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FOUR-FORMS AND THE VANISHING OF THE COSMOLOGICAL CONSTANT

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ABSTRACT

We show that the dynamics (and statics) of a four-form are more subtle than previously appreciated, especially when used in conjunction with the Baum-Hawking-Coleman mechanism. We investigate the properties of the four-forms on closed Euclidean, open Euclidean, and Lorentzian manifolds. We find that it is only under special conditions that their behaviour is such that the cosmological constant vanishes.

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1 Introduction.

The Baum [1] – Hawking [2] – Coleman [3] mechanism is the latest in a long line of attempts¹ to suggest a reason for the vanishing of the macroscopic cosmological constant regardless of the microscopic behaviour of matter. Even though this approach is still mathematically unsatisfactory it nevertheless has a certain appeal from the viewpoint of physics. The idea in itself is rather straightforward: the cosmological constant defies its name by varying over a range of values. All that is required is a theory which provides a measure of the probability for a particular value. If the probability is strongly peaked around a value of zero this would suggest that the universe we inhabit has vanishing cosmological constant.

The theories that attribute such a probabilistic behaviour to the cosmological constant are unadulterated quantum cosmology, as in the approach of references [1,2,4], or its extension to include topological fluctuations due to wormholes, as in references [3,5]. Both rely on the precepts of Euclidean quantum gravity. In both cases one has to believe that a rotation to complex time is valid for gravitational systems and that (semiclassically) the Feynman path integral is dominated by configurations which are stationary points of the Euclidean action. With these assumptions the argument proceeds as follows:

1. In Euclidean space solve the Einstein equations (stationary configurations of the Euclidean action) to identify the true cosmological constant. The equation of motion for the background geometry is

$$<G_{\mu\nu}> = 8\pi G_N <T_{\mu\nu}> - \Lambda_0 <g_{\mu\nu}>$$

where Λ_0 is a 'bare' cosmological constant, arising from a term

$$\frac{\Lambda_0}{8\pi G_N} \int \sqrt{g} d^4x$$

in the original Euclidean action. If the matter background, obtained from the matter equations of motion, is such that

$$<T_{\mu\nu}> = -\frac{\Lambda_m}{8\pi G_N} <g_{\mu\nu}>$$

then $\Lambda_{\text{cos}} = \Lambda_0 + \Lambda_m$ becomes the 'true'

¹For an introduction to the history of the cosmological constant problem consult the recent review by Weinberg [4].

cosmological constant which determines the global or large-scale structure of the universe *i.e.* $<G_{\mu\nu}> = -\Lambda_{\text{cos}} <g_{\mu\nu}>$.

2. If $\Lambda_{\text{cos}} > 0$ the resulting Euclidean space is closed and has finite volume. The maximum volume and maximally symmetric solution is a four-sphere of volume $V = 24\pi^2/\Lambda_{\text{cos}}^2$. Otherwise, for $\Lambda_{\text{cos}} < 0$, we consider the space to be open and have infinite volume. (We shall disregard the compact, finite-volume spaces with $\Lambda_{\text{cos}} < 0$ as these have rather bizarre properties.)

3. Evaluate the Euclidean action at its stationary value $<S_E>$ by inserting the solutions to the equations of motion back into the action. The resulting expression will be a function of the bare parameters of the theory, denoted by λ . Λ_{cos} can also be expressed in terms of the λ s. Here the two theories differ:

- (a) In quantum cosmology the probability that the parameters of the theory have the values λ is [4]

$$P(\lambda) \propto \exp(-<S_E>_\lambda)$$

where $<S_E>_\lambda$ is the above extremal value of the Euclidean action with parameters λ . Some of the parameters may be regarded as fixed at the outset, such as gauge couplings, while others might be undetermined, such as a choice of some vacuum expectation values. It is this probability distribution that will determine the latter.

- (b) Alternatively, in the presence of wormholes, the expectation value of any operator $\hat{B}$ is [3,5]

$$<\hat{B}> = \int d\alpha \rho(\alpha) <\hat{B}>_{\lambda+\alpha}$$

where $<S_E>_{\lambda+\alpha}$ is understood to be the expectation in a theory where all parameters are shifted by amounts α , one for each λ . The distribution of values of the α is

$$\rho(\alpha) \sim \exp\left(-\frac{1}{2}\alpha^T D\alpha\right) X e^X$$

where $X = \exp(- < S_E >_{\lambda+\alpha})$. The matrix D encapsulates the unknown properties of the wormholes.

4. The element common to both the above probability distributions is the presence of the $\exp(- < S_E >)$ whose origin is in the semiclassical approximations to their respective path integrals. If in the limit of $\Lambda_{\text{cos}}(\lambda) \rightarrow 0_+$ the extremal value of the action diverges $< S_E >_{\lambda} \rightarrow -\infty$ then both $P(\lambda)$ and X diverge to positive infinity. There is a singularity in the parameter space as $\Lambda_{\text{cos}} \rightarrow 0_+$. In case (a) any undetermined parameters will relax to these values, and the most probable configuration will be one with vanishing cosmological constant. In case (b) the integral over the α s is dominated by those values for which the cosmological constant vanishes. Moreover, for a negative Λ_{cos} with an open, infinite-volume space the action diverges to $+\infty$ and such configurations are therefore highly suppressed.

Taken together, the four steps above constitute the argument, based on considerations of Euclidean quantum gravity, that the cosmological constant in our universe is inescapably forced to vanish.

In his original paper, Hawking [2] discusses the case of a four-form field strength as an example of a matter field whose background is undetermined by its dynamics. It can therefore serve as one of the parameters in step 3(a) above. The peculiar properties of four-forms in connection with the cosmological constant had been previously noted by several authors [6, 7, 8], albeit in a slightly different context.

Duff [9] has recently revived this discussion, claiming that proper consideration of the variational principle and subsequent evaluation of the action at its stationary point invalidates step 4 above. His point is well made: one should solve all the background equations, gravity plus matter, *simultaneously* rather than substituting the matter background configuration into the action and then varying this to obtain the gravitational background. The final stationary value of the action can differ in the two cases, and for our purposes the correct value is all-important.

In this paper we shall examine more thoroughly the dynamics (and statistics) of the four-form and in doing so we shall uncover a flaw in Duff's anal-

ysis. The next two sections consist of a discussion of the possible actions for the four-form and its behaviour in Euclidean space when incorporated into the Baum-Hawking-Coleman (BHC) mechanism. There still arises the nagging question of how to reconcile the BHC mechanism with the fact that our universe is Lorentzian. We shall keep clear of this unresolved debate and merely adopt a half-way measure which has been very useful elsewhere [10, 11]. Essentially it allows us to arrive at a Euclidean action which respects any special properties the original Lorentzian theory possesses, such as charge conservation. This is done in the fourth section. Thereafter we conclude.

2 Four-form Actions.

There are three possible actions for a four-form, each differing in the contribution of boundary terms. Denote the four-form field strength by $\mathbf{F}$. It is the curl of a three-form potential $\mathbf{A}$ i.e. $\mathbf{F} \equiv d\mathbf{A}$. On a Euclidean manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ they are respectively,

$$S_1 = \frac{1}{2} \int_{\mathcal{M}} \mathbf{F} \wedge \star \mathbf{F} \quad (1)$$

$$S_2 = \frac{1}{2} \int_{\mathcal{M}} \mathbf{F} \wedge \star \mathbf{F} - \int_{\mathcal{M}} d(\mathbf{A} \wedge \star \mathbf{F}) \quad (2)$$

$$S_3 = \frac{1}{2} \int_{\mathcal{M}} \mathbf{F} \wedge \star \mathbf{F} - c \int_{\mathcal{M}} \mathbf{F} \quad (3)$$

where c is just a constant. If instead we had chosen to examine this in a Lorentzian space with signature $(- + + +)$ they would all have acquired a minus sign. The two extra terms in (2) and (3) are readily seen to be boundary terms by Stokes' theorem.

Let us look first at S_1 . In Euclidean space S_1 is positive definite as it is the square of the field strength. In components it is simply

$$S_1 = \frac{1}{2} \frac{1}{4!} \int_{\mathcal{M}} \sqrt{g} F_{\mu\lambda\rho} F^{\mu\lambda\rho} d^4x \quad (4)$$

Under variations of the potential, δA , but keeping the background geometry fixed, the variation of this action is

$$\delta S_1 = \int_{\mathcal{M}} \delta A \wedge d \star F + \int_{\partial \mathcal{M}} \delta A \wedge \star F$$

Thus if A is fixed on the boundary², $\delta A|_{\partial \mathcal{M}} = 0$, the action is stationary when $d \star F = 0$.

Similarly, the variation of S_2 is

$$\delta S_2 = \int_{\mathcal{M}} \delta A \wedge d \star F - \int_{\partial \mathcal{M}} A \wedge \delta(\star dA)$$

and in this case if the curl is fixed on the boundary, $\delta(\star dA)|_{\partial \mathcal{M}} = 0$, we again obtain $d \star F = 0$ as the equations of motion [12].

The third action is slightly different in character [7]. Its variation is

$$\delta S_3 = \int_{\mathcal{M}} \delta A \wedge d \star F + \int_{\partial \mathcal{M}} \delta A \wedge (\star F - c)$$

There are two possibilities. If $\delta A|_{\partial \mathcal{M}} = 0$ then, as in the first action, the boundary term vanishes and the action is stationary when $d \star F = 0$. Alternatively, if the curl is fixed on the boundary, and fixed to the specific value c , $\star dA|_{\partial \mathcal{M}} = \star F|_{\partial \mathcal{M}} = c$, this still leads to the same equation of motion.

The solution to the equation of motion on the manifold, $d \star F = 0$, is that the dual of F is a constant i.e. $F = \star \phi$. This is just an integration constant whose value is *a priori* undetermined by the classical dynamics (however, see the discussion below). It was this property of the four-form that initially aroused interest. The stationary values of the first two actions are

$$\langle S_1 \rangle = \frac{1}{2} \phi^2 \int_{\mathcal{M}} \sqrt{g} d^4 x \quad (5)$$

$$\langle S_2 \rangle = -\frac{1}{2} \phi^2 \int_{\mathcal{M}} \sqrt{g} d^4 x \quad (6)$$

²To be more precise we mean the projection of A onto $\partial \mathcal{M}$ is held fixed. In mathematical parlance this is $i^*(A)$, where $i: \partial \mathcal{M} \rightarrow \mathcal{M}$ is the inclusion map.

The action $\langle S_1 \rangle$ is positive, as it ought to be. The boundary contribution in (2) over-compensates the volume term giving a negative $\langle S_2 \rangle$. (The authors of [12] instead chose to start from (2) with the opposite sign, in order that this final value be positive!)

There are two possibilities for the third action. With A fixed on the boundary its value on the stationary solution is

$$\langle S_3 \rangle = \left(\frac{1}{2} \phi^2 - c\phi \right) \int_{\mathcal{M}} \sqrt{g} d^4 x \quad (7)$$

However, if the curl is fixed then we must have $\phi = c$ to satisfy the boundary condition³. This is consistent with the equation of motion and yields a value

$$\langle S'_3 \rangle = -\frac{1}{2} c^2 \int_{\mathcal{M}} \sqrt{g} d^4 x \quad (8)$$

So we see that $\langle S_3 \rangle$ can be positive if $\phi < 0$ or $\phi > 2c$ and negative if $0 < \phi < 2c$. Due to the boundary constraint $\langle S'_3 \rangle$ is forced to be negative, and as such is similar to $\langle S_2 \rangle$ in nature.

It seems that we have a wide variety of choices for the action, each with different properties. Which is the correct one? We shall now show, in the language of quantum field theory, that the constant of integration is a label for the vacuum state $|\phi \rangle$ of the four-form under certain conditions. Regarding the boundary as initial and final hypersurfaces in Euclidean quantum field theory the amplitude to go from a configuration A_i on the initial hypersurface to a final one A_f is

$$\langle A_f | A_i \rangle = \int_{\mathcal{I}} \mathcal{D}A \exp(-S_1) \quad (9)$$

since S_1 is the action appropriate to keeping A fixed at the end points of the path. This will be dominated by the classical solution $F = \star \phi$, where ϕ is as yet undetermined. Integrating this between the hypersurfaces we find

$$\int_{\mathcal{M}} F = \int_{\partial \mathcal{M}_f} A_f - \int_{\partial \mathcal{M}_i} A_i = \phi \int_{\mathcal{M}} \eta \quad (10)$$

³In Euclidean space the double dual of a four-form is $\star \star F = F$. If the space had had a Lorentzian signature this would have been $\star \star F = -F$.

where $\eta = \star 1$ is the volume element on the manifold. With knowledge of the intervening volume we have thus determined ϕ . The semiclassical approximation to (9) is therefore $\exp(-\langle S_1 \rangle)$, but where ϕ is given by (10).

Similarly, with the curl fixed on the boundary the appropriate action to use in the path integral is S_2 . The classical solution is such that $\phi = \star d\mathbf{A}_J = \star d\mathbf{A}_i$. Therefore the semiclassical approximation would be

$$\langle d\mathbf{A}_J | d\mathbf{A}_i \rangle \sim \delta(d\mathbf{A}_J - d\mathbf{A}_i) \exp(-\langle S_2 \rangle) \quad (11)$$

We can rewrite this as

$$\langle \phi | \phi \rangle \sim \exp(-\langle S_2 \rangle) \quad (12)$$

It is in this sense that ϕ can be said to be undetermined by the dynamics of the four-form; it is a label for whatever background state we choose⁴. We

⁴The analogy with the one-dimensional quantum mechanics of a free particle is an apt one. In Euclidean time the amplitude is

$$\langle q_f | q_i \rangle = \int \mathcal{D}q \exp(-S)$$

where the action is $S = \frac{1}{2} \int_{t_i}^{t_f} \dot{q}^2 dt$. The classical solution is $\dot{q} = p$, a constant. This can be determined from the initial and final coordinates and times,

$$p = \frac{q_f - q_i}{t_f - t_i}$$

Thus the stationary value of the action is

$$\langle S \rangle = \frac{1}{2} p^2 (t_f - t_i) = \frac{1}{2} \frac{(q_f - q_i)^2}{t_f - t_i}$$

and the semiclassical approximation is $\langle q_f | q_i \rangle \sim \exp(-\langle S \rangle)$.

However, transforming to the momentum representation we have

$$\langle p_f | p_i \rangle = \int \mathcal{D}q \exp(-S^p)$$

where $S^p = S - p_f q_f + p_i q_i$ is the action appropriate to keeping the momenta fixed at the end points of the path. As the momentum is conserved we find $\langle p_f | p_i \rangle \sim \delta(p_f - p_i) \exp(-\langle S^p \rangle)$ with the stationary value of the action $\langle S^p \rangle = -\frac{1}{2} p^2 (t_f - t_i)$. The momentum is $p_f = p_i = p$, and it is up to us to specify it.

have an infinity of values to select from. However, when we move to the BHIC theory we have no choice in the matter and ϕ will be selected for us in order to zero Λ_{cos} .

For action (3) we could regard c as a coupling constant in the Lagrangian. It couples the matter to the background geometry. In the same manner as above we find the semiclassical approximation to the fixed field and fixed curl amplitudes to be

$$\begin{aligned} \langle \mathbf{A}_J | \mathbf{A}_i \rangle &\sim \exp(-\langle S_3 \rangle) \\ \langle d\mathbf{A}_J | d\mathbf{A}_i \rangle &\sim \delta(d\mathbf{A}_J - d\mathbf{A}_i) \exp(-\langle S'_3 \rangle) \end{aligned} \quad (13)$$

respectively. In the former ϕ is again given by (10). In the latter $d\mathbf{A}_J = d\mathbf{A}_i = \star c$ and the amplitude can be re-expressed as

$$\langle c | c \rangle \sim \exp(-\langle S'_3 \rangle)$$

This is superficially the same as (12). The difference lies in the fact that the value of ϕ is no longer our choice. It is constrained to equal the coupling in the Lagrangian.

We can now see which is the correct action in each circumstance. First of all it should be clear that we should specify the curl on the boundary. The Baum Hawking mechanism requires the presence of an *a priori* undetermined parameter which labels the background state of the four-form. The mechanism acts on this parameter, selecting that particular vacuum for which the cosmological constant vanishes. In this case the correct choice of initial action is (2). Coleman's mechanism, on the other hand, accomplishes the vanishing act through the determination of coupling constants. Therefore with wormholes we should use (3). The choice of coupling made by the mechanism also determines the vacuum state of the four-form.

So far we have assumed that the background geometry is static while focusing our attention on the four-form. The bare cosmological term in the gravitational action is $\frac{\Lambda_0}{8\pi G_N} \int \sqrt{g} d^4x$ and we might naively conclude that if we are in the $|\phi\rangle$ vacuum then the form of (12) and (13) would imply $\Lambda_{\text{cos}} = \Lambda_0 - 4\pi G_N \phi^2$. However, as mentioned in the introduction, this

procedure is suspect. We should not find the matter vacuum first and then vary the metric to find the geometry. What we should do is determine both simultaneously. In general relativity spacetime and matter are inseparable.

Boundary terms do not contribute to the equations of motion. If we vary the metric in (4) we find that the stress tensor for the four-form is

$$T'_{\mu\nu} = \frac{1}{4!} \left(4F_{\mu\lambda\rho\sigma} F^{\lambda\rho\sigma}_{\nu} - \frac{1}{2} g_{\mu\nu} F^{\lambda\rho\sigma\tau} F^{\lambda\rho\sigma\tau} \right) \quad (14)$$

The complete set of classical equations for the background is therefore

$$\begin{aligned} \langle G_{\mu\nu} \rangle &= 8\pi G_N \langle T'_{\mu\nu} \rangle - \Lambda_0 \langle g_{\mu\nu} \rangle \\ &\langle \mathbf{d} \star \mathbf{F} \rangle = 0 \end{aligned} \quad (15)$$

The configuration for the four-form is $\langle \mathbf{F} \rangle = \langle \star \phi \rangle$ as before. In components $\langle F_{\mu\nu\lambda\rho} \rangle = \phi \langle \sqrt{g} \rangle \epsilon_{\mu\nu\lambda\rho}$ with $\epsilon_{0123} = 1$. When inserted into (14) the background stress tensor is seen to be

$$\langle T'_{\mu\nu} \rangle = \frac{1}{2} \phi^2 \langle g_{\mu\nu} \rangle \quad (16)$$

and thus, from (15), it appears that $\Lambda_{\text{cos}} = \Lambda_0 - 4\pi G_N \phi^2$, in agreement with the naive argument of the preceding paragraph. However, this agreement occurs only because we have used the actions (2) and (3). Had we started from action (1) the term in the action would have become $\Lambda_0 + 4\pi G_N \phi^2$ due to the form of (5) and, as was shown by Duff [9], would have completely thwarted the BHC mechanism.

Duff's argument is as follows: for action (1) the background stress tensor is given by (16). Thus $\langle G_{\mu\nu} \rangle = -\Lambda_{\text{cos}} \langle g_{\mu\nu} \rangle$ where $\Lambda_{\text{cos}} = \Lambda_0 - 4\pi G_N \phi^2$. Therefore the curvature scalar is $\langle R \rangle = 4\Lambda_{\text{cos}}$. We can compute the background value of the total Euclidean action as

$$\begin{aligned} \langle S_E \rangle &= -\frac{1}{16\pi G_N} \int_{\mathcal{M}} \langle R \rangle \langle \sqrt{g} \rangle d^4x \\ &\quad + \frac{\Lambda_0}{8\pi G_N} \int_{\mathcal{M}} \langle \sqrt{g} \rangle d^4x + \langle S_i \rangle \\ &= \left(-\frac{\Lambda_{\text{cos}}}{8\pi G_N} + \phi^2 \right) V \end{aligned} \quad (17)$$

where the volume of the space is $V = \int \sqrt{g} > d^4x$. The presence of the extra ϕ^2 contribution lies at the centre of his argument. If $\Lambda_{\text{cos}} < 0$ then the space is open and has infinite volume; in this case $\exp(-\langle S_E \rangle) \rightarrow 0$. On the other hand, if $\Lambda_{\text{cos}} > 0$ and the space is considered a four-sphere then the volume is $V = 24\pi^2/\Lambda_{\text{cos}}$. In the limit $\Lambda_{\text{cos}} \rightarrow 0_+$ we again find that $\exp(-\langle S_E \rangle) \rightarrow 0$ due to the presence of the extra contribution. There is then no peaking of the probability distribution around vanishing cosmological constant.

Clearly, if we had used $\langle S_2 \rangle$ in (17) instead of $\langle S_1 \rangle$ then the ϕ^2 term would have been absent *i.e.* the effective cosmological constant in the action equals that of the equations of motion. We would find then that $\exp(-\langle S_E \rangle) \rightarrow \infty$ as $\Lambda_{\text{cos}} \rightarrow 0_+$, the BHC result. We therefore come to the conclusion that for the BHC mechanism to apply to four-forms we need to specify $\mathbf{dA}$ on the boundaries. The cosmological constant is probably zero and the proof is probably right, but for the wrong reasons.

However, the analysis we have just performed is a little at odds with one of our assumptions about the closure of the space. A closed space does not have boundaries! Moreover, if the manifold has boundaries we should have included a term in (17) for the boundary contribution to the gravity [13]. We could, of course, slice the four-sphere to provide one, specify the curl on these boundaries, and then shrink the boundaries to zero. This would recover our results. Nevertheless, we should respect the closure of the space as much as possible. In the next section we consider what happens if we assume from the very beginning that the space is closed.

3 Four-forms on a closed space.

The action for a four-form on a closed, compact, Euclidean manifold that we shall take as our starting point is

$$S_F = \frac{1}{2} \int_{\mathcal{M}} \mathbf{F} \wedge \star \mathbf{F} \quad (18)$$

There are no boundary terms as the manifold is assumed closed. Thus the arguments of the preceding section concerning the choice of action and

whether the potential or its curl should be specified on the boundary are no longer directly relevant. However, this does not mean that the problem is trivial. Recall electrodynamics on a two-sphere. There we need at least two coordinate patches to cover the sphere. We can locally define gauge potentials on each patch and which differ by a gauge transformation. The common field strength is globally defined and with a suitable choice of potential can be made a multiple of the surface element. This is familiar to us as a Dirac monopole.

The key to studying the four-form on a closed space is the Hodge decomposition. It allows us to uniquely decompose any form into an orthogonal sum of exact, co-exact and harmonic forms. Thus we can write

$$\mathbf{F} = \mathbf{d}\mathbf{A} + \phi\boldsymbol{\eta} \quad (19)$$

where $\mathbf{A}$ is a globally defined three-form and $\boldsymbol{\eta}$ is the (harmonic) volume form on the manifold. The co-exact contribution is absent in this case as there are no co-exact four-forms on a four-dimensional manifold. The coefficient ϕ is a constant on the manifold. The action (18) is therefore equivalent to

$$S_F = \frac{1}{2} \int_M \mathbf{d}\mathbf{A} \wedge \star \mathbf{d}\mathbf{A} + \frac{1}{2} \phi^2 \int_M \boldsymbol{\eta} \quad (20)$$

We treat the components of $\mathbf{A} = \frac{1}{3!} A_{\mu\nu\lambda} \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \wedge \mathbf{d}x^\lambda$ as the only dynamical degrees of freedom of $\mathbf{F}$. The parameter ϕ is taken to arise from a choice of background value for the four-form (we need not treat it as a dynamical degree of freedom as it has no kinetic term). In this sense it is akin to the undetermined parameter of the previous section and we can label the background as $|\phi\rangle$.

The stress tensor, obtained as usual from the variation of the action with respect to the metric, is now

$$T_{\mu\nu} = \frac{1}{4!} \left(4 f_{\lambda\rho\sigma\tau} f_{\nu}^{\lambda\rho\sigma} - \frac{1}{2} g_{\mu\nu} f_{\lambda\rho\sigma\tau} f^{\lambda\rho\sigma\tau} \right) - \frac{1}{2} \phi^2 g_{\mu\nu} \quad (21)$$

where $\mathbf{f}$ is the curl of $\mathbf{A}$

$$f_{\mu\nu\lambda\rho} = \partial_\mu A_{\nu\lambda\rho} - \partial_\nu A_{\lambda\rho\mu} + \partial_\lambda A_{\rho\mu\nu} - \partial_\rho A_{\mu\nu\lambda}$$

Variation of the action (20) with respect to $A_{\mu\nu\lambda}$ gives the equation of motion

$$\langle \mathbf{d} \star \mathbf{d}\mathbf{A} \rangle = 0 \quad (22)$$

which would imply that $\langle \mathbf{d}\mathbf{A} \rangle = \phi' < \boldsymbol{\eta} >$ with ϕ' a constant as before, only that the decomposition (19) is unique and thus we must have $\phi' = 0$.

The background value of the stress tensor is therefore

$$\langle T_{\mu\nu} \rangle = -\frac{1}{2} \phi'^2 \langle g_{\mu\nu} \rangle \quad (23)$$

implying that $\Lambda_{\text{cos}} = \Lambda_0 + 4\pi G_N \phi'^2$ from the gravitational equations of motion. Similarly, the stationary value of the four-form action is

$$\langle S_F \rangle = \frac{1}{2} \phi'^2 \int \langle \sqrt{g} \rangle d^4x \quad (24)$$

Contrast these with (6) and (16). All that has happened is that ϕ^2 has flipped sign, but the effective cosmological constant in the action still equals that of the equations of motion. The total Euclidean action remains $\langle S_E \rangle = -3\pi/G_N \Lambda_{\text{cos}}$. Thus the BHC mechanism is applicable in this situation.

The flaw in Duff's argument can now be seen. In order that the BHC mechanism work we need the background value of the four-form action and $\langle T_{\mu\nu} \rangle$ to have opposite signs, as was the case with (6) and (16), and also with (22) and (23). The action (20) is positive definite. The stress tensor (21) has two contributions: one from the exact form and one from the harmonic form, with opposite signs. The opposite signs arise because the first term in (20) acts like a kinetic term and the second acts like a potential. In Euclidean stress tensors the potential contribution has a negative sign. If we had ignored the harmonic part and solved (22) as $\langle \star \mathbf{d}\mathbf{A} \rangle = \phi'$ the resulting stress tensor would have been positive. However, as we have shown, this is forbidden by the uniqueness of the Hodge decomposition.

Before we close this section we mention that we can add a term $-c \int \mathbf{F}$ to our action (18), just as we did in (3). Due to the Hodge decomposition this is equivalent to a term $-c \phi \int \boldsymbol{\eta}$ in (19). We now find that the background value for the resulting action is $\langle S'_F \rangle = \langle S_3 \rangle$ as in (7). In addition,

$\langle T_{\mu\nu} \rangle = -(\frac{1}{2}\phi^2 - c\phi) < g_{\mu\nu} \rangle$. All we have done is replace $\frac{1}{2}\phi^2$ by $\frac{1}{2}\phi^2 - c\phi$, both in the action and in the gravitational equations of motion. Thus $\Lambda_{\text{cos}} = \Lambda_0 + 4\pi G_N(\phi^2 - 2c\phi)$. The BHC mechanism operates as follows: in pure quantum cosmology c is a fixed external coupling and ϕ labels the vacuum. Again ϕ relaxes to that value for which $\Lambda_{\text{cos}} \rightarrow 0_+$. With wormholes we have two choices. Either we regard ϕ as still basically undetermined, in which case the coupling c is shifted to that value for which $\Lambda_{\text{cos}} \rightarrow 0_+$, or we can regard ϕ as a dynamical degree of freedom which relaxes to the value $\phi = c$ in order to minimize the matter action $\langle S_F \rangle$ (although arguments concerning the minimization of Euclidean actions are suspect when there is gravity present). Thus $|\phi| > |c|$ is the most probable distribution for the matter. Coleman's mechanism then operates to shift the coupling to that value for which $\Lambda_{\text{cos}} \rightarrow 0_+$. Either way the cosmological constant vanishes.

4 Continuing the Lorentzian Action.

Up until now all our arguments have been purely in the arena of Euclidean quantum field theory, on a manifold with or without boundaries. However, our world is Lorentzian and one can question whether or not the above arguments have any relevance for the universe as we know it. We could, at the least, consider the Euclidean theory to be the Wick-rotated version of an originally Lorentzian theory. However, Euclidean and Lorentzian spaces have radically different topologies and it is not at all clear how to analytically continue topologies, especially if we are summing over metrics in a path integral. Recall that we found quite different behaviour between the two topologies of open and closed Euclidean spaces in the previous sections. The method we shall adopt allows us to analytically continue to Euclidean space while at the same time respecting the properties of the original Lorentzian theory.

We assume that the spacetime has closed spatial hypersurfaces which are homogeneous but not necessarily isotropic. Thus we can write the line

element as

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = -N_L^2 dt \otimes dt + h_{ij} e^i \otimes e^j \quad (25)$$

As the space is homogeneous N_L and h_{ij} are functions of time only. The one-forms e^i are taken to be the basis for a three-sphere of unit radius. The volume element of the three-sphere is $\omega_3 = e^1 \wedge e^2 \wedge e^3$ whose integral over the three-sphere is $\int_{S_3} \omega_3 = 2\pi^2$. Furthermore, the volume element of the spacetime is $\eta = N_L \sqrt{h} dt \wedge \omega_3 = \sqrt{-g} d^4x$. Since the manifold has boundaries the volume element is exact rather than harmonic [14].

The action we take for our four-form is the Lorentzian equivalent of (1),

$$S_L = -\frac{1}{2} \int_{\mathcal{M}} \mathbf{F} \wedge \star \mathbf{F} \quad (26)$$

and we have to write it in terms of a potential. As the boundaries of the spacetime are slices at constant time we may write $\mathbf{F} = d\mathbf{A}_{\parallel}$, where the subscript indicates that the three-form is tangent to the boundaries⁵. Thus $\mathbf{A}_{\parallel}$ can be written in component form as

$$\mathbf{A}_{\parallel} = \frac{1}{3!} A_{ijk} e^i \wedge e^j \wedge e^k \quad (27)$$

As the basis forms are time dependent the curl is simply

$$d\mathbf{A}_{\parallel} = \frac{1}{3!} \dot{A}_{ij*} dt \wedge e^i \wedge e^j \wedge e^k \quad (28)$$

and so the components of $\mathbf{F}$ in this basis are $F_{023} = \dot{A}_{123}$. As the space is homogeneous A_{ijk} and its derivative are functions of time only.

⁵Strictly speaking, the Hodge decompositions are only known for spaces with positive definite signature. If Ω^k is the space of k -forms on a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ then

$$\Omega^k = d\Omega_{\parallel}^{k-1} \oplus \delta\Omega_{\perp}^{k+1} \oplus \mathcal{H}^k$$

where $\mathcal{H}^k$ is the space of harmonic k -forms, $\Omega_{\parallel}^{k-1}$ is the space of $(k-1)$ -forms tangent to the boundary, and $\Omega_{\perp}^{k+1}$ is the space of $(k+1)$ -forms normal to the boundary. Further details can be found in ref. [14].

The action (26) can be rewritten as

$$S_L = \frac{1}{2}(2\pi^2) \int_{t_i}^{t_f} \frac{\sqrt{h}}{N_L} \frac{1}{3!} \dot{A}_{ijk} \dot{A}^{ijk} dt \quad (29)$$

as we can extract and perform the volume integral over the three-sphere. The raised components are $\dot{A}^{ijk} = h^{ik} h^{jm} h^{kn} \dot{A}_{lmn}$. We treat A_{ijk} as the independent degrees of freedom of the four-form.

The momenta conjugate to A_{ijk} are

$$\pi^{ijk} = (2\pi^2) \frac{\sqrt{h}}{N_L} \dot{A}^{ijk} \quad (30)$$

and are conserved. As the space is homogeneous and the momenta are conserved we must have that

$$\pi^{ijk} = (2\pi^2) p \epsilon_{ijk}$$

where p is a constant of integration (the volume of the three-sphere is included for convenience). ϵ_{ijk} is the usual Levi Civita antisymmetric symbol. This classical solution is $\langle \mathbf{F} \rangle = p < \eta >$ when re-expressed in terms of the four-form. As in section 2 we can express the constant in terms of the A_{ijk} and the times on the initial and final hypersurfaces. However, if we transform to the momentum representation then the parameter is undetermined and we have to specify it. The action appropriate to keeping the momenta fixed on the boundaries is the Legendre transform of (29),

$$S_L^p = S_L - \frac{1}{3!} \pi^{ijk} A_{ijk} \Big|_{t_i}^{t_f} + \frac{1}{3!} \pi^{ijk} A_{ijk} \Big|_{t_f}$$

and, using (30), can be rewritten as

$$S_L^p = -\frac{1}{2} \int_{\mathcal{M}} \mathbf{F} \wedge \star \mathbf{F} + \int_{\mathcal{M}} \mathbf{d} \left(\mathbf{A}_{\parallel} \wedge \star \mathbf{F} \right) \quad (31)$$

which is seen to be the Lorentzian version of (2). The stationary Lorentzian states are labelled by the value of p . The path integral is

$$\begin{aligned} \langle p | p \rangle &= \int \mathcal{D}\mathbf{A}_{\parallel} \exp(iS_L^p) \\ &\sim \delta(p' - p) \exp(i < S_L^p >) \end{aligned}$$

in the semiclassical approximation, where the stationary value of the action is $\langle S_L^p \rangle = -\frac{1}{2} p^2 \int_{\mathcal{M}} < \sqrt{-g} > d^4x$.

It should now be clear how to Wick rotate to Euclidean space in order to impose the BHC mechanism. Our Lorentzian theory must be formulated in the momentum representation; the boundary value of the four-form fixed to $\mathbf{F} = p\boldsymbol{\eta}$. We then Wick rotate the action S_L^p to S_2 , which is the Euclidean action appropriate to keeping $\mathbf{F}$ fixed on the boundary. As was shown in section 2 this action is the correct one for the application of the BHC mechanism. The analytic continuation of the boundary condition is

$$\mathbf{F} = p\boldsymbol{\eta} \rightarrow \mathbf{F} = -ip\boldsymbol{\eta}$$

since the $d\mathbf{t}$ in $\boldsymbol{\eta}$ picks up a factor $-i$. Thus we can identify $\phi = -ip$. Therefore, as in (6) and (16), the Euclidean action and stress tensor have the background values

$$\begin{aligned} \langle S_2 \rangle &= -\frac{1}{2} p^2 \int_{\mathcal{M}} < \sqrt{g} > d^4x \\ \langle T_{\mu\nu} \rangle &= -\frac{1}{2} p^2 < g_{\mu\nu} > \end{aligned}$$

and thus the effective cosmological constant is $\Lambda_{\text{eff}} = \Lambda_0 + 4\pi G_N p^2$ in both the action and gravitational equations of motion.

The Euclidean equations are an intermediary device, helping us analyse the behaviour of the Lorentzian theory [10]. It is not that the universe is Euclidean -- it is Lorentzian -- rather we can use the Wick rotated expressions as a calculational aid. The stationary states of the system are still labelled by $|p\rangle$, corresponding to a configuration of the four-form in Lorentzian space. When we move over to our pseudo-Euclidean space it appears that the Euclidean configuration is complex. This is an artifact of our procedure. An approach similar to this has been successfully used before [10, 11, 15] when resolving problems with the Giddings-Strominger wormhole [16].

The analysis above is in a sense complementary to that of section 3. There we demanded that we should firmly attach ourselves to a closed Euclidean manifold in order to circumvent the inadequacies of the analysis on

open Euclidean manifolds. Here we have attached ourselves to a Lorentzian space and have then recast the theory in a Euclidian framework. Nevertheless, the properties of the original Lorentzian theory are retained. If $\Lambda_{\text{cos}} > 0$ the pseudo-Euclidean space has positive curvature $< R > = 4\Lambda_{\text{cos}}$ and can be closed to a four-sphere. If we can close the space, by shrinking the boundaries to a point, do we negate the fact that we have specified $\mathbf{F}$ on a boundary? The answer needs only a little thought. We are actually specifying $\star\mathbf{F}$, not $\mathbf{F}$, on the boundary. This is a c-number and can be defined at a point. We are giving its values at the north and south poles of the four-sphere. Both values are identical by the equations of motion. Another way to look at this is the following: the volume of any spatial slice of the four-sphere is $2\pi^2\sqrt{h}$. The determinant $h \rightarrow 0$ as we approach the poles. However, by (30) we see that $\sqrt{h}\dot{A}\dot{U}^\star$ must remain constant in this limit by our constraint. It is easy to verify from this, and from the solution $h(t)$ to the gravitational equations of motion, that $A_{ij\star} \rightarrow 0$ faster than the volume. Thus A_{ij} and $\mathbf{F}$ vanish at the poles, as they should do.

Another reason why the $|p| >$ are the correct states to consider is that they correspond to states of fixed charge. This can be clearly seen from (29). The action is invariant under $A_{ij\star} \rightarrow A_{ij\star} + \alpha\epsilon_{ijk}$, where α is a constant. In form notation this is $A_{ij} \rightarrow A_{ij} + \alpha\omega_3$, which remains transverse to the boundary. The change in the action under this shift is

$$\begin{aligned}\delta S_L &= \alpha \int_{\mathcal{M}} d(\omega_3 \wedge \star dA_{ij}) \\ &= -\alpha \int_{\mathcal{M}} d \star j\end{aligned}\quad (32)$$

where j is the current. As the action is invariant the current is conserved, $\nabla_\mu j^\mu = 0$. The classical solution is $dA_{ij} = p\eta$ and so we find that the current one-form is $j = -p\frac{\mathcal{N}_L}{\sqrt{h}}dt$. The conserved charge is therefore

$$Q = \int d^3x (\sqrt{-g}j^0) = 2\pi^2 p \quad (33)$$

Our background value of the four-form corresponds to a fixed charge for the vacuum. Prior to rotating to Euclidean space we have to constrain

our action such that the charge is conserved throughout the motion⁶. The action which achieves this is none other than (31).

5 Summary

In order for the Baum-Hawking Coleman mechanism to operate we need the effective action in Euclidean space to behave as $< S_E > = -3\pi/G_N\Lambda_{\text{cos}}$. The value of Λ_{cos} is read off from the gravitational equations of motion, $< G_{\mu\nu} > = -\Lambda_{\text{cos}} < g_{\mu\nu} >$, and determines the large scale structure of the space. Λ_{cos} is a function of the parameters of the theory and the mechanism selects those values for which $\Lambda_{\text{cos}} \rightarrow 0_+$.

In the case of four-forms the parameter is the background value of the four-form configuration. In this paper we have established which actions to use initially in order to achieve the correct form of the final effective action. What is clear is that for open Euclidean manifolds we must start from the action appropriate to fixing $\star\mathbf{F}$ on the boundary, rather than that appropriate to fixing the potential $\mathbf{A}$. It is only in this situation that the background is undetermined. When on an *a priori* closed Euclidean manifold topological restrictions automatically yield the correct effective action.

If we begin our investigations in Lorentzian space we find that there is a conserved charge in the theory. Therefore we should constrain the system in order that the dynamics respects this. We have shown that this is equivalent to fixing the four-form on the boundary. After we Wick rotate to Euclidean space, in order to study the ramifications for the mechanism, we find that the Euclidean action is in the correct form.

The dynamics of a four-form field strength are more subtle than previously appreciated. Nevertheless, in a theory incorporating such a field the cosmological constant vanishes.

⁶This is similar to the case of the wormhole [16]. There is a conserved axion charge [17] which is carried by the wormhole. When calculating tunnelling due to the wormholes we have to maintain charge conservation [15].

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