu_l \mu_i}^{\mu_i \mu_l} = (\mu_i \leftrightarrow \rho) \quad (5.7)$$

We now proceed to show that the l.h.s. of (5.7) does not depend on μ_i . This is obviously true for $k = 1$. Suppose that it is true for $k - 1$. Fixing an index $I \neq i$ one has

$$\text{l.h.s. of (5.7)} = r_{\mu_I \mu_i}^{\mu_i \mu_I} \left(\prod_{l \neq i, I}^k r_{\mu_l \mu_I}^{\mu_I \mu_l} - \prod_{l \neq i, I}^k r_{\mu_l \mu_i}^{\mu_i \mu_l} \right) + \sum_{j \neq i, I}^k r_{\mu_j \mu_i}^{\mu_i \mu_j} \prod_{l \neq i, j}^k r_{\mu_l \mu_j}^{\mu_j \mu_l} \quad (5.8)$$

Using the hypothesis (eq. (5.7) for $k - 1$) to rewrite the parenthesis on the r.h.s of (5.8), one obtains

$$\begin{aligned} \text{l.h.s. of (5.7)} &= \sum_{j \neq i, I}^k \left(r_{\mu_I \mu_i}^{\mu_i \mu_I} \left(r_{\mu_j \mu_I}^{\mu_I \mu_j} - r_{\mu_j \mu_i}^{\mu_i \mu_j} \right) + r_{\mu_j \mu_i}^{\mu_i \mu_j} r_{\mu_I \mu_j}^{\mu_j \mu_I} \right) \prod_{l \neq i, j, I}^k r_{\mu_l \mu_j}^{\mu_j \mu_l} \\ &= -q^{\lambda_I^2 - 1} \sum_{j \neq i, I}^k \prod_{l \neq i, j, I}^k r_{\mu_l \mu_j}^{\mu_j \mu_l} \end{aligned} \quad (5.9)$$

where (A.3) was used in the last step. The result clearly is independent of μ_i . This proves (5.7), the lemma and hence the proposition 1.

PROPOSITION 2 : Let ρ be equal to one of the $\mu_1, \dots, \mu_k$, say $\rho = \mu_i$. Then

$$V^{\mu_1, \dots, \mu_k}(z_1, \dots, z_k) \mathcal{V}_{\rho}^{-}(w) \prod_{i=1}^k r_{\mu_i \rho}^{\rho \mu_i} \mathcal{V}_{\rho}^{-}(w) V^{\mu_1, \dots, \mu_k}(z_1, \dots, z_k) \quad (5.10)$$

Proof : For $k = 1$, this is just the braiding of $\mathcal{V}_{\rho}^{-}$ with itself, and for $k = 2$, it was shown in the previous section, see eq. (4.25). Suppose that eq. (5.10) is true

for k . Then, for $k + 1$ we have:

$$\begin{aligned} &V^{\mu_1, \dots, \mu_{k-1}, \rho \mu_k + 1, \dots, \mu_{k+1}}(z_1, \dots, z_{k+1}) \mathcal{V}_{\rho}^{-}(w) = \\ &\sum_{i \neq l}^{k+1} (-)^{k+1-i} \left(r_{\mu_i \rho}^{\rho \mu_i} \left(\sum_{j \neq i, l}^{k+1} r_{\mu_j \mu_i}^{\mu_i \mu_j} \prod_{r \neq i, j, l}^{k+1} r_{\mu_r \mu_i}^{\mu_i \mu_r} \mathcal{V}_{\rho}^{-}(w) V^{\mu_1, \dots, (\mu_i)_j, \dots, \mu_{k+1}}(w) \mathcal{V}_{\rho}^{-}(z_{k+1}) \right. \right. \\ &\quad \left. \left. + \bar{r}_{\rho \mu_i}^{\rho \mu_i} \prod_{r \neq i, l}^{k+1} r_{\mu_r \mu_i}^{\mu_i \mu_r} \mathcal{V}_{\rho}^{-}(w) V^{\mu_1, \dots, (\mu_i)_l, \dots, \mu_{k+1}}(w) \mathcal{V}_{\rho}^{-}(z_{k+1}) \right) \right. \\ &\quad \left. + \bar{r}_{\rho \mu_i}^{\rho \mu_i} \prod_{r \neq i, l}^{k+1} r_{\mu_r \mu_i}^{\mu_i \mu_r} \mathcal{V}_{\rho}^{-}(w) V^{\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_{k+1}}(w) \mathcal{V}_{\rho}^{-}(z_{k+1}) \right) \\ &\quad + r_{\mu_i \rho}^{\rho \mu_i} \prod_{r \neq i, l}^{k+1} r_{\mu_r \rho}^{\rho \mu_r} \mathcal{V}_{\rho}^{-}(w) V^{\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_{k+1}}(w) \mathcal{V}_{\rho}^{-}(z_{k+1}) \\ &\quad + (-)^{k+1-l} r_{\rho \rho}^{\rho \rho} \left(\sum_{i \neq l}^{k+1} \bar{r}_{\mu_i \rho}^{\rho \mu_i} \prod_{r \neq i, l}^{k+1} \bar{r}_{\mu_r \rho}^{\rho \mu_r} \mathcal{V}_{\rho}^{-}(w) V^{\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, (\rho)_l, \dots, \mu_{k+1}}(w) \mathcal{V}_{\rho}^{-}(z_{k+1}) \right. \\ &\quad \left. + \prod_{r \neq l}^{k+1} \bar{r}_{\mu_r \rho}^{\rho \mu_r} \mathcal{V}_{\rho}^{-}(w) V^{\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_{k+1}}(w) \mathcal{V}_{\rho}^{-}(z_{k+1}) \right) \end{aligned} \quad (5.11)$$

$\bar{r}$ is defined in eq. (A.4) in the appendix. Those terms in (5.11) not containing $\mathcal{V}_{\rho}^{-}(w)$ should vanish. They do, provided

$$\sum_{j \neq i, l}^k r_{\mu_j \rho}^{\rho \mu_j} r_{\mu_i \mu_j}^{\mu_j \mu_i} \prod_{r \neq i, j, l}^k r_{\mu_r \mu_j}^{\mu_j \mu_r} \bar{r}_{\mu_j \rho}^{\rho \mu_j} - r_{\mu_i \rho}^{\rho \mu_i} \prod_{j \neq i, l}^k r_{\mu_j \mu_i}^{\mu_i \mu_j} \bar{r}_{\mu_j \rho}^{\rho \mu_j} + r_{\rho \rho}^{\rho \rho} \prod_{j \neq i, l}^k \bar{r}_{\mu_j \rho}^{\rho \mu_j} \bar{r}_{\mu_i \rho}^{\rho \mu_i} = 0 \quad (5.12)$$

Using (A.5) and (5.7) (suitably modified), one shows that (5.12) equals

$$\sum_{j \neq l}^k \left(r_{\mu_i \rho}^{\rho \mu_i} \bar{r}_{\rho \mu_i}^{\rho \mu_i} (\bar{r}_{\rho \mu_j}^{\rho \mu_j} - r_{\mu_j \mu_i}^{\mu_j \mu_i}) + r_{\mu_j \rho}^{\rho \mu_j} r_{\mu_i \mu_j}^{\mu_j \mu_i} \bar{r}_{\mu_j \rho}^{\rho \mu_j} \right) \prod_{r \neq i, j, l}^k r_{\mu_r \mu_j}^{\mu_j \mu_r} \quad (5.13)$$

which vanishes by (A.8). Hence, we are only left with those terms in (5.11) containing $\mathcal{V}_{\rho}^{-}(w)$. They yield the required result due to the following lemma:

LEMMA 2 : For $\rho = \mu_l$ one has

$$\prod_{i=1}^k \bar{r}_{\mu_i \rho}^{\rho \mu_i} - \prod_{i=1}^k r_{\mu_i \rho}^{\rho \mu_i} = \sum_{i \neq l} r_{\mu_i \rho}^{\mu_i \rho} \bar{r}_{\rho \mu_i}^{\rho \mu_i} \prod_{j \neq i, l}^k r_{\mu_j \mu_i}^{\mu_j \mu_i} \prod_{j \neq i, l}^k \bar{r}_{\mu_j \mu_i}^{\mu_j \mu_i} \quad (5.14)$$

Proof : For $k = 2$, this reduces to the identity (A.6). Suppose now that (5.14) holds for $k - 1$. We will show that it holds for k . Fixing an index $I \neq l$, one has to show that the following quantity vanishes:

$$\begin{aligned} & \sum_{i \neq l, I}^k r_{\mu_i \rho}^{\mu_i \rho} \bar{r}_{\rho \mu_i}^{\rho \mu_i} \prod_{j \neq i, I, l}^k r_{\mu_j \mu_i}^{\mu_j \mu_i} r_{\mu_i \mu_j}^{\mu_i \mu_j} + r_{\mu_I \rho}^{\mu_I \rho} \bar{r}_{\rho \mu_I}^{\rho \mu_I} \prod_{j \neq I, l}^k r_{\mu_j \mu_I}^{\mu_j \mu_I} \prod_{j \neq I, l}^k \bar{r}_{\mu_j \mu_I}^{\mu_j \mu_I} \\ & - r_{\rho \rho}^{\rho \rho} \bar{r}_{\rho \rho}^{\rho \rho} \prod_{j \neq I, l}^k \bar{r}_{\mu_j \rho}^{\rho \mu_j} + r_{\rho \rho}^{\rho \rho} \prod_{j \neq I}^k r_{\mu_j \rho}^{\rho \mu_j} \end{aligned} \quad (5.15)$$

We now use the assumption that (5.14) holds for $k = 1$ in order to rewrite the third term in (5.15). Using further eqs. (A.6) and (A.7), we can factorize $\bar{r}_{\rho \mu_I}^{\rho \mu_I}$, and the vanishing of (5.15) is equivalent to the vanishing of

$$r_{\mu_I \rho}^{\mu_I \rho} \left(\prod_{j \neq I}^k r_{\mu_j \mu_I}^{\mu_j \mu_I} - \prod_{j \neq I}^k r_{\mu_j \rho}^{\rho \mu_j} \right) = \sum_{i \neq I} r_{\mu_i \rho}^{\mu_i \rho} r_{\mu_I \mu_i}^{\mu_I \mu_i} \prod_{j \neq i, I}^k r_{\mu_j \mu_i}^{\mu_j \mu_i} \prod_{j \neq i, I}^k \bar{r}_{\mu_j \mu_i}^{\mu_j \mu_i} \quad (5.16)$$

where $\sum_j$ and $\prod_j$ means that $j = I$ is excluded. But the vanishing of (5.16) is exactly the statement of the above lemma 1. This concludes the proof of lemma 2 and hence also of proposition 2.

Taking the limit $z_1, \dots, z_k \rightarrow z$ in (5.2) and (5.10), and suppressing the short distance singularities gives, according to (5.1), the braiding of the $\int V_{\alpha(k)}(z) V_{\alpha_-}^{i_1} \dots V_{\alpha_-}^{i_r}$ with $V_{\rho}^-(w)$. The factors $const(L)$ occurring in (5.1) only affect the braiding matrices (5.2,10) in a trivial way. Concerning the braiding with the symmetric tensor screened vertex operators we have the following corollary:

COROLLARY : Writing $V_{\nu_1 \dots \nu_N}^-(w) \sim V_{\nu_1}^-(w_1) \dots V_{\nu_N}^-(w_N)$ for the symmetric tensor vertex operators, it is evident that the braiding of $V^{\mu_1 \dots \mu_k}(z_1, \dots, z_k) \sim \int V_{\alpha(k)}(z) V_{\alpha_-}^{i_1} \dots V_{\alpha_-}^{i_r}$ with the $V_{\rho_1 \dots \rho_N}^-(w)$ closes on linear combinations of $V_{\sigma_1 \dots \sigma_N}^-(w) V^{\nu_1 \dots \nu_k}(z_1, \dots, z_k)$.

In the same way as for propositions 1 and 2, one could show that the braiding of a rank N antisymmetric tensor screened vertex operator $V^{\mu_1 \dots \mu_N}$ with a rank M one, $V^{\rho_1 \dots \rho_M}$ closes on a linear combination of $V^{\sigma_1 \dots \sigma_M} V^{\nu_1 \dots \nu_N}$. However, the identities between the products of the elementary r -matrix elements one has to use are more complicated than those of lemmas 1 and 2.

Screened vertex operators associated with arbitrary representations of A_n : Of course, one is interested not only in symmetric or antisymmetric tensor representations, but also in those having mixed symmetries, corresponding to arbitrary Young tableaux. It is clear that the corresponding screened vertex operators are obtained by antisymmetric fusion, as explained in section 4 and in the last subsection, from symmetric tensor screened vertex operators: for example

$$V_{\mu\nu}^-(\zeta) V_{\rho}^-(z) = V_{\rho\nu}^-(\zeta) V_{\mu}^-(z) \quad (5.17)$$

yields upon fusion a screened vertex operator corresponding to the Young tableau $\begin{array}{|c|c|} \hline \mu & \nu \\ \hline \rho & \\ \hline \end{array}$.

The braiding of such screened vertex operators with arbitrary (mixed) symmetries can be computed along the same lines as was done in the previous subsection, in order to obtain explicit formulas for the braiding matrices. Doing this in general is, however, rather cumbersome and not of much practical value. Nevertheless, we must make sure that the braiding closes, i.e.

$$V_Y(z) V_{Y'}(w) = \sum r_{YY'} V_{Y'}(w) V_Y(z) \quad (5.18)$$

where V_Y stands for the screened vertex operator having the symmetry of a given Young tableau Y . The operator (5.17), for example, corresponds to $V_{\begin{array}{|c|c|} \hline \mu & \nu \\ \hline \rho & \\ \hline \end{array}}$.

At this point we can use the results of Jimbo et al.²²⁾ on the fused A_n face model. Indeed, we have shown in section 3 that our elementary braiding matrix r coincides with a certain limit of the Boltzmann weights W of the A_n face model. In the same way as r , W corresponds to a pair of A_n representations (λ_1, λ_1) or $\square \otimes \square$. The fusion procedure of Jimbo et al. to obtain the Boltzmann weights for a pair of arbitrary Young tableaux, $Y \otimes Y'$, is the same as the one that gives us the fused r -matrices $r_{YY'}$ in (5.18) (cf. ref.¹⁾). The proof given by Jimbo et al. can be immediately transcribed for our $r_{YY'}$ -matrix :

PROPOSITION 3 : Let the braiding of $V_{\mu_1}(z_1) \dots V_{\mu_N}(z_N)$ with $V_{\rho_1}(w_1) \dots V_{\rho_M}(w_M)$ be given by

$$\begin{aligned} & V_{\mu_1}(z_1) \dots V_{\mu_N}(z_N) V_{\rho_1}(w_1) \dots V_{\rho_M}(w_M) \\ &= C_{\mu_1 \dots \mu_N \sigma_1 \dots \sigma_M}^{\nu_1 \dots \nu_N \rho_1 \dots \rho_M} V_{\sigma_1}(w_1) \dots V_{\sigma_M}(w_M) V_{\nu_1}(z_1) \dots V_{\nu_N}(z_N) \end{aligned} \quad (5.19)$$

where $C_{\dots}$ is a sum of products of elementary r -matrix elements (3.12). If we impose a mixed symmetrization / antisymmetrization on $\mu_1 \dots \mu_N$ and $\rho_1 \dots \rho_M$ corresponding to Young tableaux Y and Y' , then $C_{\dots}$ has the same symmetry / antisymmetry properties in the two sets of indices $\nu_1 \dots \nu_N$ and $\sigma_1 \dots \sigma_M$.

Proposition 3 shows that the braiding of V_Y with $V_{Y'}$ closes, i.e. that equation (5.18) holds. So far, we only considered screened vertex operators without α_+ components. More generally each screened vertex operator is labelled by a pair of Young tableaux, (Y_-, Y_+) , corresponding to the α_- and α_+ parts. However, this is not a complication, since the braiding of the α_- part with the α_+ part is trivial, see (3.28). One simply has

$$V_{Y_- Y_+}(z) V_{Y'_- Y'_+}(w) = e^{-i\pi \left(\lambda_{Y_-} \cdot \lambda_{Y'_+} + \lambda_{Y_+} \cdot \lambda_{Y'_-} \right)} \sum r_{Y_- Y'_-} r'_{Y_+ Y'_+} V_{Y'_- Y'_+}(w) V_{Y_- Y_+}(z) \quad (5.20)$$

where λ_Y denotes the highest weight of the representation corresponding to the Young tableau Y , and r' is the braiding matrix with q' replacing q , see (3.11).

6. An explicit example : the 3-state Potts model

The simplest W -algebra extended minimal conformal theory is the 3-state Potts model. It has a central charge $c = \frac{4}{5}$ and can be obtained as the coset $(A_2^{(1)})_k \oplus (A_2^{(1)})_1 / (A_2^{(1)})_{k+1}$ for $k = 1$. Note that the cosets $(A_n^{(1)})_1 \oplus (A_n^{(1)})_1 / (A_n^{(1)})_2$ yield the Z_{n+1} parafermion theories, and hence the 3-state Potts model also coincides with the Z_3 parafermion theory. It is further contained in the standard minimal series $c = 1 - 6/m(m+1)$ for $m = 5$. However, the 3-state Potts model shows its full structure when considered as the simplest W -algebra extended minimal theory. With the conventions of sect. 2, it has $\alpha_- = -\sqrt{\frac{4}{5}}$, $\alpha_+ = \sqrt{\frac{5}{4}}$.

The non-trivial primary fields are (see e.g. ref.²⁾) the spin fields σ and σ^+ of conformal dimension $\frac{1}{15}$ and Z_3 charges $+1$ and -1 , respectively, the parafermion currents ψ and ψ^+ of conformal dimension $\frac{2}{3}$ and charges $+1$ and -1 , and the energy operator ϵ of dimension $\frac{2}{5}$ and charge 0. In the Feigin-Fuchs construction they are given by²⁾

$$\begin{aligned} \sigma &= V_{-\alpha_- \lambda_1}, \quad \sigma^+ = V_{-\alpha_- \lambda_2} \\ \psi &= V_{-2\alpha_- \lambda_2}, \quad \psi^+ = V_{-2\alpha_- \lambda_1} \\ \epsilon &= V_{-\alpha_- \lambda_1 - \alpha_- \lambda_2} \end{aligned} \quad (6.1)$$

times screening operators. As in (2.17), the chiral vertex operators are also labelled by the Hilbert spaces between which they operate, for example $V_{\sigma^+ \psi^+}$ denotes the operator $\sigma^+ : \mathcal{H}_{\psi^+} \rightarrow \mathcal{H}_{\sigma^+}$, and so on. Then one has, among other relations

$$\begin{aligned} V_{\sigma^0 \sigma^0}^{\sigma^0} &= V_{-\alpha_- \lambda_1} \sim \mathcal{V}_1 \\ V_{\sigma^0 \sigma^0}^{\psi^+} &= V_{-\alpha_- \lambda_1} \sim \mathcal{V}_1 \\ V_{\sigma^+ \sigma^+}^{\epsilon} &= V_{-\alpha_- \lambda_2} \sim V^{12}/[2] \\ V_{\sigma^+ \sigma^+}^0 &= \int V_{-\alpha_- \lambda_2} V_{\alpha_-}^2 V_{\alpha_-}^1 \sim V^{23} \\ V_{\sigma^0 \sigma^0}^{\sigma^+} &= \int V_{-\alpha_- \lambda_1} V_{\alpha_-}^1 \sim \mathcal{V}_2 \end{aligned}$$

$$\begin{aligned}
V_{\sigma\sigma+}^0 &= \int V_{-\alpha-\lambda_1} V_{\alpha-}^1 V_{\alpha-}^2 \sim V_3 \\
V_{\sigma+ \sigma+}^\sigma &= \int V_{-\alpha-\lambda_2} V_{\alpha-}^2 \sim V^{13}/[3] \\
V_{\sigma+ \psi+}^\sigma &= \int V_{-\alpha-\lambda_2} V_{\alpha-}^2 V_{\alpha-}^1 \sim V^{23} \\
V_{\sigma\epsilon}^\sigma &= \int V_{-\alpha-\lambda_1} V_{\alpha-}^1 V_{\alpha-}^2 \sim V_3
\end{aligned} \tag{6.2}$$

Consider now the conformal blocks defined by

$$\mathcal{F}_p^{ijkl}(z_1, z_2, z_3, z_4) = \frac{i \begin{array}{c|c} j & (z_2) \\ \hline k & (z_3) \end{array}}{\begin{array}{c|c} p & (z_4) \\ \hline l & (z_4) \end{array}} \tag{6.3}$$

$$= \langle V_{ii}^{\bar{0}}(z_1) V_{jp}^i(z_2) V_{ki}^p(z_3) V_{l0}^i(z_4) \rangle$$

where $V_{ii}^{\bar{0}}$ means that one has to use the screened vertex operator made up with $V_{2\alpha_0-\alpha_i}$ instead of V_{α_i} and that it maps $\mathcal{H}_{\alpha_i}$ to $\mathcal{H}_{2\alpha_0}$ instead of $\mathcal{H}_0$.

One has for example

$$\begin{aligned}
\mathcal{F}_{\sigma+}^{\sigma\sigma+ \sigma\sigma} &= \langle V_{\sigma\sigma}^{\bar{0}} V_{\sigma+ \sigma+}^\sigma V_{\sigma\sigma}^{\sigma+} V_{\sigma\sigma}^\sigma \rangle \sim \langle \bar{V}_1 V^{13} V_2 V_1 \rangle / [3] \\
\mathcal{F}_{\psi+}^{\sigma\sigma+ \sigma\sigma} &= \langle V_{\sigma\sigma}^{\bar{0}} V_{\sigma+ \psi+}^\sigma V_{\sigma\sigma}^{\psi+} V_{\sigma\sigma}^\sigma \rangle \sim \langle \bar{V}_1 V^{23} V_1 V_1 \rangle \\
\mathcal{F}_0^{\sigma\sigma\sigma+ \sigma} &= \langle V_{\sigma\sigma}^{\bar{0}} V_{\sigma\sigma}^\sigma V_{\sigma+ \sigma}^0 V_{\sigma\sigma}^\sigma \rangle \sim \langle \bar{V}_1 V_1 V^{23} V_1 \rangle \\
\mathcal{F}_\epsilon^{\sigma\sigma\sigma+ \sigma} &= \langle V_{\sigma\sigma}^{\bar{0}} V_{\sigma\epsilon}^\sigma V_{\sigma+ \sigma}^\epsilon V_{\sigma\sigma}^\sigma \rangle \sim \langle \bar{V}_1 V_3 V^{12} V_1 \rangle / [2]
\end{aligned} \tag{6.4}$$

where again $\bar{V}_1 = V_{2\alpha_0+\alpha_{-\lambda_1}}$ instead of $V_1 = V_{-\alpha_{-\lambda_1}}$ has to be used. Then for each of the conformal blocks (6.4) the charge conservation condition $\sum_i \alpha_i = 2\alpha_0\rho$ is fulfilled.

The braiding of $V_{\sigma+}$ with V_σ is given by propositions 1 and 2 of the previous section. In fact, for the chiral blocks (6.4), all we need is proposition 1 (see also

(4.23)). A priori, the braiding gives a linear combination of three terms: $V^{13}V_2 = r_{12}^{12} r_{32}^{23} V_1 V^{23} + r_{32}^{32} r_{12}^{21} V_3 V^{12} + r_{12}^{21} r_{32}^{23} V_2 V^{13}$, and similar for $V^{23}V_1$. However, V^{13} gives zero when applied on $V_1(z_4)|0\rangle$. This is easily seen: V^{13} contains only one screening operator $V_{\alpha-}^2$, which is integrated around z_4 where $V_1 = V_{-\alpha_{-\lambda_1}}$ is inserted. Since the OPE of $V_{\alpha-}^2$ with $V_{-\alpha_{-\lambda_1}}$ is regular ($e_2 \cdot \lambda_1 = 0$) the integration contour can be pulled across z_4 and reduced to a point. As a result, $V^{13}(z_3)V_1(z_4)|0\rangle$ vanishes. To compute the braiding for the conformal blocks (6.4), all we need to know are the elementary r -matrix elements (3.12) for $l_1 = 2, l_2 = 1$:

$$\begin{aligned}
q &= e^{\frac{5i\pi}{5}}, [1] = 1, [2] = -2 \cos \frac{\pi}{5}, [3] = 2 \cos \frac{\pi}{5}, [4] = -1 \\
\left. \begin{aligned} l_1 &= 2, \\ l_2 &= 1 \end{aligned} \right\} : \quad \begin{aligned} r_{12}^{12} &= -\frac{e^{i\pi/3}}{[2]}, \quad r_{12}^{21} = \frac{e^{-4i\pi/15}}{[2]}, \quad r_{21}^{12} = -e^{-4i\pi/15}, \quad r_{21}^{21} = -\frac{e^{2i\pi/15}}{[2]} \\ r_{31}^{13} &= -\frac{e^{-4i\pi/15}}{[3]}, \quad r_{31}^{31} = \frac{e^{i\pi/3}}{[3]}, \quad r_{32}^{23} = e^{-4i\pi/15}[2], \quad r_{32}^{32} = e^{-i\pi/15} \end{aligned} \tag{6.5}
\end{aligned}$$

The braiding of the conformal blocks (6.4) then reads:

$$\begin{aligned}
&\left(\mathcal{F}_{\psi+}^{\sigma\sigma+ \sigma\sigma} \right) (z_1, z_2, z_3, z_4) = B \left(\mathcal{F}_{\epsilon}^{\sigma\sigma\sigma+ \sigma} \right) (z_1, z_3, z_2, z_4) \\
B &= \begin{pmatrix} \frac{1}{[3]} r_{12}^{12} r_{32}^{23} & \frac{[2]}{[3]} r_{12}^{21} r_{32}^{32} \\ r_{21}^{12} r_{31}^{13} & -[2] r_{21}^{12} r_{31}^{31} \end{pmatrix} = \begin{pmatrix} -\frac{e^{i\pi/15}}{2 \cos \frac{\pi}{5}} & \frac{e^{-i\pi/3}}{2 \cos \frac{\pi}{5}} \\ \frac{e^{-8i\pi/15}}{2 \cos \frac{\pi}{5}} & -e^{-i\pi/15} \end{pmatrix} \tag{6.7}
\end{aligned}$$

It is straightforward to work out the braiding for the other chiral blocks along the same lines as was done in this example.

APPENDIX

In this Appendix we collect several identities between products of elements of the fundamental braiding matrix $r_{\mu\nu}^{\rho\sigma}$ as given in (3.12). Since $[x]$ is a ratio of two sine-functions, the following identities are easily shown using standard trigonometric identities.

Most of the time, we do not write the argument l of $r_{\mu\nu}^{\rho\sigma}(l)$ explicitly. In these cases the argument is assumed to take on the same value in all r -matrix elements occurring in a given identity. One has:

$$r_{\mu\nu}^{\mu\nu} + r_{\mu\nu}^{\nu\mu} = q^{\frac{1}{2}}\lambda_1^2 \quad (A.1)$$

$$r_{\nu\rho}^{\nu\rho}(r_{\mu\nu}^{\nu\mu} - r_{\mu\rho}^{\rho\mu}) = r_{\mu\rho}^{\mu\rho} r_{\nu\mu}^{\nu\mu} \quad (A.2)$$

$$r_{\mu\nu}^{\nu\mu}(r_{\rho\mu}^{\mu\rho} - r_{\rho\nu}^{\nu\rho}) + r_{\rho\nu}^{\nu\rho} r_{\mu\rho}^{\rho\mu} = -q^{\lambda_1^2-1} \quad (A.3)$$

In the following, we fix the index ρ , and a bar over the r means that one should replace the argument l by

$$l_i \rightarrow l_i + \delta_{i\rho} - \delta_{i,\rho-1}$$

$$\Rightarrow \bar{r}_{\mu\rho}^{\rho\mu}(l_{\mu\rho}) = r_{\mu\rho}^{\rho\mu}(l_{\mu\rho} + 1) \quad , \quad \bar{r}_{\rho\mu}^{\rho\mu}(l_{\rho\mu}) = r_{\rho\mu}^{\rho\mu}(l_{\rho\mu} - 1) \quad , \quad \text{etc.} \quad (A.4)$$

Then one easily shows the following identities:

$$r_{\mu\rho}^{\mu\rho} \bar{r}_{\rho\mu}^{\mu\rho} = r_{\rho\rho}^{\rho\rho} \bar{r}_{\mu\rho}^{\mu\rho} \quad (A.5)$$

$$r_{\mu\rho}^{\mu\rho} \bar{r}_{\rho\mu}^{\rho\mu} = r_{\rho\rho}^{\rho\rho} (\bar{r}_{\mu\rho}^{\rho\mu} - r_{\mu\rho}^{\rho\mu}) \quad (A.6)$$

$$\bar{r}_{\rho\nu}^{\rho\nu} (\bar{r}_{\mu\rho}^{\rho\mu} - r_{\mu\nu}^{\nu\mu}) = r_{\mu\nu}^{\mu\nu} \bar{r}_{\rho\mu}^{\rho\mu} \quad (A.7)$$

We also need

$$r_{\mu\rho}^{\mu\rho} \bar{r}_{\rho\mu}^{\rho\mu} (r_{\nu\mu}^{\mu\nu} - \bar{r}_{\nu\rho}^{\rho\nu}) = r_{\nu\rho}^{\nu\rho} r_{\mu\nu}^{\mu\nu} \bar{r}_{\rho\nu}^{\rho\nu} \quad (A.8)$$

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FIGURE CAPTIONS

1. The integration contours for the screened vertex operators (2.15) are depicted. The dashed lines give the paths of analytic continuation for the integration variables from the reference configuration on the real axis.
2. The integration contours $\Gamma_1, \dots, \Gamma_5$ needed in the evaluation of (4.13) are depicted.

figure 1

figure 2