ON AN EXCEPTIONAL NON-ASSOCIATIVE SUPERSPACE

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Dedicated to Murray Gell-Mann on the occasion of his sixtieth birthday.

Abstract

The Jordan superalgebra $JF(6/4)$ is the unique exceptional Jordan superalgebra that has no realization in terms of Z_2 graded associative supermatrices. We propose it as the basis of an exceptional superspace that is non-Clifford-algebraic. We construct $JF(6/4)$ in a basis that renders itself to such an interpretation and study its derivation, reduced structure and Möbius superalgebras. These algebras are simply the Lie superalgebras of generalized rotation, Lorentz and conformal supergroups of the Jordan superalgebra $JF(6/4)$. We also comment on the implications of the exceptionality of $JF(6/4)$.

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1 Introduction

The twistor formalism in four-dimensional space-time ($d=4$) leads naturally to the representation of four vectors in terms of 2×2 Hermitian matrices over the field of complex numbers $\mathbb{C}$. In particular, the coordinate four vectors can be represented as such. In this form the actions of four-dimensional space-time symmetry groups on the Minkowski space take on particularly elegant forms. For example, the action of the conformal group on the Minkowski coordinates can be realized as a group of linear fractional transformations of the corresponding 2×2 matrices [1]. Since Hermitian matrices close under anti-commutation one can consider them as elements of a Jordan algebra with the symmetric Jordan product [2]. Then the rotation, Lorentz and conformal groups in $d=4$ can be interpreted as the automorphism, reduced structure and Möbius groups of the Jordan algebra of 2×2 complex Hermitian matrices [3]. Conversely, this interpretation allows one to define the concepts of rotation, Lorentz and conformal groups for any Jordan algebra [3, 4]. In the mathematics literature they have been studied under the names automorphism, reduced structure and superstructure groups ("super" in this case having nothing to do with supersymmetry) [5]. Denoting as J_n^A the Jordan algebra of $n \times n$ Hermitian matrices over the division algebra A , one finds the following symmetry groups:

$$\begin{array}{ccccccc} \underline{J} & & \underline{RG} & & \underline{LG} & & \underline{CG} \\ J_n^R & & SO(n) & & SL(n, \mathbb{R}) & & Sp(2n, \mathbb{R}) \\ J_n^C & & SU(n) & & SL(n, \mathbb{C}) & & SU(n, n) \\ J_n^H & & USp(2n) & & S^*(2n) & & SO^*(4n) \\ J_3^O & & F_4 & & E_{6(-26)} & & E_{7(-25)} \end{array}$$

The symbols R, G, H, O represent the four division algebras and RG, LG and CG are the rotation (automorphism), Lorentz (reduced structure), and conformal (Möbius) groups, respectively. In addition to the above Jordan algebras there is another infinite family of simple Jordan algebras [6], namely the Jordan algebras $J(d)$ of Dirac gamma matrices in d dimensions.

$$\{\Gamma_\mu, \Gamma_\nu\} = \delta_{\mu\nu} \mathbf{1} \quad ; \quad \mu, \nu, \dots = 1, 2, \dots, d \quad (1.1)$$

Their automorphism, reduced structure and Möbius groups are simply the rotation, Lorentz and conformal groups of $(d+1)$ -dimensional Minkowski spacetimes.

One finds the following special isomorphisms between the Jordan algebras of 2×2 Hermitian matrices over the four division algebras and the Jordan algebras of gamma matrices:

$$\begin{array}{ccc} J_2^R \approx \Gamma(2) & ; & J_2^C \approx \Gamma(3) \\ J_2^H \approx \Gamma(5) & ; & J_2^O \approx \Gamma(9) \end{array} \quad (1.2)$$

As has been observed by many authors, the Minkowski spacetimes they correspond to are precisely the critical dimensions for the existence of super Yang-Mills theories as well as of the classical Green-Schwarz superstrings. These Jordan algebras are all quadratic and their norm forms are precisely the quadratic invariants constructed using the Minkowski metric. The reduced structure group is defined as the invariance group of the norm form and the superstructure (conformal) group is simply the invariance group of the zero norm condition. Clearly, for the quadratic norms in question these definitions agree with the usual ones.

Jordan superalgebras are Z_2 graded algebras with a supersymmetric product and they have been classified by Kac [7]. One can similarly define their rotation, Lorentz and conformal supergroups as their automorphism, reduced structure and Möbius supergroups. A complete list of these supergroups was given in [3]. Below we list the simple Jordan superalgebras and their symmetry supergroups using a modified version of Kac's notation for labelling Jordan superalgebras. For example, the Jordan superalgebra of type X with m even elements and n odd elements is denoted as $JX(m/n)$.

$JX :$

$$\begin{array}{ccccc} SRG & & SLG & & SCG \\ \hline JA(m^2 + n^2 / 2mn) & & & & \\ \hline SU(m/n) & & SU(m/n) \times SU(m/n) & & SU(2m/2n) \\ \hline JBC(\frac{1}{2}m(m+1) + n(2n-1) / 2mn) & & & & \\ \hline OSp(m/2n) & & SU(m/2n) & & OSp(4n/2m) \\ \hline \end{array}$$

$JD(m/2n) :$

$$OSp(m-1/2n) \quad OSp(m/2n) \quad OSp(m+2/2n)$$

$JP(n^2/n^2) :$

$$P(n-1) \quad SU(n/n) \quad P(2n-1)$$

$JQ(n^2/n^2) :$

$$Q(n-1) \times U(1)_F \quad Q(n-1) \times Q(n-1) \times U(1)_F \quad Q(2n-1)$$

$JD(2/2)_\alpha :$

$$OSp(1/2) \quad SU(1/2) \quad D(2,1;\alpha)$$

$JF(6/4) :$

$$OSp(1/2) \times OSp(1/2) \quad OSp(2/4) \quad F(4)$$

$JK(1/2) :$

$$OSp(1/2) \quad SU(1/2) \quad SU(2/2)$$

Above we denoted the super rotation, Lorentz and conformal groups of Jordan superalgebras as SRC, SLG and SCG, respectively. They appear from left to right, in that order, on the line immediately below the corresponding Jordan superalgebra JX. The term $U(1)_F$ denotes the "fermionic" $U(1)$ factor generated by a single odd generator [3,8].

Some of the supergroups appearing in the above list correspond to spacetime symmetry groups in various dimensions. In those cases one may consider the underlying Jordan superalgebra as the basis of a superspace by identifying the even subspace with the space-time coordinates and the odd subspace with the Grassmann coordinates. By multiplying the odd elements with anticommuting Grassmann coefficients one can work with the symmetric Jordan product in both sectors [3,8].

In the list of simple Jordan superalgebras one is truly unique, namely the ex-

ceptional Jordan superalgebra $JF(6/4)$. It is the only simple Jordan superalgebra which has no realization in terms of Z_2 graded associative supermatrices. That it is exceptional has been proved rather recently [9]. Our aim in this article is to study $JF(6/4)$ and its symmetry supergroups. We shall propose it as the basis of an exceptional superspace. Its exceptionality implies that the corresponding supersymmetry is non-Clifford-algebraic. For a certain real form of $JF(6/4)$ the Möbius group $F(4)$ is simply the $N=2$ superconformal group in five-dimensional Minkowski space (or the $N=2$ anti-de Sitter supergroup in $d=6$). Another real form of $JF(6/4)$ leads to that real form of $F(4)$ which corresponds to an exceptional $N=8$ conformal supergroup in one dimension.

The plan of our paper is as follows. In section 2 we review the Tits-Koecher-Kantor (TKK) construction of Lie algebras from Jordan algebras [10-12] and its generalization to Jordan superalgebras [3,8]. In section 3 we formulate the Lie superalgebra of $F(4)$ in a split basis with a 3-graded (Jordan) structure. In section 4 we construct the exceptional Jordan superalgebra in a basis which is suitable for interpretation as the basis of a superspace. We then study the Lie superalgebras of the automorphism (rotation), reduced structure (Lorentz) and Möbius (conformal) supergroups of $JF(6/4)$. In the final section we discuss some of the implications of the exceptionality of $JF(6/4)$.

2 Tits-Koecher-Kantor Construction of Lie Algebras from Jordan Algebras and its Supersymmetric Generalization

A Jordan algebra J is a non-associative algebra with a symmetric product $a \cdot b = b \cdot a$ satisfying the Jordan identity:

$$a \cdot (b \cdot a^2) = (a \cdot b) \cdot a^2, \quad \forall a, b \in J \quad (2.1)$$

A derivation D of J is an endomorphism of J such that

$$D(a \cdot b) = (Da) \cdot b + a \cdot (Db) \quad (2.2)$$

Derivations form a Lie algebra under commutation which is referred to as the derivation algebra ($Der J$) of J . If we denote the left and right multiplication by an element a as L_a and R_a , respectively, we have $L_a = R_a$ since

elements of $Su(J)$. Excluding this identity operator one is left with what is referred to as the reduced structure algebra $Su(J)_0$ of J . The reduced structure algebra is simply the Lie algebra of the reduced structure group or what we called the Lorentz group (1.6) of J in the previous section.

Another way to describe the $Su(J)$ is via the Jordan triple product which is defined as

$$\{abc\} \equiv a \cdot (b \cdot c) + (a \cdot b) \cdot c - b \cdot (a \cdot c) \quad (2.10)$$

Using this triple product we can define a bilinear transformation S_{ab} on J as

$$S_{ab} c \equiv \{abc\}, \quad \forall a, b, c \in J \quad (2.11)$$

They generate $Su(J)$ under commutation:

$$[S_{ab}, S_{cd}] = -S_{\{cd\}b} + S_{a\{dcb\}} \quad (2.12)$$

This is readily seen from the fact that

$$S_{ab} = [L_a, L_b] + L_{a \cdot b} = D_{a,b} + L_{a \cdot b} \quad (2.13)$$

The TKK construction defines a 3-graded Lie algebra $\mathfrak{g}$ over a given Jordan algebra J :

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{+1} \quad (2.14)$$

The elements of grade $+1$ and grade -1 spaces are labelled by the elements of J (hence we are actually using two copies of J which are referred to as a Jordan pair [5]).

$$L_a b \equiv a \cdot b = b \cdot a \equiv R_a b \quad (2.3)$$

We can rewrite the derivation condition as

$$[D, L_a] = L_{D a} \quad (2.4)$$

Linearization of the Jordan identity leads to the identity:

$$\begin{aligned} & (a \cdot b) \cdot (c \cdot d) + (a \cdot c) \cdot (b \cdot d) + (a \cdot d) \cdot (b \cdot c) \\ &= a \cdot (b \cdot (c \cdot d)) + c \cdot (b \cdot (a \cdot d)) + d \cdot (b \cdot (a \cdot c)) \end{aligned} \quad (2.5)$$

which is equivalent to the identity:

$$[L_{a \cdot b}, L_c] + [L_{c \cdot a}, L_b] + [L_{b \cdot c}, L_a] = 0 \quad (2.6)$$

and

$$[[L_a, L_b], L_c] = L_{[L_a, L_b]} c \quad (2.7)$$

This last identity shows that $[L_a, L_b]$ is a derivation of the Jordan algebra J . In fact, one can prove that all inner derivations of a Jordan algebra can be represented in this form. Therefore we can formally write:

$$[L_f, L_g] = D_{fg} \quad (2.8)$$

Exponentiating the derivations one obtains the inner automorphisms of J

$$e^D(a \cdot b) = (e^D a) \cdot (e^D b) \quad (2.9)$$

Therefore the derivation algebra of J is simply the Lie algebra of its automorphism (rotation) group. It is obvious from above that the multiplication operators close under commutation into derivations of J . The Lie algebra generated by multiplications with elements of J and its derivations is referred to as the structure algebra $Su(J)$. Clearly, the multiplication by the identity element of J commutes with all the other

$$U_a \in \mathfrak{g}^{+1}$$

$$\tilde{U}_b \in \mathfrak{g}^{-1} \quad \forall a, b \in \mathcal{J} \quad (2.15)$$

$$S_{ab} \in \mathfrak{g}^0$$

Then one defines the Lie product among these elements as

$$\begin{aligned} [U_a, \tilde{U}_b] &\equiv S_{ab} \\ [S_{ab}, U_c] &\equiv U_{\{abc\}} \end{aligned} \quad (2.16)$$

$$[S_{ab}, \tilde{U}_c] = -\tilde{U}_{\{bac\}}$$

$$[S_{ab}, S_{cd}] = S_{\{abc\},d} - S_{c,\{bad\}}$$

with all the other products vanishing. The Jacobi identities follow from the Jordan identity. The resulting Lie algebra $\mathfrak{g}$ is referred to as the superstructure algebra and is isomorphic to the Lie algebra of the superstructure (Möbius) group of $\mathcal{J}$ which we called the conformal group (CG) in the previous section. There is a non-linear realization of the CG on $\mathcal{J}$ as a group of linear fractional transformations [13].

The TKK construction has been generalized to the construction of Lie superalgebras from Jordan superalgebras [3,8]. A Jordan superalgebra is a $\mathbb{Z}_2$ graded algebra $\mathcal{J} = \mathcal{J}^0 \oplus \mathcal{J}^1$ with a supercommutative product

$$\begin{aligned} a \cdot b &= (-1)^{\alpha\beta} b \cdot a \\ a \in \mathcal{J}^\alpha, \quad b \in \mathcal{J}^\beta; \quad \alpha, \beta = 0, 1 \end{aligned} \quad (2.17)$$

which satisfies the identity

$$(-1)^{\alpha\delta} [L_{a,b}, L_c] + (-1)^{\beta\alpha} [L_{b,c}, L_a] + (-1)^{\gamma\beta} [L_{c,a}, L_b] = 0 \quad (2.18)$$

where the mixed bracket $[\cdot, \cdot]$ denotes the usual Lie superbracket. The defining con-

ditions imply the following identity:

$$[[L_a, L_b], L_c] = (-1)^{\beta\delta} L_{\alpha.(c.b)} - (a.c).b \quad (2.19)$$

Multiplying the odd elements (grade one) with anticommuting Grassmann parameters and the even elements (grade zero) with ordinary complex parameters, the supercommutative product can be replaced with a commutative one and the TKK construction can be carried over to the super case in a straightforward manner [3,8].

One can invert the TKK construction and define Jordan (super) algebras starting from Lie (super) algebras with the appropriate 3-graded structure [7,14]. In fact, this is how Kac defined and classified Jordan superalgebras [7]. We shall also use the inverse TKK functor to write down the multiplication table of the exceptional Jordan superalgebra $\mathcal{JF}(6/4)$. We shall choose a basis that renders itself readily to a space-time interpretation, in contrast to the basis given in [7]. Our starting point will be the exceptional Lie superalgebra $F(4)$ to which we turn next.

3 The Exceptional Lie Superalgebra $F(4)$

The exceptional Lie superalgebra $F(4)$ has $\text{Spin}(7)\text{XSU}(2)$ as its even subalgebra. The odd generators transform in the spinor representation (8,2) of $\text{Spin}(7)\text{XSU}(2)$. Explicit supercommutation relations of the generators of $F(4)$ were given in references [15,16]. Denoting the generators of $\text{Spin}(7)$ and $\text{SU}(2)$ as J_{ij} and T_m , respectively, we have

$$\begin{aligned} [J_{ij}, J_{kl}] &= -\delta_{ik} J_{jl} + \delta_{jk} J_{il} - \delta_{jl} J_{ik} + \delta_{il} J_{jk} \\ J_{ij} &= -J_{ji}; \quad i, j, \dots = 1, 2, \dots, 7 \\ [T_m, T_n] &= i \epsilon_{mnp} T_p; \quad m, n, \dots = 1, 2, 3 \end{aligned} \quad (3.1)$$

The odd generators ψ_m carry the spinor indices x and a of $\text{SU}(2)$ and $\text{Spin}(7)$, respectively. The remaining non-vanishing supercommutation relations of $F(4)$ are [15]:

$$[T_m, \psi_x] = \frac{1}{2} \tau_{yz}^m \psi_y \psi_z$$

Various new quantities appearing above are defined as follows:

$$\begin{aligned}
 J_I^+ &= J_{I6} + i J_{I7} \quad ; \quad I, J, \dots = 1, 2, \dots, 5 \\
 T^+ &= T^1 + i T^2 \\
 V_\alpha^+ &= (\Gamma_6 + i \Gamma_7)_{ab} V_{1b} \\
 V_\alpha^- &= (\Gamma_6 - i \Gamma_7)_{ab} V_{2b} \\
 N &= -i J_{67} \\
 W_\alpha &= (\Gamma_6 + i \Gamma_7)_{ab} V_{2b} \\
 Z_\alpha &= (\Gamma_6 - i \Gamma_7)_{ab} V_{1b}
 \end{aligned} \tag{3.6}$$

Note also that the grading is defined by the generator $Q = N + T^3$ and only four of the eight components of $V_\alpha^+, V_\alpha^-, W_\alpha$ and Z_α are linearly independent.

In this split basis the generators of $F(4)$ satisfy the following supercommutation relations:

$$\begin{aligned}
 [J_{IJ}, J_K^+] &= -\delta_{IK} J_J^+ + \delta_{JK} J_I^+ \\
 [J_I^+, J_J^-] &= -2 J_{IJ} - 2 \delta_{IJ} N \\
 [N, J_I^+] &= + J_I^+ \\
 [N, V_\alpha^+] &= + \frac{1}{2} V_\alpha^+ \\
 [T^3, V_\alpha^+] &= + \frac{1}{2} V_\alpha^+ \\
 [N, W_\alpha] &= \frac{1}{2} W_\alpha = -[T^3, W_\alpha] \\
 [N, Z_\alpha] &= -\frac{1}{2} Z_\alpha = -[T^3, Z_\alpha] \\
 [T^+, T^-] &= 2 T^3
 \end{aligned} \tag{3.7}$$

$$\begin{aligned}
 [J_{ij}, V_{\alpha a}] &= \frac{1}{2} (\Gamma_{ij}^+)^{ba} V_{\alpha b} \\
 \{V_{\alpha a}, V_{\beta b}\} &= 2\sigma \tilde{C}_{ab} (\hat{C} \tau^m)_{xy} T_m \\
 &\quad + \frac{\sigma}{3} \hat{C}_{xy} (\tilde{C} \Gamma_{ij}^+)^{ab} J_{ij}^+
 \end{aligned} \tag{3.2}$$

where $\hat{C}$ and $\tilde{C}$ are the charge conjugation matrices in dimensions $d = 2$ and $d = 7$, respectively, and Γ_{ij}^+ are the spinor representation matrices of J_{ij} . The τ^m are the Pauli matrices. The σ is an arbitrary real parameter. We shall take $\sigma = 3/2$ for convenience later. For $\tilde{C}$ we shall choose $\tilde{C} = i\tau^2$.

The matrix $\tilde{C}$ is defined by the condition

$$\tilde{C} \Gamma_i^+ = -\Gamma_i^T \tilde{C} \quad ; \quad \tilde{C}^T = \tilde{C} \tag{3.3}$$

where Γ_i^+ are the gamma matrices in $d = 7$ satisfying

$$\{\Gamma_i^+, \Gamma_j^+\} = 2 \delta_{ij} \quad ; \quad i, j = 1, \dots, 7 \tag{3.4}$$

$$\Gamma_{ij}^+ \equiv \frac{1}{2} [\Gamma_i^+, \Gamma_j^+]$$

We shall choose Γ_i^+ all antisymmetric and $\tilde{C}$ to be the identity matrix.

For applying the inverse TKK functor to construct the exceptional Jordan superalgebra $JF(6/4)$ we need to go to a split basis for $F(4)$ with a 3-graded decomposition (Jordan structure), $F(4) = g^{-1} \oplus g^0 \oplus g^{+1}$, where the three graded subspaces contain the following generators

$$\begin{aligned}
 g^{-1} &= \{ J_I^-, T^-, V_\alpha^- \} \\
 g^0 &= \{ J_{IJ}, N, T^3, W_\alpha, Z_\alpha \} \\
 g^{+1} &= \{ J_I^+, T^+, V_\alpha^+ \}
 \end{aligned} \tag{3.5}$$

4 The Exceptional Jordan Superalgebra JF(6/4) and its Symmetry Supergroups

As we mentioned earlier one may invert the TKK construction and construct Jordan algebras [12] and Jordan superalgebras [7] from certain 3-graded Lie (super)algebras. We shall now apply this inverse TKK functor to construct the exceptional Jordan superalgebra starting from F(1). Consider now the subspace g^{+1} of F(4) as given in the previous section. Among any two elements A and B of g^{+1} we define the following product:

$$A \cdot B \equiv [\{A, P\}, B] \quad , \quad \forall A, B \in g^{+1} \quad (4.1)$$

where P is a fixed element of the g^{-1} subspace which we take to be

$$P = \frac{1}{2} (\tau^- - J_5^-) \quad (4.2)$$

From the Jacobi identity and the 3-grading it then follows that

$$(-1)^{\alpha\beta} [\{A, P\}, B] + [\{P, B\}, A] = 0 \quad (4.3)$$

This shows that the product is supercommutative and closure is a trivial consequence of the 3-grading.

$$A \cdot B = (-1)^{\alpha\beta} B \cdot A \quad (4.4)$$

Under this product we then find

$$\begin{aligned} J_I^+ \cdot J_J^+ &= -\delta_{IJ} J_5^+ + \delta_{5J} J_I^+ + \delta_{IS} J_J^+ \\ J_I^+ \cdot \tau^+ &= 0 \\ \tau^+ \cdot \tau^+ &= \tau^+ \end{aligned}$$

$$\begin{aligned} \Theta_a \cdot \Theta_b &= \frac{2\sigma}{3} \left[\Gamma_I (\Gamma_6^+ + i\Gamma_7^+) \right]_{ab} J_I^+ \\ &\quad - 2\sigma \left[\Gamma_5 (\Gamma_6^+ + i\Gamma_7^+) \right]_{ab} \tau^+ \end{aligned} \quad (4.5)$$

$$\tau^+ \cdot \Theta_a = \frac{1}{2} \Theta_a$$

$$J_I^+ \cdot \Theta_a = \frac{1}{2} \delta_{IS} \Theta_a - \frac{1}{2} (\Gamma_{IS})_{ab} \Theta_b$$

$$J_5^+ \cdot J_5^+ = J_5^+$$

where we let $\Theta_a = V_a^+$. Keeping in mind that not all the Θ_a are linearly independent let us study the derivations of JF(6/4). The superderivation algebra of JF(6/4) is generated by the following operators:

$$\begin{aligned} \mathcal{D}_{I,J} &= [L_{J_I^+}, L_{J_J^+}] = -\mathcal{D}_{J,I} \\ S_{a,b} &= \{L_{\Theta_a}, L_{\Theta_b}\} = S_{b,a} \\ F_{I,a} &= [L_{J_I^+}, L_{\Theta_a}] \end{aligned} \quad (4.6)$$

The operators $\mathcal{D}_{I,J}$ and $S_{a,b}$ are the even generators and the $F_{I,a}$ are the odd ones. The even generators $\mathcal{D}_{I,J}$ act on the elements of JF(6/4) as

$$\begin{aligned} \mathcal{D}_{I,5} J_K^+ &= 0 = \mathcal{D}_{I,I} \tau^+ \\ \mathcal{D}_{\mu,\nu} J_5^+ &= 0 \quad ; \quad \mu, \nu, \dots = 1, 2, 3, 4 \\ \mathcal{D}_{\mu,\nu} J_\lambda^+ &= \delta_{\mu\lambda} J_\nu^+ - \delta_{\nu\lambda} J_\mu^+ \\ \mathcal{D}_{\mu,\nu} \Theta_a &= \frac{1}{4} [\Gamma_\mu, \Gamma_\nu]_{ab} \Theta_b \\ \mathcal{D}_{I,5} \Theta_a &= 0 \end{aligned} \quad (4.7)$$

This implies that $\mathcal{D}_{I,5} = 0$ and the generators $\mathcal{D}_{\mu,\nu}$ close under commutation and

form the Lie algebra of $SO(4)$:

$$[D_{\mu\nu}, D_{\lambda\beta}] = \delta_{\nu\lambda} D_{\mu\beta} + \delta_{\mu\beta} D_{\nu\lambda} - \delta_{\mu\lambda} D_{\nu\beta} - \delta_{\nu\beta} D_{\mu\lambda} \quad (4.8)$$

The derivations $S_{a,b}$ turn out not to be independent of $D_{\mu\nu}$. One finds that

$$S_{a,b} = -\frac{\sigma}{3} D_{\mu\nu} [\Gamma_{\mu\nu} \Gamma_5 (\Gamma_6 + i\Gamma_7)]_{ab} \quad (4.9)$$

So far we have been using the same symbols for denoting the elements of $JF(6/4)$ and those of the subspace g^{+1} considered as elements of the Lie superalgebra $F(4)$. Let us now introduce different symbols for denoting the elements of $JF(6/4)$ and work only with the linearly independent components of Θ_a . We choose the convention of reference [16] for Γ_i i.e.

$$\begin{aligned} \Gamma_1 &= 1 \otimes \sigma_3 \otimes \sigma_2 & ; & & \Gamma_2 &= 1 \otimes \sigma_1 \otimes \sigma_2 \\ \Gamma_3 &= \sigma_2 \otimes 1 \otimes \sigma_3 & ; & & \Gamma_4 &= \sigma_2 \otimes 1 \otimes \sigma_1 \\ \Gamma_5 &= \sigma_3 \otimes \sigma_2 \otimes 1 & ; & & \Gamma_6 &= \sigma_1 \otimes \sigma_2 \otimes 1 \\ \Gamma_7 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \end{aligned} \quad (4.10)$$

where σ_1, σ_2 and σ_3 are the Pauli matrices. Then we find the following relations among the components of Θ

$$\begin{aligned} \Theta_2 &= i\Theta_1 & ; & & \Theta_4 &= i\Theta_3 \\ \Theta_6 &= -i\Theta_5 & ; & & \Theta_8 &= -i\Theta_7 \end{aligned} \quad (4.11)$$

We set $\sigma = 3/2$ and define

$$B_0 \equiv J_5^+$$

$$\begin{aligned} S &\equiv T^+ \\ B_\mu &\equiv J_\mu^+ & ; & \mu, \nu = 1, \dots, 4 \\ (Q_\alpha) &= \begin{pmatrix} \Theta_1 \\ \Theta_3 \\ \Theta_5 \\ \Theta_7 \end{pmatrix} & ; & \alpha, \beta, \dots = 1, 2, 3, 4 \end{aligned} \quad (4.12)$$

The multiplication table of $JF(6/4)$ now takes on a more transparent form

$$\begin{aligned} B_\mu \cdot B_\nu &= -\delta_{\mu\nu} B_0 \\ B_0 \cdot B_\mu &= B_\mu \\ B_0 \cdot B_0 &= B_0 \\ B_0 \cdot S &= 0 = B_\mu \cdot S \\ S \cdot S &= S \\ Q_\alpha \cdot Q_\beta &= (i\gamma_5 \gamma_\mu C)_{\alpha\beta} B^\mu + (\gamma_5 C)_{\alpha\beta} (B_0 - 3S) \\ B_\mu \cdot Q_\alpha &= \frac{i}{2} (\gamma_\mu)_{\alpha\beta} Q^\beta \\ B_0 \cdot Q_\alpha &= \frac{1}{2} Q_\alpha \\ S \cdot Q_\alpha &= \frac{1}{2} Q_\alpha \\ \mu, \nu, \dots &= 1, 2, 3, 4 & ; & \alpha, \beta, \dots = 1, 2, 3, 4 \end{aligned} \quad (4.13)$$

The B_0 and S are the two idempotents and $I = B_0 + S$ is the identity element of $JF(6/4)$. The matrices γ_μ are the four-dimensional Dirac gamma matrices and C is the charge conjugation matrix:

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \quad (4.14)$$

$$\gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu^T$$

The indices $\mu, \nu, \dots$ and $\alpha, \beta, \dots$ are the vector and spinor indices of $SO(4)$ generated by $D_{\mu\nu}$, respectively. The above convention for the Γ_i leads to the following expressions for γ_μ and Γ_i .

$$\begin{aligned} \gamma_1 &= 1 \otimes \sigma_1 & ; & & \gamma_2 &= -1 \otimes \sigma_3 \\ \gamma_3 &= -\sigma_1 \otimes \sigma_2 & ; & & \gamma_4 &= -\sigma_2 \otimes \sigma_2 \\ \gamma_5 &= \gamma_1 \gamma_2 \gamma_3 \gamma_4 & ; & & \Gamma &= \gamma_3 \end{aligned} \quad (4.15)$$

In the new basis the elements of the derivation algebra $\text{Der } JF(6/4)$ are

$$\begin{aligned} D_{\mu\nu} &= [L_{B_\mu}, L_{B_\nu}] = -D_{\nu\mu} \\ G_{\alpha\beta} &= \{L_{Q_\alpha}, L_{Q_\beta}\} = G_{\beta\alpha} \end{aligned} \quad (4.16)$$

$$F_{\alpha\alpha} = [L_{B_0}, L_{Q_\alpha}]$$

$$F_{\mu\alpha} = [L_{B_\mu}, L_{Q_\alpha}]$$

$$E_{s,\alpha} = [L_S, L_{Q_\alpha}]$$

They are however not all independent. We find that

$$\begin{aligned} F_{\mu\alpha} &= i(\gamma_\mu)_{\alpha\beta} F_{0\beta} \\ E_{s,\alpha} &= -F_{0,\alpha} \end{aligned} \quad (4.17)$$

$$G_{\alpha\beta} = -\frac{1}{4}(\gamma_{\mu\nu} \gamma_5 C)_{\beta\alpha} D_{\mu\nu}$$

They satisfy the supercommutation relations

$$\begin{aligned} [D_{\mu\nu}, D_{\lambda\beta}] &= \delta_{\nu\lambda} D_{\mu\beta} + \delta_{\mu\beta} D_{\nu\lambda} - \delta_{\mu\lambda} D_{\nu\beta} - \delta_{\nu\beta} D_{\mu\lambda} \\ [D_{\mu\nu}, F_{0,\alpha}] &= \frac{1}{2}(\gamma_{\mu\nu})_{\alpha\beta} F_{0,\beta} \\ \{F_{0,\alpha}, F_{0,\beta}\} &= -\frac{1}{8}(\gamma_5 \gamma_{\mu\nu})_{\beta\alpha} D_{\mu\nu} \end{aligned} \quad (4.18)$$

Clearly, the even subalgebra is the Lie algebra of $SO(4)$. It is straightforward to verify that the full derivation algebra is the Lie superalgebra of $OSp(1/2)XOSp(1/2)$. This is best done by decomposing the generators of $SO(4)$ into its two commuting $SU(2)$ subalgebras. Therefore, we have

$$\text{Der } JF(6/4) = OSp(1/2) \times OSp(1/2) \quad (4.19)$$

The structure algebra of $JF(6/4)$ is generated by the derivations and the multiplications by the elements of $JF(6/4)$. They satisfy the following supercommutation relations

$$\begin{aligned} [D_{\mu\nu}, L_{B_\lambda}] &= \delta_{\mu\lambda} L_{B_\nu} - \delta_{\nu\lambda} L_{B_\mu} \\ [D_{\mu\nu}, L_{Q_\alpha}] &= \frac{1}{4}[\gamma_\mu, \gamma_\nu]_{\alpha\beta} L_{Q_\beta} \\ [F_{0,\alpha}, L_S] &= \frac{1}{4} L_{Q_\alpha} \\ [F_{0,\alpha}, L_{B_0}] &= -\frac{1}{4} L_{Q_\alpha} \\ [F_{0,\alpha}, L_{B_\mu}] &= -\frac{i}{4}(\gamma_\mu)_{\alpha\beta} L_{Q_\beta} \\ [F_{0,\alpha}, L_{Q_\beta}] &= \frac{1}{2}(i\gamma_5 \gamma_\mu C)_{\alpha\beta} L_{B_\mu} + \frac{1}{2}(\gamma_5 C)_{\alpha\beta} (L_S + 3L_3) \end{aligned} \quad (4.20)$$

plus those that define the derivations in terms of multiplication operators and those of $\text{Der } JF(6/4)$. The structure algebra with eleven even and eight odd generators is simply the Lie superalgebra of $OSp(2/4)XU(1)$. The $U(1)$ generator is $L_1 = L_{B_0} + L_S$. The even subgroup of the reduced structure group $OSp(2/4)$ is $SO(2)XSp(4)$.

At this point we should stress that different real forms of $JF(6/4)$ will have different real forms of the derivation and structure algebras. For example, there exist other real forms of $JF(6/4)$ for which the even subgroup $SO(4)$ of the automorphism

group goes over to $SO(3,1)$ or to $SO(2,2)$. These different real forms of $JF(6/4)$ can be obtained via the application of the inverse TTK functor to different real forms of $F(4)$.

Finally, one can now rewrite $F(4)$ back as the Möbius (conformal) superalgebra of $JF(6/4)$ using the super generalization of the equations (2.16) as given in references [3,8]. Perhaps for physical applications, more relevant is the non-linear realization of $F(4)$ as the supergroup of linear fractional transformations of $JF(6/4)$ in the sense of reference [13]. It simply corresponds to the superconformal group action on the corresponding superspace.

5 On the Exceptionality of $JF(6/4)$

A Jordan algebra is said to be special if it can be represented in terms of associative matrices with the Jordan product defined as one half the anticommutator. Otherwise, it is said to be exceptional. In their classic work, Jordan, Von Neumann and Wigner proved that with but one possible exception all Jordan algebras are special [17]. This possible exception is the Jordan algebra J_3^O of 3×3 Hermitian octonionic matrices, whose exceptionality was proved by Albert [18]. In the Eighties Zelmanov extended these results to the infinite dimensional case and proved that there are no infinite dimensional exceptional Jordan algebras [19].

In the Jordan formulation of quantum mechanics the observables and the density matrices representing a physical system are elements of a Jordan algebra. If the underlying Jordan algebra is special, then the Jordan formulation is equivalent to the Dirac formulation of quantum mechanics over a Hilbert space (or a vector space in the finite dimensional case). The existence of an exceptional Jordan algebra raised the question as to whether or not one can formulate quantum mechanics over the exceptional Jordan algebra satisfying all the axioms of Von Neumann. This was answered in the affirmative relatively recently [20]. This so-called octonionic quantum mechanics has no Hilbert space formulation. Somewhat later the Jordan formulation was generalized to the quadratic Jordan formulation of quantum mechanics, which is applicable to the exceptional octonionic case as well [3]. The axioms of quantum mechanics as formulated by Von Neumann are equivalent to the axioms of projective geometry [21]. The projective geometry corresponding to the octonionic quantum mechanics is non-desargian [20], which implies that it cannot be embedded in a higher dimensional projective geometry. This is consistent with the exceptionality of J_3^O .

In the Eighties the exceptional Jordan algebra J_3^O has made its appearance within the framework of supergravity theories through the work of Günaydin, Sierra and Townsend (GIST) [22]. In their construction and classification of $N = 2$ Maxwell-Einstein supergravity theories, GIST showed that there exist four remarkable theories of this type that are uniquely determined by simple Jordan algebras of degree 3. These are the Jordan algebras J_3^H, J_3^C, J_3^O and the exceptional Jordan algebra J_3^O . Their symmetry groups in five, four and three space-time dimensions give the fa-

mous Magic Square of Prendenthal, Rozenfeld and Tits [23]. The exceptional theory defined by J_3^O shares all the remarkable features of the maximally extended $N = 8$ supergravity theory in the respective dimensions. More recently, several authors have speculated on the possible role J_3^O may play in the framework of string theories [24].

In going over to the Jordan superalgebras most of the concepts and definitions for ordinary Jordan algebras carry over in a natural way. For example, consider a Z_2 graded associative algebra $R = R^0 \oplus R^1$. The elements of R define a Jordan superalgebra under the superanticommutator:

$$A \cdot B \equiv \frac{1}{2} (AB + (-1)^{|A||B|} BA) ; \quad \forall A, B \in R \quad (5.1)$$

In most applications the algebra R is the algebra of operators acting on a Z_2 -graded vector space $V = V^0 \oplus V^1$. Hence one can represent them as supermatrices [25]. For space-time supersymmetry the corresponding vector space V is referred to as the superspace. We shall use this term in a more general sense.

A Jordan superalgebra is then special if it can be represented in terms of associative Z_2 graded supermatrices with the product being one half the superanticommutator. All Jordan superalgebras are special except for $JF(6/4)$. (Here we are only considering Jordan superalgebras whose odd subspaces J^1 are not empty.) The exceptionality of $JF(6/4)$ has been shown by Shierl rather recently [9]. The existence of an exceptional Jordan superalgebra raises many interesting questions. For example if we give it a space-time interpretation as the basis of an exceptional superspace then this superspace has no realization as an ordinary Z_2 graded vector space on which the supersymmetry is represented Clifford-algebraically. For special Jordan superalgebras this can always be done. In the language of ordinary quantum mechanics this means that for special Jordan algebras one may choose to work either with density matrices or with vectors in a Hilbert space and these two formulations are equivalent. One major advantage of the Hilbert or vector space formulation is the fact that one can define tensor products of vector spaces in a straightforward manner. For ordinary superspace this means that one can develop a tensor calculus. On the other hand, defining tensor products of Jordan algebras is a notoriously difficult problem; e.g. the tensor product of two Jordan algebras is in general not a Jordan algebra. Therefore, if we were to consider $JF(6/4)$ as the basis of an exceptional superspace then it is desirable to formulate it in the language of ordinary Z_2 graded vector spaces. Assume that there exists such a "generalized vector space" and let us refer to it as the exceptional supermodule MF. The action of the supersymmetry operators as well as the tensor product operation over MF will then have to be non-associative. In other words, there will be a non-associative tensor calculus corresponding to this exceptional superspace. Interestingly enough, the exceptional $N=2$ supergravity theories discovered by GIST [22] cannot be written down in the language of ordinary conformal tensor calculus [26]. It could well be that the underlying tensor calculus is a non-associative one. In fact, the exceptional $N=2$ theories originate in five space-time dimensions, and the real form of $F(4)$ with the even subgroup $SO(5,2)XSU(2)$ is

simply the $N=2$ conformal superalgebra in that dimension. Another way to approach this problem is to look at the unitary representations of $F(4)$ and try to interpret them in the language of a ten-dimensional superspace corresponding to $JF(6/4)$. A pure gauged anti-de Sitter supergravity in $d=6$ with $F(4)$ symmetry was constructed in [27]. Matter couplings of that theory as well as the conformal supergravity in $d=5$ with $F(4)$ symmetry are yet to be constructed. We hope to address these problems and related issues in the future.

We should also point out that the quadratic Jordan formulation of quantum mechanics [3] can be formally extended to Jordan superalgebras as well as Jordan algebras defined over finite fields. In the super Jordan case one takes the odd subspace over Grassmann numbers. Since the trace form is no longer a real number, one loses the usual probability interpretation. This might be remedied by interpreting the odd sector as unobservable. In any case, the formal quantum mechanics defined over $JF(6/4)$ would again not have a usual vector space formulation.

Finally, the algebra $JF(6/4)$ considered as the basis of a superspace corresponds to six bosonic and four "fermionic" (Grassmann) coordinates. Because of its potential relevance to the superstring one may want to know if there exists a larger algebraic structure, with ten bosonic elements and sixteen fermionic ones, that can be considered as the basis of a superspace. From Kac's classification we know that this algebraic structure can not be a simple Jordan superalgebra, though it may contain $JF(6/4)$ as a subalgebra. If such an algebraic structure exists then it may give us the clue as to how to generalize conformal supersymmetry to ten space-time dimensions.

References

- [1] F. Gürsey, Nuovo Cimento *3* (1956), 988.
- [2] P. Jordan, Nach. Ges. Wiss. Göttingen (1932), 209 ; *Z. Physik* *80* (1933), 285.
- [3] M. Günaydin, Ann. of Israel Physical Soc. *3* (1980), 279.
- [4] M. Günaydin, Nuovo Cimento *29A* (1975), 467.
- [5] See the excellent review by K. McCrimmon, Bull. Am. Math. Soc. *84* (1978), 612 and the references therein.
- [6] R.D. Schafer, " An Introduction to Non-associative Algebras ", Academic Press, New York 1966.
- [7] V. Kac, Comm. in Algebra *5* (1977), 1375.
- [8] I. Bars and M. Günaydin, J. Math. Phys. *20* (1979), 1977.
- [9] A.S. Shtern, Funkt. Analiz. i Ego Prilozheniya *21* (1987), 93.
- [10] J. Tits, Nederl. Akad. van Wetens. *65* (1962), 530.
- [11] M. Koecher, Amer. J. Math. *89* (1967) 787.
- [12] I.L. Kantor, Sov. Math. Dok. *5* (1964), 1404.
- [13] M. Koecher, Inv. Math. *3* (1967), 136.
- [14] M. Günaydin, J. Math. Phys. *30* (1989), 937.
- [15] W. Nahm, V. Rittenberg and M. Scheunert, J. Math. Phys. *9* (1976), 1640.
- [16] B.S. De Witt and P. van Nieuwenhuizen, J. Math. Phys. *23* (1982), 1953.
- [17] P. Jordan, J. von Neumann and E. Wigner, Ann. Math. *36* (1934), 29.
- [18] A.A. Albert, Ann. Math. *36* (1934), 65.
- [19] E.I. Zelmanov, Sib. Math. Jour. *24* (1983), 89.
- [20] M. Günaydin, C. Piron and H. Ruegg, Comm. Math. Phys. *61* (1978), 69.
- [21] C. Piron, " Foundations of Quantum Physics ", Benjamin Press, (Mass. 1976).
- [22] M. Günaydin, G. Sierra and P.K. Townsend, Phys. Lett. *133B* (1983), 72; Nucl. Phys. *B242* (1984), 244; ibid *B253* (1985), 573.
- [23] H. Freudenthal, Nederl. Akad. Wetensch. Proc. *A62* (1959), 447; B.A. Rozenfeld, Dokl. Akad. Nauk. SSSR *106* (1956), 600; J. Tits, Mem. Acad. Roy. Belg. Sci. *29* (1955) fasc.3.
- [24] D.B. Fairlie and C. Manouge, Phys. Rev. *D34* (1986), 1832; G. Sierra, Class. Quant. Grav. *4* (1987), 385; P. Goddard, W. Nahm, D. Olive, H. Ruegg and A. Schwimmer, Comm. Math. Phys. *112* (1987), 385; F. Gürsey, Mod. Phys. Lett. *A2* (1987), 967; G. Chapline and M. Günaydin, UCRL Preprint 95290 (1986), unpublished; E. Corrigan and T.J. Hollowood, Phys. Lett. *B203* (1988), 47; M. Günaydin and S.J. Hynn, Phys. Lett. *B209* (1988), 498.
- [25] M. Scheunert, " The Theory of Lie Superalgebras ", Lecture Notes in Math. *716*, Springer Verlag, New York, 1979.
- [26] B. de Wit and A. van Proeyen, private communication.
- [27] L.J. Romans, Nucl. Phys. *B269* (1986), 691.