CERN/TH-5307/89

Chern-Simons Quantum Mechanics

P.S. Howe, *

CERN, Geneva, Switzerland

and

P.K. Townsend,

DAMTP, Univ. of Cambridge, Cambridge, U.K.

Abstract

The Chern-Simons action for three-dimensional gauge theories is a special case of a particle mechanics "sigma model Chern-Simons" action with a symplectic target space. We discuss the classical and quantum mechanics of this action and its (N -extended) supersymmetric generalizations.

We show further that the $N = 1$ supersymmetric Chern-Simons term can be used to cancel the global anomaly that arises in the standard supersymmetric sigma model when the target space is not a spin manifold. A new global anomaly arises, however, if the target space does not admit a spin^c -structure.

* Permanent address: Dept. of Mathematics, King's College, London, U.K.

The topological invariant $\int F \wedge F$ plays an important part in our understanding of four-dimensional gauge theories. For a four-manifold with boundary it equals the integral over the boundary of the associated Chern-Simons (CS) three-form, and this CS term is important for our understanding of three-dimensional gauge theories. Two-dimensional sigma models are analogous in many respects to four-dimensional gauge theories; in particular, if the 2n-dimensional target space M is symplectic, with symplectic two-form Ω (closed and non-degenerate), the integral over a two-dimensional spacetime of the pullback of Ω is a topological invariant analogous to $\int F \wedge F$. Since Ω is closed it can be written, locally, as dB , and the integral over a two-manifold with boundary S^1 then equals, locally, the integral over S^1 of the pullback of B . That is, in local coordinates $\{x^i, i = 1, \dots, 2n\}$ for M and coordinate t for S^1 , we have the "one-dimensional Chern-Simons" term

$$S_{CS} = \frac{1}{2\pi} \int_0^{2\pi} dt \, \dot{x}^i B_i(x(t)) \quad (1)$$

The variables $x^i(t)$ are periodic in t with period 2π . In this paper we study the classical and quantum mechanics of various actions containing this term.

Motivated by Witten's recent study of the pure three-dimensional CS theory [1] we shall first study the model for which the action is just S_{CS} , which we call "Chern-Simons Quantum Mechanics" (CSQM)*. The variation of S_{CS} under a general variation δx^i of $x^i(t)$ is

$$\delta S_{CS} = \frac{1}{2\pi} \int_0^{2\pi} dt \, \delta x^i \dot{x}^j \Omega_{ij} \quad (2)$$

Since Ω is non-degenerate the equation of motion for x^i is simply**

$$\dot{x}^i = 0 \quad (3)$$

Identifying the phase space with the space of classical solutions, we see that the former is just the target space M itself.

Observe that

$$S_{CS}[x; B] = S_{CS}[x + v; B + \mathcal{L}_v B] \quad (4)$$

where v is a vector field on M and $\mathcal{L}_v$ is the Lie derivative with respect to it. Since also

$$S_{CS}[x; B + dF] = S_{CS}[x; B] \quad (5)$$

it follows that S_{CS} is invariant under symplectomorphisms (symplectic diffeomorphisms) of M , i.e. for v satisfying

$$\mathcal{L}_v \Omega = 0 \quad (6)$$

* This model has been discussed in another context in ref. [2].

** The inclusion of the "potential" term $-\frac{1}{2\pi} \int dt H(x)$ would instead produce Hamilton's equations with Hamiltonian H .

which implies that $\mathcal{L}_v B = dF$ for some function F on M . A vector field satisfying (6) has components $v^j = \Omega^{ij} \partial_i \phi$ for some function ϕ on M , and one then verifies from (2) that $\delta S_{CS} = 0$. In addition, the action is invariant under the time reparametrizations

$$\delta x^i = a(t) \dot{x}^i \quad (7)$$

induced by a diffeomorphism of S^1 .

The symplectomorphism invariance of S_{CS} is analogous to those rigid invariances of the usual sigma model induced by diffeomorphisms of the target space that are isometries of the sigma model metric. If $\pi_1(M) = 0$ the generators of the symplectomorphisms are the C^∞ functions on M . Given a C^∞ function ϕ we verify that

$$\delta x^i = -[\phi, x^i]_{PB} = \Omega^{ij} \partial_j \phi \quad (8)$$

The Poisson bracket is defined by

$$[f, g]_{PB} = \Omega^{ij}(x) \partial_i f \partial_j g \quad (9)$$

where Ω^{ij} is the matrix inverse of Ω . If $\pi_1(M) \neq 0$ there will be additional generators of the group of symplectomorphisms.

Just as one can gauge a subgroup of the isometry group of the target space of the usual sigma model, one can gauge a subgroup of the symplectomorphism group of the target space of a CSQM model. Let H be a subgroup with Lie algebra $\mathcal{H}$ spanned by generators T_r with

$$[T_r, T_s]_{\mathcal{H}} = f_{rs}{}^t T_t \quad (10)$$

This Lie algebra is represented by vector fields v_r on M and this Lie algebra homomorphism can be lifted to a Lie algebra homomorphism from $\mathcal{H}$ to $C^\infty(M)$ provided that $H^1(\mathcal{H}) = H^2(\mathcal{H}) = 0$. The image of T_r under this homomorphism is the function ϕ_r . These functions define the *moment map*, $\mu: M \rightarrow \mathcal{H}^*$, by

$$x \mapsto \phi_r(x) T^r \quad (11)$$

where the $\{T^r\}$ span $\mathcal{H}^*$. The gauged CSQM action is then

$$S = \frac{1}{2\pi} \int_0^{2\pi} dt [\dot{x}^i B_i - u^r \phi_r(x)] \quad (12)$$

where the gauge fields are the Lagrange multipliers $\{u^r(t)\}$ which impose the constraints

$$\phi_r(x) = 0 \quad (13)$$

These equations define the subspace $\mu^{-1}\{0\}$ of M (which may be disconnected). The physical phase space is $\mu^{-1}\{0\}/H$ and is called the *symplectic quotient* of M by H . A standard theorem ensures that this is also a symplectic manifold [3].

An example of a gauged CSQM model is Witten's three-dimensional CS gauge theory on a three manifold of the type $S^1 \times \Sigma$, with Σ a compact two-manifold without boundary. Let P be a principal G -bundle over Σ ; then the target space of the CSQM model is $\mathcal{A}$, the space of connections on P . Let E be the vector bundle associated to P via the adjoint representation of G , and let $E^* \equiv \Lambda^2 \Sigma \otimes E$; then the tangent space for $\mathcal{A}$ is the space of sections, $\Gamma(E^1)$, of E^1 . For any two tangent vectors α, β , the symplectic form Ω is [1,4]

$$\Omega(\alpha, \beta) = \int_{\Sigma} \text{tr}(\alpha \wedge \beta) \quad (14)$$

Clearly Ω is closed, and can be written locally as $\Omega = dB$ with

$$B(\alpha) = \int_{\Sigma} \text{tr}(\alpha \wedge A) \quad (15)$$

The non-gauged CSQM action is therefore

$$S = \int dt \int_{\Sigma} \text{tr}(A \wedge \dot{A}) \quad (16)$$

The symplectomorphisms to be gauged (i.e. to be promoted to t -dependent symmetries) are just the two-dimensional gauge transformations. The group $\mathcal{G}$ of these gauge transformations consists of the vertical automorphisms of P . Under an infinitesimal gauge transformation with parameter $u \in \mathcal{G} = \Gamma(E)$ we have

$$\delta A = D_A u \quad (17)$$

where D_A is the covariant derivative with respect to the connection A . The corresponding generator ϕ_u is

$$\phi_u = \int_{\Sigma} \text{tr}(u F_A) \quad (18)$$

where F_A is the Yang-Mills field-strength two-form for A . The Lie algebra $\mathcal{G}$ of $\mathcal{G}$ is $\Gamma(E)$ and its dual space $\mathcal{G}^*$ is $\Gamma(E^2)$, so that the moment map in this case is simply $A \mapsto F_A$ [4]. Following the prescription for the construction of a gauged CSQM we obtain the action

$$S = \int dt \int_{\Sigma} \text{tr}(A \wedge \dot{A} + A_0 F^1) \quad (19)$$

where A_0 is a Lagrange multiplier. This is the action of the three-dimensional CS gauge theory. The associated symplectic quotient is the space of flat G -bundles on Σ , i.e. the space of flat connections modulo (two-dimensional) gauge transformations [1]. The well-known quantization condition for the coefficient of the CS term in three-dimensional gauge theories [5] is therefore a special case of the quantization condition that we shall find below for CSQM.

To pass to the quantum theory, we consider the path-integral

$$Z = \int [dx] e^{iqS_{CS}} \quad (20)$$

For this to be well defined, the "charge" q must be an integer. To see this recall that we introduced S_{CS} as the integral of Ω over a two-chain of M with boundary S^1 . It follows that this S^1 is a homologically trivial one-cycle of M and can be contracted to a point. If we view the S^1 as a great circle passing through the north pole of an S^2 then it is clear that the S^1 can be contracted to the north pole in essentially two different ways, leading to actions which differ by the addition of the term

$$\Delta S_{CS} = \frac{1}{2\pi} \int_{S^2} \Omega \quad (21)$$

In general, therefore, S_{CS} will not be well defined, but if q is an integer and Ω is an integral two-form, such that $\int_{S^2} \Omega = 2\pi k$ ($k \in \mathbb{Z}$), then $e^{iqS_{CS}}$ will be. This is essentially the Dirac quantization condition. If $\pi_1(M) \neq 0$ one may consider a generalisation of the CSQM model introduced above for which S^1 is not the boundary of any two-chain. The proof of the quantization condition is then more involved [6].

The requirement that Ω be integral is precisely the condition that M be "quantizable", in the terminology of geometric quantization*. This is the only requirement for the consistency of the path-integral for the CS action with periodic boundary conditions. Such boundary conditions arise in the computation of the partition function at finite temperature. The computation of other quantities, such as the propagator, requires different boundary conditions, in which case, since the CS action leads to first-order equations of motion, it is necessary to choose a *polarisation* of the phase space. This allows the construction of a quantum mechanical Hilbert space. An example would be M Kähler with Ω the Kähler form related to a complex structure I , in which case I defines a complex polarization. In this case the condition that Ω be integral is the condition that M be Hodge.

The CSQM action (1) is easily supersymmetrized. For $N=1$ we have the superspace action

$$S_{CS}^{N=1} = \frac{1}{2\pi} \int_0^{2\pi} d\theta d\theta \, D X^i B_i(X(t, \theta)) \quad (22)$$

where $P^2 = i\partial_t$. In components this is

$$S = \frac{1}{2\pi} \int_0^{2\pi} dt \left[\dot{x}^i B_i - \frac{i}{2} \lambda^i \lambda^j \Omega_{ij} \right] \quad (23)$$

where** $x^i = X^i$ and $\lambda^i = DX^i$. This action is in fact *locally* supersymmetric.

Any additional supersymmetry transformation of X^i must have the form

$$\delta X^i = i\zeta F_j(X) D X^j \quad (24)$$

where I is a complex structure on M (this follows from the $N=2$ supersymmetry algebra alone). The anticommuting parameter ζ is a function of t and θ with $\zeta|$ the parameter of

* See for example [7]

** The vertical bar indicates the $\theta = 0$ component of a superfield.

the second local supersymmetry and $D\zeta$ the parameter of the local $SO(2)$ invariance. The transformation (24) is an invariance of the action (22) if and only if Ω is a $(1,1)$ form with respect to I . Then, since Ω is closed it can be written locally, in terms of a real potential F , as

$$\Omega = -2i\partial\bar{\partial}F, \quad (25)$$

where ∂ and $\bar{\partial}$ are the holomorphic and anti-holomorphic projections of the exterior derivative operator d . Thus, when the action (22) is $N = 2$ supersymmetric it can be written in the $N = 2$ superspace form

$$S_{CS}^{N=2} = \frac{1}{2\pi} \int dt d\theta d\bar{\theta} F(X, \bar{X}) \quad (26)$$

where the complex superfields X^α ($\alpha = 1, 2, \dots, n$) are “chiral”, i.e.

$$\bar{D}X^\alpha = 0 \quad (27)$$

with $D^2 = \bar{D}^2 = 0$, $\{D, \bar{D}\} = i\partial_t$. The symplectomorphisms which leave the $N = 2$ action invariant are generated by *holomorphic* vector fields, i.e. vector fields u for which $\mathcal{L}_u I = 0$.

In the $N = 2$ case it is possible to introduce a metric which is Kähler with respect to I such that Ω is the corresponding Kähler two-form. The situation is different when we have $N = 4$ supersymmetry. In this case there are three complex structures obeying the algebra of the unit imaginary quaternions, and Ω must be $(1,1)$ with respect to each of them. On a hyperKähler manifold there are three Kähler forms, corresponding to the three complex structures, each one of which is $(1,1)$ with respect to its associated complex structure but $(2,0) \oplus (0,2)$ with respect to each of the other two. On a hyperKähler manifold, therefore, one cannot identify Ω with any of the Kähler two-forms. A target space for an $N = 4$ CSQM may therefore be called “*Quaternionic Hamiltonian*”. The symplectomorphisms which are invariances of the $N = 4$ action are those which are triholomorphic, i.e. holomorphic with respect to all three complex structures.

The $N = 1$ supersymmetrization of the gauged CSQM model is straightforward. The superfield action is

$$S = \frac{1}{2\pi} \int_0^{2\pi} dt d\theta [DX^i H_i + \Psi^\tau \phi_\tau(X)] \quad (28)$$

where the Lagrange multipliers $\{\Psi^\tau\}$ are Grassman odd superfields. Gauging the $N = 2$ model is more complicated. In order to carry it out in $N = 2$ superfields one has to take into account the structure of $N = 2$ superspace gauge theories. This structure is formally similar to $N = 1$ in four dimensions or $(2,0)$ in two dimensions. Geometrically, the action of the Lie group G of transformations of M generated by holomorphic Hamiltonian vector fields can be extended to an action of the complexified group $G_\mathbb{C}$. Modulo some technical points, the symplectic quotient $\mu^{-1}\{0\}/G$ is the same as the complex quotient $M/G_\mathbb{C}$. We refer to [8] for details and for the associated superfield construction.

We now introduce a metric on M and consider the $N = 1$ CSQM action in conjunction with the usual supersymmetric sigma model. The combined action is

$$S = \frac{1}{2\pi} \int_0^{2\pi} dt d\theta \left[\frac{1}{2} g_{ij} \dot{X}^i \dot{X}^j + q DX^i B_i \right] \quad (29)$$

where, for the moment, we assume that the charge q is an integer. Of course, the phase space of *this* model is the cotangent bundle of M and not M itself. In components the action reads

$$S = \int_0^{2\pi} dt \left[\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \frac{i}{2} g_{ij} \lambda^i \nabla_t \lambda^j + \frac{q}{2\pi} (\dot{x}^i B_i - \frac{i}{2} \lambda^i \lambda^j \Omega_{ij}) \right] \quad (30)$$

In Hamiltonian form this becomes

$$S = \int_0^{2\pi} dt \left[p_i \dot{x}^i + \frac{i}{2} g_{ij} \lambda^i \dot{\lambda}^j - i \frac{q}{4\pi} \lambda^i \lambda^j \Omega_{ij} - \frac{1}{2} g^{ij} \left(p_i - \frac{q}{2\pi} B_i - \frac{i}{2} \omega_{iab} \lambda^a \lambda^b \right) \left(p_j - \frac{q}{2\pi} B_j - \frac{i}{2} \omega_{jcd} \lambda^c \lambda^d \right) \right], \quad (31)$$

where $\lambda^a = \lambda^i e_i{}^a(x)$ with $e_i{}^a$ a vielbein matrix ($e_i{}^a e_j{}^b \delta_{ab} = g_{ij}$), and ω_{iab} is the standard spin connection. The supercharge Q is

$$Q = \lambda^a e_a{}^i \left(p_i - \frac{q}{2\pi} B_i - \frac{i}{2} \omega_{iab} \lambda^a \lambda^b \right). \quad (32)$$

Upon quantization

$$Q \rightarrow \hat{Q} = -\frac{i}{\sqrt{2}} \gamma^a e_a{}^i \left(\hat{p}_i - i \frac{q}{2\pi} B_i + \frac{1}{4} \omega_{iab} \gamma^{ab} \right) \equiv \hat{\mathcal{P}}(\omega, B) \quad (33)$$

and the space of wave functions is the space of sections of $V_q \equiv V \otimes L^q$ where V is the spin bundle of M and L is a complex line bundle over M . Since M is even dimensional $V = V_+ \oplus V_-$ where $V_\pm$ are the chiral spin bundles. Therefore the Witten index of the model is the difference between the number of positive and negative chirality zero modes of the Dirac operator $\hat{\mathcal{P}}$, i.e.

$$\Delta_W = \text{Index}(\hat{\mathcal{P}}) \equiv n_+ - n_- \quad (34)$$

This index can be calculated by path-integral methods [9, 10]. We have

$$\begin{aligned} \Delta_W &= \text{str}(e^{-\beta H}) \\ &= \int_{PBC} [dx] [d\lambda] e^{-\int_0^\beta dt L_E}, \end{aligned} \quad (35)$$

where $\tilde{H} = \hat{Q}^2$, and L_E is the classical Euclidean Lagrangian

$$L_E = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} g_{ij} \lambda^i \nabla_t \lambda^j + \frac{q}{2\pi} (-i \dot{x}^i B_i + \frac{i}{2} \lambda^i \lambda^j \Omega_{ij}) \quad (36)$$

corresponding to (30). The path integral is to be taken for periodic $x(t), \lambda(t)$, with period β . After the rescaling $t \rightarrow \beta t, \lambda \rightarrow \beta^{-\frac{1}{2}} \lambda$ we obtain the Euclidean action

$$S_E = \beta^{-1} \int_0^1 dt \left[\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} g_{ij} \lambda^i \nabla_t \lambda^j + \frac{\beta q}{2\pi} (-i \dot{x}^i B_i + \frac{i}{2} \lambda^i \lambda^j \Omega_{ij}) \right]. \quad (37)$$

Since ΔW is independent of β the path-integral can be evaluated in the $\beta \rightarrow 0$ limit. In this limit the path integral is dominated by the constant configurations. The only term that survives for constant paths is the $\lambda^2 \Omega$ term. For the fluctuations, β acts as a loop-counting parameter, except that diagrams with insertions of the $\dot{x}B$ or $\lambda^2 \Omega$ terms are suppressed by additional factors of β . The terms independent of β are thus the constant path contributions from $\lambda^2 \Omega$, and the one-loop contribution from the conventional signal model terms. The latter has been evaluated previously [9, 10] and the combined result is

$$\Delta W = \int_M \dot{\lambda}(M) \wedge \text{ch}(L^q) \quad (38)$$

with

$$\begin{aligned} \dot{\lambda}(M) &= \prod_{p=1}^n \frac{\sinh x_p}{x_p} \\ \text{ch}(L^q) &= \exp\left(\frac{iq\Omega}{2\pi}\right), \end{aligned} \quad (39)$$

where the x_p 's are the formal two-form skew-eigenvalues of the curvature tensor. (As usual, in the formula (38) the integrand is understood to be the 2n-form component.)

The validity of the formula (38) depends on the path integral being well defined. Integrating over the $\lambda(t)$ variables in (35) we obtain

$$\Delta W = \int [dx] \det^{\frac{1}{2}}(\delta_{ab} \partial_t + A_{ab}(t)) \exp\left(-\frac{1}{2\beta} \int_0^1 dt g_{ij} \dot{x}^i \dot{x}^j + \frac{iq}{2\pi} \int_0^1 dt \dot{x}^i B_i\right), \quad (40)$$

where the quantity

$$A_{ab} \equiv \dot{x}^i \omega_{iab} + \frac{iq\beta}{4\pi} \Omega_{ab} \quad (41)$$

is an $SO(n)$ gauge connection on the circle. The sign of the square root of the determinant in (40) must be chosen for a representative loop in each homology class; the sign for all other loops in this class should then be determined by continuity. If we consider a sequence of loops that starts and terminates at a given loop we trace out a closed path in the space of loops. This defines, via the the $SO(n)$ holonomy of each loop, a closed path in $SO(n)$. Since $\pi_1(SO(n)) = \mathbb{Z}_2$ this closed path may be non-contractible, and hence not a closed path in $Spin(n)$. In such circumstances it is obvious that a spinor cannot be consistently defined on M . On the other hand the explicit evaluation of the determinant in (40) shows that for just such a sequence of loops the sign of the square root flips and therefore cannot be consistently defined [10, 11].

Previously we considered the change in the Chern-Simons term under a particular such sequence of loops. We saw firstly that Ω is an integral two-form and secondly that q should be an integer. If q is half-integral then in passing through this sequence of loops the factor $e^{iq\delta_{CS}}$ may change sign. This raises the question of whether one can arrange for the change in sign of $\det^{\frac{1}{2}}(\partial_t + A)$ to be compensated by the change in sign of $e^{iq\delta_{CS}}$ or, rather, whether one can find a closed two-form Ω on M such that this cancellation of signs occurs for every two-cycle defined by all closed sequences of loops in M . It is easy to

see that this can always be arranged to happen if $H_2(M, \mathbb{Z})$ is a free group. In this case, let ω_r be a basis for $H^2(M, \mathbb{Z})$ and c_r the dual basis of $H_2(M, \mathbb{Z})$ so that

$$\int_{c_r} \omega_r = \delta_{r,s} \quad (42)$$

The general Ω will have the form

$$\Omega = \sum_r \alpha_r \omega_r \quad (43)$$

Associated with each two-cycle c_r we have a sign η_r associated with the square root of the determinant (i.e. η_r is $+1$ or -1 depending on whether $\det^{\frac{1}{2}}(\partial_t + A)$ does or does not change sign, respectively). This sign is compensated by a change in sign of $e^{iq\delta_{CS}}$ provided we choose

$$\alpha_r = \begin{cases} even, & \eta_r = +1 \\ odd, & \eta_r = -1 \end{cases} \quad \text{and} \quad q + \frac{1}{2} \in \mathbb{Z} \quad (44)$$

In the case that $H_2(M, \mathbb{Z})$ has torsion the above mechanism may not work. In fact, as we now discuss, the anomaly cancellation mechanism will work if and only if M admits a spin^c-structure. We recall that spin structures can be studied via the short exact sequence of groups,

$$0 \rightarrow \mathbb{Z}_2 \rightarrow Spin(n) \rightarrow SO(n) \rightarrow 0 \quad (45)$$

which induces an exact sequence in sheaf cohomology,

$$\rightarrow H^1(M, Spin(n)) \xrightarrow{\Delta} H^1(M, SO(n)) \xrightarrow{\Delta} H^2(M, \mathbb{Z}_2). \quad (46)$$

Now $H^1(M, G)$ is the set of G -bundles over M ; if $\xi \in H^1(M, SO(n))$ it can be lifted to a $Spin(n)$ -bundle ζ , $\beta\zeta = \xi$, if and only if $\delta\xi = 0$. If ξ is $O^+(M)$, the principal $SO(n)$ -bundle of orthonormal frames^{*}, then $\delta\xi$ is $w_2(M)$, the second Stiefel-Whitney class of M , and hence the condition on M for it to be a spin manifold is that $w_2(M) = 0$.

Similarly, spin^c-structures [12, 13] can be studied via the short exact sequence [14],

$$0 \rightarrow U(1) \rightarrow Spin^c(n) \rightarrow SO(n) \rightarrow 0 \quad (47)$$

where

$$Spin^c(n) = Spin(n) \times_{\mathbb{Z}_2} U(1) \quad (48)$$

This is the group of equivalence classes in $Spin(n) \times U(1)$, the equivalence relation being

$$(\lambda, a) \sim (-\lambda, -a), \quad (\lambda, a) \in Spin(n) \times U(1) \quad (49)$$

In sheaf cohomology we then have the exact sequence

$$\rightarrow H^1(M, Spin^c(n)) \xrightarrow{\Delta} H^1(M, SO(n)) \xrightarrow{\Delta} H^2(M, U(1)) \cong H^3(M, \mathbb{Z}). \quad (50)$$

^{*} We assume throughout that M is orientable.

Thus if $\xi \in H^1(M, SO(n))$ it can be lifted to a bundle $\zeta \in H^1(M, Spin^c(n))$ if $\delta\xi = 0$. For $\xi = O^+(M)$, $\delta\xi = W_3(M)$, the third integral Stiefel-Whitney class of M . Thus the condition for M to admit a spin^c-structure is $W_3(M) = 0$.

The short exact sequence of Abelian groups,

$$0 \rightarrow Z \rightarrow Z \rightarrow Z_2 \rightarrow 0 \quad (51)$$

gives rise to a long exact sequence in cohomology

$$\rightarrow H^2(M, Z) \xrightarrow{\Delta} H^2(M, Z_2) \xrightarrow{\Delta} H^3(M, Z) \rightarrow \quad (52)$$

and $W_3(M) = \delta w_2(M)$, where δ is the Hockstein homomorphism for this sequence. Hence $W_3(M) = 0$ if and only if $w_2(M) = \beta\eta$, where η is some class in $H^2(M, Z)$ and β is the reduction of coefficients from Z to Z_2 . Thus the condition for M to admit a spin^c structure can be rephrased as the requirement that the second Stiefel-Whitney class of M be the reduction mod 2 of an integral cohomology class.

Recalling the cancellation mechanism discussed previously we see that it will work when M admits a spin^c-structure, as the Z_2 anomaly in the fermion determinant is cancelled by the sign arising from the Chern-Simons term, and the normalised integral of Ω is an integer. To see how this mechanism can fail we recall that the universal coefficients theorem states that

$$H^n(M, A) = Hom(H_n, A) \oplus Ext(H_{n-1}, A) \quad (53)$$

where $H^n(M, A)$ is the n th cohomology group with coefficients in the Abelian group A , H_n is the n th singular homology group (integer coefficients) and $Ext(H_{n-1}, A)$ is the group of all extensions of A by H_{n-1} . This theorem implies that

$$(H^n(M, Z))_{torsion} \cong (H_{n-1}(M))_{torsion} \quad (54)$$

If M is not a spin^c-manifold, $W_3(M) \neq 0$, which implies that there is a Z_2 torsion element in $H^3(M, Z)$ and hence, from (54), a Z_2 torsion element in $H_2(M)$, i.e. a two-cycle c such that $c = \frac{1}{2}\partial c'$ for some three-chain c' . (Then $\partial c = 0$, but $c \neq \partial(\cdot)$ in Z ; however, $2c = \partial c'$, and so $2c = 0$ in cohomology.) In this case we have

$$\int_{Z_2} \Omega = 2 \int_{\Omega} = \int_{\partial c'} \Omega = \int_{c'} d\Omega = 0 \quad (55)$$

and so a minus sign arising from the fermion determinant from a sequence of loops which traces out c cannot be compensated by the Chern-Simons term.

In the case that M is four-dimensional the index formula (38) becomes

$$n_+ - n_- = \frac{1}{192\pi^2} \int_{M_4} tr(R \wedge R) + \frac{q^2}{8\pi^2} \int_{M_4} \Omega \wedge \Omega \quad (56)$$

As an example [13] consider $M_4 = \mathbb{CP}^2$ for which the first term on the r.h.s. of this equation equals $-\frac{1}{8}$. For $q = 0$ equation (56) is therefore not satisfied by $\mathbb{CP}^2$; this is a

manifestation of the fact that $\mathbb{CP}^2$ is not a spin manifold. However, for a line bundle over $\mathbb{CP}^2$ the second term in (45) equals $\frac{1}{2}q^2$, so that for $q = m + \frac{1}{2}$, the r.h.s. of (45) equals $\frac{1}{2}m(m+1)$ which is always an integer, as it should be because $\mathbb{CP}^2$ has a spin^c-structure. Indeed, any four-manifold admits a spin^c-structure [14], although there are examples of manifolds which do not admit such structures for dimensions greater than four.

The non-Abelian generalization of a spin^c-structure is a spin G -structure [15]. The existence of such structures is relevant to the global anomaly problem for a supersymmetric sigma model whose quantum supercharge is the twisted Dirac operator acting on wavefunctions which are sections of $V \otimes E$ where V is again the spin bundle of M and E is a vector bundle with a non-Abelian structure group, G . For a discussion of these models and of the anomalies of general supersymmetric sigma models with supercharges corresponding to arbitrary elliptic operators the reader is referred to ref. [16].

Acknowledgements

We are grateful to C.J. Isham, T. Killingback, I. Leary and G. Papadopoulos for helpful discussions.

References

- [1] E. Witten, *Quantum field theory and the Jones polynomial*, to appear in the proceedings of the IAMP Congress, Swansea, July 1988, eds. I. Davis, B. Simon, and A. Truman.
- [2] R. Floreanini, R. Percacci, and E. Sezgin, *Sigma-models with purely Wess-Zumino-Witten terms*, Trieste preprint ICTP-88-363.
- [3] J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys. **5** (1974) 121.
- [4] T. Killingback, *Quantum Chern-Simons theory*, preprint DAMTP/88-31.
- [5] S. Deser, R. Jackiw, and S. Templeton, *Topologically massive gauge theory*, Ann. Phys. (N.Y.) **140** (1984) 372.
- [6] O. Alvarez, *Topological quantization and cohomology*, Commun. Math. Phys. **100** (1985) 279.
- [7] D.J. Simons and N.M.J. Woodhouse, *Lectures on geometric quantization*, (Springer Verlag 1976).
- [8] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, *HyperKähler metrics and supersymmetry*, Commun. Math. Phys. **108** (1987) 535.
- [9] L. Alvarez-Gaumé, *Supersymmetry and the Atiyah-Singer index theorem*, Commun. Math. Phys. **90** (1983) 161.
- [10] D. Friedman and P. Winfrey, *Supersymmetric derivation of the Atiyah-Singer index and the chiral anomaly*, Nucl. Phys. **235** [FS11] (1984) 395.
- [11] E. Witten, *Global anomalies in string theory*, in 'Symposium on Geometry, Anomalies and Topology', eds. W.A. Bardeen and A.R. White, (World Scientific 1985).
- [12] M.F. Atiyah, R. Bott, and V.K. Patodi, *On the heat equation and the index theorem*, Inv. Math. **19** (1973) 279.
- [13] S.W. Hawking and C.N. Pope, *Generalized spin structures in quantum gravity*, Phys. Lett. **73B** (1977) 42.
- [14] T.P. Killingback and E.G. Rees, *Spin^c-structures on manifolds*, Class. Quantum Grav. **2** (1985), 433.
- [15] S.J. Avis and C.J. Isham, *Generalized spin-structures on four-dimensional space-times*, Commun. Math. Phys. **72** (1980) 103.
- [16] P.S. Howe, *Elliptic operators and supersymmetry*, to appear in the proceedings of the 1989 Trieste workshop on supermembranes and physics in 2 + 1 dimensions, eds. M. Duff, E. Sezgin and C.N. Pope, (World Scientific).