CERN-TH.5512/89

KUL-TF-89/25

Anomalies and the Batalin-Vilkovisky Lagrangian formalism

W. Troost¹,

Instituut voor theoretische fysica
Universiteit Leuven, B-3030 Leuven, Belgium

P. van Nieuwenhuizen²

Institute for Theoretical Physics,
State University of New York at Stony Brook
Stony Brook, NY 11794, U.S.A.

and

A. Van Proeyen³

Theory Division, CERN
CH-1211 Geneva 23, Switzerland

ABSTRACT

Anomalies can be studied in the framework of the BV Lagrangian formalism. They are present whenever there are terms of order $\hbar$ (or higher) in the master equation which cannot be removed by a local counterterm. Regularisation is essential to define those BV expressions which correspond to Jacobians and determinants in other approaches. We use a regularisation scheme, motivated by Pauli-Villars regularisation, which allows one to use the Fujikawa method without being restricted to Fujikawa variables, and which regularises also non-propagating fields. It leads to finite and local, but in general noncovariant, terms in the master equation at order $\hbar$. Several explicit examples illustrate the relation between counterterms, canonical transformations and different consistent regulators.

¹ Bevoegdverklaard navorser, N.F.W.O. Belgium, Bitnet FGBDA19 at BLEKUL11

² Supported in part by NSF grant No. PHY 89-08495

³ Onderzoekseleider, N.F.W.O. Belgium; on leave of absence from K.U. Leuven; Bitnet FGBDA19 at BLEKUL11

CERN-TH.5512/89

KUL-TF-89/25

August 1989

1 Introduction

For the quantisation of Lagrangian gauge theories the scheme of Batalin and Vilkovisky (BV) [1,2] is a useful general framework. It yields a uniform description of theories with ghosts for ghosts and/or open algebras. For superparticles and superstrings this scheme has recently been used to covariantly quantize them [3]. Curiously, no discussion of the role of anomalies in this scheme has appeared in the literature.

In gauge theories a gauge has to be chosen to define the quantum action. The physical quantities should be independent of this choice. If no functional integral can be defined with this requirement, then one has ‘anomalies’.

It has been known for several years [4] that anomalies are related to the non-invariance of the measure of the path integral under BRST symmetries. We discuss in the next section the implications for the BV formalism. We will see that the anomalies are related to a violation of the master equation

$$i\hbar\Delta W - \frac{1}{2}(W, W) = \hbar a_c c'' + \mathcal{O}(\hbar^2) \quad (1.1)$$

where a_c are the anomalies and c'' the ghost fields. (Definitions of brackets, ΔW , etc., are given in the next section). However to define the quantities appearing in these expressions the issue of regularisation is crucial. The applications that have appeared in the literature up to now deal with the construction of S , the action to zeroth order in $\hbar$,

$$W(\Phi, \Phi^*) = S + \hbar M_1 + \hbar^2 M_2 + \dots \quad (1.2)$$

For these applications no regularisation was needed; rather, purely algebraic manipulations were sufficient to determine S from $(S, S) = 0$. On the other hand, the terms at higher order in $\hbar$ need a regularisation to be well-defined. In their pioneering work, Batalin and Vilkovisky comment ([2], top of page 2571) “... The only quantity that remains undefined in the functional integral ... is the measure. ... in a local basis of the gauge algebra, the right hand side of [the master equation] is proportional to $\delta(0)$. In the framework of a regularisation which annihilates such divergences one may put $M_p = 0$, $p \geq 1$, ... In nonlocal bases the measure may generate all sorts of Feynman diagrams; only in a local basis is the measure inessential ... in a local basis the measure may be ignored”.

We hope to show that, rather than dismiss $\delta(0)$ terms, one may use regulators, as first introduced by Fujikawa in configuration path-integrals, and recently extended to phase space path integrals in [5]. The effect of these regulators is that they replace the terms with the singular delta functions by local finite but in general noncovariant expressions depending on the fields which are present in the regulator and in the unregularised

Jacobians. It may be that properties of S in order $\hbar$ equations restrict the allowed S . In that case anomalies would lead to a restriction on the class of actions S satisfying $(S, S) = 0$. Whether or not this is true, we can now study the master equation at order $\hbar$, and repeat (part of) the analysis which until now has been performed at level $\hbar = 0$. We thus intend to show that the BV formalism provides a suitable setting to discuss anomalies. For simple gauge theories it is equivalent to the usual formalisms. An advantage of the BV formalism is that the gauge has only to be fixed at the very end. A hidden dependence on the gauge choice sits however in the regularisation scheme, which in the more usual approaches depends on the gauge chosen. If one could find a regularisation scheme which can be used before the antifields are eliminated using the gauge choices, then one would be able to compute all quantities in eq.(1.1) with the antifields still present, and one might try to solve the hierarchy of equations which constitute the master equation by algebraic methods. We have not found such a scheme, hence in this article we will perform the calculations of regularised expressions after having eliminated all antifields by the prescription given by BV.

We will discuss issues as counterterms, Fujikawa variables, In particular, we shall establish connections between canonical transformations and counterterms which remove anomalies. Canonical transformations in the BV scheme are transformations between fields and antifields which preserve the brackets. Such transformations include all usual field redefinitions but also e.g. adding equation of motion symmetries to transformation laws.

We will only treat consistent anomalies. These are obtained when one chooses a regularisation scheme which regularises the full path integral, e.g. Pauli-Villars regularisation or putting the path integral on a lattice. Within the Fujikawa scheme, a class of regulators which gives consistent anomalies were determined in [6]. The formulation of the consistency conditions can in the BV framework be done in more generality. As the formalism allows e.g. open gauge algebras, we will have the consistency conditions also for these cases.

It will also become clear that the Fujikawa scheme itself can be extended. In terms of ‘Fujikawa variables’ the Fujikawa regulator takes a simpler form, but also other variables can be used. To this purpose we extend our prescription from [6] and define a suitable Fujikawa-like regularisation for consistent anomalies starting from a Pauli-Villars [7] regularisation. In examples we will see that the difference between the anomalies obtained from two such regularisations can be absorbed in the variation of a local counterterm.

In fact, after we have explained the place of anomalies in the BV setting, first neglecting regularisation issues in section 2, and then taking regularisation into account in section 3, we will give several explicit examples in section 4. First we discuss the vector and chiral anomalies produced by

fermion loops. Different regulators are used and the relation between the resulting expressions agrees with general results. Then we will discuss the scalar field in a gravitational field. We consider the Einstein and Weyl anomalies with different variables and different regulators. We show that one can also use Fujikawa regularisation when one chooses other variables than the Fujikawa variables. The third example deals with the particle action. Although there cannot be anomalies in that case, we find here a good laboratory to illustrate the general ideas discussed before by looking at the infinite terms. Other examples of the use of the modified Fujikawa prescription were given in [5]. There it was shown that due to extra terms in the Jacobian one can prove that both the Liouville and the Gaussian measure for the phase space path integral of a scalar field coupled to external gravity are free from Einstein anomalies. In [6] our regularisation prescription was illustrated by the bosonic string. Section 5 gives some conclusions and open problems. In the appendix we review the derivation of properties of canonical transformations between fields and antifields.

2 The BV Lagrangian formalism without regularisation.

2.1 The definition of the functional integral.

The aim of Batalin and Vilkovisky is to define the functional integral which then determines all physical quantities. This functional integral is given by

$$Z_\Phi(J) = \int D\Phi \exp\left(\frac{i}{\hbar} W_\Phi(\Phi) + J(\Phi)\right), \quad (2.1)$$

where $J(\Phi)$ are sources for the fields Φ . The standard choice is $J(\Phi) = J \cdot \Phi$. The complete set of fields, to be determined later, includes the classical fields, but also ghosts, antighosts, ghosts for ghosts, ... The functional integral for gauge theories requires gauge fixing. The latter will be obtained from a 'gauge fermion' ψ .

The basic question therefore is : given a set of classical fields $\phi^i(x)$ ¹ and a classical action $S(\phi^i(x))$, what is the full quantum action $W_\Phi(\Phi)$? To answer this question, one first extends the set of fields to include the usual ghost fields, $\{\Phi\} = \{\phi^i, c^\mu\}$. The reason for this will be given later. Next, BV introduce antifields, an antifield Φ_A^* corresponding to each field Φ^A , with opposite statistics. In addition the sum of the ghost numbers of any field and its antifield is -1 . Then they define a generating (bosonic) function

¹We write α, i when the space-time coordinates are separately given. Capitals A, I are combined indices denoting at once α, i and the space-time coordinates.

$W(\Phi, \Phi^*)$. This generating function should satisfy the master equation :²

$$\Delta \exp\left(\frac{i}{\hbar} W\right) = 0, \quad \Delta = \frac{\partial_r}{\partial \Phi^A} \frac{\partial_l}{\partial \Phi_A^*} (-)^{\gamma_A+1} \quad (2.2)$$

(where $\epsilon_A = 0$ if Φ^A is bosonic and 1 if it is fermionic) for reasons to be explained. When Δ acts on bosons (as on W) the last sign factor can be removed by changing the last right derivative into a left derivative. Introducing the 'antibrackets'

$$(X, Y) \equiv \frac{\partial_r X}{\partial \Phi^A} \frac{\partial_l Y}{\partial \Phi_A^*} - \frac{\partial_r X}{\partial \Phi_A^*} \frac{\partial_l Y}{\partial \Phi^A} = -(Y, X) (-)^{(\epsilon_X+1)(\epsilon_Y+1)}, \quad (2.3)$$

the **master equation** (2.2) can be written as

$$(W, W) = 2i\hbar \Delta W. \quad (2.4)$$

Expanding in orders of $\hbar$:

$$W = S + \sum_{p=1}^{\infty} \hbar^p M_p, \quad (2.5)$$

the master equation gives

$$(S, S) = 0 \quad (2.6)$$

$$(M_1, S) = i\Delta S \quad (2.7)$$

$$(M_p, S) = i\Delta M_{p-1} - \frac{1}{2} \sum_{q=1}^{p-1} (M_q, M_{p-q}) ; \quad p \geq 2. \quad (2.8)$$

Two more requirements have to be imposed : the nondegeneracy of the functional integral and the correctness of the classical limit. These give conditions on S . The first one is translated in the requirement that S should be a 'proper' solution of the master equation eq.(2.6). By that we mean that the matrix of second derivatives of S (the Hessian) should be of maximal rank at the stationary point (where all first derivatives vanish). As the derivative of $(S, S) = 0$ already implies that this Hessian has as many zero modes as the number of fields Φ^A , this maximal rank is also equal to this number. The second condition, the **classical limit**, is the requirement

$$S(\Phi, \Phi^* = 0) = S(\phi^i) \quad (2.9)$$

The variables ϕ^i , corresponding to the classical fields, are in general a subset of the variables Φ^A : ghost fields are not included.

If there are no gauge invariances we can take $S(\Phi, \Phi^*) = S$. Gauge invariance just means that the Hessian is not of maximal rank, so that this

²Right and left derivatives are denoted by $\frac{\partial_r}{\partial \Phi^A}$ and $\frac{\partial_l}{\partial \Phi_A^*}$, and defined by $\frac{\partial_r}{\partial \Phi^A}(\eta \Phi^B) = \eta \delta_A^B$ and $\frac{\partial_l}{\partial \Phi_A^*}(\eta \Phi^B) = +\eta \delta_A^B$ if η or Φ^A are bosonic, and $-\eta \delta_A^B$ if η and Φ^A are fermionic.

solution is not 'proper'. It implies that there are transformations $\delta\phi^I = R^I_U$ (not proportional to field equations)³ such that

$$\frac{\partial_r S}{\partial\phi^I} R^I_U = 0. \quad (2.10)$$

A proper solution of the master equation can be given by introducing ghost fields c^U with statistics opposite to the parameter of the corresponding transformation ($\epsilon(c^U) = \epsilon_U + 1$), and demanding that

$$\left. \frac{\partial_r S(\Phi, \Phi^*)}{\partial\phi^I \partial c^U} \right|_{\Phi^* = 0} = R^I_U(\phi). \quad (2.11)$$

Treating all gauge invariances this way, one obtains the minimal set of fields in which to express a proper solution of the master equation with the boundary condition eq.(2.9).

$$\{\Phi^A\}_{\min} = \{\phi^I, c^U\} \quad (2.12)$$

A solution for S now always exists [8] with zero ghost number, where ghost numbers are assigned to ϕ^* (-1), to c (1), and to c^* (-2).

We now discuss the quantum action $W_\Psi(\Phi)$ in eq.(2.1). As already mentioned we will need gauge fixing to define it. The gauge is fixed by choosing an arbitrary 'gauge fermion' $\Psi(\Phi)$ of ghost number -1. The quantum action is then given by restricting W to a surface Σ

$$W_\Psi(\Phi) = W(\Phi^A, \Phi_\lambda^*)|_\Sigma, \quad (2.13)$$

where Σ is the surface defined by

$$\zeta_\lambda \equiv \Phi_\lambda^* - \frac{\partial\Psi(\Phi)}{\partial\Phi^\lambda} = 0. \quad (2.14)$$

The remaining problem is now how to arrange for the **nondegeneracy of the functional integral**, i.e. to define a gauge fermion such that Z_Ψ is well defined. With the fields in the minimal set (eq.(2.12)) we have no fields with negative ghost number, so we cannot define such a Ψ . For this reason, Batalin and Vilkovisky introduce the usual antighosts ($\bar{c}$) and their antifields, and Nakanishi-Lautrup fields (π). Note that these antighosts are not the antifields of the ghosts introduced previously: the usual Faddeev-Popov ghosts and antighosts each have their own antifields. The statistics and ghost numbers are given by

$$\begin{aligned} \epsilon(\bar{c}_U) &= \epsilon_U + 1 & gh(\bar{c}_U) &= -1 \\ \epsilon(\pi_U) &= \epsilon_U & gh(\pi_U) &= 0 \end{aligned} \quad (2.15)$$

³We will not repeat here the solution for the case that these gauge transformations have zero modes (see [2]).

A new proper solution of the master equation, including now the antighosts, is then given by

$$S(\Phi, \Phi^*) = S(\Phi_{\min}, \Phi_{\min}^*) + \bar{c}^{*U} \pi_U. \quad (2.16)$$

In order to completely fix the gauge invariances, the gauge fermion should satisfy at the classical stationary point

$$rank \left[\frac{\partial_r \partial_r \Psi}{\partial \bar{c}_U \partial \phi^I} R^I \right] = m \quad (2.17)$$

where m is the number of gauge invariances. Of course this number is infinite in the case of local gauge symmetries, but the equality should hold at each point in spacetime. The simplest form for the gauge fermion is

$$\Psi(\Phi) = \bar{c}_U \Psi^U(\phi) \quad (2.18)$$

in which case the gauge fixing conditions become $\Psi^U = 0$.

A final remark. We also want to study theories with rigid symmetries. To use this framework to study their anomalies, we introduce also a ghost mode for these symmetries. So this is a ghost field with only one mode, i.e. it does not depend on the space-time coordinates.

2.2 BRST transformations.

We now want to make the connection to the BRST formalism. BRST transformations are defined on functions of fields only (not antifields). The definition of the BRST transformation is

$$\delta_{BRST} f(\Phi) = \frac{\partial_r f}{\partial\Phi^\lambda} \frac{\partial_r S}{\partial\Phi^\lambda} \Big|_\Sigma \Lambda = (f, S)|_\Sigma \Lambda, \quad (2.19)$$

where Λ is a constant anticommuting parameter. From the above definition we can obtain the BRST transformation of a gauge fixed quantity

$$\begin{aligned} \delta_{BRST} (F(\Phi, \Phi^*))|_\Sigma &= \left[\frac{\partial_r F}{\partial\Phi^\lambda} \frac{\partial_r S}{\partial\Phi^\lambda} + \frac{\partial_r F}{\partial\Phi_\lambda^*} \frac{\partial_r \Psi}{\partial\Phi^\lambda} \frac{\partial_r S}{\partial\Phi_\lambda^*} \right] \Lambda \\ &= \left[(F, S) + \frac{\partial_r F}{\partial\Phi_\lambda^*} \frac{\partial_r S}{\partial\Phi^\lambda} + \frac{\partial_r F}{\partial\Phi_\lambda^*} \frac{\partial_r \Psi}{\partial\Phi^\lambda} \frac{\partial_r S}{\partial\Phi_\lambda^*} \right] \Lambda \\ &= (F, S)|_\Sigma \Lambda + \frac{\partial_r F}{\partial\Phi_\lambda^*} \Big|_\Sigma \frac{d_r}{d\Phi^\lambda} S_\Psi \Lambda \end{aligned} \quad (2.20)$$

where the last factor is the field equation of the gauge fixed action. (In the terminology of [1,2] S contains minimal and non-minimal terms). We have the following theorem :

Theorem 1 *The quantum action $S^q(\Phi) = S(\Phi, \Phi^*)|_\Sigma$ is invariant under BRST transformations if and only if $(S, S)|_\Sigma = 0$*

The proof follows from the identity

$$\begin{aligned} \delta S(\Phi, \Phi^*)|_\Sigma &= \frac{\partial_r S}{\partial \Phi^A}|_\Sigma \delta \Phi^A + \frac{\partial_r S}{\partial \Phi_A^*} \frac{\partial_r S}{\partial \Phi^B} \frac{\partial \Psi}{\partial \Phi^A} \delta \Phi^B \Big|_\Sigma \\ &= \frac{1}{2} (S, S)|_\Sigma \Lambda + \frac{\partial_r S}{\partial \Phi_A^*} \frac{\partial_r S}{\partial \Phi^B} \frac{\partial \Psi}{\partial \Phi^A} \frac{\partial S}{\partial \Phi^B} \Big|_\Sigma \Lambda. \end{aligned} \quad (2.21)$$

In the last term the symmetry factor from interchanging Φ^A and Φ^B is opposite to that from interchanging Φ_A^* and Φ_B^* , and therefore the last term vanishes.

The action S is at the same time the BRST generator Q . In the usual formulations for open algebras the equation $Q^2 = 0$ is only true on-shell [9]. In the BV formalism this fact emerges very clearly from eq.(2.20) and the Jacobi identities for the antibrackets [8]

$$\delta_2 \delta_1 \Phi^A = \left[\frac{1}{2} (\Phi^A, (S, S)) + \frac{\partial_r \partial_l S(\Phi, \Phi^*)}{\partial \Phi_B^* \partial \Phi_A^*} \frac{d_l}{d \Phi^B} S_\Psi(\Phi) \right]_\Sigma \Lambda_2 \Lambda_1. \quad (2.22)$$

The first term vanishes if $(S, S) = 0$ (not only $(S, S)|_\Sigma = 0$). S contains terms quadratic in antifields if the algebra closes only modulo classical field equations ('open algebra'). Thus the first factor of the second term is only non-zero if the algebra is open. So one finds immediately that $Q^2 = 0$ exactly when the algebra is closed, and for open algebras up to field equations.

2.3 Dependence on the gauge choice

Anomalies manifest themselves if the classical symmetries are not satisfied by the full quantum action. Then the path integral depends on the gauge chosen. We will consider only gauges connected by infinitesimal transformations (for a more general discussion see [10]). Consider the dependence of the functional integral on the gauge fermion:

$$Z_{\Psi+\delta\Psi}(J) = Z_\Psi + \int D\Phi e^{(\frac{1}{\hbar}W(\Phi, \Phi^*)+\mathcal{U}(\Phi))} \frac{\partial \delta\Psi}{\partial \Phi^A} \frac{i}{\hbar} \frac{\partial W(\Phi, \Phi^*)}{\partial \Phi_A^*} \Big|_\Sigma. \quad (2.23)$$

To evaluate the last term, we first consider a change of variables in the functional integral for Z_Ψ . Changing variables from Φ to $\Phi + \delta\Phi$ leaves it invariant. We take

$$\delta\Phi^A = \frac{\partial_l W(\Phi, \Phi^*)}{\partial \Phi_A^*} \Big|_\Sigma \Lambda(\Phi) \quad (2.24)$$

where Λ is an infinitesimal fermionic function. The transformation of the measure is

$$\int D\Phi (-)^a \frac{\partial \delta\Phi^A}{\partial \Phi^A} = \int D\Phi \left(\Delta W(\Phi, \Phi^*) \Lambda - \frac{\partial \Lambda(\Phi)}{\partial \Phi^A} \frac{\partial W(\Phi, \Phi^*)}{\partial \Phi_A^*} \right) \Big|_\Sigma, \quad (2.25)$$

while the transformation of the integrand gives, using the same arguments as in eq.(2.21),

$$\int D\Phi e^{\frac{1}{\hbar}W(\Phi, \Phi^*)} \left(\frac{i}{2\hbar} (W, W) + (J, W) \right) \Big|_\Sigma \Lambda \quad (2.26)$$

The sum of the contributions of eq.(2.25) and eq.(2.26) vanishes

$$0 = \int D\Phi e^{\frac{1}{\hbar}W(\Phi, \Phi^*)} \left(\frac{-i}{2\hbar} [2i\Delta W - (W, W)] + (J, W) \right) \Big|_\Sigma \Lambda - (\Lambda, W) \Big|_\Sigma. \quad (2.27)$$

For constant Λ this expression is the generator of BRST Ward identities. Adding eq.(2.27) to eq.(2.23) with $\Lambda = \frac{i}{\hbar} \delta\Psi$ we find

$$\begin{aligned} Z_{\Psi+\delta\Psi}(J) &= Z_\Psi + \int D\Phi e^{\frac{1}{\hbar}W(\Phi, \Phi^*)} \times \\ &\quad \left(\frac{-i}{2\hbar} [2i\Delta W - (W, W)] + (J, W) \right) \Big|_\Sigma \frac{i}{\hbar} \delta\Psi. \end{aligned} \quad (2.28)$$

So the path integral is independent of the gauge choice for BRST-invariant sources, $(J, W) = 0$, if and only if the master equation eq.(2.4) is satisfied.

We stress that this discussion was formal. If S is local, which we will always assume, ΔS is not well-defined. This shows the necessity of a regularisation scheme.

2.4 Solving the master equation to order $\hbar$.

In the previous subsection we have shown that the path integral with BRST invariant sources is independent of the gauge choice iff the master equation is satisfied. We now want to solve the master equation. Batalin and Vilkovisky proved [8] that there is always a solution to the equation at lowest order in $\hbar$, eq.(2.6), for finite stage theories (series with a finite number of ghosts for ghosts). For infinite stage theories also solutions have been found [3]. This determines the lowest order of W which is $S(\Phi, \Phi^*)$. Note that at this stage no choice of the gauge fixing function Ψ is needed.

If $\Delta S = 0$ then the master equation is completely satisfied by putting $M_p = 0$, $p \geq 1$, and we have finished. No anomalies occur. We now suppose $\Delta S \neq 0$. There are then two strategies :

1. Look for a local function M_1 such that $(M_1, S) - i\Delta S = 0$.

2. Look for new variables Φ such that $\Delta S = 0$. In this case eq.(2.7) is satisfied with $M_1 = 0$. At the same time we want to keep $(S, S') = 0$.

Canonical transformations can be used for the second strategy. Suppose one has a theory described by a path-integral as in eq.(2.1), satisfying the master equation in eq.(2.4). Then this theory has no anomalies. Let us now perform a canonical transformation. A canonical transformation in the BV scheme preserves the BV brackets, and if the matrix $\frac{\partial \Phi^p}{\partial \psi^a}$ at fixed Φ^q is invertible, it may be shown that the new variables are obtained from an anticommuting functional $F(\Phi, \Phi^*)$ (see the appendix). The operator Δ is not invariant under such canonical transformations; rather, it transforms as in eq.(A.47). The master equation will remain satisfied, and anomalies will remain absent, if one changes W as in eq.(A.56). This change amounts to adding a counterterm to the action to remove anomalies. More generally, if some symmetries have anomalies, then adding M_p -terms may shift the anomalies to other symmetries.

As examples of canonical transformations we mention :

- ordinary field redefinitions $\Phi'(\Phi)$ (point transformation) can be extended to a canonical transformation. In this case

$$F(\Phi, \Phi^*) = \Phi_A^* \Phi'^A(\Phi). \quad (2.29)$$

- adding equation of motion terms to symmetries

$$\delta_{new} \phi^A = \delta_{old} \phi^A + U^{AB}_\alpha c^B - \frac{\partial_t S(\phi)}{\partial \phi^B} \Lambda \quad (2.30)$$

can also be interpreted as canonical transformations. In these cases F is of the form

$$F = \phi_A^* \phi^A + c_\alpha^* c^\alpha - \frac{1}{2} \phi_A^* \phi_B^* U^{BA}_\alpha(\phi) c^\alpha. \quad (2.31)$$

The relation between the two approaches is as follows. If ΔS is cancelled by adding a counterterm M_1 , then $M_1 = -\frac{i}{2} \ln J$ gives the Jacobian for the canonical transformation to Φ variables such that $\Delta S' = 0$.

2.5 Anomalies

We now finally turn to anomalies. If the master equation cannot be solved to order $\hbar$ with a local M_1 , we will say that the system has anomalies. As we showed in subsection 2.3, this implies a dependence of Z on the gauge choice. Since eq.(2.7) has ghost number 1, its violation necessarily contains the ghost fields. They act as a book-keeping device for the anomalies related to the different symmetries. Indeed, suppose

$$iA(\Phi, \Phi^*) \equiv i\Delta W - \frac{1}{2\hbar}(W, W) = a_\nu c^\nu \quad (2.32)$$

and choose a gauge fermion of the form eq.(2.18). Then eq.(2.28) gives for the dependence on the gauge fixing :

$$Z_{\Psi+\delta\psi}(J) = Z_\Psi + \int D\Phi e^{\frac{1}{\hbar}W_\Psi + J(\Phi)} \left(\frac{1}{\hbar} a_\nu c^\nu \dot{c}_\nu \delta\psi^\nu \right). \quad (2.33)$$

In the case of a closed algebra with Ψ of the form eq.(2.18), W_Ψ will contain only terms linear in $\dot{c}$ and in c :

$$W_\Psi = \dots + \dot{c}_U A^U_{V,c^V} \quad (2.34)$$

with

$$A^U_V = \frac{\partial_\psi \Psi^U}{\partial \phi^V} R^U_V, \quad (2.35)$$

and

$$\frac{\partial_t Z_\Psi}{\partial \Psi^U} = i \int D\Phi a_U (\Lambda^{-1})^V_U c^{\frac{1}{2}W_\Psi + J(\Phi)} \quad (2.36)$$

for sources which do not contain the ghosts. As usual the c , $\dot{c}$ integration is performed using the identity

$$0 = \int Dc D\dot{c} \frac{\partial}{\partial \dot{c}_U} \dot{c}_U c^{\frac{1}{2}W_\Psi + J(\Phi)}. \quad (2.37)$$

The one-loop form of the anomaly equation is thus

$$iA(\Phi) = i\Delta S - (S, M_1)|_\Sigma = a_\nu c^\nu|_\Sigma \quad (2.38)$$

Even when the r.h.s. of eq.(2.38) is not zero, it may be possible to arrange for certain ghost fields to be absent, by a judicious choice of M_1 . The corresponding symmetries have no anomalies. Changing M_1 will in general shift the anomalies from one symmetry to another. This process will be called 'adding a counterterm'.

Which symmetries one wants to keep anomaly-free may be a matter of physical preference, or of necessity (for example to preserve renormalisability). In gauge theories with vector and axial gauge invariances, one may choose to preserve the vector current conservation. In theories with reparametrisation and Weyl invariance, one may impose that the general coordinate transformations are kept anomaly-free. In models with local supersymmetry [11] one can choose M_1 such that the local supersymmetry transformations are also without anomaly.

In the paragraph above we only considered M_1 . In general, once M_1 is found, one should look at eq.(2.8) to determine whether there are anomalies in higher orders of $\hbar$. If M_1 depends only on Φ and not on Φ^* , eq.(2.8) is solved immediately by $M_p = 0$ for $p \geq 2$, and we have finished. This will always be the case in the examples below. In fact if the algebra is closed, S has only linear terms in Φ^* , and therefore ΔS has no Φ^* . Therefore

(and because the transformations are not proportional to field equations) we expect M_1 to be only a function of Φ , and no higher orders of $\hbar$ are necessary. This shows that for closed algebras anomalies occur only at one loop. However, M_p might also be related to renormalisation terms. If the transformations get renormalised, then it is possible that $\Delta M_p \neq 0$, and then one should analyse eq.(2.8) to discuss higher loop anomalies.

The anomaly consistency equations [12] get a simple form in this framework. From $(S, S) = 0$ one derives, by taking two derivatives and using symmetry properties to cancel various terms, that $(S, \Delta S) = 0$. Using the Jacobi identity $(S, (S, M_1)) = 0$ one then finds $(S, \mathcal{A}(\Phi, \Phi^*)) = 0$. Using eq.(2.20) we have for the one-loop anomalies (we will further restrict ourselves to one loop)

$$\delta_{BRS}(\mathcal{A}(\Phi, \Phi^*))|_S = \left[(\mathcal{A}, S) + \frac{\partial_t \mathcal{A}}{\partial \Phi_a^*} d_t \Phi_a^* S_\Phi \right]_S. \quad (2.39)$$

The previous arguments imply the vanishing of the first term. In the examples the anomalies are independent of antifields, and thus the right hand side vanishes. Therefore we find that $i\mathcal{A}(\Phi) = a_U c^U|_S$ is BRS invariant, and that statement contains all the consistency equations.

The general result is that the first term still vanishes, and then eq.(2.39) still gives the consistency equations in the case of open algebras or with ghosts for ghosts. This discussion presupposes a regularisation scheme that acts in the space of fields and antifields so that one determines $\mathcal{A}(\Phi, \Phi^*)$ and that respects $(S, \Delta S) = 0$.

We now turn to regularisation.

3 Regularisation

The expressions in the previous section are only formal expressions. For a local function S , the expression ΔS is proportional to $\delta(0)$. Indeed, our sums over $\mathcal{A}$ include integrations over the space-time points, so

$$\Delta S = \int dx \frac{\partial_t}{\partial \phi^a(x)} \frac{\partial_t}{\partial \phi_a^*(x)} \int dy \mathcal{L}(\phi(y), \phi^*(y)) \quad (3.1)$$

which leads to two $\delta(x - y)$ factors and hence $\delta(0)$. So to make sense out of these expressions we have to introduce a regularisation scheme.

In a previous publication [6] we have shown how several theories can be regularised at one loop by a Pauli-Villars (PV) regularisation, which after integrating over the PV fields becomes a Fujikawa regulator for the measure. We will now repeat this using the BV language, and meanwhile give an extension of the formulas presented in [6]. The expressions in the BV formalism correspond of course to concepts known before under other names. For the convenience of the reader, we now first give a vocabulary.

3.1 Vocabulary

- The expression $(S, S)|_S = 0$ corresponds to the BRS invariance of the quantum action. This was shown in subsection 2.2. If it is not zero, then it has ghost number 1. Usually it is then proportional to ghosts, and we can say that the gauge invariances which correspond to these ghosts are broken.
- $\Delta S|_S$ corresponds to the logarithm of the Jacobian of the infinitesimal BRS transformation even for open algebras. The remark that it is proportional to $\delta(0)$ corresponds to the well known fact that we need a regularisation to define the of the measure. Fujikawa [4] introduced such regularisations. We will explain and extend his procedure in the next subsection.
- M_1 corresponds to a local counterterm. If the Jacobian is non-zero, then there is only an anomaly if it cannot be absorbed in the variation of a local counterterm. The BRS variation of the counterterm is in BV language $(S, M_1)|_S$ (if M_1 does not depend on antifields), and the anomaly comes from $(M_1, S) - i\Delta S$. In other words, adding an M_1 corresponds to the addition of local counterterms to remove candidate anomalies, or to shift anomalies from one symmetry to another [12].

3.2 The Fujikawa regularisation and its relation to Pauli-Villars

As mentioned above, we need regularisation to define ΔS . The procedure advocated by Fujikawa replaces it with a regulated expression of the form

$$\mathcal{F} = \text{Tr } F \exp \mathcal{R} / \mathcal{M}^2 \quad (3.2)$$

where F is the logarithm of the Jacobian matrix, $\mathcal{R}$ is a regulating matrix and $\mathcal{M}$ is the regulating mass. Tr is a trace over function space. It includes an integral over the $\mathbf{x}$ -variables. If the result is expanded in powers of the mass

$$\mathcal{F} = \sum_{n=-n_0}^{\infty} f_n (\mathcal{M}^2)^{-n}, \quad (3.3)$$

the renormalisation procedure should eliminate, if necessary, terms like $\ln \mathcal{M}$. The positive values of n disappear because one considers a limit $\mathcal{M} \rightarrow \infty$, while the negative values are dropped as part of the renormalisation procedure. We define

$$\Delta^{(R)} S \equiv f_0. \quad (3.4)$$

The ‘regulator’ $\mathcal{R}$ is some operator related to the action. We will now indicate which operator we have to use for $\mathcal{R}$, and define the Jacobian matrix F . To do so, we start from a PV scheme. In PV an anomaly does not originate from a Jacobian factor but from a non-BRST-invariant mass term introduced for the regularisation. In the BV language, the PV scheme is defined such that

$$\Delta(S + S^{PV}) = 0. \quad (3.5)$$

Anomalies correspond to a violation of the master equation. When the PV fields are still present, the anomaly does not manifest itself in a violation of eq.(2.7) but of eq.(2.6): then $(S + S^{PV}, S + S^{PV}) \neq 0$. On the other hand, if we integrate out the PV fields, we will find that the limit $\mathcal{M} \rightarrow \infty$ corresponds to the result obtained from the Fujikawa scheme. The anomalies appear after this integration in a completely different form. In fact $(S, S) = 0$, and the anomalies appear as ΔS in the $\hbar$ part of the master equation. The identification of the result of the PV procedure will fix the Jacobian F and the regulator $\mathcal{R}$ to be used in the Fujikawa expression corresponding to ΔS .

3.3 The Pauli-Villars scheme

First we have to construct a PV regularisation. For each field Φ which can occur in a loop one introduces a PV partner χ . It has the same statistics as Φ , but the path integral is formally defined such that a minus sign is

produced in loops (see [6]). The PV fields should have the same couplings as their partners. Also the propagators should agree apart from the mass terms. The latter should be such that the PV particles disappear in the limit $\mathcal{M} \rightarrow \infty$. The above requirements should be met for the gauge fixed action $S_{\Sigma}(\Phi) \equiv S(\Phi, \Phi^*)|_{\Sigma}$. Note that a particular PV regularisation may be only suitable for some class of gauge fermions. In this way the regularisation possibly introduces a new gauge dependence. The proof in section 2.3 is then only valid for the gauge dependence within the class of gauges for which this regularisation applies.

The BRST transformations of the PV fields are defined such that the total measure is invariant, and such that the massless part of the PV action is invariant. If the mass terms are not invariant, their variation may give rise to anomalies.

In formulas this leads to the following generating function :

$$S^{PV} = \frac{1}{2} \chi^A (T\mathcal{O})_{AB} \chi^B - \frac{i}{2} M \chi^A T_{AB} \chi^B + \chi_{\lambda}^A K^A{}_{\bar{b}} \chi^{\bar{b}} + \chi^{\bar{a}} \chi^{\bar{a}} \dots \quad (3.6)$$

M is related to the regulator mass. For example, for the usual bosons it is M^2 , while for the Dirac fermions it is equal to M . The matrix T_{AB} is in principle arbitrary except that it must be invertible (at least on the fields for which $T\mathcal{O} \neq 0$). The invertibility assures that the PV fields disappear in the limit $\mathcal{M} \rightarrow \infty$. Also, T may depend on the fields Φ . We take it symmetric in the sense $T_{BA} = (-)^{A+B+AB} T_{AB}$ and then the inverse satisfies $(T^{-1})^{BA} = (-)^{AB} (T^{-1})^{AB}$.

The matrices $T\mathcal{O}$ (symmetric in the same sense) and K are determined by the previous requirements. In particular,

$$(T\mathcal{O})_{AB} = \frac{\partial_i}{\partial \Phi^A} \frac{\partial_r}{\partial \Phi^B} S_{\Sigma}. \quad (3.7)$$

For K we cannot write a general expression. The requirement of invariance of the massless action gives

$$\chi^A \left[(T\mathcal{O}K)_{AB} + \frac{i}{2} \delta(T\mathcal{O})_{AB} (-)^B \right] \chi^B = 0 \quad (3.8)$$

where δ stands for the operation $\delta X = (X, S)$.

In addition we have to impose invariance of the measure, eq.(3.5). The cancellation of the contributions to the measure is obtained because of an extra minus sign in the definition of the path integral over PV fields (or extra factors c_i if several copies of PV fields occur [7,6]). Going through our derivation of the dependence on the gauge choice, subsection 2.3, we see that this implies a modification of the definition of the operation Δ on PV fields by these extra factors.)

We then satisfy eq.(3.5) by a mode-by-mode cancellation between the contributions to ΔS and $\Delta S^{PV} = -K^A{}_A$. The PV regularisation consists

thus in first summing over a pair of modes of an ordinary field and a PV field, before making the limit to an infinite number of modes. These pairs do not have to coincide with the naive pairing of fields used before in constructing the action. This was illustrated in [6]. There eq.(3.5) has been satisfied by dividing the fields into subsets (e.g. the matter fields, the gauge fields, the ghosts, ...). In some subsets one had a (field independent) invertible matrix P^λ (with statistics $c_A + c_B$) such that

$$\left(\frac{\partial_\lambda}{\partial \Phi^\lambda} \frac{\partial_\tau S}{\partial \Phi^\tau} \right)_\Sigma = P^\lambda_C K^C_D (P^{-1})^D_B. \quad (3.9)$$

The matrix P defines here the pairing of fields Φ with PV fields χ whose modes have to be added first to guarantee eq.(3.5).

For other subsets, we had a relation

$$\left(\frac{\partial_\lambda}{\partial \Phi^\lambda} \frac{\partial_\tau S}{\partial \Phi^\tau} \right)_\Sigma = P^{\lambda C} (K^T)_C^D (P^{-1})_{DB}, \quad (3.10)$$

where the subscript T denotes the supertranspose in the sense

$$(K^T)_C^D = K^D_C (-)^{P(C+1)}. \quad (3.11)$$

P_{AB} has now statistics $c_A + c_B + 1$. The connection which P defines is now between modes of fields and dual modes of the PV fields (modes of the antifields, see [6]).

We now consider the master equation at the surface Σ . The terms with more than one antifields in S^{PV} can be omitted. Indeed, as the gauge fermion was chosen not to depend on PV fields, the PV antifields vanish on Σ . The violation of the master equation manifests itself in

$$\frac{1}{2}(S + S^{PV}) \Big|_\Sigma = M \left(\chi^A T_{AC} K^C_B \chi^B + \frac{1}{2} \chi^A \delta T_{AB} \chi^B (-)^B \right). \quad (3.12)$$

This is the form in which anomalies show up in the PV scheme.

We now perform the functional integral over the PV fields. Then the χ -fields in eq.(3.12) are replaced by their propagator, and we get (from eq.(2.32))

$$\begin{aligned} \mathcal{A} &= \left((TK)_{AB} + \frac{1}{2} \delta T_{AB} (-)^B \right) \left(\frac{M}{TM - T\mathcal{O}} \right)^{AB} \\ &= \left(K + \frac{1}{2} T^{-1} \delta T (-)^B \right)_B^A \left(\frac{M}{M - \mathcal{O}} \right)^B_A. \end{aligned} \quad (3.13)$$

If the propagators are linear in momenta, it is expedient to multiply on the right by $\frac{M+\mathcal{O}}{M-\mathcal{O}}$ in order to recognise the regulators. Then we get

$$\begin{aligned} \mathcal{A} &= \left(K + \frac{1}{2} T^{-1} \delta T (-)^B \right)_B^A \left(\frac{M^2}{M^2 - \mathcal{O}^2} \right)^B_A \\ &\quad + \frac{1}{2} \left(K + T^{-1} K^T T + T^{-1} \delta T (-)^B \right)_B^A \left(\frac{M\mathcal{O}}{M^2 - \mathcal{O}^2} \right)^B_A \end{aligned} \quad (3.14)$$

where we have used symmetry properties of T and $\mathcal{O}$ to write the second line. This term can be simplified by observing that from the invariance of the massless PV action one has, from eq.(3.8), the identity

$$\begin{aligned} 0 &= (T\mathcal{O}K)_{AB} + (T\mathcal{O}K)_{BA} (-)^{AB} + \delta(T\mathcal{O})_{AB} (-)^B \\ &= (\mathcal{O}K)_B^A + (T^{-1} K^T T\mathcal{O})_B^A + (T^{-1} \delta T)_B^A (-)^C \mathcal{O}_B^C + \delta \mathcal{O}_B^A (-)^B. \end{aligned} \quad (3.15)$$

Using this identity in eq.(3.14) we find

$$\mathcal{A} = \left(K + \frac{1}{2} T^{-1} \delta T (-)^B - \frac{1}{2M} \delta \mathcal{O} (-)^B \right)_B^A \left(\frac{M^2}{M^2 - \mathcal{O}^2} \right)^B_A \quad (3.16)$$

The results in both cases eq.(3.13) and eq.(3.16) have thus the common form

$$\mathcal{A} = T^\tau F \frac{1}{1 - \mathcal{R}/M^2}. \quad (3.17)$$

If $\mathcal{O}$ is quadratic in momenta, we will use

$$\begin{aligned} M^2 &= M \\ \mathcal{R} &= \mathcal{O} \\ F^A_B &= \left(K + \frac{1}{2} T^{-1} \delta T (-)^B \right)_B^A. \end{aligned} \quad (3.18)$$

If $\mathcal{O}$ is only linear, we will use

$$\begin{aligned} M &= M \\ \mathcal{R} &= \mathcal{O}^2 \\ F^A_B &= \left(K + \frac{1}{2} T^{-1} \delta T (-)^B - \frac{1}{2M} \delta \mathcal{O} (-)^B \right)_B^A. \end{aligned} \quad (3.19)$$

3.4 Results for the Fujikawa regularisation of ΔS

We now show that the result eq.(3.17) is equivalent to a Fujikawa regularisation.

Using

$$\frac{1}{1 - \mathcal{R}/M^2} = \int_0^\infty \exp(\lambda \mathcal{R}/M^2) \exp(-\lambda) d\lambda \quad (3.20)$$

and eqs. (3.2), (3.3), we obtain

$$\mathcal{A} = \int_0^\infty \left\{ \sum_{n=-n_0}^\infty \int_n \left(\frac{\lambda}{M^2} \right)^n \right\} e^{-\lambda} d\lambda \quad (3.21)$$

In the PV scheme the divergent terms are eliminated by taking several copies of the PV fields, weighted with factors c_i which satisfy

$$\begin{aligned} \sum_i c_i &= 1 \\ \sum_i c_i M_i^{2n} &= 0 \quad (n = 1, 2, \dots, n_0) \end{aligned} \quad (3.22)$$

The limit $\mathcal{M} \rightarrow \infty$ of the result of the PV procedure gives then

$$\mathcal{A} = f_0. \quad (3.23)$$

This is clearly the result using Fujikawa regularisation of $\mathcal{A} = \Delta S$, if we identify F , $\mathcal{R}$ and $\mathcal{M}$ as above. Thus we have used the PV regularisation to arrive at an expression for the anomaly which coincides with Fujikawa's regularised expression, with in addition a specific prescription for a regularised Jacobian and a regulator $\mathcal{R}$.

The most important feature of eq.(3.18) and eq.(3.19) is that if we start from such a PV action, then we obtain a regulator which gives automatically rise to consistent anomalies. The consistency is guaranteed because of the procedure: first a definition of the path integral is given (with the PV prescription) and then its full variation is computed. This is to be contrasted with the procedure where first a formal expression is obtained for a Jacobian, and then a regulator is added by hand. The anomalies one obtains in this way may not be consistent. In fact by choosing other expressions for $\mathcal{R}$ in eq.(3.2) one can obtain 'covariant anomalies' which do not satisfy the consistency conditions. The only way to check the consistency is then a posteriori, after computing the anomalies explicitly. We do not know if a similar general prescription can be given which guarantees one to obtain a covariant anomaly.

The second new feature is the modified form of the Jacobian of the transformation eq.(3.19). Extra terms have turned up when T or $\mathcal{O}$ are field-dependent. In the examples we present in this paper and in [5], the $\delta\mathcal{O}$ term does not contribute because $\delta\mathcal{O}$ is off-diagonal while $\mathcal{O}^2$ is diagonal. However, the second term allows us now to consider also variables and regulators such that T is field dependent. It was argued before that in a gravitational background one should only use certain 'Fujikawa variables' in order to have no anomalies in general coordinate transformations. As we will show in the examples, the $T^{-1}\delta T$ modification allows one to circumvent that restriction. When one does not want to add extra terms to the naive Jacobian, one may still use other quantum variables than those of Fujikawa. We claim that the anomaly one seems to produce is not genuine: a counterterm M_1 will remove it when one uses consistent regulators.

Concerning the regulator an extra remark is in order. The indices A , B in eq.(3.13) and eq.(3.16) referred to the PV fields. We can translate these equations to the ordinary fields using eq.(3.7) and eq.(3.9) or eq.(3.10). So therefore the regulator is not necessarily related to the kinetic term of the field of which we calculate the Jacobian. An example of this was given in [6]. The PV fields, associated to the antighosts as far as the action is concerned (reproducing the same propagators and couplings), were paired to the gravitational field as far as the measure is concerned (for obtaining eq.(3.5)). We used in that case a relation of the form eq.(3.10). Thus in the

expression for the anomaly, F referred to the Jacobian of the transformation of the gravitational field while the regulator was connected to the kinetic terms of the antighosts. The matrix P makes this connection. In this way a non-propagating field can also be regularised. Examples of this were given in [5].

3.5 The choice of the regulator

An important advantage of our prescription is that a large amount of freedom is left for the choice of T , the mass operator in the PV scheme. The only requirement is that T is invertible (and even a non-invertible T can sometimes be used [6]). Each choice of T corresponds to a different regulator. Each choice gives rise to consistent anomalies. However, some choices of T may make life easier. If the PV scheme can use a mass term such that it is invariant under a subset of the gauge transformations, then this subset will automatically be anomaly free. These are the 'preferred symmetries'. So the choice of T can shift anomalies from one sector of symmetries to another one.

In some cases the massive action can be made invariant under some symmetries by modifying the transformation rules of the PV fields by terms proportional to M . If one thinks of such a procedure, one has of course first to study eq.(3.5) again. In this case one will have to connect modes of different fields (identify the modes and sum over these connected modes before taking the limit to an infinite number of modes). We will give an example of this in subsection 4.3. Suppose that this is done. There are then extra terms in S^{PV} of the form

$$S^{PV} = \dots + M \chi_A^* L^A \chi^B. \quad (3.24)$$

We have introduced them to have an invariant massive action (at least for the terms proportional to the ghosts occurring in L). So the cancellation of the M^2 terms implies

$$TL + L^T T = 0. \quad (3.25)$$

These terms would also contribute to the terms linear in M in eq.(3.12) and then in eq.(3.13) we get extra terms

$$\begin{aligned} \mathcal{A}^L &= -\frac{1}{2} \text{Tr} (T\mathcal{O}L + L^T T\mathcal{O}) \frac{M}{TM - T\mathcal{O}} \\ &= -\frac{1}{2} \text{Tr} (\mathcal{O}L - L\mathcal{O}) \frac{M}{M - \mathcal{O}} \end{aligned} \quad (3.26)$$

using eq.(3.25). The cyclicity of the trace puts this to zero if the trace of the 2 terms separately is well regularised. This shows that, if in the PV scheme the massive action can be made invariant by changing the transformations

of the PV fields, then our previous expression for the anomaly in the Fujikawa scheme also automatically gives no anomaly for these symmetries. We present examples of this in subsection 4.3.

3.6 Canonical transformations and transformations of the regulator

In subsection 2.4 we explained how a canonical transformation formally changes the value of ΔS . This change can be absorbed by adding a counterterm $M_1 = -i \ln J^{\frac{1}{2}}$, where $J^{\frac{1}{2}}$ is the Jacobian for the transformation of the fields Φ . However now we have to combine this with our regularisation prescription. When we perform a canonical transformation on the fields Φ , then we must perform this transformation also on the PV fields χ in order that the couplings of pairs of fields remain the same. As PV fields contribute with an extra minus sign to the logarithm of the Jacobian, and so to ΔS^{PV} , the result is now that $\Delta(S + S^{PV})$ does not change. So we have the following theorem :

Theorem 2 *Canonical transformations which are duplicated on the PV fields keep $\Delta(S + S^{PV})$ unchanged.*

In the following section this will be demonstrated on the particle action. However, we make the following conjecture.

Conjecture 1 *By canonical transformations on the PV fields only, a PV system which regularises the path integral is transformed in another PV system which also regularises the path integral.*

In that case these canonical transformations change ΔS , and so an M_1 is generated. This would imply that we can also perform a canonical transformation on the fields only, which would then also lead to an M_1 , as it is a combination of the previous two types of canonical transformations.

As T is arbitrary, a natural question is : what is the relation between the anomalies calculated for two different regulators corresponding to different matrices T_1 and T_2 ? As both regulators give a consistent form of the anomaly, we expect from the general ideas about anomalies [12] that they are related by a local counterterm. In the examples we have constructed this counterterm by interpreting the change of regulator also as a canonical transformation. We conjecture that our formula for a Jacobian of a BRST transformed Jacobian eq.(3.18) and eq.(3.19) also applies for calculating a regularised Jacobian of a canonical transformation. The change of regulator is a canonical transformation which does not change the fields Φ , but because of the $T^{-1}\delta T$ term in eq.(3.18) the Jacobian of the transformation is not zero. Suppose now that there is a continuous path $T_1 : T_1 \rightarrow T_2$ from the first to the second regulator. At each point of the path we have a different

regulator, and therefore the total Jacobian of the transformation can only be calculated by integrating the small changes in T where we know the regulator. In this way we obtain the Jacobian, and therefore M_1 . So we formulate the following conjecture :

Conjecture 2 *The regularised value of ΔS using two different matrices T differ by*

$$i\Delta^{(2)}S = i\Delta^{(1)}S + (S, M_1) \quad (3.27)$$

where M_1 is a local counterterm. If there is a continuous path from one regulator ($T(t = t_1)$) to another ($T(t = t_2)$) then M_1 can be obtained from

$$M_1 = \frac{i}{2} \int_{t_1}^{t_2} dt \text{Tr} \left(T^{-1} \frac{\partial T}{\partial t} \right)^A (-)^B \left(\exp \mathcal{R}_c / M^2 \right)^B_A \quad (3.28)$$

(to be corrected by the $\delta\mathcal{O}$ term in some cases as explained at the end of subsection 3.3)

In the examples we will illustrate this.

4 Examples

4.1 Chiral anomalies in two different PV regulators.

The importance of specifying a regulator to define ΔS is illustrated in this subsection with an example. We will present two computations of ΔS in a specific theory, using two different regulators, which are both based on PV ideas. The results will turn out to be different. The difference will then be written in the form (S, M_I) , demonstrating that the change of regulator can be compensated by a counterterm M_I .

4.1.1 Regularisation for chiral couplings.

Let us first consider the anomaly in a theory where a gauge field V_μ^L couples only to left-handed fermions ψ_L ,

$$S = \int d^4x \bar{\psi}_L \not{D}_L \psi_L \quad (4.1)$$

where $\not{D}_L = (\not{\partial} + \not{A}^{\prime L})\frac{(1+\gamma_5)}{2}$ and $V_\mu^L = V_\mu^{La}\lambda_a$, and λ_a are the (anti-Hermitian) representation matrices (γ_5 is Hermitian, the metric is $(+, +, +, +)$).

The anomaly calculation for this case is of course well known, but we want to rephrase it in the BV language, at the same time also providing a systematic way of constructing consistent regulators.

The gauge invariance leads in the present language to a generating function

$$S = S + \psi_L^* c_L \psi_L - \bar{\psi}_L^* c_L^* (\bar{\psi}_L)^* + V_\mu^{*a} \left(\partial_\mu c_L + [V_\mu^L, c_L] \right) + c_L^* c_L c_L \quad (4.2)$$

(an integral $\int d^4x$ is understood here and in further formulae), where also the ghost is taken to be Lie algebra valued ($c_L = c_L^a \lambda_a$) and the antifields have values in its dual, such that, e.g. the last term is equal to $\frac{1}{2} c_L^a c_L^b c_L^c f_{bc}^a$.

To lowest order, the master equation is satisfied, but the next order in $\hbar$, eq.(2.7), necessitates a regulator. We will consider here the ψ -loops. A regulator is provided in [6] by looking at a PV action for a massive fermion, having the same coupling as ψ_L in eq.(4.1). This can be taken as

$$S^{PV} = \bar{\chi}_L \not{D}_L \chi_L + \bar{\rho}_R \not{\partial}_R \rho_R - M(\bar{\chi}_L \rho_R + \bar{\rho}_R \chi_L) + \chi_L^* c_L \chi_L - \bar{\chi}_L^* c_L^* (\bar{\chi}_L)^* \quad (4.3)$$

where $\not{\partial}_R = \not{\partial} \frac{1-\gamma_5}{2}$. We have introduced here extra PV fields ρ_R . They have no interactions in the massless sector and do not transform. Therefore they are allowed in the context of our requirements on PV regulators. Their only purpose is the construction of the mass terms.

The mass term, on the basis $\chi_L, \bar{\chi}_L, \rho_R, \bar{\rho}_R$ provides us with the twist matrix

$$T = \begin{pmatrix} & & -1 \\ & 1 & \\ -1 & & \\ 1 & & \end{pmatrix} \quad (4.4)$$

whereas the kinetic energy matrix is written as

$$T\mathcal{O} = \begin{pmatrix} & -\psi_L^i & \\ \psi_L & & \\ & & -\phi_R^i \\ & & & \phi_R \end{pmatrix} \quad (4.5)$$

where $\chi_L^i \phi_L^i \bar{\chi}_L^i = -\bar{\chi}_L \not{D}_L \chi_L$ defines the transpose of ϕ_L . The regulator is then constructed out of

$$\mathcal{O} = \begin{pmatrix} & \phi_R & \\ \psi_L & & \\ & \phi_R^i & \end{pmatrix} \quad (4.6)$$

We now use eq.(3.19) for the regularisation in Fujikawa's scheme

$$\mathcal{R} = \mathcal{O}^2 = \begin{pmatrix} \phi_R \phi_L & & \\ \phi_R^i \phi_L^i & \psi_L \phi_R & \\ \psi_L^i \phi_R^i & \psi_L^i \phi_R^i & \end{pmatrix} \quad (4.7)$$

The matrix describing the transformations is

$$K^L = \begin{pmatrix} c_L & & \\ -c_L^i & & \\ & 0 & \end{pmatrix} \quad (4.8)$$

so that we find ($\delta\mathcal{O}$ is off diagonal, and thus can be forgotten with this diagonal regulator)

$$\begin{aligned} \Delta S^{(L)} &= \lim_{M^2 \rightarrow \infty} \text{Tr} \begin{pmatrix} c_L & & \\ -c_L^i & & \\ & 0 & \end{pmatrix} \frac{\mathcal{O}^2}{M^2} \\ &= \lim_{M^2 \rightarrow \infty} \text{Tr} \begin{pmatrix} 1 & & \\ -1 & c_L \exp \frac{1}{M^2} & \\ & & \psi_L \phi_R \end{pmatrix} \end{aligned} \quad (4.9)$$

This leads in the usual way to (including divergent terms) [13, 14]

$$\Delta S^{(L)} = \frac{M^2}{8\pi^2} \text{Tr} c_L \partial^\mu V_\mu^L$$

which can be rewritten in terms of the vector and axial symmetries as

$$i\mathcal{A}^{(1)} = c_V a^{(1)V} + c_A a^{(1)A} \quad (4.19)$$

where

$$c_L = c_V + c_A, \quad c_R = c_V - c_A \quad (4.20)$$

$$a^{(1)V} = a^R + a^L, \quad a^{(1)A} = a^L - a^R. \quad (4.21)$$

Note that $a^{(1)V}$ does not vanish.

On the other hand, one can choose a different PV regulator, by regularising the L and R sectors together with a single PV-field

$$S^{PV} = [\bar{\chi}^{(2)}(\not{p} + \not{Y} + \not{A}\gamma_5)\chi^{(2)} - M\bar{\chi}^{(2)}\chi^{(2)}] \chi^*(c_V + c_A\gamma_5)\chi + \bar{\chi}^{(2)*}(-c_V + c_A\gamma_5)^t(\bar{\chi})^t. \quad (4.22)$$

There is now a different twist-matrix (on the basis $\chi^{(2)}, (\bar{\chi}^{(2)})^t$)

$$T^{(2)} = \begin{pmatrix} 1 & -1 \end{pmatrix}. \quad (4.23)$$

Furthermore,

$$O^{(2)2} = \begin{pmatrix} \not{p}\not{p} & \\ & \not{p}^t\not{p}^t \end{pmatrix} \quad (4.24)$$

$$K = \begin{pmatrix} c_V + c_A\gamma_5 & \\ & (-c_V + c_A\gamma_5)^t \end{pmatrix}, \quad (4.25)$$

and this second regulator gives

$$\begin{aligned} \Delta^{(2)}S &= \lim_{M^2 \rightarrow \infty} \text{Tr } K^{(2)} \exp \frac{O^{(2)2}}{M^2} \\ &= \lim_{M^2 \rightarrow \infty} \text{Tr } \begin{pmatrix} c_V + c_A\gamma_5 & \\ & -c_V + c_A\gamma_5 \end{pmatrix} \exp \frac{1}{M^2} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \not{p}\not{p}. \end{aligned} \quad (4.26)$$

The trace now results in the expression

$$i\Delta^{(2)}S = c_A \left(a^{(2)A} + \text{normal parity terms} \right) \quad (4.27)$$

where

$$\begin{aligned} a_a^{(2)A} &= i \frac{M^2}{2\pi^2} \text{Tr } \lambda_a (\partial_\mu A^\mu + \{V_\mu, A^\mu\}) \\ &+ \frac{i}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr } \lambda_a \left\{ F_{\mu\nu} F_{\rho\sigma} + \frac{1}{3} G_{\mu\nu} G_{\rho\sigma} \right. \\ &\quad \left. - \frac{8}{3} (F_{\mu\nu} A_\rho A_\sigma + A_\mu F_{\nu\rho} A_\sigma + A_\mu A_\nu F_{\rho\sigma}) + \frac{32}{3} A_\mu A_\nu A_\rho A_\sigma \right\}. \end{aligned} \quad (4.28)$$

$$\begin{aligned} &+ \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr } c_L \partial_\mu (V_\nu^L \partial_\rho V_\sigma^L + \frac{1}{2} V_\nu^L V_\rho^L V_\sigma^L) \\ &+ \frac{1}{48\pi^2} \text{Tr } c_L \partial^\mu (\partial^2 V_\mu^L + [\partial_\mu V_\nu^L, V^{\mu\nu}] + 2[V_\mu^L, \partial_\nu V^{\mu\sigma}] \\ &\quad + V_\mu^L V_\nu^L V^{\mu\nu} + V_\mu^L V_\nu^L V^{\mu\sigma} - V_\mu^L V_\nu^L V^{\mu\sigma}) \end{aligned} \quad (4.11)$$

(Tr denotes a trace over group indices only). As is well known, there is no polynomial in the V^L fields which can compensate this expression. In the BV language, this corresponds to the statement that no M_1 can be found such that the master equation is satisfied including the order $\hbar$, eq.(2.7). Choosing for M_1

$$\begin{aligned} M_1^{(L)} &= -i \frac{M^2}{16\pi^2} \text{Tr } V^L V_\mu^L \\ &+ \frac{i}{48\pi^2} \left[(\partial V^L)^2 - \frac{1}{2} (\partial_\mu V_\nu^L)^2 - \frac{1}{2} (V^L)^2 (V^L)^2 + \frac{1}{4} (V_\mu^L V_\nu^L)^2 \right] \end{aligned} \quad (4.12)$$

we find back the usual expression for the anomaly

$$\begin{aligned} i\Delta S^{(L)} &= (M_1^{(L)}, S) = c_L a^L \\ a_a^L &= \frac{i}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\mu \text{Tr } \lambda_a (V_\nu^L \partial_\rho V_\sigma^L + \frac{1}{2} V_\nu^L V_\rho^L V_\sigma^L) \end{aligned} \quad (4.13)$$

This analysis may be repeated for a gauge field V^R coupling only to a right-handed fermion ψ_R , reading off a regulator from a new PV action where only the right-handed component couples to V^R . One then finds with this regulator, and $M_1^{(R)}$ as in eq.(4.12) with $L \rightarrow R$,

$$\Delta S^{(R)} = \lim_{M^2 \rightarrow \infty} \text{Tr } \lambda_a \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \exp \frac{1}{M^2} \begin{pmatrix} \not{p}_L \not{p}_R & \\ & \not{p}_R \not{p}_L \end{pmatrix} \quad (4.14)$$

$$i\Delta S^{(R)} = (M_1^{(R)}, S) = c_R a^R \quad (4.15)$$

$$a_a^R = -\frac{i}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\mu \text{Tr } \lambda_a (V_\nu^R \partial_\rho V_\sigma^R + \frac{1}{2} V_\nu^R V_\rho^R V_\sigma^R) \quad (4.16)$$

4.1.2 Two regularisations

Now we come to the model we want to regularise in two different ways. It consists simply of a nonchiral fermion with both vector and axial couplings. This can be treated as the sum of the L and R models before:

$$\bar{\psi}(\not{p} + \not{Y} + \not{A}\gamma_5)\psi = \bar{\psi}_L(\not{p} + \not{Y}^L)\psi_L + \bar{\psi}_R(\not{p} + \not{Y}^R)\psi_R \quad (4.17)$$

with $\psi_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\psi$, $V = \frac{1}{2}(V^R + V^L)$ and $A = \frac{1}{2}(V^L - V^R)$. The L and R sector can either be regularised separately, or together. In the former case, one obtains, as explained above,

$$i\mathcal{A}^{(1)} = i\Delta^{(1)}S - (M_1^{(L)} + M_1^{(R)}, S) = c_L a^L + c_R a^R \quad (4.18)$$

is the expression obtained by Bardeen [15] and the normal parity terms constitute a lengthy expression which we have checked to agree with the I_2 terms given in [14]. They can be removed from the anomaly by an appropriate counterterm $M_1^{(2)}$ which is invariant under vector transformations :

$$i\Delta^{(2)}S - (M_1^{(2)}, S) = c_A a^{(2)\mu} \quad (4.29)$$

We will not give the explicit form of $M_1^{(2)}$ here, since the divergent terms and c -terms are sufficient to illustrate our ideas. The vector anomaly vanishes identically in eq.(4.27). This is of course due to the fact that the mass term is invariant under vector transformations, so these are preferred symmetries.

We have seen that ΔS can be given a definite meaning, but its value depends on the regulator. The difference between $\Delta S^{(1)}$ and $\Delta S^{(2)}$ however can be absorbed in a change of M_1 . This corresponds to the statement that the anomaly calculation is regularisation-dependent, but the difference is the variation of a local counterterm. In the present case, the vector-invariant choice $M_1^{(2)}$ makes the vector anomaly vanish. In the case of the first regulator, the same anomaly can be obtained by adding to M_1 an extra term:

$$i\Delta^{(2)}S = i\Delta^{(1)}S + (\tilde{M}_1^{(1)}, S) \quad (4.30)$$

where now

$$\begin{aligned} \tilde{M}_1^{(1)} = & M_1^{(2)} - M_1^{(1)} - M_1^{(R)} - i\frac{M^2}{4\pi^2} A_\mu A^\mu \\ & + \frac{i}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} \lambda_a (2A_\mu V_\nu \partial_\rho V_\sigma - 2V_\mu A_\nu \partial_\rho V_\sigma \\ & + A_\mu A_\nu A_\rho V_\sigma + 3A_\mu V_\nu V_\rho V_\sigma) \end{aligned} \quad (4.31)$$

Note that none of the counterterms M_1 discussed in this section has a dependence on the antifields. As a consequence they have no influence on the higher order (in $\hbar$) parts of the master equation. Note also that no local M_1 can be found which would allow one to satisfy the master equation at order $\hbar$. The violation of the master equation thus represents a genuine anomaly.

4.1.3 Derivation of the counterterm.

Now we want to construct the counterterm $\tilde{M}_1^{(1)}$ from a canonical transformation as conjectured in subsection 3.6. Let us introduce a continuous set of T matrices interpolating between the 2 regulators. We choose the following mass terms

$$-M(\bar{\chi}_L(c_\theta \rho_R + s_\theta \chi_R) + \bar{\rho}_R(c_\theta \chi_L - s_\theta \rho_L)) + (L \leftrightarrow R) \quad (4.32)$$

where

$$c_\theta = \cos \theta ; \quad s_\theta = \sin \theta. \quad (4.33)$$

For $\theta = 0$, this corresponds to the first regularisation. For $\theta = \pi/2$, the terms with ρ -fields survive but do not couple, so this is equivalent to the second regularisation above. Now we can calculate the matrices T and $\mathcal{O}$. There are 8 entries for each (we use the basis $\chi_L, \rho_L, \chi_R, \rho_R, \bar{\chi}_L, \bar{\rho}_L, \bar{\chi}_R, \bar{\rho}_R$), but $T^{-1}\delta T$ and $\mathcal{O}^2$ split in 4 two by two matrices. ($\delta\mathcal{O}$ does not contribute). We find

$$T^{-1} \frac{\partial}{\partial \theta} T = \mathbb{1}_4 \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.34)$$

and the regulator $\mathcal{R} = \mathcal{O}^2$ is

$$\mathcal{R} = \begin{pmatrix} \sigma_1^\theta \tilde{D}_R \sigma_1^\theta \tilde{D}_L & \sigma_1^\theta \tilde{D}_L \sigma_1^\theta \tilde{D}_R & \sigma_1^\theta \tilde{D}_R \sigma_1^\theta \tilde{D}_L & \sigma_1^\theta \tilde{D}_L \sigma_1^\theta \tilde{D}_R \\ \sigma_1^\theta \tilde{D}_R \sigma_1^\theta \tilde{D}_L & \sigma_1^\theta \tilde{D}_L \sigma_1^\theta \tilde{D}_R & \sigma_1^\theta \tilde{D}_R \sigma_1^\theta \tilde{D}_L & \sigma_1^\theta \tilde{D}_L \sigma_1^\theta \tilde{D}_R \end{pmatrix} \quad (4.35)$$

where we have introduced the notations

$$\tilde{D} = \begin{pmatrix} \psi & \\ & \theta \end{pmatrix} \quad \sigma_1^\theta = \begin{pmatrix} s_\theta & c_\theta \\ c_\theta & -s_\theta \end{pmatrix} = \sigma_1 \cos \theta + \sigma_3 \sin \theta \quad (4.36)$$

Comparing the previous results with eq.(3.27) and eq.(3.28) we expect to obtain

$$\tilde{M}_1^{(1)} = -\frac{i}{2} \int_0^{\frac{\pi}{2}} d\theta \, T \tau^{-1} \frac{\partial}{\partial \theta} T \exp \mathcal{R}/M^2 \quad (4.37)$$

One can check that the 4 lower entries give the same contributions as the upper 4, and we can rewrite eq.(4.37) as

$$\tilde{M}_1^{(1)} = -\int_0^{\frac{\pi}{2}} d\theta \, T \tau \sigma_2 \mathcal{E} \quad (4.38)$$

where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and

$$\mathcal{E} = \exp(\sigma_1^\theta \tilde{D}_R \sigma_1^\theta \tilde{D}_L)/M^2 - \exp(\tilde{D}_L \sigma_1^\theta \tilde{D}_R \sigma_1^\theta)/M^2. \quad (4.39)$$

This should lead to eq.(4.31).

Using the previous expressions we can also write down a regularised ΔS for all values of θ :

$$\Delta^{(\theta)} S = \text{Tr} (\tilde{c}_L \mathcal{E} + \tilde{c}_R \mathcal{E}') \quad (4.40)$$

where $\mathcal{E}'$ is $\mathcal{E}$ with left and right interchanged, and $\tilde{c} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$. We will obtain the same value for the anomaly $\mathcal{A} = \Delta^{(\theta)} S + i(S, M_1^{(\theta)})$ for all θ , provided the counterterm $M_1^{(\theta)}$ calculated from

$$\frac{dM_1^{(\theta)}}{d\theta} = -\text{Tr} \sigma_2 \mathcal{E}, \quad (4.41)$$

satisfies the following equation :

$$i \frac{d}{d\theta} \text{Tr} (\tilde{c}_L \mathcal{E} + \tilde{c}_R \mathcal{E}') = \left(\frac{dM_1^{(\theta)}}{d\theta}, S \right). \quad (4.42)$$

We have evaluated these expressions using

$$\begin{aligned} \text{Tr} \quad & X \left[\exp \frac{\mathcal{P}_1 \mathcal{P}_2}{M^2} \frac{1 + \gamma_5}{2} - \exp \frac{\mathcal{P}_2 \mathcal{P}_1}{M^2} \frac{1 - \gamma_5}{2} \right] \\ = \quad & -\frac{M^2}{8\pi^2} \text{tr} \quad X \cdot (D_1 \cdot D_2 - D_2 \cdot D_1) \\ & + \frac{1}{48\pi^2} \text{tr} \quad X^{\mu\nu\rho\sigma} [D_{1\mu} D_{1\nu} D_{1\rho} D_{2\sigma} + D_{1\mu} D_{2\nu} D_{2\rho} D_{2\sigma} + D_{1\mu} D_{2\nu} D_{1\rho} D_{2\sigma} \\ & \quad + 1 \leftrightarrow 2] \\ & + \frac{1}{192\pi^2} \text{sp} \quad X [\mathcal{P}_1 \mathcal{P}_1 \mathcal{P}_1 \mathcal{P}_2 + \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_2 \mathcal{P}_2 + \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_1 \mathcal{P}_2 \\ & \quad + 2\mathcal{P}_2 D_2 \cdot D_1 \mathcal{P}_1 - 1 \leftrightarrow 2]. \end{aligned} \quad (4.43)$$

The Tr is as usual a trace over function space, tr is the trace is over Yang-Mills (or other) matrices, and sp includes a trace over Dirac matrices as well. We use this equation with $D_1 \frac{1-\gamma_5}{2}$ and $D_2 \frac{1+\gamma_5}{2}$ replaced by $\sigma_1^\mu D_{1\mu} \sigma_1^\nu$ and D_2 .

We then obtain from eq.(4.40)

$$\Delta^{(\theta)} S = \frac{M^2}{4\pi^2} \text{tr} \left\{ c_1 \partial^\mu V_\mu + c_A \partial^\mu A_\mu - (c_V \partial^\mu V_\mu - c_A \partial^\mu A_\mu) \sin^2 \theta \right. \\ \left. + 2c_A [V_\mu, A^\mu] \sin^2 \theta \right\} + \dots \quad (4.44)$$

The ... stand for the finite terms which constitute a long expression. We have checked that for $\theta = 0$ it agrees with eq.(4.11) $+L \leftrightarrow R$, and for $\theta = \pi/2$ it agrees with eq.(4.27), (4.28) and the expression in [14]. Also from eq.(4.41) and eq.(4.43) we obtain

$$\frac{dM_1^{(\theta)}}{d\theta} = \frac{iM^2}{4\pi^2} \text{tr} (V_\mu V^\mu - A_\mu A^\mu) \sin \theta \cos \theta + \dots, \quad (4.45)$$

which can be seen to reproduce eq.(4.31) by integration. We have checked that also the finite terms agree with this equation and with eq.(4.42), but we do not reproduce the (lengthy) expressions here, since they do not provide additional insight.

4.2 Einstein and Weyl anomalies.

As a second example we consider the Einstein and Weyl symmetries of a scalar field coupled to an external gravitational field in two dimensions. In this case the master equation can never be satisfied, and we thus have

anomalies. For the regularisation of the measure Fujikawa [4,16] introduced preferred variables and claimed that these should be used if the theory has to be free of Einstein anomalies. In the first part we will show that the modifications of the definition of the Jacobian as explained at the end of subsection 3.3 avoid this restriction. So we can use the Fujikawa scheme without Fujikawa variables if we use this new regularised definition. In the next part we will show that even without this modification, thus without $T^{-1}\delta T$ terms, we can use other variables than the Fujikawa ones and still obtain the same anomalies if we add a counterterm (M_1) to the theory. The former part illustrates changes of variables as in theorem 2, the second part illustrates conjecture 2.

4.2.1 Fujikawa regularisation without Fujikawa variables.

The solution to the lowest order master equation with the minimal set of fields is ($g = -\det g_{\alpha\beta}$)

$$\begin{aligned} S = \int d^2x \left[-\frac{1}{2} \sqrt{g} g^{\alpha\beta} \partial_\alpha X \partial_\beta X + X^* c^\alpha \partial_\alpha X \right. \\ \left. + g^{*\alpha\beta} \left(c^\gamma \partial_\gamma g_{\alpha\beta} + 2g_{\gamma(\alpha} \partial_{\beta)} c^\gamma + c g_{\alpha\beta} \right) \right. \\ \left. - c_\beta c^\alpha \partial_\alpha c^\beta - c^* c^\alpha \partial_\alpha c \right]. \end{aligned} \quad (4.46)$$

where c^α is the ghost for general coordinate transformations and c is the ghost for local dilatations. X^* is the antifield of X , $g^{*\alpha\beta}$ is the antifield of $g_{\alpha\beta}$ (supposed to be symmetric), etc.

We will treat the X -loops in this subsection. We introduce PV fields Y with (again we omit the integrations over x in the notation)

$$S_{(1)}^{PV} = \frac{1}{2} Y \square Y + Y^* c^\alpha \partial_\alpha Y - \frac{1}{2} \sqrt{g} \mathcal{M}^2 Y^2 \quad (4.47)$$

where $\square = \partial_\alpha \sqrt{g} g^{\alpha\beta} \partial_\beta$ (derivatives continue to work to the right unless we write brackets). We perform a canonical transformation as in theorem 2 by defining

$$X_a = X g^a; \quad Y_a = Y g^a. \quad (4.48)$$

For $a = \frac{1}{4}$ these are the Fujikawa variables. Now we have

$$\begin{aligned} T &= g^{\frac{1}{2}-2a}; \quad T\mathcal{O} = g^{-a} \square g^{-a} = T^{\frac{1}{2}} \square T^{\frac{1}{2}} \\ \mathcal{O} &= T^{-\frac{1}{2}} \square T^{\frac{1}{2}} \\ K &= c^\alpha \partial_\alpha + 2a(\partial_\alpha c^\alpha) + 2ca \end{aligned} \quad (4.49)$$

where $\bar{\square} = g^{-\frac{1}{2}} \square g^{-\frac{1}{2}}$. We use eq.(3.18) in eq.(3.2) to find

$$\begin{aligned} \mathcal{F} &= \text{Tr} \left[c^\alpha \partial_\alpha + \frac{1}{2} T^{-1} (c^\alpha \partial_\alpha T) + \frac{1}{2} (\partial_\alpha c^\alpha) + \frac{1}{2} T^{-\frac{1}{2}} \bar{\square} / M^2 T^{\frac{1}{2}} \right] \\ &= \text{Tr} \left[c^\alpha \partial_\alpha + \frac{1}{2} (\partial_\alpha c^\alpha) + \frac{1}{2} c \bar{\square} / M^2 \right] \end{aligned} \quad (4.50)$$

where we used the cyclicity of the regularised trace. This expression is already independent of a . The c^α terms cancel, which can be seen by choosing as basis the eigenfunctions of the symmetric operator $\bar{\square}$, while the c terms give the usual Weyl anomaly

$$\Delta^{(1)}S = \frac{1}{8\pi} \left(\mathcal{M}^2 \sqrt{g} - \frac{1}{6} \sqrt{g} R \right) c \quad (4.51)$$

(We kept here also the divergent term).

4.2.2 The difference between 2 Fujikawa regularisations.

As another illustration of the fact that different regulators lead to different ΔS , but to the same anomalies if one changes M_1 , we now consider the same model with another mass term for the PV fields :

$$S_{(2)}^{PV} = \frac{1}{2} Y \square Y + Y^\dagger c^\alpha \partial_\alpha Y - \frac{1}{2} \mathcal{M}^2 Y^2. \quad (4.52)$$

The mass term of the previous subsection is invariant under Einstein but not Weyl transformations, while the new one is invariant under Weyl but not Einstein transformations. The regulator for this second mass term is (again with variables X_a)

$$\mathcal{R}_{(2)} = g^\alpha \square g^{-\alpha} \quad (4.53)$$

One can then evaluate the regularised results for ΔS as before, and using the result of the appendix of [5] we find again that ΔS is independent of a , and given by

$$\Delta^{(2)}S = \frac{1}{48\pi} \left[\sqrt{g} R - \frac{1}{2} \square \ln g \right] \partial_\gamma c^\gamma. \quad (4.54)$$

Both expressions for ΔS are BRST invariant ($(\Delta S, S) = 0$) which proves that our regularisation did respect the consistency conditions.

One may then check that there is no local M_1 such that $(S, M_1) = i\Delta S$ in either of the two cases. However, one can move the anomaly from the Einstein to the Weyl sector by counterterms M_1 . There exists a counterterm M_1 such that

$$\mathcal{A}^{(1)} = i\Delta^{(1)}S = i\Delta^{(2)}S + (S, M_1) = \mathcal{A}^{(2)} \quad (4.55)$$

To derive the form of M_1 we interpolate between the two regularisations by a mass term $-\frac{1}{2} \mathcal{M}^2 \gamma^2 g^{\alpha\alpha}$, so that $t = \frac{1}{4}$ gives regulator I , while $t = 0$ gives regulator II . From

$$T^{-1} \frac{\partial}{\partial t} T = 2 \ln g \quad (4.56)$$

and eq.(3.28) we get

$$M_1 = i \int_0^1 dt \ln g \exp(g^{-2t} \square / \mathcal{M}^2) \quad (4.57)$$

and using the appendix of [5] this gives

$$M_1 = \frac{i}{4\pi} \left[\frac{1}{2} \mathcal{M}^2 (\sqrt{g} - 1) - \frac{1}{24} (\ln g) \sqrt{g} R + \frac{1}{96} \ln g \square \ln g \right]. \quad (4.58)$$

We can check that this satisfies eq.(4.55), thus providing an example of conjecture 2.

Note that we included the $\mathcal{M}^2$ terms in the above discussion although in our general treatment we eliminated these terms e.g. in PV by eq.(3.22). We have no general argument why these divergent terms as calculated in the Fujikawa scheme should follow the rules for consistent anomalies, as our proof that the finite terms follow from a PV regularisation, leading to eq.(3.23), does not apply for the infinite terms.

Putting the two parts of this subsection together we can also choose $a = \frac{1}{4}$ in the regularisation (1), and $a = 0$ in regularisation (2) (as we did in [5]). Then we always have $T = 1$, and do not have to use the $T^{-1} \delta T$ term. So we have in both cases a non-modified Fujikawa expression for the Jacobian. This confirms that it is not necessary to use a specific set of preferred variables : the result is always the same, provided we add the appropriate counterterm.

4.3 The particle and canonical transformations.

We consider here the particle action. For the particle there are no genuine anomalies, but the general ideas can be illustrated by looking at the divergent terms. In the previous examples we saw that these follow the same rules as the finite terms which describe the real anomalies in other systems. The particle provides an easy laboratory to study some of the features mentioned in the previous sections. In particular we will illustrate our remarks in subsection 3.5 about M -dependent transformations, and we will present other examples of canonical transformations.

We first look at the following action

$$S_1 = p\dot{x} - \frac{1}{2} g p^2 + x^\dagger c p + g^\dagger \dot{c} + \dot{c}^\dagger \pi \quad (4.59)$$

ΔS_1 seems obviously zero, but in fact one should first regularise.

Before going to Pauli-Villars regularisation we choose a gauge. With the gauge fermion

$$\Psi = \dot{c}(g - 1), \quad (4.60)$$

the gauge fixed action becomes

$$S_{gf1} = p\dot{x} - \frac{1}{2}gp^2 + \dot{c}c + \pi(g-1). \quad (4.61)$$

We now introduce a PV regulator. We do not have to introduce partners for the gravitational field g and for π as they do not appear in loops. The massless part is

$$S_{PV1}(0) = P\dot{X} - \frac{1}{2}gP^2 + \dot{\eta}\eta + X^*cP. \quad (4.62)$$

We can choose invariant mass terms as follows

$$S_{PV1}(m) = -mP^*cP + mX^*Xc - mgPX - M\dot{\eta}\eta. \quad (4.63)$$

As these mass terms are invariant the coordinate transformations are ‘preferred symmetries’. However, the total action including the mass terms is only invariant if we introduce mass dependent terms in the transformations. These are terms of the form eq.(3.21). Also, to save eq.(3.5) we have to impose that the modes of P and X , and therefore p and x are related.

In the ghost-antighost sector K and δT are zero, so the anomaly gets no contributions from there. The interesting part is the (px) sector. We have the following matrices.

$$\begin{aligned} T &= \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix}; & \mathcal{O} &= \begin{pmatrix} -g^{-1}d & 0 \\ -1 & g^{-1}d \end{pmatrix} \\ \mathcal{R} &= \mathcal{O}^2 = g^{-1}dg^{-1}d\mathbb{1} \\ K &= \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}; & L &= \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix} \end{aligned} \quad (4.64)$$

where d denotes a derivative which works on everything to the right. These lead to

$$2K + T^{-1}\delta T = \mathcal{O}L - L\mathcal{O} = \begin{pmatrix} g^{-1}\dot{c} & 0 \\ 2c & g^{-1}\dot{c} \end{pmatrix}. \quad (4.65)$$

$\delta\mathcal{O}$ does not contribute to eq.(3.19) because the 1-1 entry cancels the 2-2 entry in the trace. The same is true for $(\mathcal{O}L - L\mathcal{O})\mathcal{O}$ when eq.(3.26) is multiplied with $\frac{M+Q}{M+O}$ to arrive at the same $\mathcal{O}^2$ regulator. So the total anomaly from eq.(3.19) and eq.(3.26) vanishes. But we also find that both are separately zero, as follows from the cyclicity argument in subsection 3.5. Indeed

$$\int dx \int_{-\infty}^{\infty} \frac{dk}{2\pi} g^{-1}\dot{c} \exp[g^{-1}(d+ik)g^{-1}(d+ik)/m^2] = \int dx \frac{m}{2\sqrt{\pi}}\dot{c} + \dots = 0. \quad (4.66)$$

This is the result if we regularise without the first 2 terms in eq.(4.63). So whether or not we include the L -terms, we find $\Delta S = 0$.

We now perform a canonical transformation on the ghost sector to get to the more usual form of GCT. We use the generating function F_{12} (the variables previously used are now primed)

$$F_{12} = x^*x + p^*p + g^*g + c^*cg + \dot{c}^*\dot{c} + \pi^*\pi. \quad (4.67)$$

This leads to

$$S_2 = p\dot{x} - \frac{1}{2}gp^2 + x^*gcp - \dot{g}^*gc - c^*\dot{c}\dot{c} + \dot{c}^*\pi. \quad (4.68)$$

Using the same gauge fermion, the gauge fixed action is

$$S_{gf2} = p\dot{x} - \frac{1}{2}gp^2 - \dot{c}gc + \pi(g-1). \quad (4.69)$$

We then want to use the PV action (without the mass terms)

$$\begin{aligned} S_{PV2}(0) &= P\dot{X} - \frac{1}{2}gP^2 - \dot{\eta}g\eta + X^*gcP \\ &\quad + \eta^*(\dot{c}\eta - c\dot{\eta}) - \dot{\eta}^*\dot{c}\dot{\eta}. \end{aligned} \quad (4.70)$$

However, when just copying the canonical transformation from the ordinary fields to the PV sector the last line of eq.(4.70) is replaced by $-\eta^*\eta g^{-1}d(gc)$. In that case it is not obvious that eq.(3.5) is satisfied. So we see that just copying the canonical transformation to the PV sector does not necessary transform a suitable PV regulator to a suitable PV regulator for the new action.

The difference between the PV action eq.(4.70) and the one from the copied canonical transformation is an equation of motion symmetry. In accordance with the discussion in section 2, we add a canonical transformation of the form eq.(2.31) to make the transition from eq.(4.62) to eq.(4.70). The total generating function is then

$$F_{12,PV} = F_{12} + P^*P + X^*X + \eta^*\eta g + \dot{\eta}^*\dot{\eta} - \dot{\eta}^*\eta^*c. \quad (4.71)$$

The transformation rules for this canonical transformation are (nothing has changed in the (px) sector)

$$\begin{aligned} g' &= g & g^* &= g^* + c^*c + \eta^*\eta \\ c' &= gc & c^* &= c^*g - \dot{\eta}^*\eta^* \\ \dot{c}' &= \dot{c} & \dot{c}^* &= \dot{c}^* \\ \eta' &= g\eta - \dot{\eta}^*c & \eta^* &= \eta^*g \\ \dot{\eta}' &= \dot{\eta} - \eta^*c & \dot{\eta}^* &= \dot{\eta}^*. \end{aligned} \quad (4.72)$$

Without the last term in eq.(4.71) the canonical transformation is identical for the ordinary fields and the PV fields, so according to theorem 2 ΔS does not change and thus remains zero in this ghost-antighost sector. The

5 Conclusions and outlook

In this article we have argued that one need not, and should not, discard the terms of order $\hbar$ and higher in the Batalin-Vilkovisky formalism for the quantisation of Lagrangian field theories. On the contrary, they have important physical content: they describe the anomalies in the theory. The $\hbar = 0$ terms (S) of the quantum action W are BRST invariant by themselves: $(S, S) = 0$. The program of constructing actions S satisfying $(S, S) = 0$ is purely algebraic: nowhere does one need regularisation to make sense of ill-defined expressions. And in fact in [8] it was shown that a solution always exists for finite stage theories.

For the order $\hbar$ and higher terms, the situation is radically different; here one needs regularisation to obtain finite, local and well-defined expressions. If the master equation $(W, W) - 2i\hbar\Delta W = 0$ is satisfied, the S -matrix is independent of the gauge choices. At order $\hbar$ one finds the requirement

$$(S, M_i) = i\Delta S \quad (5.1)$$

where $W = S + \hbar M_1 + \dots$. The counterterms needed for the renormalisation of quantum field theories are an example of a local, BRST invariant, M_p . However, also in nonrenormalisable theories one can study the presence of M_p .

The term ΔS is formally the naive Jacobian for BRST transformations, since $\Delta \sim \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi^*}$ and $\frac{\partial}{\partial \phi^*} S \equiv \delta \phi^A(\phi, \phi^*)$ is the transformation rule of the field ϕ^A under BRST transformations. The master equation offers therefore a framework to study anomalies for gauge theories with open algebras or with ghosts for ghosts, their consistency conditions and their relation to counter terms. However, the naive Jacobian ΔS is divergent, and still depends on the antifields ϕ_B^* . Thus, regularisation at order $\hbar \neq 0$ is unavoidable.

Let us briefly speculate on the possibility to construct a regulator in the space of fields and antifields. This may well be inconsistent in itself. Since the antifields appear without kinetic terms, it may be difficult to construct such regulators. (However, recently we solved a similar problem in phase space, where we constructed regulators for the nonpropagating canonical momenta, and reobtained results for the Einstein and Weyl anomalies which agreed with results previously obtained in configuration space [5]). Still, if one were to have such a regulator, one could compute such terms as ΔM_p , and then proceed by purely algebraic methods to solve the master equation, similar to the analysis of $(S, S) = 0$.

In this article we have constructed regulators in the space of fields ϕ_A after eliminating antifields by projection onto the hypersurfaces $\Sigma = \{\phi^* - \frac{\partial W}{\partial \phi^*} = 0\}$. In a series of examples, we have shown that different regulators yield different ΔS , but that using a different regulator is equivalent to

last term seems naively also to give no contribution to ΔS . We now want to check this.

First we see that to have eq.(3.5), we can relate the K for the PV ghost to the transformation of the ghost using eq.(3.9) where P_B^A then just relates their modes. For the PV antighosts $K = -cd$ is related to the transformation law for the gravitational field g by eq.(3.10). This equation reads then $dc = (-cd)^T$. This is similar to the example treated in [6].

The canonical transformation on the mass terms gives

$$-M(g\dot{\eta}\eta - \eta^*c\dot{\eta} + \dot{\eta}^*c\dot{\eta}). \quad (4.73)$$

So we get here a formulation with M -dependent transformations. Using these, the mass term is invariant, and we can calculate as before that $\Delta S = 0$. If we omit the last two terms in eq.(4.73) then we also find $\Delta S \propto \int dx M \dot{c} = 0$. To avoid M -dependent transformations already in PV, we could have started with a mass term $-Mg^2\dot{\eta}\eta$. It gives

$$K = \begin{pmatrix} \dot{c} - cd & 0 \\ 0 & -cd \end{pmatrix}; \quad T^{-1}\delta T = 2g^{-1}dgc\mathbb{1} \quad (4.74)$$

$$\tilde{Q}^2 = \begin{pmatrix} g^{-2}dg^{-1}dg & 0 \\ 0 & g^{-1}dg^{-1}d \end{pmatrix} = \begin{pmatrix} g^{-\frac{1}{2}} & 0 \\ 0 & g^{-\frac{1}{2}} \end{pmatrix} \tilde{Q}^2 \begin{pmatrix} g^{\frac{1}{2}} & 0 \\ 0 & g^{\frac{1}{2}} \end{pmatrix}$$

where $\tilde{Q}^2$ is the symmetric operator $g^{-\frac{1}{2}}dg^{-1}dg^{-\frac{1}{2}}\mathbb{1}$. As the indices refer to fermionic fields, the $(-)$ sign has to be used in eq.(3.19). The $cd(\ln g)$ terms cancel, and ΔS vanishes.

adding a local counterterm M_1 to the action. We also showed that when there is no solution to the master equation, anomalies are present. In these examples we used a pV regularisation to construct Fujikawa regulators and thus these were consistent regulators. For an arbitrary theory the main problem will be to find a suitable pV regularisation, or another scheme which can be applied to the full path integral.

We can extend our analyses further. For example, the BV formalism is the appropriate setting to study consistency conditions for gauge theories with open algebras or with ghosts for ghosts. Also, we might study whether at two-loop level new anomalies are allowed as far as the master equation is concerned (present day thinking excludes the existence of higher-loop anomalies which are not a direct consequence of one-loop anomalies). Also higher-loop consistency conditions can be studied. We have shown how consistency equations can be derived from the master equation in the one-loop case.

Acknowledgements

It is a pleasure to thank the institutes of theoretical physics at Leuven and Stony Brook, where part of this work was done, for their hospitality.

Appendix : Canonical transformations

We present here results of [17]. Canonical transformations are defined as transformations from Φ and Φ^* to Φ' and Φ'^* such that the antibrackets are left invariant. Therefore they conserve the lowest order master equation eq.(2.6).

Canonical transformations can always be obtained from a fermionic generating function $F(\Phi, \Phi'^*)$ for which the matrix⁴

$$M_B^A = \frac{\partial \partial F(\Phi, \Phi'^*)}{\partial \Phi_A^* \partial \Phi'^B}, \quad N_B^A = \frac{\partial \partial F(\Phi, \Phi'^*)}{\partial \Phi'^B \partial \Phi_A^*} \quad (\text{A.1})$$

(N is the supertranspose of M) is nonsingular. (This is the "F₂" in the terminology of canonical transformations [18]). The new variables are defined by

$$\Phi'^A = \frac{\partial F(\Phi, \Phi'^*)}{\partial \Phi_A^*}, \quad \Phi_A^* = \frac{\partial F(\Phi, \Phi'^*)}{\partial \Phi'^A}. \quad (\text{A.2})$$

That this is always possible is proven as follows.

A.1 From invariant brackets to generating function.

Suppose the brackets in terms of the variables $\{\Phi'^A, \Phi_A'^*\}$ (denoted by $(\cdot, \cdot)'$ henceforth) of the variables $\{\Phi^A, \Phi_A^*\}$ are given by

$$(\Phi^A, \Phi'^B)' = 0; \quad (\Phi_A^*, \Phi_B'^*)' = 0; \quad (\Phi^A, \Phi_B'^*)' = \delta_B^A. \quad (\text{A.3})$$

Furthermore, we assume that the matrix

$$\left. \frac{\partial \Phi'^B}{\partial \Phi'^A} \right|_{\Phi'^*} \quad (\text{A.4})$$

is nonsingular. Then we shall show that a fermionic generating function $F(\Phi, \Phi'^*)$ exists satisfying eq.(A.2).

We begin by writing Φ'^A as a function of the "natural variables" Φ and Φ'^* , but re-express Φ as a function of Φ', Φ'^* . Then with the chain rule

$$\begin{aligned} \left. \frac{\partial \Phi'^A(\Phi(\Phi', \Phi'^*), \Phi'^*)}{\partial \Phi'^B} \right|_{\Phi'^*} &= 0 = \left. \frac{\partial \Phi'^A}{\partial \Phi'^B} \right|_{\Phi'} + \left. \frac{\partial \Phi'^C}{\partial \Phi'^B} \right|_{\Phi'} \left. \frac{\partial \Phi'^A}{\partial \Phi'^C} \right|_{\Phi'^*} \\ \left. \frac{\partial \Phi'^B(\Phi(\Phi', \Phi'^*), \Phi'^*)}{\partial \Phi'^A} \right|_{\Phi'^*} &= 0 = \left. \frac{\partial \Phi'^B}{\partial \Phi'^A} \right|_{\Phi'} + \left. \frac{\partial \Phi'^C}{\partial \Phi'^A} \right|_{\Phi'} \left. \frac{\partial \Phi'^B}{\partial \Phi'^C} \right|_{\Phi'^*}. \end{aligned} \quad (\text{A.5})$$

It then follows that

$$\left. \frac{\partial \Phi'^A}{\partial \Phi'^B} \right|_{\Phi'} = \left. \frac{\partial \Phi'^B}{\partial \Phi'^A} \right|_{\Phi'} \quad (\text{A.6})$$

⁴All partial derivatives of a function are always defined such that the other fields on which the function depends are kept fixed.

since the difference, according to eq.(A.5), is given by

$$\begin{aligned} & \frac{\partial_r \Phi^H}{\partial \Phi^C} \Big|_{\Phi^{**}} - \frac{\partial_r \Phi^C}{\partial \Phi^A} \Big|_{\Phi^{**}} - \frac{\partial_r \Phi^C}{\partial \Phi^A} \Big|_{\Phi^{**}} \frac{\partial_r \Phi^A}{\partial \Phi^C} \Big|_{\Phi^{**}} \\ &= - \frac{\partial_r \Phi^H}{\partial \Phi^C} \Big|_{\Phi^{**}} (\Phi^C, \Phi^D)' \frac{\partial_r \Phi^A}{\partial \Phi^D} \Big|_{\Phi^{**}} = 0 \end{aligned} \quad (\text{A.7})$$

which vanishes according to the first relation in eq.(A.3)

The eq.(A.6) is the integrability condition for the existence of the function F in the first relation of eq.(A.2). This function is only determined so far up to a term depending on Φ . The matrix M in eq.(A.1) can now be written as

$$M_B^A = \frac{\partial_r \Phi^A}{\partial \Phi^B} \Big|_{\Phi^{**}}, \quad (\text{A.8})$$

and is clearly the inverse of eq.(A.4).

To proceed it is advantageous to change variables from Φ' and Φ'' to Φ and Φ^* since $F(\Phi, \Phi^*)$ depends on these variables. Hence it will be natural to express the derivatives w.r.t. Φ^A and Φ_A^* in terms of derivatives w.r.t. Φ^A at constant Φ^* , and Φ_A^* at constant Φ . From the first relation in eq.(A.2) one has

$$d\Phi^A = \left(\frac{\partial_r}{\partial \Phi^B} \frac{\partial}{\partial \Phi^A} F \right) d\Phi^B + \left(\frac{\partial_r}{\partial \Phi^B} \frac{\partial}{\partial \Phi^A} F \right) d\Phi_B^* \quad (\text{A.9})$$

and using eq.(A.1) and defining $F^{AB} \equiv \frac{\partial_r}{\partial \Phi^B} \frac{\partial}{\partial \Phi^A} F$ one obtains

$$d\Phi^A = M_B^A d\Phi^B + F^{AB} d\Phi_B^* \quad (\text{A.10})$$

For a function $G = G(\Phi, \Phi^*)$ one has in general

$$dG = \frac{\partial_r G}{\partial \Phi^A} \Big|_{\Phi^{**}} d\Phi^A + \frac{\partial_r G}{\partial \Phi_A^*} \Big|_{\Phi^{**}} d\Phi_A^* \quad (\text{A.11})$$

The derivatives of any G w.r.t. Φ_A^* at fixed Φ' then follow from the relation $M d\Phi + F d\Phi^* = 0$ in eq.(A.10) and using it in eq.(A.11)

$$dG = \left\{ \frac{\partial_r G}{\partial \Phi^A} \Big|_{\Phi^{**}} (-M^{-1})_B^A F^{BC} + \frac{\partial_r G}{\partial \Phi_C^*} \Big|_{\Phi^{**}} \right\} d\Phi_C^* \text{ at } d\Phi' = 0 \quad (\text{A.12})$$

Hence, removing G from the left

$$\frac{\partial_r}{\partial \Phi_A^*} \Big|_{\Phi^{**}} = - \frac{\partial_r}{\partial \Phi^C} \Big|_{\Phi^{**}} (M^{-1})_B^C F^{BA} + \frac{\partial_r}{\partial \Phi_A^*} \Big|_{\Phi^{**}} \quad (\text{A.13})$$

(Recall that these right derivatives act on functions G to the left of them, not onto $M^{-1}F$). Similarly one finds for the derivatives w.r.t. Φ' at fixed

Φ''

$$\begin{aligned} d\Phi^A &= M_B^A d\Phi^B \text{ at } d\Phi'' = 0 \\ dG &= \frac{\partial_r G}{\partial \Phi^A} \Big|_{\Phi^{**}} (M^{-1})_B^A d\Phi^B \\ \frac{\partial_r}{\partial \Phi^A} \Big|_{\Phi^{**}} &= \frac{\partial_r}{\partial \Phi^B} \Big|_{\Phi^{**}} (M^{-1})_A^B \end{aligned} \quad (\text{A.14})$$

We also need left derivatives w.r.t. Φ' and Φ'' . One finds along similar lines as before

$$\begin{aligned} d\Phi^A &= d\Phi^B N_B^A + d\Phi_B^* F^{BA} \\ dG &= (d\Phi^B N_B^{-1A}) \frac{\partial_r G}{\partial \Phi^A} \Big|_{\Phi^{**}} \text{ at } d\Phi'' = 0 \\ \frac{\partial_r}{\partial \Phi^A} \Big|_{\Phi^{**}} &= N^{-1B}_A \frac{\partial_r}{\partial \Phi^B} \Big|_{\Phi^{**}} \end{aligned} \quad (\text{A.15})$$

Furthermore

$$dG = d\Phi^C \frac{\partial_r G}{\partial \Phi^C} \Big|_{\Phi^{**}} + d\Phi_A^* \frac{\partial_r G}{\partial \Phi_A^*} \Big|_{\Phi^{**}} \quad (\text{A.16})$$

$$\begin{aligned} d\Phi^C &= -d\Phi_A^* F^{AB} N_B^{-1C} \text{ at } d\Phi' = 0 \\ \frac{\partial_r}{\partial \Phi_A^*} \Big|_{\Phi^{**}} &= \frac{\partial_r}{\partial \Phi_A^*} \Big|_{\Phi^{**}} - F^{AB} N_B^{-1C} \frac{\partial_r}{\partial \Phi^C} \Big|_{\Phi^{**}} \end{aligned} \quad (\text{A.17})$$

From the last equation of eq.(A.3) we obtain, using the above formulas to express all primed right derivatives in terms of 'natural right derivatives'

$$\begin{aligned} \delta_B^A &= (\Phi^A, \Phi_B^*)' = \frac{\partial_r \Phi^A}{\partial \Phi^B} \Big|_{\Phi^{**}} - \frac{\partial_r \Phi^A}{\partial \Phi_C^*} \Big|_{\Phi^{**}} \frac{\partial_r \Phi_B^*}{\partial \Phi_C^*} \Big|_{\Phi^{**}} \\ &= \frac{\partial_r \Phi^A}{\partial \Phi^D} \Big|_{\Phi^{**}} (M^{-1})_C^D \frac{\partial_r \Phi_B^*}{\partial \Phi_C^*} \Big|_{\Phi^{**}} \\ &\quad + \left\{ \frac{\partial_r \Phi^A}{\partial \Phi^D} \Big|_{\Phi^{**}} (M^{-1})_E^D F^{EC} - \frac{\partial_r \Phi^A}{\partial \Phi_C^*} \Big|_{\Phi^{**}} \frac{\partial_r \Phi_B^*}{\partial \Phi_C^*} \Big|_{\Phi^{**}} \right\} \\ &= (M^{-1})_C^A \frac{\partial_r \Phi_B^*}{\partial \Phi_C^*} \Big|_{\Phi^{**}} + (M^{-1})_E^A F^{EC} \frac{\partial_r \Phi_B^*}{\partial \Phi_C^*} \Big|_{\Phi^{**}} \end{aligned} \quad (\text{A.18})$$

Inserting into this intermediate expression the results for the primed left derivatives in terms of natural left derivatives, the terms with F^{AB} cancel, and we obtain

$$\delta_B^A = (M^{-1})_C^A \frac{\partial_r \Phi_B^*}{\partial \Phi_C^*} \Big|_{\Phi^{**}} \quad (\text{A.19})$$

or

$$\frac{\partial_r \Phi_B^*}{\partial \Phi_A^*} \Big|_{\Phi^{**}} = M_B^A \quad (\text{A.20})$$

From eq.(A.20) it follows that

$$\Phi_n^* = \frac{\partial F(\Phi, \Phi^*)}{\partial \Phi^B} + g_B(\Phi), \quad (\text{A.21})$$

where the functions $g_B(\Phi)$ are so far undetermined. To fix this freedom, we consider the second equation in eq.(A.3). By brute substitutions one finds in terms of natural derivatives that the F^{AB} terms again cancel

$$\begin{aligned} 0 &= \frac{\partial_r \Phi_A^*}{\partial \Phi^C} \Big|_{\Phi^*} \frac{\partial_l \Phi_B^*}{\partial \Phi_C^*} \Big|_{\Phi^*} - \frac{\partial_r \Phi_A^*}{\partial \Phi_C^*} \Big|_{\Phi^*} \frac{\partial_l \Phi_B^*}{\partial \Phi^C} \Big|_{\Phi^*} \\ &= \frac{\partial_r \Phi_A^*}{\partial \Phi^C} \Big|_{\Phi^*} (M^{-1})_B^C \frac{\partial_l \Phi_B^*}{\partial \Phi_n^*} \Big|_{\Phi^*} - \frac{\partial_r \Phi_A^*}{\partial \Phi_C^*} \Big|_{\Phi^*} (N^{-1})_C^D \frac{\partial_l \Phi_B^*}{\partial \Phi^D} \Big|_{\Phi^*} \end{aligned} \quad (\text{A.22})$$

Substituting the result for Φ^* obtained in eq.(A.21), all F , M and N factors cancel, and we find

$$\frac{\partial_r g_A}{\partial \Phi^B} = \frac{\partial_l g_B}{\partial \Phi^A}. \quad (\text{A.23})$$

The general solution reads

$$g_A = \frac{\partial_r f(\Phi)}{\partial \Phi^A} \quad (\text{A.24})$$

where $f(\Phi)$ is a definite fermionic function.

By redefining F in eq.(A.21), we obtain also the second relation in eq.(A.2). Hence, eq.(A.3) implies eq.(A.2).

For later use, we derive a similar result, but interchanging everywhere primed and unprimed variables. From $(\Phi_A^*, \Phi_B^*) = 0$ we obtain a result analogous to eq.(A.6), from which we deduce again

$$\Phi_A^* = \frac{\partial}{\partial \Phi^A} F(\Phi, \Phi^*). \quad (\text{A.25})$$

A.2 From generating function to invariant brackets.

Conversely, suppose that one has given the transformations in eq.(A.2). Then we want to show that they are canonical, i.e. that eq.(A.3) holds. To this purpose we recall the definition of F^{AB} and define a further object

$$F_{AB} = \frac{\partial_l}{\partial \Phi_A^*} \frac{\partial}{\partial \Phi_B^*} F(\Phi, \Phi^*) ; \quad F_{AB} = \frac{\partial_l}{\partial \Phi^A} \frac{\partial}{\partial \Phi^B} F(\Phi, \Phi^*). \quad (\text{A.26})$$

In addition to relations (A.13), (A.14), (A.15) and (A.17) which transform derivatives from Φ' , Φ'^* to Φ , Φ^* , we now also need to transform from Φ , Φ^* to Φ , Φ^* . From the second equation in eq.(A.2) one has

$$\begin{aligned} d\Phi_A^* &= d\Phi^B \frac{\partial_l}{\partial \Phi^B} \frac{\partial F}{\partial \Phi^A} + d\Phi^* \frac{\partial_l}{\partial \Phi_B^*} \frac{\partial F}{\partial \Phi^A} \\ &= d\Phi^B F_{BA} + d\Phi_B^* M_A^B. \end{aligned} \quad (\text{A.27})$$

Using this in eq.(A.16) it follows for $d\Phi = 0$ that

$$\frac{\partial_l}{\partial \Phi_A^*} \Big|_{\Phi^*} = (M^{-1})_B^A \frac{\partial_l}{\partial \Phi_B^*} \Big|_{\Phi^*}. \quad (\text{A.28})$$

Similarly one finds at $d\Phi^* = 0$

$$\frac{\partial_l}{\partial \Phi^A} \Big|_{\Phi^*} = \frac{\partial_l}{\partial \Phi^A} \Big|_{\Phi^*} - F_{AB} (M^{-1})_C^B \frac{\partial_l}{\partial \Phi_C^*} \Big|_{\Phi^*}. \quad (\text{A.29})$$

The following relations now hold

$$\frac{\partial_r \Phi^A}{\partial \Phi^B} \Big|_{\Phi^*} = \frac{\partial_l \Phi_B^*}{\partial \Phi_A^*} \Big|_{\Phi^*} = (M^{-1})_B^A \quad (\text{A.30})$$

(use eq.(A.14) and eq.(A.8))

$$\frac{\partial_r \Phi^A}{\partial \Phi_B^*} \Big|_{\Phi^*} = -\frac{\partial_l \Phi^B}{\partial \Phi_A^*} \Big|_{\Phi^*} = -(M^{-1})_C^A F^{CB} \quad (\text{A.31})$$

(use eq.(A.13), eq.(A.28) and the first in eq.(A.2))

$$\frac{\partial_r \Phi_A^*}{\partial \Phi^B} \Big|_{\Phi^*} = -\frac{\partial_l \Phi_B^*}{\partial \Phi^A} \Big|_{\Phi^*} = F_{AC} (M^{-1})_B^C \quad (\text{A.32})$$

(use eq.(A.14) and eq.(A.29))

$$\frac{\partial_r \Phi_A^*}{\partial \Phi_B^*} \Big|_{\Phi^*} = \frac{\partial_l \Phi^B}{\partial \Phi^A} \Big|_{\Phi^*} = N_A^B - F_{AC} (M^{-1})_D^C F^{DB}, \quad (\text{A.33})$$

(use eq.(A.13) and eq.(A.29))

From these results we can now prove that eq.(A.3) holds. The first relation in eq.(A.3) follows from eq.(A.30), (A.31), (A.15) and (A.17). The second relation in eq.(A.3) follows from eq.(A.32) and eq.(A.33), using also eq.(A.17), (A.15). Finally, the last relation in eq.(A.3) follows from eq.(A.30), (A.31) followed by eq.(A.17) and eq.(A.15).

A.3 Berezinians

For evaluating the Berezinian

$$J = s \det \frac{\partial(\Phi, \Phi^*)}{\partial(\Phi', \Phi'^*)} \quad (\text{A.34})$$

it is useful to decompose the transformation matrix as

$$\frac{\partial(\Phi, \Phi^*)}{\partial(\Phi', \Phi'^*)} = \begin{pmatrix} \delta_C^A & 0 \\ F_{AC} & N_A^C \end{pmatrix} \begin{pmatrix} (M^{-1})_B^C & -(M^{-1})_D^C F^{DB} \\ 0 & \delta_B^C \end{pmatrix}. \quad (\text{A.35})$$

As mentioned, N is the supertranspose of M , which means

$$N^B = M^B_A (-1)^{A(B+1)}; \quad (N^{-1})^B_A = (M^{-1})^B_A (-1)^{A(B+1)} \quad (\text{A.36})$$

(see e.g. [19] for definitions of supermatrices and their determinants). Both matrices have equal superdeterminants⁵:

$$\begin{aligned} J^{\frac{1}{2}} &= \text{sdet } M^{-1} \\ &= \text{sdet } \frac{\partial \Phi}{\partial \Phi'} \Big|_{\Phi''} \\ &= \text{sdet}^{-1} \frac{\partial \Phi''}{\partial \Phi'} \Big|_{\Phi} = \text{sdet} \frac{\partial \Phi'}{\partial \Phi''} \Big|_{\Phi}. \end{aligned} \quad (\text{A.37})$$

From eq.(A.35) it follows that the change of variables can be done in two steps with equal Berezinians:

$$\begin{pmatrix} \Phi \\ \Phi'' \end{pmatrix} \Rightarrow \begin{pmatrix} \Phi \\ \Phi' \end{pmatrix} \Rightarrow \begin{pmatrix} \Phi' \\ \Phi'' \end{pmatrix}. \quad (\text{A.38})$$

One can also see that the operator Δ is not invariant under canonical transformations. First define an intermediate operator $\tilde{\Delta}$ with "natural derivatives" for any function X

$$\tilde{\Delta} X = (-1)^{B+1} \left(\frac{\partial_r}{\partial \Phi''^A} \left(\frac{\partial_r X}{\partial \Phi''^A} \right) \Big|_{\Phi} \right) (M^{-1})^A_B. \quad (\text{A.39})$$

It is useful for the following to note that the Φ^A derivative can also be taken at fixed Φ'' rather than at fixed Φ' . Indeed the difference is according to eq.(A.29) (One can in these relations always change from left to right derivatives by changing the order of the factors and replacing M by its transpose N)

$$\left(\frac{\partial_r}{\partial \Phi''^A} \left(\frac{\partial_r X}{\partial \Phi''^A} \right) \Big|_{\Phi} \right) (N^{-1})^D_C F_{DA} (M^{-1})^A_B (-)^{B+1} = 0. \quad (\text{A.40})$$

The vanishing of this expression follows from using eq.(A.32) and eq.(A.30) and the symmetry of the first factor, to rewrite the last three factors as the vanishing antibracket (Φ''^A, Φ''^A) . Using now eq.(A.28) and eq.(A.30) we obtain

$$\Delta X = \tilde{\Delta} X + \frac{\partial_r X}{\partial \Phi''^A} \Big|_{\Phi} \Delta \Phi''^A. \quad (\text{A.41})$$

With eq.(A.29) the last Δ can be evaluated. The contribution from the second term in eq.(A.29) vanishes using

$$\begin{aligned} \frac{\partial_r}{\partial \Phi''^A} \left(\frac{\partial_r \Phi''^A}{\partial \Phi''^B} \right) \Big|_{\Phi} &= \frac{\partial_r}{\partial \Phi''^A} (N^{-1})^B_A \Big|_{\Phi} \\ &= -(N^{-1})^B_A \frac{\partial_r}{\partial \Phi''^A} \frac{\partial_r \Phi''^B}{\partial \Phi''^A} \Big|_{\Phi} = (N^{-1})^B_F (-)^{C+1} F_{FB} \end{aligned} \quad (\text{A.42})$$

⁵Note that in *-variables bosons and fermions are interchanged, as a result of which the determinants in the last line are replaced by their inverse.

and eq.(A.40) with X replaced by Φ''_B . For the remaining part we use an equation similar to eq.(A.43) for Φ derivatives, interchange the double derivatives and using

$$(-)^B \frac{\partial_r}{\partial \Phi''^C} N^D_B \Big|_{\Phi''} (N^{-1})^B_D = \frac{\partial_r}{\partial \Phi''^C} (\ln J^{-\frac{1}{2}}) \Big|_{\Phi''}. \quad (\text{A.43})$$

we get

$$\begin{aligned} \Delta X &= \tilde{\Delta} X - \frac{1}{2} \frac{\partial_r X}{\partial \Phi''^A} (N^{-1})^B_A \frac{\partial_r}{\partial \Phi''^B} \ln J \Big|_{\Phi''} \\ &= \tilde{\Delta} X - \frac{1}{2} \frac{\partial_r X}{\partial \Phi''^A} \Big|_{\Phi} \left(\frac{\partial_r}{\partial \Phi''^A} \ln J \Big|_{\Phi} + F_{AB} \frac{\partial_r}{\partial \Phi''^B} \ln J \Big|_{\Phi} \right). \end{aligned} \quad (\text{A.44})$$

For the last line we used again eq.(A.28) and eq.(A.29). Now we perform similar steps for the primed derivatives. Instead of eq.(A.40) we now obtain

$$\left(\frac{\partial_r}{\partial \Phi''^D} \left(\frac{\partial_r X}{\partial \Phi''^C} \right) \Big|_{\Phi''} \right) (M^{-1})^D_A (M^{-1})^C_B F^{BA} (-)^{AC+A+1} = 0 \quad (\text{A.45})$$

using now $(\Phi''^D, \Phi''^C) = 0$. We find then

$$\begin{aligned} \Delta' X &= \tilde{\Delta} X - \frac{1}{2} \frac{\partial_r X}{\partial \Phi''^A} \Big|_{\Phi''} \Delta' \Phi''^A \\ &= \tilde{\Delta} X - \frac{1}{2} \frac{\partial_r X}{\partial \Phi''^A} \Big|_{\Phi''} (M^{-1})^A_B \frac{\partial_r}{\partial \Phi''^B} \ln J \Big|_{\Phi} \\ &= \tilde{\Delta} X - \frac{1}{2} \left(\frac{\partial_r X}{\partial \Phi''^A} \Big|_{\Phi} + \frac{\partial_r X}{\partial \Phi''^A} F_{BA} \right) \frac{\partial_r}{\partial \Phi''^B} \ln J \Big|_{\Phi}, \end{aligned} \quad (\text{A.46})$$

from which it follows that

$$\Delta X - \Delta' X = \frac{1}{2} (X, \ln J). \quad (\text{A.47})$$

Replacing X by $\ln J$, one has also that

$$\tilde{\Delta} \ln J = 0. \quad (\text{A.48})$$

(To prove this, use an equation similar to eq.(A.43). Taking the second derivative of this result, one finds two terms

$$\partial[(\partial^* M) M^{-1}] = (\partial \partial^* M) M^{-1} - (\partial^* M) M^{-1} (\partial M) M^{-1}. \quad (\text{A.49})$$

Using eq.(A.1), each term vanishes separately due to the opposite symmetry of pairs of indices.

From eq.(A.48) one obtains

$$\Delta' \ln J = -\Delta \ln J = -\frac{1}{4} (\ln J, \ln J) \quad \text{or} \quad \Delta' \sqrt{J} = 0. \quad (\text{A.50})$$

A.4 Application for the functional integral.

The functional integral was taken on the surface Σ defined in eq.(2.14). More generally, one can define a surface Σ by the constraints $\zeta_A(\Phi, \Phi^*) = 0$, where

$$(\zeta_A, \zeta_B) = 0, \quad \frac{\partial \zeta_A}{\partial \Phi_B^*} \Big|_{\Phi} = \text{invertible.} \quad (\text{A.51})$$

The ζ_A in eq.(2.14) satisfy these two conditions. We can now interpret ζ_A as new coordinates Φ_A^* . As we saw at the end of section A.1, the brackets $(\zeta_A, \zeta_B) = 0$ imply that a function F exists, satisfying the second relation in eq.(A.2). If one now defines the variable Φ' conjugate to $\Phi_A^* = \zeta_A$ by the first equation in eq.(A.2), one has performed a canonical transformation, and hence ζ_A and Φ'^A satisfy again the canonical brackets. There is some freedom in the choice of Φ'^A since F is only determined by $(\zeta_A, \zeta_B) = 0$ up to functions of Φ'^* .

So when $\zeta = 0$, we have

$$\Phi_A^* = \frac{\partial \Psi(\Phi)}{\partial \Phi'^A}, \quad (\text{A.52})$$

where Ψ is defined by $F(\Phi, 0)$. Hence, hypersurfaces defined by $\zeta_A(\Phi, \Phi^*) = 0$ where ζ_A are in involution, can always be characterised by $\Phi_A^* - \frac{\partial \Psi}{\partial \Phi'^A} = 0$. Using eq.(A.37) we have

$$\delta(\zeta) = \delta(\Phi^* - \frac{\partial \Psi(\Phi)}{\partial \Phi'}) J^{\frac{1}{2}} \quad (\text{A.53})$$

The functional integral in terms of Φ and Φ^* is then given by

$$\begin{aligned} Z_\Psi &= \int D\Phi D\Phi^* e^{\frac{i}{\hbar} W} \delta(\Phi^* - \frac{\partial \Psi}{\partial \Phi'}) \\ &= \int D\Phi D\Phi^* \delta(\zeta) \left(\exp \frac{i}{\hbar} W' \right) J^{-\frac{1}{2}} \end{aligned} \quad (\text{A.54})$$

After a canonical transformation from (Φ, Φ^*) to $(\Phi', \Phi'^*) = (\zeta)$ we pick up a factor J , and obtain

$$\begin{aligned} Z_\Psi &= \int D\Phi' D\zeta \delta(\zeta) \left(\exp \frac{i}{\hbar} W'(\Phi, \Phi^*) \right) J^{\frac{1}{2}} \\ &= \int D\Phi' D\zeta \delta(\zeta) \left(\exp \frac{i}{\hbar} W'(\Phi', \zeta) \right). \end{aligned} \quad (\text{A.55})$$

where

$$W'(\Phi', \Phi'^*) = W'(\Phi, \Phi^*) + \frac{\hbar}{i} \ln J^{\frac{1}{2}}. \quad (\text{A.56})$$

From eq.(A.47) and eq.(A.50) it follows that the new W' satisfies again the master equation in the new variables

$$\Delta' e^{\frac{i}{\hbar} W'} = 0. \quad (\text{A.57})$$

References

- [1] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **69B** (1977) 309; **102B** (1981) 27;
- I.A. Batalin and E.S. Fradkin, Phys. Lett. **122B** (1983) 157.
- [2] I.A. Batalin and G.A. Vilkovisky, Phys. Rev. **D28** (1983) 2567.
- [3] M. Bochicchio, Phys. Lett. **B188** (1987) 330;
- C.B. Thorn, Nucl. Phys. **B287** (1987) 61;
- R.E. Kallosh, Phys. Lett. **B195** (1987) 369;
- I.A. Batalin, R.E. Kallosh and A. Van Proeyen, Symmetries of superparticle and superstring action, in "11th Seminar on Quantum Gravity", eds. M. Markov, V. Berезin and F. Frolov, World Scientific, 1988;
- A.H. Diaz and F. Toppan, Phys. Lett. **B211** (1988) 285;
- R.E. Kallosh, W. Troost and A. Van Proeyen, Phys. Lett. **B212** (1988) 428;
- U. Lindström, M. Roček, W. Siegel, P. van Nieuwenhuizen and A.E. van de Ven, Phys. Lett. **B224** (1989) 285; preprints ITP-SB-89-38, ITP-SB-89-45;
- S.J. Gates, Jr., M.T. Grisaru, U. Lindström, M. Roček, W. Siegel, P. van Nieuwenhuizen and A.E. van de Ven, Phys. Lett. **B225** (1989) 44;
- R. Kallosh, Phys. Lett. **B224** (1989) 273; **B225** (1989) 49;
- M.B. Green and C.M. Hull, Phys. Lett. **B225** (1989) 57, preprint QMC-89-11;
- M. Roček, W. Siegel, P. van Nieuwenhuizen and A.E. van de Ven, Phys. Lett. **B227** (1989) 87;
- J. Fisch and M. Henneaux, preprint ULB-T12/89-04;
- F. Bastianelli, G.W. Delius and E. Laenen, preprint ITP-SB-89-51;
- E.A. Bergshoeff and R.E. Kallosh, CERN-TH.5462/89.
- [4] K. Fujikawa, Phys. Rev. Lett. **42**(1979)1195; **44**(1980) 1733; Phys. Rev. **D21**(1980)2848 (E : **D22** (1980) 1499).
- [5] M. Hatsuda, P. van Nieuwenhuizen, W. Troost and A. Van Proeyen, Stony Brook preprint ITP-SB-89-43.
- [6] A. Diaz, W. Troost, P. van Nieuwenhuizen and A. Van Proeyen, JMPA **4** (1989) 3959.
- [7] W. Pauli and F. Villars, Rev. Mod. Phys. **21** (1949) 434.
- [8] I.A. Batalin and G.A. Vilkovisky, J. Math. Phys. **26** (1985) 172.
- [9] E.S. Fradkin and M.A. Vasiliev, Phys. Lett. **72B** (1977) 70;
- G. Sterman, P.K. Townsend and P. van Nieuwenhuizen, Phys. Rev.

- D17** (1978) 1501;
R.E. Kalosh, Zh. Eksp. Teor. Fiz. Pis'ma **26** (1977) 575; Nucl. Phys. **B141** (1978) 141;
B. de Wit and J.W. van Holten, Phys. Lett. **79B** (1978) 389;
B. de Wit, P. van Nieuwenhuizen and A. Van Proeyen, Phys. Lett. **104B** (1981) 27.
- [10] J. Govaerts, preprint CERN-TII.5010/88, to be published in JMPA; "*Hamiltonian Quantisation and Constrained Systems*", Leuven Notes in Mathematical and Theoretical Physics, Leuven University Press, to be published.
- [11] U. Lindström, N.K. Nielsen, M. Roček and P. van Nieuwenhuizen, Phys. Rev. **D37** (1988) 3588.
- [12] W.A. Bardeen, B. Zumino, Nucl. Phys. **B244** (1984) 421.
- [13] M.B. Einhorn and D.R.T. Jones, Phys. Rev. **D29** (1984) 331;
L. Alvarez-Gaumé and P. Ginsparg, Nucl. Phys. **B243** (1984) 449;
P. van Nieuwenhuizen, "*Anomalies in Quantum Field Theory*", Leuven Notes in Mathematical and Theoretical Physics, Leuven University Press, to be published.
- [14] Shi-Ke Hu, Bing-Lin Young and Douglas W. Mc Kay, Phys. Rev. **D30** (1984) 836.
- [15] W.A. Bardeen, Phys. Rev. **184** (1969) 1848.
- [16] K. Fujikawa, Phys. Rev. **D25** (1982) 2584, Nucl. Phys. **B226** (1983) 437.
- [17] I.A. Batalin and G.A. Vilkovisky, Nucl. Phys. **B234** (1984) 106.
- [18] H. Goldstein, "*Classical Mechanics*", Addison-Wesley Publ. co. , 1950, p. 237.
- [19] F.A. Berezin, "*The Method of Second Quantization*", Academic Press, 1966;
P. van Nieuwenhuizen, Phys. Rep. **68** (1981) 189 (appendices F-II);
B. DeWitt, '*Supermanifolds*', Cambridge Univ. Press, 1984 (ch. 1.5 and 1.6).