SPATIALLY FLAT QUANTUM COSMOLOGY

Jorma Louko*

*Department of Theoretical Physics
University of Helsinki
Siltavuorenpenger 20 C
SF-00170 Helsinki, Finland*

and

Peter J. Ruback**

*CERN
1211 Geneva 23, Switzerland*

Abstract

We investigate the quantum cosmology of spatially homogeneous cosmological models with flat compact orientable spatial sections in three and four space-time dimensions. In three dimensions the only possible spatial topology is the two-torus, whereas in four dimensions there are six possible spatial topologies. The metric ansatz is kept in its most general form compatible with Hamiltonian minisuperspace dynamics. We find that the Hartle-Hawking proposal is applicable, at the semiclassical level, in the three-dimensional model and in two of the four-dimensional models. In each case the Hartle-Hawking wave function has a countable infinity of possible semiclassical contributions, all of which can be generated by operating on a finite subset by the group of spatial coordinate transformations. For a positive cosmological constant, the Hartle-Hawking wave function tends to avoid the singularities of the Lorentzian solutions in the three-dimensional model, and completely succeeds in avoiding them in the four dimensional models.

* Bitnet address: Louko@FINUHCB. Address after September 1, 1989: Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alberta, T6G 2J1, Canada

** Bitnet address: Ruback@CERNVM

1. INTRODUCTION

Over the last few years, much work in the path integral approach to quantum cosmology has been done in the context of the boundary condition proposal by Hartle and Hawking [1-3]. Combining ideas of canonical and path integral quantisation of general relativity, this proposal defines the wave function of the Universe by the Euclidean path integral

$$\Psi_{HH}(h_{ij}, \phi) = \int d[g_{\mu\nu}] d[\phi] e^{-I} \quad (1.1)$$

where I is the Euclidean action of the gravitational field $g_{\mu\nu}$ and the matter fields ϕ . The integration is taken to extend over Euclidean metrics $g_{\mu\nu}$ and matter fields ϕ on compact four-manifolds with a boundary, such that the induced metric and the values of the matter fields on the boundary are given by the arguments of the wave function. Reviews of the appealing properties of this proposal are given in Ref. [4].

By general properties of path integrals, one expects Ψ_{HH} to take the semiclassical form

$$\Psi_{HH}(h_{ij}, \phi) \approx \sum_n P_n(h_{ij}, \phi) \exp[-I_n^c(h_{ij}, \phi)] \quad (1.2)$$

where $I_n^c(h_{ij}, \phi)$ are the actions of the (possibly complex-valued) classical solutions satisfying the boundary conditions of the integral, the prefactors $P_n(h_{ij}, \phi)$ are slowly varying compared with the exponentials, and n is an index labelling the different classical solutions. As Ψ_{HH} is constructed to satisfy the quantum constraint equations arising from the Dirac quantisation of general relativity, most importantly the Wheeler-DeWitt equation [5], one can get some information about the functional forms of the prefactors $P_n(h_{ij}, \phi)$ from a semiclassical expansion of the Wheeler-DeWitt equation. But one would need to know more about the definition of the path integral in order to determine the relative weights of the P 's. For example, in some simple models it can be explicitly demonstrated that different choices for the contour of integration lead into drastically different wave functions, corresponding to different weights of the P 's in (1.2) [6]. Knowing just the exponential

factors in (1.2) is therefore not sufficient for making a firm statement about the predictions of the Hartle-Hawking (HH) proposal. One would need a careful definition of the path integral which would enable one to estimate also the semiclassical prefactors.

In situations where one is able to find the exponential factors in (1.2) but the definition of the path integral remains problematic, one's possibilities of making predictions are more limited. When interpreting the wave function of the form (1.2) in terms of classical space-times, one usually argues that the different semiclassical components may be treated separately, at least if the I_n^c are sufficiently different from each other [7,8] (see also Ref. [9]). Adopting this view, one can separately find the classical space-times corresponding to each of the exponentials in (1.2) and then regard these space-times as predictions modulo the ignorance about the P 's. More cautiously, one would at least expect that knowing the general semiclassical form of the wave function could give some insight into the issue of giving a proper definition of the integral.

In this paper we shall investigate the exponential factors in the semiclassical III wave function (1.2) in three- and four-dimensional spatially homogeneous cosmological models with flat spatial sections. The Lorentzian/Euclidean metric ansatz in these models is given by

$$ds^2 = \mp N^2(t) dt^2 + h_{ij}(t) dx^i dx^j \quad (1.3)$$

where $\{x^i\}$ are a set of local spatial coordinates. We shall take the cosmological constant to be positive, but we shall not include matter. It is well known that every classical solution in these models can, by a suitable choice of the local coordinates $\{x^i\}$, be brought into a form in which $h_{ij}(t)$ is diagonal, and from the point of view of the local dynamics of the classical solutions it is therefore no loss of generality to set $h_{ij}(t)$ diagonal already in the ansatz [10]. If the spatial slices are chosen to have the topology of $\mathbb{R}^2$ ($\mathbb{R}^3$, respectively), the coordinates $\{x^i\}$ give a global coordinate system on these slices, and in this case making h_{ij} diagonal does not restrict even the global properties of the classical solutions in any way. However, if one wishes to apply the III proposal the spatial slices must be compact, and different local coordinate systems are then in general distinguishable from each other by the associated global properties [11]. The special case where the spatial slices

are tori and h_{ij} is taken diagonal in a basis corresponding to the natural angle coordinates on the torus is discussed in Refs. [12–14]. Our aim here is to investigate the most general case with compact orientable spatial slices. The assumption of orientability is needed for unambiguous integration over the spatial slices and thus for recovering an unambiguous definition of the Einstein action.

In Section 2 we investigate the case of three space-time dimensions. It is well known that all Lorentzian solutions of three-dimensional Einstein gravity are locally flat, de Sitter or anti-de Sitter, depending on the value of the cosmological constant, and the classical theory is in this sense locally trivial [15]. The global structure of the solutions is, however, highly non-trivial [16]. With spatial flatness, the assumptions of compactness and orientability fix the spatial slices to have the topology of the two-torus [11]. We exhibit first all such Lorentzian solutions and analyse in some detail their global properties and singularity structure. We then find the Euclidean solutions with the III boundary data and evaluate the possible exponential contributions to the semiclassical IIII wave function. The III wave function is found to correspond to those Lorentzian solutions whose singularities are in a sense the mildest that any Lorentzian solution in the model may have.

In Sections 3 and 4 we turn to the more interesting case of four space-time dimensions. Here both the local and global properties of the Lorentzian solutions are non-trivial, and one therefore expects the III wave function to give predictions about both local and global properties of these space-times. Also the issue of the spatial topologies is more complicated than in three dimensions, as there now exist six different topologies compatible with spatial flatness, compactness and orientability [11]. It turns out, however, that only two of these topologies admit classical Euclidean solutions satisfying the III boundary data. We shall find that for the two admissible topologies the III wave function corresponds to those classical space-times that have vanishing anisotropy in the comoving coordinate system given by the ansatz (1.3). However, the wave function ceases to correspond to these space-times near their singularities. We can therefore say that these space-times fade quantum mechanically out of existence near the region where the singularity classically should be. The III proposal avoids thus all the singularities of the classical Lorentzian solutions.

Section 5 contains conclusions and a discussion. We discuss in some detail the fact that the sum in the semiclassical wave function contains a countable infinity of terms, both in three and four dimensions. The origin of this infinite sum is not dynamical but purely kinematical: all these terms can be generated by operating on a finite subset by the group of spatial coordinate transformations. We also comment on some aspects of our results which may be of interest in view of defining the III integral beyond minisuperspace models.

The global structure of the Lorentzian solutions in the three-dimensional model is analysed in Appendix A. Appendices B, C and D contain results needed for investigating the regularity of the Euclidean solutions.

2. 2+1-DIMENSIONAL MODEL

In this section we shall work with the spatially flat ansatz (1.3) in three space-time dimensions. As mentioned in the Introduction, the assumptions of spatial compactness and orientability fix the spatial slices to be two-tori. We can therefore choose the coordinates $\{x^i\} = \{x, y\}$ to be a pair of natural angle coordinates on the torus, each identified periodically with period 2π . Note that here and from now on we refer to variables with global identifications loosely as “coordinates,” trusting that the meaning be clear from the context.

We can adopt a parametrisation of h_{ij} in terms of three independent variables $\{q^a\}$, insert the ansatz into the 2+1 dimensional Einstein action [17]

$$S = \frac{1}{16\pi G} \int_M d^3x \sqrt{-g} (R - 2\Lambda) + \frac{1}{8\pi G} \int_{\partial M} d^2x \sqrt{h} K \quad (2.1)$$

and integrate over the spatial slices. The resulting minisuperspace action can be written in its Hamiltonian form as

$$S = \int dt (p_a \dot{q}^a - N\mathcal{H}) \quad , \quad (2.2)$$

where the superHamiltonian $\mathcal{H}$ is a function of the p 's, the q 's and Λ . This minisuperspace action reproduces correctly the Einstein equations, and we can therefore take it as the starting point for the minisuperspace analysis. If we choose the q 's to be just the metric components h_{xx} , h_{yy} and h_{xy} , the explicit form of $\mathcal{H}$ is given by

$$\mathcal{H} = \frac{2G}{\pi} \sqrt{h} (p_{xy}^2 - 4p_{xx}p_{yy}) + \frac{\pi}{2G} \Lambda \sqrt{h} \quad (2.3)$$

where p_{xx} , p_{yy} and p_{xy} are the conjugate momenta of h_{xx} , h_{yy} and h_{xy} , respectively, and $h = \det(h_{ij}) = h_{xx}h_{yy} - h_{xy}^2$.

To find the Lorentzian solutions, one could vary the action (2.2), write down the equations of motion, and solve them in a straightforward way. It is, however, known that every solution can be expressed in a local spatial coordinate system in which h_{ij} is diagonal [10]. We can therefore directly write the general non-diagonal solution by combining the general diagonal solution given, for example, in Ref. [14] to a linear time-independent transformation of the spatial basis forms $\{dx, dy\}$.

It is convenient to divide the solutions into two qualitatively different classes. The first class are the solutions which are spatially anisotropic, in the sense that the world lines comoving in the coordinate system of the ansatz have non-vanishing shear. The general form of these solutions in the gauge $N = 1$ is

$$ds^2 = -dt^2 + \frac{1}{\Lambda} \cosh^2(\sqrt{\Lambda}t) (\bar{A}dx + \bar{B}dy)^2 + \frac{1}{\Lambda} \sinh^2(\sqrt{\Lambda}t) (\bar{C}dx + \bar{D}dy)^2 \quad (2.4)$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are four constants of integration. The second class are the isotropic solutions, whose general form in the gauge $N = 1$ is

$$ds^2 = -dt^2 + e^{2\sqrt{\Lambda}t} h_{ij}(0) dx^i dx^j \quad (2.5)$$

where $h_{ij}(0)$ are constants of integration. There are just two essential constants in $h_{ij}(0)$, as $\det[h_{ij}(0)]$ can be scaled away by changing the zero-point of t . The isotropic solutions can be thought of as a singular limiting case of the anisotropic ones.

As mentioned in the Introduction, it is known from general properties of three dimensional Einstein gravity that the solutions (2.4) and (2.5) must be locally diffeomorphic to three-dimensional de Sitter space. We shall show in Appendix A that these solutions cannot be analytically extended to cover full de Sitter space. In particular, the apparent singularities of these solutions, at $t = 0$ for (2.4) and at $t = -\infty$ for (2.5), always describe a region of space-time where something in the geometry becomes singular. The nature of these singularities depends on the constants of integration in a way discussed in Appendix A.

We would now like to find the possible exponential contributions to the HII wave function. For this, we must find the compact Euclidean solutions which match with a given two-metric on their boundary. We take the boundary to be connected, as usual, so that the boundary data consists of just one set of the two-metric components h_{ij} .

The general Euclidean classical solution is given in the gauge $N = 1$ by

$$ds^2 = dt^2 + \frac{1}{\Lambda} \sin^2(\sqrt{\Lambda}t) (A dx + B dy)^2 + \frac{1}{\Lambda} \cos^2(\sqrt{\Lambda}t) (C dx + D dy)^2 \quad (2.6)$$

where A , B , C and D are constants of integration. The "bottom" of the three-geometry may be chosen to be at $t = 0$. We must now determine the constants in (2.6) such that the metric can be extended regularly to $t = 0$ and that the two-metric components on some final slice $t = t_f$ match with the given boundary data.

Let us consider first the regularity. We have the following theorem.

Theorem 1. The metric (2.6) with $0 < t < \pi/(2\sqrt{\Lambda})$ and with real A , B , C and D can be extended to $t = 0$ in a regular way if and only if there exist coprime integers (m, n) such that

$$Am + Bn = 1 \quad (2.7a)$$

$$Cm + Dn = 0 \quad (2.7b)$$

Here by integers we understand both positive, negative and zero. If one of the integers is zero, we understand coprime to imply that the remaining one equals ± 1 .

The proof of this theorem is given in Appendix B.

With the help of Theorem 1, it is straightforward to find the classical solutions with the III boundary data. Given coprime integers (m, n) and imposing (2.7), one can solve for A , B , C , D and t_f from the condition that the metric components on the slice $t = t_f$ match with the given h_{ij} . One can then insert this solution into the Euclidean Einstein action

$$I = \frac{1}{16\pi G} \int_M d^3x \sqrt{g} (-R + 2\Lambda) - \frac{1}{8\pi G} \int_{\partial M} d^2x \sqrt{h} K \quad (2.8)$$

and evaluate the classical action as a function of the boundary data. The result is

$$I_{(m,n)}^c = -\frac{\pi}{2G} \sqrt{\frac{(1 - \Lambda(m^2 h_{xx} + n^2 h_{yy} + 2mn h_{xy}))h}{m^2 h_{xx} + n^2 h_{yy} + 2mn h_{xy}}} \quad (2.9)$$

For given boundary data, there are thus a countable infinity of classical solutions, labelled by the ordered pairs (m, n) of coprime integers. The only degeneracy in this labelling is that the pairs (m, n) and $(-m, -n)$ give the same solution and must thus be identified.

Let us now fix (m, n) and consider the corresponding contribution to the semiclassical III wave function. For $\Lambda(m^2 h_{xx} + n^2 h_{yy} + 2mn h_{xy}) < 1$, $I_{(m,n)}^c$ given by (2.9) is real and comes from a real-valued solution of the Euclidean Einstein equations. We have chosen the sign of the square root so that (2.9) agrees with (2.8) in the naïve sense of a real Euclidean metric. If the integral is defined by a non-trivial contour integral over complex metrics, it might be possible to obtain instead a contribution with the opposite sign of the square root [6,18]; this would come from the same real-valued solution but with the Euclidean action defined by the “wrong” sign Wick rotation. In either case, the resulting $I_{(m,n)}^c$ is real and does not give a rapidly oscillating contribution to the wave function.

For $\Lambda(m^2 h_{xx} + n^2 h_{yy} + 2mn h_{xy}) > 1$, the quantity under the square root in (2.9) becomes negative, and one obtains a complex conjugate pair of values for $I_{(m,n)}^c$. Although these values are purely imaginary, they do not come from purely real Lorentzian solutions. In terms of the coordinates used in (2.6), the solutions have real values of A and B ,

purely imaginary values of C and D , and t ranging from $t = 0$ to a complex t_f with $\text{Re}(t_f) = \pi/(2\sqrt{\Lambda})$. One can rewrite these solutions in terms of a real-valued “time” coordinate τ defined by $t = t_f \tau$ and verify that these solutions are complex-valued non-degenerate metrics on a real compact manifold. We therefore see that the conditions (2.7), which we derived as regularity conditions for real Euclidean metrics, do lead to metrics with the desired properties also when the solutions are complex.

For $\Lambda(m^2 h_{xx} + n^2 h_{yy} + 2mn h_{xy}) = 1$ there are no solutions, real or complex, satisfying the III boundary data. The value obtained for $I_{(m,n)}^c$ from (2.9) is a singular limit case between the domains of real and complex solutions.

We can now find the Lorentzian space-times corresponding to these components of the wave function. We follow the interpretation presented, for example, in Refs. [3,19]. The straightforward way to find these space-times would be to solve the Hamilton-Jacobi equations of motion

$$p_\alpha = \frac{\partial S}{\partial q^\alpha} \quad (2.10a)$$

$$\dot{q}^\alpha = N \frac{\partial \mathcal{H}}{\partial p_\alpha} \quad (2.10b)$$

for the particular Hamilton-Jacobi function obtained from (2.9),

$$S = \pm \frac{\pi}{2G} \sqrt{\frac{(\Lambda(m^2 h_{xx} + n^2 h_{yy} + 2mn h_{xy}) - 1)h}{m^2 h_{xx} + n^2 h_{yy} + 2mn h_{xy}}} \quad (2.11)$$

It is, however, possible to use a short-cut which is essentially similar to the argument sometimes used at this point in the de Sitter minisuperspace model [2,6]. As explained above, the classical solutions giving the purely imaginary actions are not purely Lorentzian but complex. A direct substitution of these solutions in the Einstein action (2.8), integration over the spatial slices and an integration by parts gives the classical actions in the form

$$I = \int_\gamma dt f(t) + \kappa \quad (2.12)$$

where f is a complex analytic function of the Euclidean proper time t , the contour γ is a straight line in the complex t plane from $t = 0$ to $t = t_f$, and κ is a boundary term from $t = 0$ [20]. Recalling that $\text{Re}(t_f) = \pi/(2\sqrt{\Lambda})$, we can deform the contour and break the integral into two parts as

$$I = I_1 + I_2 \quad (2.13)$$

$$I_1 = \int_0^{\frac{t_f}{2\sqrt{\Lambda}}} dt f(t) + \kappa \quad (2.14a)$$

$$I_2 = \int_{\frac{t_f}{2\sqrt{\Lambda}}}^{t_f} dt f(t) \quad , \quad (2.14b)$$

where the first integral is taken along the real t axis and the second one parallel to the imaginary t axis. An explicit calculation shows that I_1 vanishes. I_2 , on the other hand, is purely imaginary, and comparison of (2.4) and (2.6) shows that $\pm iI_2$ is equal to the Lorentzian action of a purely Lorentzian solution of the form (2.4) satisfying the conditions

$$\bar{A}m + \bar{B}n = 1 \quad (2.15a)$$

$$\bar{C}m + \bar{D}n = 0 \quad , \quad (2.15b)$$

having a vanishing initial $\det(h_{ij})$, and having the same final values of h_{ij} as the complex solution. One can further verify that the two-parameter family of solutions given by (2.4) with (2.15) intersects the surface $\det(h_{ij}) = 0$ on a curve which is orthogonal to this family of trajectories in minisuperspace. Here orthogonality is defined in terms of the DeWitt metric whose inverse appears in the kinetic term in (2.3). It then follows from the general Hamilton-Jacobi theory that the solutions of equations (2.10)–(2.11) are just the two-parameter family of space-times given by (2.4) with (2.15). Both signs in (2.11) give essentially the same family of solutions, the only difference being in the direction of the coordinate time.

We have thus shown that the component in the wave function coming from fixed (m, n) corresponds to the two-parameter family of Lorentzian space-times given by (2.4) with the conditions (2.15). Although the conditions (2.15) depend explicitly on (m, n) , the resulting two-parameter set of space-times is in fact the same for all (m, n) . To see

this, we make in (2.4) a change of spatial coordinates given by equations (B4)–(B5) in Appendix B. The new spatial coordinates (u, v) are periodic with period 2π , and the family of solutions given by (2.4) with (2.15) reduces in the coordinates (t, u, v) into the form

$$ds^2 = -dt^2 + \frac{1}{\Lambda} \cosh^2(\sqrt{\Lambda}t) (du + \bar{\alpha}dv)^2 + \frac{\bar{\beta}^2}{\Lambda} \sinh^2(\sqrt{\Lambda}t) dv^2 \quad (2.16)$$

where $\bar{\alpha}$ and $\bar{\beta}$ are two independent parameters. In this form, this family of space-times is manifestly independent of the integers (m, n) .

The apparent singularity of the solutions (2.16) at $t = 0$ is a topological singularity. As discussed in Appendix A, these solutions can be regarded as the least singular of all the Lorentzian solutions in the model.

3. 3+1-DIMENSIONAL MODEL WITH TOROIDAL TOPOLOGY

We turn now to the case of four space-time dimensions. As mentioned in the Introduction, the assumptions of spatial compactness and orientability allow now six different choices for the spatial topology. In this section we make the most obvious choice and take the spatial slices to be three-tori. The analysis will then be closely analogous to that in the three-dimensional model above. The remaining five topologies will be discussed in the next section.

Following the analysis of the three-dimensional model, we choose the coordinates $\{x^i\} = \{x, y, z\}$ to be a set of natural angle coordinates on the three-torus, each identified periodically with period 2π . One can parametrise h_{ij} in terms of six independent variables $\{q^a\}$, insert the ansatz into the 3+1-dimensional Einstein action [17]

$$S = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g} (R - 2\Lambda) + \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{h} K \quad , \quad (3.1)$$

integrate over the spatial slices and write the resulting minisuperspace action in its Hamiltonian form (2.2). This action will correctly reproduce the Einstein equations [10]. The

explicit form of the minisuperspace action in a particular parametrisation can be found, for example, in Ref. [21].

To find the general Lorentzian solutions, we can again combine the general diagonal solutions to a time-independent linear transformation of the spatial basis $\{dx, dy, dz\}$ [10]. As in the three-dimensional case, also here it is convenient to divide the solutions into the anisotropic and isotropic ones.

The general anisotropic solutions can be found as generalisations of the diagonal anisotropic solutions given in Ref. [22]. These general anisotropic solutions join an asymptotically locally de Sitter-like region to an asymptotically locally Kasner-like region. When the asymptotic Kasner exponents are not a permutation of $(1, 0, 0)$, the solution has a standard Kasner curvature singularity at vanishing three-volume. When the asymptotic Kasner exponents are a permutation of $(1, 0, 0)$, on the other hand, the local curvature remains regular at vanishing three-volume. If the spatial topology were $\mathbb{R}^3$, the surface with vanishing three-volume would be a null three-surface analogous to the Schwarzschild horizon, and one could continue the space-time past this horizon in a regular way. The maximal analytic extension would have a curvature singularity on the other side of the horizon, and the Penrose-Carter diagram with two spatial dimensions suppressed would be similar to that in the locally rotationally symmetric Bianchi type III model with uncompactified spatial topology [23]. However, as our spatial topology is that of a torus, we now have a topological singularity at the surface of vanishing three-volume. Furthermore, one cannot in general draw a two-dimensional Penrose-Carter diagram of the resulting non-manifold in a way which is possible in the compactified Bianchi type III model, as the periodic identifications would in general mix the two dimensions shown in the Penrose-Carter diagram with the two suppressed dimensions.

The general isotropic solutions are given by the four-dimensional version of (2.5). In the gauge $N = 1$ they take the form

$$ds^2 = -dt^2 + e^{2\sqrt{\Lambda/3}t} h_{ij}(0) dx^i dx^j \quad (3.2)$$

where $h_{ij}(0)$ are constants of the integration. There are five essential constants, as $\det[h_{ij}(0)]$ can be scaled away by changing the zero-point of t . These solutions are locally diffeomorphic to four-dimensional de Sitter space, and the global structure is analogous to that of the isotropic solutions in the three-dimensional model. In particular, the apparent singularity at $t = -\infty$ is a topological singularity.

We wish now to find the Euclidean solutions which give the possible exponential contributions to the semiclassical HH wave function. We shall again take the boundary to be connected, so that the boundary data consists of one set of the three-metric components h_{ij} .

The general Euclidean classical solution is in the gauge $N = 1$ given by

$$ds^2 = dt^2 + \frac{4}{3\Lambda} \sin^{2p_1} \left(\frac{\sqrt{3\Lambda}}{2} t \right) \cos^{\frac{4}{3}-2p_1} \left(\frac{\sqrt{3\Lambda}}{2} t \right) (Adx + Bdy + Cdz)^2 \\ + \frac{4}{3\Lambda} \sin^{2p_2} \left(\frac{\sqrt{3\Lambda}}{2} t \right) \cos^{\frac{4}{3}-2p_2} \left(\frac{\sqrt{3\Lambda}}{2} t \right) (Ddx + Edy + Fdz)^2 \\ + \frac{4}{3\Lambda} \sin^{2p_3} \left(\frac{\sqrt{3\Lambda}}{2} t \right) \cos^{\frac{4}{3}-2p_3} \left(\frac{\sqrt{3\Lambda}}{2} t \right) (Hdx + Jdy + Kdz)^2 \quad (3.3)$$

where (p_1, p_2, p_3) and $A, \dots, K$ are constants, the p 's satisfying the Kasner relations

$$1 = p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 \quad (3.4)$$

Without loss of generality, we can take the regular "bottom" of the geometry to be at $t = 0$. Near $t = 0$, the solution (3.3) is asymptotically the Euclidean Kasner solution and (p_1, p_2, p_3) are just the Kasner exponents. Demanding that there be no curvature singularity at $t = 0$ therefore tells us that the p 's must be a permutation of $(1, 0, 0)$, and without loss of generality we can take $p_1 = 1, p_2 = p_3 = 0$. The remaining conditions for regularity at $t = 0$ are then given by the following theorem.

Theorem 2. The metric (3.3) with $p_1 = 1$, $p_2 = p_3 = 0$, $0 < t < \pi/\sqrt{3\Lambda}$ and real $A, \dots, K$ can be extended to $t = 0$ in a regular way if and only if there exist coprime integers (m, n, p) such that

$$Am + Bn + Cp = 1 \quad (3.5a)$$

$$Dm + En + Fp = 0 \quad (3.5b)$$

$$Hm + Jn + Kp = 0 \quad (3.5c)$$

Here by integers we understand both positive, negative and zero. If one of the integers is zero, we understand coprime to imply that the remaining two are coprime in the sense of Theorem 1.

The proof of this theorem is similar to that of Theorem 1 and is outlined in Appendix C.

For given (m, n, p) , one can now find the classical solutions which match with the given h_{ij} on the boundary, and one can find their actions as a function of the boundary data by inserting these solutions into the four-dimensional Euclidean Einstein action

$$I = \frac{1}{16\pi G} \int_M d^4x \sqrt{g} (-R + 2\Lambda) - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{h} K \quad (3.6)$$

The result is

$$I_{(m,n,p)}^c = \frac{\pi^2}{2G} \sqrt{\frac{\Lambda h}{3}} \left(T - \frac{3}{T} \right) \quad (3.7)$$

where T is a solution of the equation

$$\frac{T^3}{1 + T^2} = \left(\frac{3\Lambda}{4} \right)^{\frac{3}{2}} (m^2 h_{xx} + n^2 h_{yy} + p^2 h_{zz} + 2mn h_{xy} + 2np h_{yz} + 2pm h_{zx})^{\frac{3}{2}} \quad (3.8)$$

The solutions are thus labelled by the ordered triples of coprime integers (m, n, p) , each triple giving three possible values for T . In analogy with the three-dimensional model, also here the triples (m, n, p) and $(-m, -n, -p)$ give the same Euclidean solutions and must therefore be identified.

Of the three solutions of (3.8) for T , one is always real and positive, corresponding to a real Euclidean metric. The component coming to the wave function from this solution will not be rapidly oscillating and will therefore not correspond to Lorentzian space-times.

The two remaining solutions of (3.8) for T are a complex conjugate pair. As with the three-dimensional model, one can verify that these solutions come from complex-valued metrics on a real compact manifold. When

$$\Lambda(m^2 h_{xx} + n^2 h_{yy} + p^2 h_{zz} + 2mn h_{xy} + 2np h_{yz} + 2pm h_{zx}) \gg 1 \quad (3.9)$$

the complex values for $I_{(m,n,p)}^c$ are asymptotically given by

$$I_{(m,n,p)}^c \approx \pm i \frac{2\pi^2}{G} \sqrt{\frac{\Lambda h}{3}} \quad (3.10)$$

These components of the wave function therefore correspond to those classical space-times that solve the equations (2.10) with the Hamilton-Jacobi function

$$S = \mp \frac{2\pi^2}{G} \sqrt{\frac{\Lambda h}{3}} \quad (3.11)$$

Using the explicit form of the super-Hamiltonian [21], one can verify that these space-times are given by the five-parameter family (3.2) of isotropic solutions. However, when one attempts to follow these solutions toward their singularities at $t = -\infty$, the condition (3.9) used in deriving the HJ function (3.11) eventually becomes invalid. The first correction term to $I_{(m,n,p)}^c$ (3.10) is purely real, and at the limit

$$\Lambda(m^2 h_{xx} + n^2 h_{yy} + p^2 h_{zz} + 2mn h_{xy} + 2np h_{yz} + 2pm h_{zx}) \ll 1 \quad (3.12)$$

the real and imaginary parts of $I_{(m,n,p)}^c$ are proportional to each other. This means that when (3.9) is violated, these components of the wave function do not correspond to classical Lorentzian space-times. We can therefore say that these components of the wave function correspond to the isotropic solutions (3.2) far from the singularity but cease to correspond to these space-times when one approaches the region where the singularity classically should be.

The condition (3.9) characterising the domain of classical Lorentzian space-times depends explicitly on the integers (m, n, p) . However, the restriction which (3.9) imposes on the five-parameter set of classical space-times described by (3.2) is in fact independent of (m, n, p) . This can be seen by making in (3.2) a transformation from (x, y, z) to new spatial coordinates (u, v, w) defined by equation (C2) in Appendix C. Like x, y and z , also the new coordinates u, v and w are periodic with period 2π . One finds that the combination multiplying Λ in (3.9) and (3.12) is just the du^2 component of the spatial metric in the coordinates (u, v, w) . Therefore, for the five-parameter family of space-times described by (3.2), the condition (3.9) has precisely the same content as (for example) the condition

$$\Lambda h_{zz} \ll 1 \quad (3.13)$$

The difference between the two conditions is just a transformation of the spatial coordinates.

4. 3+1 DIMENSIONAL MODEL WITH NON-TOROIDAL TOPOLOGIES

In this section we investigate the five non-toroidal spatial topologies in four space-time dimensions. The metric ansatz for each of these topologies can be obtained from the toroidal ansatz by making further identifications of the angle coordinates $\{x, y, z\}$ and by imposing restrictions of the metric components h_{ij} in the associated basis [11]. The different possibilities are tabulated in the Table. We follow the notation of Ref. [11] and label the topologies from $\mathcal{G}_1$ to $\mathcal{G}_6$, $\mathcal{G}_1$ being the three-torus.

It is straightforward to repeat the analysis of the previous section for the non-toroidal topologies. The general Lorentzian solution is obtained from the general toroidal solution by imposing the restrictions and identifications given in the Table and the global properties of these solutions can be investigated in the corresponding manner. Similarly, the general Euclidean solution is obtained from (3.3) by the restrictions and identifications of the

Table. The issue of finding Euclidean solutions with the III boundary data is more subtle, however. We shall show in Appendix D that the only non-toroidal topology admitting such solutions is $\mathcal{G}_2$. We show further that the III solutions for $\mathcal{G}_2$ are obtained from those of Theorem 2 by the $\mathcal{G}_2$ identification

$$(x, y, z) \sim (-x, -y, z + \pi) \quad (4.1)$$

and setting

$$C = F = H = J = p = 0 \quad (4.2)$$

The classical actions of these solutions are then obtained from (3.7) and (3.8) by setting

$$h_{xx} = h_{yy} = p = 0 \quad (4.3)$$

and multiplying the right-hand side of (3.7) by $\frac{1}{2}$. These actions are thus labelled by ordered pairs of coprime integers (m, n) , each pair giving three possible values, with the degeneracy that the pairs (m, n) and $(-m, -n)$ are identified.

The predictions of the III proposal for Lorentzian $\mathcal{G}_2$ solutions can now be discussed as in the case of the toroidal topology. For given (m, n) , the two oscillating components correspond to the three-parameter family of isotropic Lorentzian solutions when the condition (3.9) (with (4.3)) is satisfied but cease to correspond to these solutions when (3.9) is violated. One can also verify that the predictions for Lorentzian space-times are independent of the pair (m, n) by performing the $p = 0$ version of the coordinate transformation discussed at the end of Section 3.

5. CONCLUSIONS AND DISCUSSION

We have investigated the semiclassical III wave function in three- and four-dimensional spatially homogeneous cosmological models with flat compact orientable spatial sections. In three dimensions the only possible choice for the spatial topology was the two-torus. In four dimensions there were six possible choices for the spatial topology, but it turned out

that the III proposal is applicable at the semiclassical level for only two of them. The predictions of the III wave function for Lorentzian space-times were found to be qualitatively very similar to those in the previously investigated special cases in which the spatial metric is both in three and four dimensions taken to be diagonal in a basis of natural angle coordinates on the torus [12-14]. In our three dimensional model the Lorentzian space-times corresponding to the III wave function have topological singularities, and these singularities can be regarded as the mildest of all possible singularities occurring in the Lorentzian solutions. In our two allowed four-dimensional models the Lorentzian space-times corresponding to the III wave function have vanishing anisotropy, but the wave function ceases to correspond to these space-times near their singularities. In both three and four dimensions, the III proposal therefore tends to eliminate the singularities of the Lorentzian solutions. This is the kind of prediction one would expect to emerge from a proper theory of quantum gravity.

The Euclidean path integral construction in the four-dimensional spatially flat model has been discussed also by Duncan and Jensen [24] in the special case in which the spatial slices are taken to be tori and h_{ij} is taken to be diagonal in a set of natural angle coordinates on the torus. These authors specify the boundary conditions for the integral in terms of some of the minisuperspace coordinates and some of the momenta, and they choose to do this in a way which does not incorporate the compactness condition of the III proposal. The object constructed in Ref. [24] is therefore different from the wave function discussed in this paper and in Refs. [12,13].

In all of this paper, we have taken the cosmological constant to be positive. It is possible to repeat the analysis for a vanishing or negative cosmological constant, and the evaluation of the semiclassical contributions to the III wave function proceeds then in essentially the same way as above. One finds, however, that the resulting wave function is never rapidly oscillating. The III wave function in these spatially flat models for a non-positive cosmological constant is therefore not very interesting from the point of view of Lorentzian space-times.

We saw that some, and indeed the most interesting, contributions to the wave function came from complex-valued solutions of the Euclidean Einstein equations. This may not be surprising in view of the fact that the Euclidean action of general relativity on real-valued metrics is not bounded from below and one would thus expect to define the integral over some non-trivial contour in the space of complex-valued metrics [2,6,18,25-28]. It should be emphasised, however, that the fundamental reason why the possibility of complex-valued extrema arises is not the contour of integration but the boundary conditions of the integral. At the semiclassical level, the contour of integration becomes an issue only when one wishes to discuss which of the possible semiclassical contributions actually do contribute to the integral.

In each of the three models in which the III proposal was applicable, we found a countable infinity of possible semiclassical contributions to the III wave function. The origin of this infinite sum is in the spatial diffeomorphism structure of the models. Consider first the three- and four-dimensional models with toroidal spatial topology. Recall that we fixed the coordinates $\{x^i\}$ in the ansatz (1.3) to be a set of natural angle coordinates on the torus, each periodic with period 2π . There are a countable infinity of such coordinate systems, related to each other by the transformations

$$x^i = M^i_j \tilde{x}^j \quad (5.1)$$

where M^i_j is a 2×2 (3×3 , respectively) matrix with integer entries and unit determinant (we consider only transformations which preserve the orientation) [11]. (For the two-torus, this is known as the modular group.) The spatial metric components h_{ij} and

$$\tilde{h}_{ij} = M^k_i h_{kl} M^l_j \quad (5.2)$$

can therefore be viewed as describing the same geometry in different coordinate systems. Furthermore, if we have a Lorentzian/Euclidean solution $h_{ij}(t)$, we can from this generate a new solution

$$\tilde{h}_{ij}(t) = M^k_i h_{kl}(t) M^l_j \quad (5.3)$$

which can be viewed as just describing the same Lorentzian/Euclidean space-time in a different coordinate system. It is straightforward to verify that the sum over the ordered pairs (triples) of coprime integers in the semiclassical III wave function can be rewritten as fixing the integers but summing over all possible transformations (5.2) of the boundary data. If the semiclassical prefactors are chosen in a suitable way, the effect of this sum is to make the wave function invariant under transformations of the spatial coordinates. For predictions about Lorentzian space-times it is sufficient to consider just one term in this sum, as was explicitly shown in Sections 2 and 3. The situation is essentially similar also in the four dimensional-model with G_2 spatial topology. The only difference there is that the transformation (5.1) is restricted to operate non-trivially only on the coordinates x and y .

It is interesting to contrast the role of the spatial coordinate transformations in our models to that in the spatially homogeneous models with the homogeneity group $SO(3)$ or $SU(2)$, known as Bianchi type IX models. The general form of the Lorentzian metric ansatz for Bianchi type IX is

$$ds^2 = -N^2(t) dt^2 + h_{ij}(t) (\omega^i + N^i(t) dt) (\omega^j + N^j(t) dt) \quad (5.4)$$

where $\{\omega^i\}$ are a set of one-forms satisfying

$$d\omega^i = -\frac{1}{2} \epsilon^i_{jk} \omega^j \wedge \omega^k \quad (5.5)$$

and ϵ^i_{jk} is the permutation symbol [10]. An obvious difference between this ansatz and the spatially flat ansatz (1.3) is that the Bianchi type IX ansatz contains non-vanishing shift functions N^i . In the Hamiltonian form of the action, these shift functions appear as the Lagrange multipliers associated with the constraints generating transformations of h_{ij} by $SO(3)$ matrices [10, 21]. As the most general transformations of the basis $\{\omega^i\}$ that leave the relations (5.5) invariant are just $SO(3)$ transformations, one can regard $SO(3)$ as the spatial coordinate transformation group of this model. One can thus say that the ansatz (5.4) and the corresponding action are invariant under time-dependent spatial coordinate transformations, and one can regard these transformations as a local gauge symmetry.

This has the consequence that the path integral constructed from the Lorentzian action contains an explicit averaging over the spatial coordinate transformations, and the resulting “propagation amplitude” $K(h_{ij}^{(2)}, h_{ij}^{(1)})$ is naturally invariant under spatial coordinate transformations in both its arguments [21].

The III path integral in Bianchi type IX is more problematic than a Lorentzian path integral for at least two reasons. Firstly, a path integral formulated in terms of the Euclideanised spatially homogeneous ansatz is naturally adapted to a situation where one fixes data on two three-surfaces, whereas for the III path integral we would like to fix data on just one connected three-surface. Although some of this apparent incompatibility can be reconciled by a careful analysis of what one expects to happen at the “bottom” of the four-geometry [20], it does not appear possible to write a path integral which would incorporate simultaneously topologically different four-geometries. For example, if one takes the three-surfaces to be the group manifold $SU(2)$, there are two topologically different ways in which the four-geometry may close [29], and the corresponding initial conditions in the path integral would have to be chosen differently for the two cases. Secondly, the Euclidean path integral is not convergent in a naive sense, and one ought to carefully investigate possible contours for defining this integral. In spite of these problems, however, the Euclidean action has the same manifest gauge invariance under time dependent spatial coordinate transformations as the Lorentzian action, and one would therefore expect also the Euclidean path integral to give naturally a wave function which would be invariant under transformations of the spatial coordinates.

In a semiclassical evaluation of the III wave function in Bianchi type IX one could of course find the classical Euclidean solutions in a gauge in which $N^i = 0$ and h_{ij} is diagonal. The expression for the semiclassical III wave function, invariant under spatial coordinate transformations, would then be a sum of six terms related to each other by permutations of the diagonal elements of h_{ij} . The point is, however, that this discrete sum would only appear at the level of the semiclassical approximation but not in the full expression for the wave function.

Now compare this to the spatially flat models. As the shift functions in our ansatz (1.3) are identically zero, this ansatz is clearly not invariant under time dependent transformations of the spatial coordinates. Moreover, if one demands that the dynamics be derivable from a Hamiltonian action with finitely many degrees of freedom, it is not possible to extend the ansatz to include such transformations [10]. For compactified spatial topology this is evident already from the fact that the group of spatial coordinate transformations is not continuous but discrete. Therefore, when one wishes to give in these models a definition of the III path integral such that the resulting wave function be invariant under spatial coordinate transformations, one cannot expect this invariance to emerge from a local gauge symmetry of the action in the way of Bianchi type IX. We saw, however, that the spatially flat ansatz is invariant under time-independent transformations (5.3) of the spatial coordinates, which can be regarded as global gauge symmetries. To recover a wave function invariant under spatial coordinate transformations, it should therefore be possible to explicitly introduce in the definition of the path integral a discrete sum over these transformations. Such a discrete sum may lead to questions of interest when relating the wave function to the Wheeler-DeWitt equation [30,31].

The discussion in this paper has so far been strictly in the context of spatially homogeneous minisuperspace models. We would like to finish by commenting on three aspects of our results that may be of interest in view of the full space-time dynamics.

The first point has to do with the spatial diffeomorphisms. There exist compact manifolds with a boundary such that not every diffeomorphism on the boundary can be extended to a diffeomorphism in the interior. If one considers such manifolds in the III path integral and wishes to recover a wave function invariant under spatial diffeomorphisms, the extendable and nonextendable diffeomorphisms have to be treated in different fashions in the construction of the integral [31]. As an example of such a manifold, let us consider $D^2 \times T^2$ (the product of the disc and the two-torus), whose boundary is the three-torus T^3 . From the discussion of Hartle and Witt [31] it follows that some of the diffeomorphisms of the form (5.2) on the T^3 boundary can be extended to the interior but

others cannot. In the full III integral one would thus expect to treat some of these diffeomorphisms differently from the others. In our 3+1-dimensional minisuperspace model with T^3 spatial topology, however, all the diffeomorphisms (5.2) stood on equal footing at the semiclassical level, and we argued that the same should also hold for a genuine minisuperspace path integral. The resolution of this apparent inconsistency is that although our Euclidean minisuperspace geometries have the topology $D^2 \times T^2$, the coordinate system of the ansatz is singular at the T^2 “bottom” of these manifolds. This means that the time-independent coordinate transformations (5.3), which were essential in the minisuperspace reasoning, cannot as such be interpreted as diffeomorphisms on $D^2 \times T^2$, and the discussion of Hartle and Witt shows that some of them indeed cannot be extended to such diffeomorphisms. We see therefore that the way in which spatial coordinate transformations appear in our models is not directly relevant for what one expects to happen with the diffeomorphisms in the full theory. The underlying reason is that the spatially homogeneous ansatz is based on a slicing which is necessarily singular for III type geometries with a connected boundary.

The second point concerns the ultralocal limit of general relativity, which is obtained by dropping the spatial curvature term in the Hamiltonian form of the action. This limit describes the classical dynamics of general relativity at small time intervals, and it is particularly relevant in analysing certain kinds of space-time singularities [32]. At the ultralocal limit all dynamical couplings between different points on the spatial slices disappear, and the dynamics is like that of a spatially flat minisuperspace model at each point on these slices. The classical theory is completely solvable [33], as is evident in view of our discussion of the spatially flat classical solutions. It is also possible to quantise the theory much like a conventional free field theory, including a Fock space formulation and an explicit construction of the propagator [34–36]. It might therefore seem natural to consider also the implementation of the III proposal in the ultralocal theory. However, in the spatially flat models we saw that applying the III proposal was very much an issue of the global properties of the spatial slices. Although these global properties were not directly determined by the local Hamiltonian dynamics obtained from the minisuperspace action, the situation remained under control by the assumption of spatial homogeneity. In the

ultralocal theory, on the other hand, nothing in the form of the general classical solution gives information about the relation between different points on the spatial slices. To find Euclidean classical solutions with the III boundary data, one would have to restrict the form of the metric by global considerations while still using the ultralocal description of the dynamics. It may be of interest to investigate conditions under which this program could be consistently implemented, but one would expect to encounter here in a more severe guise the issues which led us above to question the relevance of the spatially homogeneous minisuperspace III integral for the III integral in the full theory. The heart of the matter is again that the ultralocal theory is based on slicing the space-time in a Hamiltonian split which is necessarily singular for III type geometries with a connected boundary.

The last point is about the class of geometries to be summed over in the path integral. As general relativity is not expected to be a fundamental theory of gravity, it is likely that boundary condition proposals for the quantum state of the Universe should eventually be formulated within the framework of some underlying more fundamental theory, such as supersstring theory [37] or some kind of a pregeometric theory [38,39]. But even if one sticks to the framework of general relativity, it is possible to consider path integrals where the configurations summed over are not necessarily metrics on manifolds but slightly more general geometrical objects [40,41]. This issue is made pressing by the fact that the standard definition of the III wave function contains a sum over manifolds, and compact four-dimensional manifolds are known not to be classifiable [42]. It has in particular been suggested that the class of geometries summed over in the III proposal could be meaningfully extended to contain also some non-manifolds [41]. Our spatially flat 3+1-dimensional models might serve as a simple arena for investigating the consequences of extensions of this kind. For example, it might be possible to enlarge the III integral so that the wave function for the toroidal topology models would remain unchanged but the topological singularities brought in by the additional identifications for the non-toroidal topologies $\mathcal{G}_2 - \mathcal{G}_6$ would now be regarded as allowed. The predictions for Lorentzian space-times of such an extended III proposal would in these models be perfectly reasonable. On the other hand, one can clearly drastically lose predictive power in these models if one extends the class of allowed geometries too much, for example to contain singularities

of the conical type. Spatially flat models may therefore give insight not only into the topological aspects of the III proposal in its standard form, but also into the possibilities of meaningfully extending this proposal to more general configurations.

ACKNOWLEDGEMENTS

We are grateful to Kristin Schleich for bringing the work of Ref. [41] to our knowledge. J. L. would like to thank Petr Hajicek for stimulating discussions and for his kind hospitality at the University of Bern, where part of this work was carried out. He would also like to thank Malcolm Duncan for an enlightening discussion on Ref. [24].

APPENDIX A. LORENTZIAN SOLUTIONS IN 2+1 DIMENSIONS

In this Appendix we analyse the global properties of the classical Lorentzian solutions in the 2+1-dimensional model. As explained in the main text, these solutions are locally diffeomorphic to 2+1-dimensional de Sitter space, which can be thought of as the hyperboloid

$$\frac{1}{\Lambda} = -U^2 + V^2 + W^2 + Z^2 \quad (A1)$$

embedded in four-dimensional Minkowski space with the metric

$$ds^2 = -dU^2 + dV^2 + dW^2 + dZ^2 \quad (A2)$$

We shall describe how the global properties of the solutions (2.4) and (2.5) can be understood in terms of global identifications on this hyperboloid.

Let us first consider the region $U + V > 0$ of the de Sitter hyperboloid (A1) in the coordinates (T, X, Y) defined by

$$\begin{aligned} U &= \frac{1}{\sqrt{\Lambda}} \sinh(\sqrt{\Lambda}T) + \frac{\sqrt{\Lambda}}{2} e^{\sqrt{\Lambda}T} (X^2 + Y^2) \\ V &= \frac{1}{\sqrt{\Lambda}} \cosh(\sqrt{\Lambda}T) - \frac{\sqrt{\Lambda}}{2} e^{\sqrt{\Lambda}T} (X^2 + Y^2) \\ W &= e^{\sqrt{\Lambda}T} X \\ Z &= e^{\sqrt{\Lambda}T} Y \end{aligned} \quad (A3)$$

In these coordinates the metric takes the standard spatially flat form

$$ds^2 = -dT^2 + e^{2\sqrt{\Lambda}T} (dX^2 + dY^2) \quad (A4)$$

Here X , Y and T take values from $-\infty$ to ∞ . The spatial topology is thus $\mathbb{R}^2$.

The metrics (2.5) with toroidal spatial topology are obtained from (A4) via global identifications generated by two linearly independent combinations of the Killing vector fields $e_1 = \partial/\partial X$ and $e_2 = \partial/\partial Y$. In the coordinates (A1) these vector fields take the form

$$e_1 = \sqrt{\Lambda} \left[W \left(\frac{\partial}{\partial U} - \frac{\partial}{\partial V} \right) + (U + V) \frac{\partial}{\partial W} \right] \quad (A5a)$$

$$e_2 = \sqrt{\Lambda} \left[Z \left(\frac{\partial}{\partial U} - \frac{\partial}{\partial V} \right) + (U + V) \frac{\partial}{\partial Z} \right], \quad (A5b)$$

and it is clear that the expressions (A5) define the analytic extension of e_1 and e_2 from the region $U + V > 0$ to the full de Sitter hyperboloid. Therefore, the maximal analytic extension of the space described by (2.5) is obtained by dividing the full de Sitter hyperboloid by the exponential maps of two linear combinations of e_1 and e_2 . The resulting quotient space is not a manifold, since any linear combination of e_1 and e_2 will vanish on a hyperbola belonging to the surface $U + V = 0$ and the discrete group generated by these exponential maps is thus not discontinuous. Note also that all linear combinations of e_1 and e_2 are spacelike for $U + V \neq 0$ and lightlike for $U + V = 0$. This means that the

quotient space has closed lightlike loops in the topologically singular domain arising from the surface $U + V = 0$.

Let us next consider the region $U + V > 0$ in the coordinates $(\tau, \tilde{u}, \tilde{v})$ defined by

$$\begin{aligned} U &= \frac{1}{\sqrt{\Lambda}} \sinh(\sqrt{\Lambda}\tau) \cosh(\tilde{v}) \\ V &= \frac{1}{\sqrt{\Lambda}} \sinh(\sqrt{\Lambda}\tau) \sinh(\tilde{v}) \\ W &= \frac{1}{\sqrt{\Lambda}} \cosh(\sqrt{\Lambda}\tau) \cos(\tilde{u}) \\ Z &= \frac{1}{\sqrt{\Lambda}} \cosh(\sqrt{\Lambda}\tau) \sin(\tilde{u}) \end{aligned} \quad (A6)$$

In these coordinates the metric takes the form

$$ds^2 = -d\tau^2 + \frac{1}{\Lambda} \cosh^2(\sqrt{\Lambda}\tau) d\tilde{u}^2 + \frac{1}{\Lambda} \sinh^2(\sqrt{\Lambda}\tau) d\tilde{v}^2 \quad (A7)$$

Here $\tau > 0$, $\tilde{u}$ is periodic with period 2π , and $-\infty < \tilde{v} < \infty$.

In the main text we showed that those anisotropic solutions (2.4) for which the relations (2.15) hold can be written in the form (2.16). We see now that the metrics (2.16) can be obtained from (A7) by making a global identification generated by a linear combination of the Killing vector fields $e_3 = \partial/\partial \tilde{v}$ and $e_4 = \partial/\partial \tilde{u}$. In the coordinates (A1) these vector fields take the form

$$e_3 = U \frac{\partial}{\partial V} + V \frac{\partial}{\partial U} \quad (A8a)$$

$$e_4 = W \frac{\partial}{\partial Z} - Z \frac{\partial}{\partial W}, \quad (A8b)$$

and the expressions (A8) clearly define the analytic extension of e_3 and e_4 from the region $U + V > 0$ to the full de Sitter hyperboloid. Therefore, the maximal analytic extension of the space described by (2.16) is obtained by dividing the de Sitter hyperboloid by the

exponential map of a linear combination of e_3 and e_4 . In terms of the parameters $\bar{\alpha}$ and $\bar{\beta}$ in (2.16), this linear combination is given by

$$\xi = 2\pi(\bar{\beta}e_3 + \bar{\alpha}e_4) \quad , \quad (A9)$$

where non-degeneracy of (2.16) implies $\bar{\beta} \neq 0$. The resulting quotient space is not a manifold, since the action of this discrete group is not discontinuous near the circle $U = V = 0$, $W^2 + Z^2 = 1/\Lambda$. Note that if $\bar{\alpha} = 0$, ξ vanishes on this circle. The quotient space has also closed lightlike and timelike loops, as there are regions on the hyperboloid where ξ is respectively lightlike or timelike.

What remains to be studied are the anisotropic solutions (2.4) when the conditions (2.15) do not hold. When (2.15b) holds but (2.15a) does not, we can still obtain the metric from a metric of the form (A7) by making a global identification generated by ξ (A9), but with the difference that the coordinate $\bar{u}$ in (A7) is now periodic with a period different from 2π . From the coordinate transformations (A6) one then sees that the resulting space cannot be obtained from de Sitter space by global identifications and the singularity at $t = 0$ is analogous to a conical singularity. Finally, if (2.15b) does not hold, the singularity structure is yet more complicated. To obtain the metric from a metric of the form (A7), the coordinate $\bar{u}$ must be not periodic but fully infinite, and one must divide by identifications generated by two linearly independent combinations of $\partial/\partial\bar{u}$ and $\partial/\partial\bar{v}$ such that both combinations have a non-vanishing component from $\partial/\partial\bar{v}$. The singularity at $t = 0$ can be compared to that of the Riemann sheet of the logarithm at the origin.

APPENDIX B. PROOF OF THEOREM 1

(i) Necessity.

Suppose the metric (2.6) can be extended to $t = 0$ in a regular way. As non-degeneracy of the metric implies $AD - BC \neq 0$, one can define local coordinates (θ, z)

by

$$d\theta = Adx + Bdy \quad (B1a)$$

$$dz = Cdx + Ddy \quad (B1b)$$

In the coordinates (t, θ, z) , the metric takes the form

$$ds^2 = dt^2 + \frac{1}{\Lambda} \sin^2(\sqrt{\Lambda}t) d\theta^2 + \frac{1}{\Lambda} \cos^2(\sqrt{\Lambda}t) dz^2 \quad (B2)$$

It is straightforward to verify that the two dimensional submanifolds on which $dz = 0$ are totally geodesic, i.e., every geodesic on these submanifolds in the induced metric

$$ds_{ind}^2 = dt^2 + \frac{1}{\Lambda} \sin^2(\sqrt{\Lambda}t) d\theta^2 \quad (B3)$$

is also a geodesic on the three-dimensional manifold in the metric (B2). As (B2) can be regularly extended to $t = 0$ by assumption, it follows that also the induced metric (B3) must be regularly extendable to $t = 0$. We shall show that regularity of (B3) at $t = 0$ implies (2.7).

Let Γ be a two-dimensional submanifold on which $dz = 0$. The constant t slices of Γ are clearly geodesics on the two-torus. If these geodesics do not close, one can extend the coordinate θ defined locally by (B1a) into a global coordinate on Γ , with $-\infty < \theta < \infty$. In this case the induced metric (B3) cannot be extended regularly to $t = 0$. Hence the constant t slices of Γ must be closed loops, which implies (2.7b).

If (2.7b) is satisfied, the constant t slices of Γ close after wrapping m times around the x -direction and n times around the y -direction. This means that the coordinate θ defined locally by (B1a) must be identified periodically with period $2\pi|Am + Bn|$. By regularity of (B3) at $t = 0$, this period must be equal to 2π , which implies $Am + Bn = \pm 1$, and one can choose the upper sign by changing (m, n) to $(-m, -n)$ if necessary. This implies (2.7a).

(ii) Sufficiency.

Suppose (2.7) holds. Let M be a 2×2 matrix given by

$$M = \begin{pmatrix} m & p \\ n & q \end{pmatrix} \quad (B4)$$

where p and q are integers and $\det M = 1$ (such matrices exist by Euclid's algorithm). Define new coordinates (u, v) by

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} u \\ v \end{pmatrix} \quad (B5)$$

As x and y are identified by 2π , it follows that also u and v are identified by 2π . In the coordinates (t, u, v) the metric (2.6) takes the form

$$ds^2 = dt^2 + \frac{1}{\Lambda} \sin^2(\sqrt{\Lambda}t) (du + \alpha dv)^2 + \frac{\beta^2}{\Lambda} \cos^2(\sqrt{\Lambda}t) dv^2 \quad (B6)$$

where

$$\alpha = Ap + qB \quad (B7a)$$

$$\beta = Cp + qD \quad (B7b)$$

Define new coordinates (U, V) by

$$U = t \cos u \quad (B8a)$$

$$V = t \sin u \quad (B8b)$$

In the coordinates (U, V, v) the metric is manifestly regular at $U = V = 0$. This proves the claim.

APPENDIX C. PROOF OF THEOREM 2

The proof of Theorem 2 is similar to that of Theorem 1 in Appendix B. Necessity of the conditions (3.5) follows from regularity of the the totally geodesic two-dimensional submanifolds on which

$$Ddx + E dy + F dz = H dx + J dy + K dz = 0 \quad (C1)$$

To show the sufficiency of (3.5), we make a coordinate transformation by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad (C2)$$

where M is a 3×3 matrix with integer entries, unit determinant and first column $(m, n, p)^T$. (The existence of such matrices can be shown by Euclid's algorithm.) As x, y and z are periodic with period 2π , also u, v and w are periodic with period 2π . Finally we make the coordinate transformation given by (B8) and express the metric in the coordinates (U, V, w) . The resulting metric is manifestly regular at $U = V = 0$. This proves the claim.

APPENDIX D. EUCLIDEAN SOLUTIONS WITH NON-TOROIDAL SPATIAL TOPOLOGIES

In this Appendix we examine Euclidean solutions with the III boundary data in the $3+1$ -dimensional model with the non-toroidal topologies $\mathcal{G}_2 - \mathcal{G}_6$. The general Euclidean solution for these topologies is obtained from the general toroidal solution (3.3) by the identifications and restrictions displayed in the Table. The bottom of the four-geometry may be chosen to be at $t = 0$. Absence of a curvature singularity at $t = 0$ implies that the asymptotic Kasner exponents (p_1, p_2, p_3) be a permutation of $(1, 0, 0)$, and we can without loss of generality take $p_1 = 1, p_2 = p_3 = 0$. We must now examine whether it is

possible to choose the remaining constants such that the metric can be extended to $t = 0$ in a regular way.

Consider first $\mathcal{G}_2$. There are two different ways of imposing on h_{ij} the restrictions of the Table. The first possibility is to set

$$A = B = F = K = 0 \quad (D1)$$

With this choice it is not possible to extend the metric to $t = 0$ in a regular way. To see this, consider the totally geodesic two-dimensional submanifolds on which $dx = dy = 0$. On most of such submanifolds the coordinate z is identified with period 2π , but there are some (e.g., the one on which $x = y = 0$) on which z is identified with period π . There is therefore no choice of the constant C such that all these submanifolds could be extended to $t = 0$ in a regular way. Hence the four-metric cannot be extended to $t = 0$.

The second way of imposing the $\mathcal{G}_2$ restrictions on h_{ij} is to set

$$C = F = H = J = 0 \quad (D2)$$

To derive the regularity conditions with this choice, consider the totally geodesic two-dimensional submanifolds on which

$$Ddx + Edy = dz = 0 \quad (D3)$$

Proceeding as in the proof of Theorem 1 in Appendix B, one can show that the regularity of these submanifolds implies the conditions stated in Theorem 2 with $p = 0$ (and with the restrictions (D2)). These conditions are therefore necessary for regularity of the four-metric at $t = 0$.

To see that these conditions are also sufficient for regularity, assume that they hold and make a coordinate transformation given by (C2) such that the matrix M takes the block diagonal form

$$M = \begin{pmatrix} m & p & 0 \\ n & q & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (D4)$$

Make then the transformation (B8) and express the metric in the coordinates (U, V, v, w) . Without the $\mathcal{G}_2$ identification the metric is manifestly regular at $U = V = 0$. The $\mathcal{G}_2$ identification is in these coordinates given by

$$(U, V, v, w) \sim (U, -V, -v, w + \pi) \quad (D5)$$

which corresponds to quotienting by a discrete group whose action is clearly properly discontinuous. Hence the metric with the $\mathcal{G}_2$ identification is regular, which proves the claim.

Consider then $\mathcal{G}_3$, $\mathcal{G}_4$ and $\mathcal{G}_5$. For these topologies the restrictions of the Table on h_{ij} imply (D1) as well as further conditions on D , E , H and J . None of these metrics can be extended to $t = 0$ in a regular way. This can be seen as for the first case of $\mathcal{G}_2$ above, by considering the totally geodesic two-dimensional submanifolds on which $dx = dy = 0$.

Consider finally $\mathcal{G}_6$. There are now three possible ways of imposing the restrictions of the Table on h_{ij} . They are given by setting to zero all the constants $A, \dots, K$ except, respectively,

$$\begin{array}{ll} \text{(i)} & A, E \text{ and } K \\ \text{(ii)} & B, F \text{ and } H \\ \text{(iii)} & C, D \text{ and } J \end{array} \quad (D6)$$

It is not possible to extend the metric to $t = 0$ in a regular way in any of these cases. This can be seen as above by considering the totally geodesic two-dimensional submanifolds on which, respectively,

$$\begin{array}{ll} \text{(i)} & dx = dy = 0 \\ \text{(ii)} & dy = dz = 0 \\ \text{(iii)} & dz = dx = 0 \end{array} \quad (D7)$$

REFERENCES

- [1] S. W. Hawking, in: *Astrophysical Cosmology: Proceedings of the Study Week on Cosmology and Fundamental Physics*, ed. H. A. Brück, G. V. Coyne and M. S. Longair (Pontificiae Academiae Scientiarum Scripta Varia, Vatican City, 1982).
- [2] J. B. Hartle and S. W. Hawking, *Phys. Rev. D* **28** (1983) 2960.
- [3] S. W. Hawking, *Nucl. Phys. B* **239** (1984) 257.
- [4] J. B. Hartle, in: *Gravitation in Astrophysics: Cargèse 1986*, ed. B. Carter and J. B. Hartle (Plenum, New York, 1987);
L. Z. Fang and Z. C. Wu, *Int. J. Mod. Phys. A* **1** (1986) 887.
- [5] B. S. DeWitt, *Phys. Rev.* **160** (1967) 1113;
J. A. Wheeler, in: *Battelle Rencontres 1967*, ed. C. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).
- [6] J. J. Halliwell and J. Louko, *Phys. Rev. D* **39** (1989) 2206; "Steepest descent contours in the path integral approach to quantum cosmology II: Microsuperspace," *Phys. Rev. D* (1989), to be published; "Steepest descent contours in the path integral approach to quantum cosmology III," in preparation (1989).
- [7] J. J. Halliwell, *Phys. Rev. D* **39** (1989) 2912.
- [8] A. Vilenkin, *Phys. Rev. D* **39** (1989) 1116.
- [9] C. Kiefer, *Phys. Rev. D* **38** (1988) 1761.
- [10] R. T. Jantzen, *Commun. Math. Phys.* **64** (1979) 211; *Nuovo Cim.* **55B** (1980) 161; in: *Cosmology of the Early Universe*, ed. L. Z. Fang and R. Ruffini (World Scientific, Singapore, 1984).
- [11] J. A. Wolf, *Spaces of Constant Curvature* (McGraw-Hill, New York, 1967).
- [12] R. Laflamme, PhD thesis (University of Cambridge, 1988) (unpublished).
- [13] J. Louko, *Ann. Phys. (N. Y.)* **181** (1988) 318.
- [14] J. Louko and P. A. Tuckey, "Regge calculus in anisotropic quantum cosmology," in preparation (1989).
- [15] S. Deser, R. Jackiw and G. 't Hooft, *Ann. Phys. (N. Y.)* **152** (1984) 220;
S. Deser and R. Jackiw, *Ann. Phys. (N. Y.)* **153** (1984) 405.
- [16] E. Witten, *Nucl. Phys. B* **311** (1988) 46.
- [17] J. M. York, *Phys. Rev. Lett.* **28** (1972) 1082;
G. W. Gibbons and S. W. Hawking, *Phys. Rev. D* **15** (1977) 2752.
- [18] J. B. Hartle, *J. Math. Phys.* **30** (1989) 452.
- [19] J. J. Halliwell, *Phys. Rev. D* **36** (1987) 3626.
- [20] J. Louko, *Phys. Lett. B* **202** (1988) 201.
- [21] J. Louko, *Class. Quantum Grav.* **4** (1987) 581.
- [22] Ø. Grøn, *Phys. Rev. D* **32** (1985) 2522.
- [23] J. Louko, *Phys. Rev. D* **35** (1987) 3760.
- [24] M. J. Duncan and L. G. Jensen, *Nucl. Phys. B* **312** (1989) 662; "Is the Universe Euclidean?", preprint CERN-TH.5311/89.
- [25] J. B. Hartle and K. Schleich, in: *Quantum Field Theory and Quantum Statistics*, ed. I. A. Batalin, C. J. Isham and G. A. Vilkovisky (Adam Hilger, Bristol, 1987), Vol. 2;
K. Schleich, *Phys. Rev. D* **36** (1987) 2342.
- [26] K. Schleich, *Phys. Rev. D* **39** (1989) 2192.
- [27] G. W. Gibbons, S. W. Hawking and M. J. Perry, *Nucl. Phys. B* **138** (1978) 141.
- [28] P. O. Mazur and E. Mottola, "The gravitational measure, solution of the conformal factor problem and stability of the ground state of quantum gravity," University of Florida preprint (1989).
- [29] G. W. Gibbons and C. N. Pope, *Commun. Math. Phys.* **66** (1979) 267.
- [30] C. J. Isham, *Phys. Lett. B* **106** (1981) 188.
- [31] J. B. Hartle and D. M. Witt, *Phys. Rev. D* **37** (1988) 2833.
- [32] V. A. Belinsky, I. M. Khalatnikov and E. M. Lifschitz, *Adv. Phys.* **19** (1970) 525;
J. Denaret, M. Henneaux and P. Spindel, *Phys. Lett. B* **164** (1985) 27.
- [33] A. Helfer, M. S. Hickman, C. Kozamech, C. Lucey and E. T. Newman, *Gen. Rel. Grav.* **20** (1988) 875.
- [34] M. Piliati, *Phys. Rev. D* **26** (1982) 2645; *Phys. Rev. D* **28** (1983) 729;
G. Francisco and M. Piliati, *Phys. Rev. D* **31** (1985) 241.

TABLE

Class	Additional identifications	Constraints on h_{ij}
$\mathcal{G}_1$	None	None
$\mathcal{G}_2$	$(x, y, z) \sim (-x, -y, z + \pi)$	$h_{xz} = h_{yz} = 0$
$\mathcal{G}_3$	$(x, y, z) \sim (-y, x - y, z + 2\pi/3)$	$h_{xz} = h_{yz} = 0$ $h_{xx} = h_{yy} = -4h_{xy}$
$\mathcal{G}_4$	$(x, y, z) \sim (y, -x, z + \pi/2)$	h_{ij} diagonal $h_{xx} = h_{yy}$
$\mathcal{G}_5$	$(x, y, z) \sim (-y, x + y, z + \pi/3)$	$h_{xz} = h_{yz} = 0$ $h_{xx} = h_{yy} = 4h_{xy}$
$\mathcal{G}_6$	$(x, y, z) \sim (-x, -y, z + \pi)$ $\sim (x + \pi, -y + \pi, -z)$ $\sim (-x + \pi, y + \pi, -z + \pi)$	h_{ij} diagonal

Table.

Topological classification of compact connected orientable flat three dimensional Riemannian manifolds [11]. The coordinates x , y and z are identified periodically with period 2π , and the metric is written in these coordinates as $h_{ij} dx^i dx^j$. The second column gives the identifications of the coordinates in addition to the periodic ones, and the third column gives the restrictions for the symmetric positive definite matrix h_{ij} . Class $\mathcal{G}_1$ is the three-torus.

- [35] C. Teitelboim, *Phys. Rev. D* **25** (1982) 3159; *Phys. Rev. D* **28** (1983) 297.
- [36] M. Heineaux, M. Pilati and C. Teitelboim, *Phys. Lett. B* **110** (1982) 123.
- [37] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, Vols. I and II (Cambridge University Press, Cambridge, 1987).
- [38] M. Lehto, H. B. Nielsen and M. Ninomiya, *Nucl. Phys. B* **272** (1986) 213; *Nucl. Phys. B* **272** (1986) 228; *Nucl. Phys. B* **289** (1987) 684.
- [39] C. J. Isham, in: *Proceedings of the Osgood Hill Conference on Conceptual Problems in Quantum Gravity*, ed. A. Ashtekar and J. Stachel (Birkhäuser, Boston, in press).
- [40] J. B. Hartle, *Class. Quantum Grav.* **2** (1985) 707.
- [41] K. Schleich and D. M. Witt, private communication.
- [42] J. B. Hartle, *J. Math. Phys.* **26** (1985) 804.