FIELD IDENTIFICATION FIXED POINTS IN THE COSET CONSTRUCTION

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ABSTRACT

We discuss two related problems in conformal field theory. The first is the construction of the modular transformation matrix S for integer spin modular invariants in which some characters appear with multiplicity larger than 1. The second problem is the relation between the characters and the branching functions in coset theories in which the field identification identifies some fields with themselves ("fixed points"). We find that these problems are closely related, and that the solution is remarkably interesting. The fixed points of any conformal field theory seem always to define a new (not necessarily unitary) conformal field theory whose primary fields are in one-to-one correspondence with the fixed points. The characters of this conformal field theory are needed to modify the coset branching functions.

^{*} Work supported in part by the US-Israel Binational Science Foundation, and the Israel Academy of Science.

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1. Introduction

The coset construction [1] provides one of the most powerful ways of obtaining rational conformal field theories. One associates such a theory with every pair of Kac-Moody algebras $\mathcal{G}$ and $\mathcal{H}$, such that $\mathcal{H}$ is a sub-algebra of $\mathcal{G}$. For every such pair one obtains an energy momentum tensor satisfying the Virasoro algebra by simply taking the difference of the Sugawara energy momentum tensors of $\mathcal{G}$ and $\mathcal{H}$. The central charge of the resulting theory is larger than or equal to zero, where the lower bound is saturated by the conformal sub-algebras. Apart from the latter class (which consists of trivial coset theories), all other coset theories are expected to correspond to well-defined, unitary conformal field theories with a finite number of primary fields.

In principle one might hope to derive all properties of coset theories (such as characters and correlation functions) from those of $\mathcal{G}$ and $\mathcal{H}$, but in practice this is not so simple. The only properties that are easy to obtain are the central charge and the fractional part of the conformal dimensions of the primary fields. As we will see, even the matrix S that represents the modular transformation $\tau \rightarrow -\frac{1}{\tau}$ on the characters can in general not be derived in a straightforward way from the S -matrices of $\mathcal{G}$ and $\mathcal{H}$. This problem, and the related problem of finding the characters and modular invariant partition functions, is the subject of this paper.

To appreciate the problem, consider the usual method of defining the characters of the coset theory in terms of the branching functions of the Kac-Moody embedding $\mathcal{G} \supset \mathcal{H}$. We write the characters of $\mathcal{G}$ as χ_A and those of $\mathcal{H}$ as χ_λ , where A and λ denote primary fields of these algebras. The branching functions b satisfy

$$\chi_A(\tau) = \sum_{\lambda} b_{\lambda}^A(\tau) \chi_{\lambda}(\tau). \quad (1.1)$$

These functions can in principle be computed by decomposing the Kac-Moody representation A level-by-level into representations of the Lie-algebra of $\mathcal{H}$, and recombining these representations into primary fields of $\mathcal{H}$ and their descendants. From (1.1) we can read off the modular transformation properties of the branching functions in terms of those of the characters of $\mathcal{G}$ and $\mathcal{H}$.

$$\begin{aligned} b_{\lambda}^A\left(\tau + 1\right) &= e^{2\pi i(h_A - h_{\lambda})} b_{\lambda}^A(\tau) \\ b_{\lambda}^A\left(-\frac{1}{\tau}\right) &= \sum_{\lambda' \in \lambda'} S_{\lambda \lambda'}^{\mathcal{G}} (S_{\lambda \lambda'}^{\mathcal{H}})^* b_{\lambda'}^A(\tau), \end{aligned} \quad (1.2)$$

in a self-explanatory notation. Obviously the following combination of branching functions is then modular invariant

$$\sum_{\lambda, \lambda'} |b_{\lambda}^A(\tau)|^2. \quad (1.3)$$

The branching functions are thus obvious candidates for the characters of the coset theory. Unfortunately there are some serious problems with this interpretation. The first problem that typically arises is that some branching functions vanish because some $\mathcal{H}$ -representation λ never appears in the decomposition of the $\mathcal{G}$ representation A . In that case one has to conclude that the coset theory does not have primary fields corresponding to these vanishing branching functions. Although (1.3) is still a modular invariant partition function, the modular transformation matrix defined in (1.2) is now unacceptable, since it has non-vanishing matrix elements between good primary fields and non-existing fields. Of course one cannot simply restrict the matrix to the proper primary fields without destroying its unitarity.

It was observed in [2] that the vanishing of branching functions is always accompanied by a “field identification”, which means that the non-vanishing branching functions are group-wise equal to each other. By identifying N characters that are equal with just one primary field rather than N different ones one can hope to obtain a unitary modular matrix, and in some simple cases this does indeed solve the problem.

However one still has to worry about the modular invariant partition function (1.3). The fact that the branching functions are group-wise equal implies in particular that the identity appears more than once. The solution proposed in [2] is to divide the entire partition function by the multiplicity of the identity. This is however only allowed if *all* characters appear with the same multiplicity. If this is not true, dividing by the multiplicity of the identity yields an obviously unacceptable partition function with fractional coefficients. For reasons that become clear in the next section, we refer to characters that appear with a smaller multiplicity than the identity as *fixed points*.

One might hope that (a) this problem never occurs or (b) if it does occur the corresponding characters already have a “built-in” multiplicity factor, making the division possible, or (c) there exists always a different, non-diagonal combination of branching functions which does not have such a problem. It is however not difficult to construct examples where neither (a), (b) or (c) is realized. Of course one could try to argue that perhaps those examples

do simply not correspond to sensible conformal field theories, but this already unattractive option becomes even less appealing once one realizes that this would eliminate, for example, some coset theories that can be identified with the $N = 1$ minimal models.

The main purpose of this paper is to provide a way out of this dilemma. The solution is surprisingly non-trivial, and interesting in its own right. We find that one can associate a new “conformal field theory” with the fixed points. This conformal field theory may be non-unitary but it must have characters with integer coefficients in their q -expansions (note that we are not insisting on *positive* integer coefficients). One can add these characters to the characters of the coset theory in such a way that the new characters allow a division by the multiplicity of the identity. Although we do not have a proof that this procedure will always work, we do have a set of non-trivial examples to support this conjecture.

This paper is organized as follows. In the next section we will explain the field identification problem. The discussion can be simplified considerably by using the concept of simple currents introduced in [3]. It will be immediately obvious why there is an identification of characters, why the field identification yields a new, unitary matrix S and, most importantly, why this matrix is inadequate if fixed points are present.

The field identification procedure turns out to be completely analogous to writing down an integer spin modular invariant (for example the D -invariants of $SU(2)$ at level $4l$) by means of the method of [3]. Such modular invariants have a feature that at first sight appears to be exactly the opposite of the field identification problem: some characters may appear with a multiplicity that is *larger* than that of the identity. Although this is not a problem for the partition function, it is now well understood that such higher multiplicities usually imply that there are several distinct primary fields with respect to the extended algebra whose characters transform in the same way. A proper modular matrix with correct fusion rules can be obtained if one enlarges the set of primary fields, so that all distinct fields are represented once. This procedure will be explained in section 3.

The conditions we find for the modular transformations among the fixed points are very reminiscent of some of the conditions for modular transformations of characters of rational conformal field theories. We make this more precise in section 4, where we find, empirically but for a large set of examples, that the h -values of fixed points are equal to those of

* Sometimes higher multiplicities have a different interpretation. We will give some examples in Appendix B.

some rational conformal field theory up to a multiple of $\frac{1}{12}$. This fixed point CFT may be non-unitary even if it corresponds to fixed points of a unitary CFT.

In section 5 we explain how this provides an answer to the field identification problem. It turns out that the characters of the fixed point CFT can be added, with certain coefficients, to the branching functions, in such a way that the sum can be divided by the multiplicity of the identity. The modified branching functions are the correct characters of the coset theory. We demonstrate how this works for some examples in section 6.

Many of our results are based on examples rather than general proofs, and a lot of work remains to be done. Many new and interesting discoveries are appearing on the horizon. We summarize some of the remaining problems and future prospects in section 6.

In appendix A we give a complete derivation of all the fixed point CFT's for all $SU(N)$ Kac-Moody algebras. In Appendix B we discuss some examples of fixed point resolution for exceptional invariants. An interesting curiosity is a new automorphism invariant for $SU(5)$ level 5, which is part of a series consisting of the E_7 invariant of $SU(2)$ level 16 and the invariant found in [4] for $SU(3)$ level 9.

2. Field Identification

A branching function b_λ^Λ of a coset theory vanishes if the primary field λ never appears in the decomposition of the primary field Λ and its descendants. In principle this allows two possibilities: (a) the Lie-algebra representation λ never appears at all or (b) the Lie-algebra representation λ does appear, but only as an $\mathcal{H}$ -descendant of some other primary field, or as a null state. Therefore we should investigate which selection rules one might have in Lie-algebra and Kac-Moody embeddings.

Lie-algebra representations can be classified into conjugacy classes, which are in one-to-one correspondence with the center of the algebra. Denote the Lie-algebras belonging to the Kac-Moody algebras $\mathcal{G}$ and $\mathcal{H}$ as G and H respectively. In general, representations belonging to some conjugacy class of G contain in their decomposition with respect to H only representations in some subset of the conjugacy classes of H . In this way a Lie-algebra embedding $H \subset G$ defines a homomorphism Φ between the centers of G and H .

Kac-Moody representations are built by acting with the modes J_n^a on the highest weight states, which belong to some representation Λ of the Lie-algebra. Hence any KM-

representation will inevitably contain, at some excitation level, any Lie-algebra representation that can be obtained from Λ by tensor-products with the adjoint representation. In other words, the KM-representation must contain all Lie-algebra representations that belong to the same conjugacy class as Λ .

For coset theories this implies that b_λ^Λ vanishes if the conjugacy class of λ does not appear in the image of the conjugacy class of Λ under Φ . We expect this to be the *only* reason why a branching function can vanish, but there are two possible caveats. First of all we have ignored null states in the foregoing argument. Even though a representation must appear for group-theoretical reasons, it might be that it is always a null-state. Unlikely as this may seem, it is in fact quite common in conformal embeddings, i.e. trivial coset theories. The second potential problem is that all occurrences of a given representation λ might be used up as descendants of other primary fields, so that there is never a representation λ left to serve as a primary field. We cannot rule out either possibility, but neither seems very likely in non-trivial coset theories. We assume from now on that conjugacy class selection rules are the only reason why a branching function can vanish.

The simple currents (i.e. primary fields whose fusion with any other primary field yield just one primary field) of the KM-algebras $\mathcal{G}$ and $\mathcal{H}$ can play a useful rôle here because their charges precisely label the conjugacy classes. The highest weights of the relevant* simple currents are given by the highest weight vectors

$$\begin{array}{ll}
 A_n & J_l = (0, \dots, 0, \overbrace{k, 0, \dots, 0}^l, 0), l = 1, \dots, N \\
 B_n & J = (k, 0, \dots, 0) \\
 C_n & J = (0, \dots, 0, k) \\
 D_n & J_v = (k, 0, \dots, 0) \\
 & J_s = (0, 0, \dots, k, 0) \\
 E_8 & J_c = (0, 0, \dots, 0, k) \\
 & J_1 = (k, 0, \dots, 0) \\
 & J_2 = (0, 0, \dots, 0, k) \\
 E_7 & J = (0, 0, \dots, k, 0)
 \end{array}$$

* In most cases these are in fact *all* simple currents of the KM-algebras. The only known exception is E_8 , level 2 [5, 6].

Here k is the level of the KM-algebra, and the simple roots are ordered in the standard way (see e.g. [7]). Note that in general the currents themselves are not in different conjugacy classes. However it is easy to check that the monodromy of each primary field with respect to the simple currents (or, more precisely, with respect to the generators of each orbit) corresponds precisely to the conjugacy class of its representation. It is convenient to define the monodromy phase of a field ϕ with respect to a simple current J as the charge Q of that field, i.e. $J(z)\phi(w) \propto (z-w)^{-Q}\phi'(w)$, as was done in [3]. Obviously this charge is defined modulo integers.

For each coset theory one can now find a simple current so that vanishing branching functions have non-zero charge with respect to this current. The idea is now to use this current to project out the vanishing branching functions from the partition function. We will call this simple current the *identification current* of the coset theory (in general there may be a set of several such currents, but the idea is the same).

To see how one finds the identification currents of a coset theory consider two algebras $\mathcal{G}$ and $\mathcal{H}$. The embedding defines a homomorphism Φ from the conjugacy classes of $\mathcal{G}$ into those of $\mathcal{H}$. Alternatively, one may consider the tensor product of the centers $C_{\mathcal{G}}$ and $C_{\mathcal{H}}$ of $\mathcal{G}$ and $\mathcal{H}$. Clearly the elements $(c, \Phi(c)) \in C_{\mathcal{G}} \times C_{\mathcal{H}}$ form a subgroup C . (a similar description of the conjugacy class restrictions may be found in [8]).

The product $C_{\mathcal{G}} \times C_{\mathcal{H}}$ is isomorphic to $\mathbb{Z}_{N_1} \times \dots \times \mathbb{Z}_{N_K}$, for some integer K . Consider the lattice Γ consisting of the points $(\frac{m_1}{N_1}, \dots, \frac{m_K}{N_K})$, $m_i \in \mathbb{Z}$. Each primary field is characterized by a set of charges defined modulo 1. The possible charges are thus in one-to-one correspondence with the coset $\Gamma/\mathbb{Z}^K$. Then the subgroup C defines a sublattice $\Gamma_C \subset \Gamma$, which is built out of some subset of $\Gamma/\mathbb{Z}^K$ -cosets.

By the arguments given above, there exists a set of currents $J_{(1)} \dots J_{(K)}$ whose monodromies with each combination of primary fields of $\mathcal{G}$ and $\mathcal{H}$ yield precisely the aforementioned $\mathbb{Z}_N$ charges. Consider now the dual lattice Γ_C^* of Γ_C . Since $\Gamma_C \supset \mathbb{Z}^K$ it follows that all vectors in Γ_C^* have integral components. Hence we can consider current combinations $J_{\vec{n}} = J_{(1)}^{n_1} \dots J_{(K)}^{n_K}$, where $\vec{n} = (n_1, \dots, n_K) \in \Gamma_C^*$. Let (Λ, λ) denote a combination of $\mathcal{G}$ and $\mathcal{H}$ representations whose charges belong to C (i.e. we expect b_λ^Λ to be a non-vanishing branching function). The fact that Γ_C^* is the dual of Γ_C means precisely that all such representations have integral charge with respect to the current $J_{\vec{n}}$. Conversely, if (Λ, λ) is a combination of representations whose charges do not belong to C (i.e. the corresponding

branching function vanishes), then duality of Γ_C with respect to Γ_C^* implies that there must be a current $J_{\vec{n}}, \vec{n} \in \Gamma_C^*$ with respect to which this combination has non-integral charge. Hence the set of identification currents is $J_{\vec{n}}, \vec{n} \in \Gamma_C^*$ (this is of course a finite set since $J_{(i)}^{N_i} = 1$).

The presence of a simple current among the primary fields of a conformal field theory implies relations among the matrix elements of the modular transformation matrix S [9]

$$S_{J^a a, J^b b} = e^{2\pi i \beta Q(a)} e^{2\pi i \alpha Q(b)} e^{2\pi i \frac{q\beta^2}{N}} S_{ab} . \quad (2.1)$$

Here J^a denotes the primary field obtained by acting a times on the field ϕ_a with the current J , $Q(a)$ is the charge of ϕ_a , r is the monodromy parameter of the current and N its order ($J^N = 1, p = 1, \dots, N-1$). To apply this formula to the field identification problem, regard a as a pair (Λ, λ) of primary fields of $\mathcal{G}$ and $\mathcal{H}$, and denote b_λ^Λ as b_a . Then we find

$$\begin{aligned} \frac{b_a(-1/\tau)}{b_{J^a}(-1/\tau)} &= \frac{\sum_c S_{a,c} b_c(\tau)}{\sum_c S_{J^a,c} b_c(\tau)} \\ &= \frac{\sum_c S_{a,c} e^{2\pi i Q(c)} b_c(\tau)}{\sum_c S_{a,c} e^{2\pi i Q(c)} b_c(\tau)} \end{aligned}$$

where J is the identification current. Because $b_c = 0$ if (and only if) $Q(c) \neq 0$ it follows that the modular transform of b_a/b_{J^a} is equal to 1, so that this ratio has to be equal to 1 itself. Hence $b_a = b_{J^a}$, so that all branching functions that are related to each other by the action of the identification current are identical. The identification current itself is linked to the identity, so that it must have $h = 0$. Thus in particular it follows that the identification current has integer conformal spin, which implies that the monodromy parameter $r = 0$. Consequently (2.1) becomes simply

$$S_{J^a a, J^b b} = S_{ab}, \quad \text{if } Q(a) = Q(b) = 0 , \quad (2.2)$$

i.e. when S is restricted to the non-vanishing branching functions. This formula has been obtained before in [2] by means of explicit calculations starting with the Kac-formula [11] for S in Kac-Moody algebras. Note that (2.2) implies that some rows and columns of S - restricted to the non-vanishing branching functions - are identical, so that this restriction is not unitary.

The solution to this problem is to define a new matrix that acts on just one field in each set of identified ones. To achieve this we may use the method of [3] to write down a new modular invariant partition function that only involves the fields with zero charge. This partition function has the form

$$\sum_{Q(a)=0 \text{ orbits}} \frac{N}{N_a} |b_a + b_{J^a} + \dots + b_{J^{N_a-1}a}|^2 , \quad (2.3)$$

where N_a is the length of the orbit of the field ϕ_a and N is the length of the identity orbit; N_a is always a divisor of N . Now the vanishing branching functions are completely absent, but the identity character of the new theory (whose chiral algebra contains the current J) has a leading coefficient equal to N instead of 1. Hence the partition function has to be divided by N^2 in order to get the partition function of a theory with just one vacuum. This is possible if $N_a = N$ for all orbits, but if shorter orbits are present the division leads, in general, to fractional coefficients. The presence of an orbit of length $N_a \neq N$ implies that $J^{N_a} \phi_a = \phi_a$, so that ϕ_a is a fixed point under the action of the current $J^{N_a} \neq 1$.

The matrix $\hat{S}$ that acts on the identified characters can be computed by going to a basis

$$\chi_a = \sqrt{\frac{N}{N_a}} (b_a + b_{J^a} + \dots + b_{J^{N_a-1}a}) . \quad (2.4)$$

All these arguments are exactly the same as for integral spin modular invariants, and with that analogy in mind we have included an overall factor $\sqrt{N}$ in the normalization of the right-hand side. Using (2.2) one finds that on the characters (2.4) the modular transformation $\tau \rightarrow -1/\tau$ is represented by the matrix

$$\hat{S}_{ab} = \sqrt{N_a N_b} S_{ab} . \quad (2.5)$$

Because of the relative normalizations of the character combinations (2.4) the basis transformation on the space of characters is orthogonal (up to an overall normalization). It is straightforward to show that therefore the matrix $\hat{S}$ is unitary.

3. Fixed Point Resolution

The discussion in this section applies in general to integer spin modular invariants generated by means of simple currents (and even in some cases to exceptional invariants, although we will not explore this systematically). The field identification in coset theories is just a special case. The currents generating such invariants very often have fixed points, and whenever this happens special care is needed in constructing the matrix S of the theory.

There are several ways of seeing that the matrix $\hat{S}$ constructed above is inadequate if there are fixed points, even though it is unitary. One problem is that it acts on characters (2.4) whose leading coefficient is not an integer. With the overall normalization chosen in (2.4) the identity character has integral coefficients. If there are no fixed points, the other characters have integral coefficients as well. However if fixed points are present it is usually not possible to obtain integer ground state multiplicities for all characters by multiplying with a common factor.

A related problem is that of finding the fusion rules of the new theory. If we apply Verlinde's formula [12] to $\hat{S}$ we get the following expression for the fusion rule coefficients $\hat{N}_{abc}$ of the new theory

$$\hat{N}_{abc} = \sum_{\tilde{n}} \frac{\hat{S}_{a\tilde{n}} \hat{S}_{b\tilde{n}} \hat{S}_{c\tilde{n}}}{\hat{S}_{0\tilde{n}}}.$$

Here $\tilde{n}$ indicates one representative of each orbit with $Q(n) = 0$. Substituting (2.5) we get

$$\hat{N}_{abc} = \sqrt{\frac{N_b N_c}{N_a N_0}} \sum_{\alpha} \sum_{n, Q(n)=0} \frac{S_{J_{\alpha a, n}} S_{b n} S_{c n}^*}{S_{0 n}},$$

where we have replaced the factors $\sqrt{N_n}$ by a sum over the orbit, so that $\tilde{n}$ is replaced by n . Furthermore we have replaced $S_{a\tilde{n}}$ by $\sum_{\alpha} S_{J_{\alpha a, n}}/N_a$, a replacement which is allowed since $Q(\tilde{n}) = Q(n) = 0$. Having done this, we may extend the sum on n to include also the orbits with non-zero charge, since the sum on α makes their contributions cancel. Since we now have a sum over all primary fields of the original theory, we can express $\hat{N}_{abc}$ in terms of the fusion coefficients N_{abc} of the original theory

$$\hat{N}_{abc} = \sqrt{\frac{N_b N_c}{N_a N_0}} \sum_{\alpha} N_{J'_{\alpha a}, b, c}$$

If there are no fixed points $N_a = N_b = N_c = N_0 = N$ the square root is equal to 1, and the

result is clearly an integer. However if one of the orbit lengths is smaller than N and the ratio is not a square then one will get non-integer fusion coefficients.

Several authors (see e.g. [13] [14]) have observed that the appearance of coefficients $\frac{N}{N_a} \neq 1$ as in (2.3) indicates that this contribution to the partition function is due to $\frac{N}{N_a}$ primary fields rather than just one. A more precise statement, proved in [4] is that any modular invariant partition can always be regarded as a diagonal invariant with respect to some extended algebra (up to a fusion rule automorphism). If $\frac{N}{N_a}$ is not a square, this can only be true if there is more than one primary field per fixed point. After splitting the corresponding rows and columns of $\hat{S}$ into $\frac{N}{N_a}$ different ones, one should obtain a new matrix $\tilde{S}$ which satisfies all relevant consistency conditions, including that of having correct fusion rules. We will refer to this procedure as "resolving the fixed points". It is by no means obvious that fixed points can always be resolved in a satisfactory way, and indeed there are examples where this is not possible (see Appendix B). It is also not obvious that there should always be a unique answer.

The resolution of fixed points is a sensible procedure for integer spin modular invariants, but it seems disastrous for the mathematically analogous situation appearing in the field identification problem. After dividing out the multiplicity of the identity, the coefficient in front of a fixed point character, which was previously $\frac{1}{N_a N}$, now becomes $\frac{1}{N_a \tilde{N}}$, which is even smaller. Nevertheless we have no choice, because the aforementioned problems with $\hat{S}$ are identical for integral spin modular invariants and coset theories.

Thus we are led to define a new matrix, $\tilde{S}$, which acts on characters

$$\chi_{a_i} = \sqrt{\frac{N_a}{N}} \chi_a, \quad i = 1, \dots, N/N_a, \quad (3.1)$$

with χ_a defined as in (2.4). Note that all these new characters have integral coefficients. The matrix elements of $\tilde{S}$ are related to those of $\hat{S}$ by the known transformation of the characters χ_a . However there is some freedom in defining $\tilde{S}$ since $\hat{S}$ does not tell us how $\hat{S}$ acts on the labels i . For the transformation of the new characters we find

$$\chi_{a_i}(-1/\tau) = \sqrt{\frac{N_a}{N}} \sum_b \tilde{S}_{ab} \chi_b(\tau). \quad (3.2)$$

* If $\frac{N}{N_a}$ is a square (or contains a factor that is a square) there might be another way out, namely to absorb a factor $\frac{N}{N_a}$ into the character. This does indeed happen in some examples (see Appendix B).

On the other hand the right hand side should be equal to

$$\sum_{b,j} \hat{S}_{a,b} \chi_b(\tau) = \sum_{b,j} \hat{S}_{a,b} \sqrt{\frac{N_b}{N}} \chi_b(\tau), \quad (3.3)$$

so that

$$\sum_j \hat{S}_{a,b_j} = \sqrt{\frac{N_a}{N_b}} \hat{S}_{ab}. \quad (3.4)$$

The most general parametrization of $\hat{S}$ that is consistent with (3.4) is

$$\begin{aligned} \hat{S}_{a,b_j} &= \sqrt{\frac{N_a N_b}{N^2}} \hat{S}_{ab} E_{ij} + \Gamma_{ij}^{ab} \\ &= \frac{N_a N_b}{N} S_{ab} E_{ij} + \Gamma_{ij}^{ab}, \end{aligned} \quad (3.5)$$

where $E_{ij} = 1$ (independent of i and j) and Γ must satisfy

$$\sum_i \Gamma_{ij}^{ab} = \sum_j \Gamma_{ij}^{ab} = 0. \quad (3.6)$$

and

$$\Gamma_{ij}^{ab} = \Gamma_{ji}^{ba}.$$

The matrices Γ are restricted by the consistency conditions for $\hat{S}$, namely $\hat{S}\hat{S}^\dagger = 1$, $\hat{S}^2 = C$ (with $C^2 = 1$), $(\hat{S}T)^3 = C$ plus the condition that the fusion coefficients should be integers. The first two conditions lead to

$$\begin{aligned} \sum_{b,j} \Gamma_{ij}^{ab} \Gamma_{jk}^{bc*} &= \delta_{ac} \delta_{il} P_{lk}^a \\ \sum_{b,j} \Gamma_{ij}^{ab} \Gamma_{jk}^{bc} &= C_{ac} C_{il}^a P_{lk}^a, \end{aligned} \quad (3.7)$$

where P_{ij}^a is a projection operator

$$P_{ij}^a = \delta_{ij} - \frac{N_a}{N} E_{ij}.$$

The appearance of this operator is related to (3.6), since it projects on the non-zero modes of Γ . The matrices C appearing in (3.7) are the charge conjugation matrix of the original theory and a new matrix C_{ij}^a , which defines the charge conjugation matrix of the new theory.

The derivation of (3.7) is analogous to the calculation presented above for the fusion rules. The aim is always to relate all products of S to the known expression for the original theory. In the new theory, defined by $\hat{S}$, all summations are over indices $\hat{n}_i, i = 1, \dots, N_a/N$, where $\hat{n}$ labels all orbits with $Q(n)=0$. This sum can be replaced by a sum over all $J^\alpha_n, \alpha = 1, \dots, N_n, Q(n) = 0$, since the normalization factors in (3.5) precisely provide the required multiplicity factors, and since S is constant on the zero charge orbits. The final step is to average over the orbit of one of the external indices. This makes it possible to extend the sum on n also over the orbits with $Q(n) \neq 0$, and hence over all primary fields in the original theory.

In exactly the same way one can derive a relation from $(ST)^3 = C$. It is convenient to first use $\hat{S}^2 = C$ to derive $STS = T^{-1}ST^{-1}$. For Γ we find then essentially the same relation

$$\Gamma_{ij}^{ab} T_{bb} \Gamma_{jk}^{bc} = T_{aa}^{-1} \Gamma_{ik}^{ac} T_{cc}^{-1}. \quad (3.8)$$

The discussion becomes rather complicated if there are fixed points of different order in the problem. This can obviously not happen if N is prime, since the only allowed orbit lengths are then 1 or N . Presumably not much generality is lost by assuming that N is prime, since one can always build up an integer spin modular invariant for N not prime in steps, each step corresponding to a prime factor of N . Thus if $N = pq$, p and q prime one can first build an invariant by using the current J^q , which has order p . This organizes all charge zero fields (which includes the current J) into orbits of length p or 1. After splitting the fixed point fields and computing $\hat{S}$ one gets a theory in which J is an integer spin simple current of order q . Now one can apply the same procedure to the current J (we will often refer to the result as a *secondary* modular invariant). If the fixed point resolution problem always has a unique answer, then this stepwise procedure will produce that answer. On the other hand, one could imagine the existence of additional solutions that can only be obtained by looking directly at the final modular invariant partition function. Some examples of secondary modular invariants are discussed in Appendix B.

From now on we will assume that N is prime. If there is just one fixed point field, ϕ_a , we can solve the problem explicitly. From (3.8) we get

$$\Gamma_{ij} = (T_{aa})^3 (\Gamma^2)_{ij}, \quad (3.9)$$

which, combined with (3.7) yields

$$\Gamma_{ij} = (T_{aa})^3 C_{ik} P_{kj} . \quad (3.10)$$

Here we have used $C_{aa} = 1$, since a single fixed point field can only be conjugate to itself. It seems that we need to specify C_{ij} in order to solve the problem completely. However, by substituting (3.10) back into the right hand side of (3.9) we find

$$\Gamma_{ij} = (T_{aa})^9 P_{ij} . \quad (3.11)$$

Now that we know Γ we can use (3.7) to determine the charge conjugation matrix C_{ij}

$$C_{ij} = T_{aa}^6 \delta_{ij} + \frac{1}{N} (1 - T_{aa}^6) E_{ij} .$$

From this equation we get by imposing the condition $C^2 = 1$ a restriction on T_{aa}

$$T_{aa}^{12} = 1 . \quad (3.12)$$

This allows two possible values for T_{aa}^6 and hence two possible forms of the charge conjugation matrix:

$$\begin{aligned} T_{aa}^6 = +1 &\rightarrow C_{ij} = \delta_{ij} \\ T_{aa}^6 = -1 &\rightarrow C_{ij} = \frac{2}{N} E_{ij} - \delta_{ij} \end{aligned}$$

The second possibility is however in general not acceptable, since the entries of the matrix C must be equal to 0 or 1 (note that the fusion coefficient N_{a0} is equal to C_{ab}). This is true only if $N = 2$, in which case the two fields that one gets after resolving the fixed point are each other's conjugate. Thus we conclude that T_{aa}^6 must be equal to 1 if N is a prime number larger than 2, and may be equal to ± 1 if $N = 2$. In the next section we will check that this condition is indeed respected.

What made the previous derivation possible is the fact that T and Γ commute on the fixed points. This will in general not be true if there is more than one fixed point. The foregoing discussion suggests the following *ansatz*:

$$\Gamma_{ij}^{ab} = \gamma^{ab} P_{ij} . \quad (3.13)$$

This is certainly correct if $N = 2$ (where, as before, N is the order of the simple current) as a consequence of (3.6), but for larger values of N more general expressions can be considered.

This ansatz leads to the following conditions for γ

$$\begin{aligned} \gamma \gamma^\dagger &= 1 \\ \gamma \rho \gamma &= \rho^{-1} \gamma \rho^{-1} , \end{aligned} \quad (3.14)$$

where ρ is equal T , restricted to the fixed points. For the third condition we find

$$(\gamma^2)_{ab} P_{ij} = C_{ab} C_{ik}^b P_{kj} .$$

This can only be satisfied if

$$C_{ik}^b P_{kj} = \eta_b P_{ij} \quad (3.15)$$

with

$$(\gamma^2)_{ab} = \eta_b C_{ab} . \quad (3.16)$$

The first equation, combined with the knowledge that C must be a permutation of order two, can be satisfied only if $C_{ik}^b = \delta_{ik}$, or if $N = 2$ and $C^b = \sigma_1$. The value of η_b is $+1$ and -1 respectively in these two cases. Note that it is not necessary to consider the choice $C^b = \sigma_1$ unless ϕ_b is self-conjugate, since otherwise one can always rearrange the basis to make C^b diagonal.

4. Fixed Point Conformal Field Theories

Since (according to (3.14) and (3.16)) γ and ρ seem to behave very much like S and T matrices of a conformal field theory, it is tempting to consider their fusion rules and characters. At first sight, there appears to be no reason (other than curiosity) to do this. The characters of the new theory corresponding to the integral spin modular invariant appear to have nothing to do with those of γ and ρ . The fusion rules of the new theory do depend on γ and ρ , but not in a very straightforward way. In particular, there seems to be no direct relation between the fusion rules obtained from γ and ρ and those of the new theory.

Nevertheless something quite interesting can be learned by studying the fusion rules and characters. Nothing we have derived so far shows that there should even exist good fusion rules and characters associated with γ and ρ . It is easy to find matrices γ and ρ with

non-integer fusion coefficients. It is a surprising and so far empirical fact that all matrices γ we have found lead to positive integer fusion coefficients, and that characters can be defined provided one generalizes the notion of a character. Our observations can be summarized as follows. For every set of matrices γ and ρ as defined above there exists a rational conformal field theory with as many primary fields as there are fixed points. The modular matrices S and T of this conformal field theory are related to γ and ρ in one of the following ways

$$\begin{aligned}
 (a) \quad & \gamma = S & \rho = T \\
 (b) \quad & \gamma = -S & \rho = -T \\
 (c) \quad & \gamma = iS & \rho = iT \\
 (d) \quad & \gamma = -iS & \rho = -iT
 \end{aligned} \tag{4.1}$$

Note that in the latter two cases $\eta_c = -1$, for all c . Although *a priori* a combination of signs for η might be possible, we have not found any examples where this happens. It is obvious that in all these cases the fusion rules are not affected by the extra factors $(i)^n$. Something more interesting happens to the characters.

Consider first the single character theories. There are just three choices for S and T that correspond to conformal field theories in the usual sense. These values and the corresponding characters are

$$\begin{aligned}
 S = 1, \quad T = 1 & \quad \chi = P(j) \\
 S = 1, \quad T = e^{4\pi i/3} & \quad \chi = \frac{G_4}{\eta^8} P(j) \\
 S = 1, \quad T = e^{2\pi i/3} & \quad \chi = \frac{G_8}{\eta^{16}} P(j)
 \end{aligned} \tag{4.2}$$

Here G_n are the Eisenstein functions and $P(j)$ is some polynomial in the absolute modular invariant j . Examples of theories having such characters are respectively the Niemeier lattices, E_8 level 1, and $E_8 \times E_8$ or D_{16} , level 1. By combining these possibilities for S and T with the factors listed in (4.1) we get precisely all values for T that satisfy $T^{12} = 1$, which, as we saw in the previous section, are the only values allowed for a single fixed point. (In the following S and T refer generically to representation matrices of the modular group, which may play the rôle of matrices γ and ρ in fixed point resolution.)

Can one associate "characters" with *all* these values of S and T ? Since we have just seen that it can be done if T is a cubic root of unity, it is sufficient to investigate whether it is possible for $S = i^n, T = i^n$.

It is easy to find a modular function transforming with $S = -1$ and $T = -1$, namely

$$\chi = \frac{G_6}{\eta^{12}} P(j).$$

The function G_6 has a series expansion starting with $1 - 504q$ and is clearly not an acceptable character of a conformal field theory. There is no reason why the matrix γ associated with the fixed points of a modular invariant partition function should have any character associated with, so the negative sign should not worry us. The importance of integer coefficients will become clear in a moment.

Finally we would like to find a character representation for $S = T = \pm i$. Formally, a realization is provided by powers of $P_1 = \theta_1/\eta$. This combination has $S = e^{3\pi i/2}$ and $T = e^{i\pi/6}$, so that by taking powers one precisely realizes all possibilities of (3.11) in agreement with (3.12) (including some cases for which we already found other solutions above). Of course P_1 vanishes, but from experience with superstring partition functions it is well known that the appearance of this function is usually meaningful. Examples demonstrating this will be discussed in section 6.

Consider now theories with more than one fixed point. Our main conjecture can be rephrased as follows: The values of $h - c/24$ for the fixed points of any integer spin modular invariant of prime order differ from those of some unitary or non-unitary conformal field theory by some common constant, which is equal to $\frac{\pi}{12}$, $n \in \mathbb{Z}$. The shift in the h -values can always be accounted for by tensoring with some number of E_8 , level 1 theories, and by factors i^n as shown in (4.1). This equation can then be used to determine the matrix γ . A set of "characters" that form a representation of the modular group with matrices γ and ρ can be obtained by multiplying the characters of the fixed point conformal field theory with a character of the single character theory with $T = \exp(2\pi i n/12)$. The second part of our conjecture is then that this matrix γ yields a matrix $\tilde{S}$ for the new theory which has integer fusion coefficients.

An immediate consequence of this conjecture is that a similar one must hold for fixed points of half-integer spin currents, which are the only fractional spin ones (of prime order) that can have fixed points. A tensor product of two theories with half-integer spin currents has an integer spin current which is the product of the two, and which should satisfy the conjecture. For one of the theories one can take one with a single fixed point (for example the Ising model). It follows that the values of $h - c/24$ of the fixed point of any half-integer

spin current must differ from those of some CFT by $\frac{m}{24}$, where m is either always odd or always even. It is sufficient to determine m for just one CFT to find out which of these two possibilities is chosen, and (as one might have expected) it turns out that m must always be odd for half-integer spin currents, and, as we have seen, even for integer spin ones.

We have checked our conjecture for several Kac-Moody algebras with integral or half-integral simple currents. Unfortunately the classification of conformal field theories is far from complete, so that it is not always easy to positively identify a theory from its h -values. In fact, the only chance of finding irrefutable counter-examples to our conjecture is for theories with one or two fixed points, since (to our knowledge) there does not exist a complete classification of (not necessarily unitary) theories with three characters or more. We have succeeded in identifying all fixed point CFT's with less than 6 characters, but for six and more characters there are some unidentified fixed point CFT's. In view of the large number of positive identifications, we believe that these unidentified fixed point theories correspond to good conformal field theories as well, but this remains to be proved.

In some cases we can not identify the matrix T , but we can identify T^* . Clearly one finds then a solution for the matrix S by taking the complex conjugate of the solution for S^* . In [15] we have called such conformal field theories with mutually conjugate S and T matrices *complementary* conformal field theories, because a pair of such theories can be combined to a meromorphic $c = 8n$ theory. The latter characterization does not seem to make much sense for non-unitary theories however. Since we do not know an explicit realization of the complement of non-unitary models, we define them here by simply changing the sign of c and h . Note that the fusion rules are not affected by these complex conjugations.

Our results are as follows (here δ_h is the shift in the $h - \frac{c}{24}$ -values)

— A_n . All fixed point theories have been identified, as explained in appendix A. The fixed point CFT's are always $SU(N)$ Kac-Moody algebras.

— B_n . For odd levels, the current has half-integer spin, and we find the following fixed point theories

$$B_n \text{ level } 2m+1 \rightarrow C_{n-1} \text{ level } m, \quad \delta_h = \frac{2n+1}{24} + \frac{m}{4}$$

For even levels we have only partial results, and they are as follows

$$B_n \text{ level } 2 \rightarrow (p=2, p'=2n+1)^c, \quad \delta_h = \frac{2n+1}{12}$$

Here $(p=2, p'=2n+1)$ refers to a (non-unitary) element of the minimal series of Virasoro representations [16,17] (we will discuss this in more detail in section 6). The superscript “c” indicates that we do not find these minimal models themselves, but rather their complement. The characters for the complement of the $(p=2, p'=5)$ theory have been obtained in [18] (these authors rejected their solution however, because the identity appears with multiplicity 57). A second series that we have identified is

$$SO(5) \text{ level } 2m \rightarrow (p=2, p'=2m+3), \quad \delta_h = \frac{m}{4}$$

(For $m=1$ this agrees with the previous result, since one has to switch the assignment of the identity character.) For larger values of m and n the identification is more problematic. The h -values for the next few cases are

$$\begin{aligned} SO(7) \text{ level } 4 &\rightarrow (A) : \frac{7}{9}, 1, \frac{10}{9}, \frac{4}{9}, \frac{13}{9}, \frac{15}{9} \\ SO(7) \text{ level } 6 &\rightarrow (B) : \frac{12}{11}, \frac{14}{11}, \frac{15}{11}, \frac{17}{11}, \frac{18}{11}, \frac{20}{11}, \frac{21}{11}, \frac{24}{11}, \frac{27}{11} \\ SO(9) \text{ level } 4 &\rightarrow (C) : \frac{9}{11}, \frac{12}{11}, \frac{14}{11}, \frac{15}{11}, \frac{16}{11}, \frac{18}{11}, \frac{19}{11}, \frac{21}{11}, \frac{27}{11} \end{aligned} \quad (4.3)$$

Note that the spectrum of the complement of (C) is the same as that of (B) , up to a constant.

— C_n . Here our results are still somewhat fragmentary

$$C_3 \text{ level } 2m \rightarrow SU(2) \text{ level } m, \quad \delta_h = \frac{3m}{8},$$

with a half-integer spin current for odd m , and integer spin for even m . For C_4 we find

$$C_4 \text{ level } 2m+1 \rightarrow SO(5) \text{ level } m, \quad \delta_h = \frac{6m+5}{12}.$$

The current has always integer spin, also for even levels. However, the conformal field theories associated with the fixed point at even levels are quite different:

$$\begin{aligned} C_4 \text{ level } 2 &\rightarrow (p=2, p'=7)^c, \quad \delta_h = \frac{5}{6} \\ C_4 \text{ level } 4 &\rightarrow (A) + \frac{1}{2} \\ C_4 \text{ level } 6 &\rightarrow (B) + \frac{3}{4} \end{aligned}$$

We find thus exactly the same set of unidentified theories as previously for $SO(7)$.^{*} In the last two cases we cannot identify δ_h since we do not know the correct h -values. Instead we have indicated the shift in the h -values with respect to (4.3).

— D_n Here we only have results for the current J_v . For odd levels, this current has half-integer spin and no fixed points. For even levels, we find

$$D_n, \text{ level } 2m \rightarrow C_{n-2} \text{ level } m, \quad \delta_h = \frac{m}{4}.$$

— Exceptional invariants. Here we have only studied cases where the exceptional invariant is not a secondary invariant of some other integral spin invariant. With one exception, all known integral spin exceptional invariants are conformal sub-algebras. Their fixed points (by which we mean character combinations with fewer characters than the identity) are usually resolved in terms of two conjugate representations of a Kac-Moody algebra. Among the theories with one fixed point, one finds conformal embeddings in $SO(2n)$, E_6 and $SU(3)$ (all level 1). To check our conjecture one can simply compute the $h - \frac{c}{24}$ value of the complex representation of these algebras, and verify that it is always equal to an integer divided by 12. Something far more spectacular seems in fact to be true: the non-self-conjugate primary fields at higher levels seem to satisfy our conjecture as well. For example, it appears that the $h - \frac{c}{24}$ -values of the non-self-conjugate $SU(3)$, level k representations differ by a common constant $\frac{n}{12}$ from those of G_2 , level $k - 1$. A similar relation exists between $SO(2N)$ and $SO(2N + 1)$. It is an intriguing question whether this might be true for all KM-algebras, or perhaps even for any conformal field theories, but this takes us in an entirely different direction which we will not explore here.

The one known integer spin invariant which is not a conformal sub-algebra occurs for F_4 , level 6, and its fixed point resolution was discussed in [15]. There are two fixed points, and the corresponding fixed point CFT is F_4 level 1 with $\delta_h = \frac{13}{12}$.

^{*} The spectrum of theories (A) and (B) was fixed to be $h - \frac{c}{24}$ for $SO(7)$, without additional constant $\frac{1}{12}$. The general pattern seems to be that the fixed point CFT of C_n , level $2m$ is that of B_3 , level $2m$ plus $\frac{m}{4}$.

5. Character Modification

Up to now we have implicitly assumed that all characters of distinct fields belonging to the same fixed point are identical. However the process of resolving the fixed points allows us in principle to choose different characters for each distinct field. Instead of (3.1) one could define

$$\chi_a = \sqrt{\frac{N_a}{N}} \chi_a + \tilde{\chi}_a, \quad i = 1, \dots, N/N_a, \quad (5.1)$$

with

$$\sum_i \tilde{\chi}_a = 0, \quad (5.2)$$

for each fixed point a . This condition guarantees that the functions $\tilde{\chi}$ do not affect the modular transformations between the fixed points and other fields. The functions $\tilde{\chi}$ have to satisfy additional constraints to ensure correct modular transformations of the fixed point characters among each other. For the reasons explained in section 3 we will restrict ourselves to prime N , so that for fixed points $N_a = 1$. The most general solution of (5.2) involves $N - 1$ independent functions, and one can parametrize the answer as follows

$$\tilde{\chi}_a = \sum_{j=1}^{N-1} v_a^j \Psi_a^j, \quad (5.3)$$

where v_a^j are $N - 1$ orthonormal and mutually orthogonal vectors that are orthogonal to $(1, 1, 1, \dots, 1)$. This parametrization is useful if one makes the ansatz (3.13), since in that case the matrices Γ do not mix the vectors v^j . The condition that the functions $\Psi_a^j(\tau)$ have to satisfy is then simply that they should transform as characters with respect to the matrices γ and ρ , for any value of j .

It should now be clear why previously we were interested in having characters of γ and ρ with integer, but not necessarily positive coefficients. Since the characters of γ and ρ are added to or subtracted from the characters of the original CFT in order to get the characters of the new CFT, all one really has to require is that the sums and differences have positive integral coefficients.

The procedure discussed above is far from unambiguous. Even if the characters of the theory defined by γ and ρ are unambiguous, one could multiply them with any common integral coefficient (including 0) before using them. These coefficients are obviously restricted

by the requirement that the characters of the new theory have positive integral coefficients, but it is not obvious that this is sufficiently strong condition.

If one is considering an integer spin modular invariant of some CFT, there is no obvious need to add anything to the characters, although it may be useful to keep in mind that the possibility exists. Although the “naive” characters $\mathcal{X}_a$ have integer coefficients and transform correctly under modular transformations, they might turn out not to be the correct characters of the theory for other reasons. Some evidence that fixed point characters of integer spin modular invariants may require modification will be given in the next section.

On the other hand, in coset theories the fixed point characters $\mathcal{X}_a$ are in general not acceptable, since they have to be divided by the multiplicity of the identity. Here character modification is obviously needed and provides, at least potentially, a way out of the fixed point dilemma. The hope is that by adding an extra function to a branching function that is not divisible by N , one obtains a new function that is divisible by N , and corresponds (after division) to the true character of the coset theory. We will see some examples where this is indeed the correct solution, and it seems very likely that it will be true in general.

The final answer to the question whether and how the fixed point characters have to be modified can presumably be found by investigating the null-state conditions of the appropriate chiral algebra.

Note that all cross-terms between the characters of the original CFT and those of the fixed point CFT cancel in the sum of squares of the fixed point characters. Thus on the partition function the net effect of the character modification is that a modular invariant partition function of the fixed point theory is added to it.

6. Examples

In this section we will present some examples of the modification of fixed point characters. Unfortunately the exact computation of characters is possible only in a few cases, so that non-trivial examples are hard to find. The examples which we present here all have the common property that the theory we get after resolving the fixed points (which may be an integer spin modular invariant or a coset theory) is an already known and understood one, so that we know what the answer should be.

The extended algebra of integer spin modular invariants of Kac-Moody algebras is only well understood if it is itself a Kac-Moody algebra [19]. Thus we might consider the set of integer spin modular invariants that corresponds to conformal sub-algebras [20] of other Kac-Moody algebras. There are many examples of this kind, but in all cases we know the two or more fields coming from the same fixed points have the same character.

Consider for example all conformal embeddings into $SO(2N)$. If $\mathcal{H}$ is conformally embedded into $SO(2N)$, the diagonal invariant of $SO(2N)$ is equal to some integer spin modular invariant of $\mathcal{H}$. If there are resolvable fixed points, they can usually only correspond to the spinors of $SO(2N)$. The fixed point is resolved into two conjugate spinors, and the fixed point CFT is the identity, shifted by $N/12$. For the characters of this fixed point theory one can choose $(\theta_1/\eta)^N$ (other choices may sometimes be possible, but are not relevant in this context). The relevant integer spin modular invariant partition function always has the form

$$|\text{Ch}_0^{D_N}|^2 + |\text{Ch}_v^{D_N}|^2 + 2|\text{Ch}_S^{D_N}|^2. \quad (6.1)$$

The characters $\text{Ch}_i^{D_N}$ are sums of characters of $\mathcal{H}$, with $\text{Ch}_S^{D_N}$ containing half as many characters of $\mathcal{H}$ as $\text{Ch}_0^{D_N}$ and $\text{Ch}_v^{D_N}$. Since the resulting partition function corresponds to D_N , level 1, we know that these sums can be expressed in terms of θ -functions

$$\begin{aligned} \text{Ch}_0^{D_N} &= \frac{1}{2}(\theta_3^N + \theta_4^N)/\eta^N \\ \text{Ch}_v^{D_N} &= \frac{1}{2}(\theta_3^N - \theta_4^N)/\eta^N \\ \text{Ch}_S^{D_N} &= \frac{1}{2}\theta_2^N/\eta^N \end{aligned}$$

The factor of 2 in (6.1) tells us that the fixed point field corresponds to two distinct fields, and the character of the fixed point CFT allows us to replace $\text{Ch}_S^{D_N}$ by two characters

$$\text{Ch}_S^{D_N} \rightarrow \begin{cases} \text{Ch}_s^{D_N} = \frac{1}{2}(\theta_2^N + i^N \theta_1^N)/\eta^N \\ \text{Ch}_c^{D_N} = \frac{1}{2}(\theta_2^N - i^N \theta_1^N)/\eta^N \end{cases} \quad (6.2)$$

The partition function (6.1) can now be written as

$$|\text{Ch}_0^{D_N}|^2 + |\text{Ch}_v^{D_N}|^2 + |\text{Ch}_s^{D_N}|^2 + |\text{Ch}_c^{D_N}|^2, \quad (6.3)$$

Obviously the modification of the characters by θ_1 has a formal meaning, even though $\theta_1(0|\tau) = 0$. This can undoubtedly be made explicit by considering correlation functions that

are sensitive to the difference of the spinor representations. In other words, the vanishing characters introduced above have non-vanishing character-valued generalizations (in this case $\theta_1(z|\tau)$) which give a meaning to the difference between the two conjugate spinor characters. We will however not pursue that here.

Before giving examples where the character modifications are non-vanishing, we would like to discuss one more example that is similar in spirit to the previous one, namely $SU(3)$ level 3, which has a modular invariant partition function

$$|\chi_{00} + \chi_{30} + \chi_{03}|^2 + 3|\chi_{11}|^2, \quad (6.4)$$

where the subscripts are $SU(3)$ Dynkin labels. The fixed point CFT is now the identity, with h -values shifted by $+\frac{1}{3}$. Possible choices of the characters of the theory are

$$\begin{aligned} \chi_A &= \left(\frac{\theta_1}{\eta} \right)^4 \\ \chi_B &= \frac{(\theta_3^4 - \theta_2^4 - \theta_4^4)}{\eta^4}. \end{aligned}$$

Since we know that the invariant (6.4) should correspond to the diagonal invariant of $SO(8)$, level 1, a sensible interpretation of the character modification problem is obtained by writing the fixed point character as follows

$$\chi_{11} = \frac{1}{3}(\text{Ch}_v^{D_4} + \text{Ch}_s^{D_4} + \text{Ch}_c^{D_4}) = \frac{1}{6}(\theta_3^4 - \theta_4^4 + 2\theta_2^4)/\eta^4.$$

After resolving the fixed point, we modify this character in the following way

$$\chi_{11} \rightarrow \begin{cases} \chi_{11} + \frac{1}{3}\chi_B & = \text{Ch}_v^{D_4} \\ \chi_{11} - \frac{1}{6}\chi_B + \frac{1}{2}\chi_A & = \text{Ch}_s^{D_4} \\ \chi_{11} - \frac{1}{6}\chi_B - \frac{1}{2}\chi_A & = \text{Ch}_c^{D_4} \end{cases} \quad (6.5)$$

This example demonstrates the rôle of the vectors v_i introduced in (5.3). In this example $N = 3$, and the two vectors are $(\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6})$ and $(0, 1, -1)$. As in the previous example, the character modification is far from unique, and one could have used non-vanishing functions like the $E_8 \times E_8$ partition function. However, from the properties of the $SO(8)$ Kac-Moody

algebra we know that (6.5) is the correct answer. (Other choices often yield negative coefficients. For example, the $E_8 \times E_8$ partition function starts with $q^{-2/3}$, whereas the $SO(8)$ vector and spinor characters start with $q^{1/3}$. Modifying them with the $E_8 \times E_8$ partition function clearly does not yield a sensible answer. We do not know whether there is always a unique sensible answer.)

For a less trivial example, consider the class of coset theories

$$\frac{SU(N)_k \times SU(N)_l}{SU(N)_{k+l}}$$

with a diagonal embedding (here $SU(N)_k$ means $SU(N)$, level k). The identification current is easily seen to be $J_l = J_k J_l J_{k+l}^\dagger$, where J_k is the primary field with highest weight $(k, 0, \dots, 0)$ in $SU(N)_k$, and the superscript “ $\dagger$ ” denotes the conjugate. In each $SU(N)_k$ factor the current J_k has fixed points if the level is a multiple of N , so that the identification current has fixed points if and only if both k and l are multiples of N . We will only consider those cases where there is just one fixed point, which is equivalent to requiring that N is prime. An interesting subclass of these models is the $N = 1$ super-*Virasoro* discrete series, which has $l = 2$, $N = 2$ and k arbitrary.

According to the general formula for $SU(N)$ fixed points, A_1 , the value of $h - c/24$ for the fixed point of each $SU(N)$ is proportional to the level, and hence it vanishes for the coset theory. The functions one can use in the character modification procedure are thus constants, or polynomials in j . The latter are not suitable, since they add poles where previously there were none.

The fixed point branching function is defined by the combination of $SU(N)_k$, $SU(N)_l$ and $SU(N)_{k+l}$ representations $(\frac{k}{N}\delta, \frac{l}{N}\delta; \frac{k+l}{N}\delta)$, where δ is half the sum of the positive roots. It is easy to see that the leading coefficient in the fixed point branching function is 1, which forbids division of this function by N . The solution to this problem suggested by the foregoing discussion is to modify the N identical branching functions obtained from the fixed point resolution by adding $N - 1$ to one of them, and subtracting 1 from the $N - 1$ remaining ones. This certainly solves the problem for the ground state, but it is not at all obvious whether it will work also at higher levels. If our prescription is correct, all higher coefficients in the fixed point branching function should already be proportional to N . By explicit calculations one can check that this is true in the simplest cases, but we do not have a general proof.

The character modification does however have verifiable interpretation for the superconformal models, characterized by $l = 2$, $c = \frac{3}{2}(1 - \frac{8}{(k+2)(k+4)})$ [1]. The branching functions are in this case

$$b_{j_3}^{j_1, j_2} \leftrightarrow b_{k+2-j_3}^{2-j_1, k-j_2},$$

where j_i denotes twice the spin of an $SU(2)$ representation and the arrow indicates the field identification. The branching functions are non-vanishing if and only if $j_1 + j_2 = j_3 \bmod 2$. The supercurrent corresponds to the branching function $b_0^{0,2}$, so that the Ramond states have $j_2 = 1$ and the Neveu Schwarz states have $j_2 = 0$ (related to a $j_2 = 2$ character by field identification). A fixed point exists only for even k , and its branching function is $b_{k/2+1}^{k/2,1} \equiv \chi_R$. The corresponding highest weight state has $h = \frac{k}{24}$ and is obviously a Ramond ground state (it is known that the minimal $N = 1$ theories have such a state only for even k).

After resolving the fixed point, we are left with two characters, which are modified by adding or subtracting 1, and divided by two:

$$\chi_R \rightarrow \begin{cases} \frac{1}{2}(\chi_R + 1) \\ \frac{1}{2}(\chi_R - 1) \end{cases}$$

Note that the first character has an expansion $1 + \mathcal{O}(q)$, whereas the second one starts at the first excited level. Using the fusion rules obtained after resolving the fixed point one may check that the two fields are mapped into each other by the supercurrent (whereas all other Ramond states are fixed points of the supercurrent). Hence they are two members of an $N = 1$ supermultiplet. It is then clear that only one of the two characters can start with a constant. If one denotes the ground state corresponding to the constant term in the first character by $|0\rangle$, then $G_0|0\rangle$, the would-be ground state for the second character, is a null-state. The first state in the second module is in fact $G_{-1}|0\rangle$.

In the diagonal partition function of the coset theory these manipulations have the following effect

$$\frac{1}{4} \left[|b_0^{0,0} + b_{l+2}^{2,l}|^2 + \dots + 2|\chi_R|^2 \right] \rightarrow \frac{1}{2} (b_0^{0,0} + b_{l+2}^{2,l})^2 + \dots + \left| \frac{1}{2}(\chi_R + 1) \right|^2 + \left| \frac{1}{2}(\chi_R - 1) \right|^2.$$

Thus the net effect is to add a constant $\frac{1}{4}$ to the partition function. The interpretation of this constant is well understood [21]: it is simply the partition function in the PP-sector of the supercurrent, which is separately modular invariant. The “naive” modular

invariant partition function obtained from the coset only involves the other 3 combinations of boundary conditions, and the existence of a fourth one becomes apparent by splitting the fixed point.*

Finally we will discuss a class of examples with character modification to all orders in q . Consider the coset theories

$$\frac{SO(N)_1 \times SO(N)_1}{SO(N)_2}, \quad N \text{ odd}$$

These theories all have $c = 1$ so that they should correspond to a 1-dimensional orbifold or torus [22]. The identification current of these models is labelled by the representations (v, v, T) and the fixed points are labelled by (s, s, r) , where v denotes the vector representation, T the rank-2 symmetric tensor, s the spinor and r denotes the anti-symmetric tensor of rank r , with $r \geq 1$. The identification current is the same for even N , but has fixed points only for odd N . Since all representations r occur just once in the tensor product of two spinor representations, it is obvious that all fixed point branching functions start with a coefficient 1, and are hence not divisible by 2.

The fixed point CFT is easily obtained from those of $\mathcal{G}$ and $\mathcal{H}$. Each of the two $SO(N)$ factors in G contributes a constant $\frac{N}{24}$, whereas the fixed point CFT of $\mathcal{H}$ is the complement of the non-unitary minimal models with $p = 2$ and $p' = N$, shifted by $\frac{N}{12}$. Since the matrix T of the coset theory depends on the complex conjugate of the matrix T of $\mathcal{H}$, we find that the shifts in the h -values cancel, and that the fixed point CFT is precisely the $p = 2, p' = N$ non-unitary model.[†]

The spectrum of these models is

$$h_{rs} = \frac{(sp' - rp)^2 - (p - p')^2}{4pp'}, \quad s = 1, \dots, p-1, \quad r = 1, \dots, p-1.$$

and an explicit formula for the characters exists [23]. Define the following partition function

* One may also subtract $\frac{1}{4}$ from the original partition function. This corresponds to a non-diagonal modular invariant of the $N = 1$ model, generated by the supercurrent (which is a simple current). The only non-diagonal pairing in this partition functions is between the two characters obtained from the fixed point.
† Note that the fixed point CFT of a coset theory is the coset theory of the fixed point CFT's of $\mathcal{G}$ and $\mathcal{H}$.

for a shifted 1-dimensional lattice

$$\mathcal{W}_q^P = \sum_{m \in \mathbb{Z}} e^{2\pi i r \frac{1}{2} (\frac{P}{2P} + m\sqrt{2P})^2}, \quad q \text{ defined mod } 2P. \quad (6.6)$$

Then the partition function for the ground states (r s) is

$$\mathcal{X}_{r,s} = \mathcal{W}_{sp'-rp}^{pp'} - \mathcal{W}_{sp'+rp}^{pp'} \quad (6.7)$$

The fixed point of the coset model that is labelled by a rank r anti-symmetric tensor corresponds to the state with the same value of r .

We can also compute the branching functions explicitly, by observing that $SO(N)_2$ can be conformally embedded in $SU(N)$. This embedding can be used to compute some of the $SO(N)_2$ characters. The modular invariant partition function corresponding to this conformal embedding is

$$|\mathcal{X}_0^{(2)} + \mathcal{X}_T^{(2)}|^2 + 2 \sum_{r=1}^{\frac{1}{2}(N-1)} |\mathcal{X}_r^{(2)}|^2,$$

where the superscript denotes the level. Note that the spinor representations do not appear, so that the relation with $SU(N)_1$ is of little help for computing them, but fortunately we do not need them. By resolving the fixed points and interpreting each term as an $SU(N)$ character we get

$$\begin{aligned} \mathcal{X}_0^{(2)} + \mathcal{X}_T^{(2)} &= \mathcal{Y}_0 \\ \mathcal{X}_r^{(2)} &= \mathcal{Y}_r = \mathcal{Y}_{N-r} \end{aligned} \quad (6.8)$$

where $\mathcal{Y}_r$ is the character of the rank r anti-symmetric tensor of $SU(N)_1$.

In order to compute the fixed point branching functions we have to consider the product of two $SO(N)_1$ spinor characters and decompose it into $SO(N)_2$ characters. This problem can be solved as follows. The $SO(N)_1$ spinor character is

$$\mathcal{X}_s^{(1)} = \frac{1}{2} \sqrt{2} \left(\frac{\theta_2}{\eta} \right)^{N/2},$$

and its square is the character of one of the spinors of $SO(2N)_1$. We can decompose the latter into $SO(N)_2$ characters by first decomposing to $SU(N)_1$, and then use (6.8) to write the

result in terms of $SO(N)_2$ characters. It follows that the branching functions for $s \times s \rightarrow r$ in $SO(N) \times SO(N)/SO(N)$ are related to the branching functions of $SO(2N)/SU(N)$. However the latter coset is simply a $U(1)$ theory with definite charge quantization, whose characters can easily be computed.

To make this more precise, consider the decomposition $SO(2N) \supset SU(N) \times U(1)$. One can think of the $U(1)$ factor in terms of a root lattice with roots $2m\sqrt{N}$, $m \in \mathbb{Z}$ and a dual lattice with weights $\frac{s}{2N}\sqrt{N}$, $q \in \mathbb{Z}$. We will call the integer q the $U(1)$ charge. The weight lattice decomposes into $4N$ conjugacy classes with respect to the root lattice. The characters for these classes are $\mathcal{W}_q^{2N}$, defined in (6.6). The $U(1)$ conjugacy classes are in one-to-one correspondence with the $4N$ combinations of $SO(2N)$ and $SU(N)$ conjugacy classes, so that in the decomposition of a representation (x) of $SO(2N)$ every $SU(N)$ representation in conjugacy class (y) has the same charge, modulo $4N$.

The $SO(2N)$ conjugacy class of most interest to us, the spinor, decomposes as follows into $SU(N) \times U(1)$ conjugacy classes

$$(s) \rightarrow \sum_{r=0, \text{ even}}^{N-1} (r, N-2r) + \sum_{r=1, \text{ odd}}^N (r, -N-2r),$$

where the first entry denotes the rank of the $SU(N)$ tensor, and the second the charge, modulo $4N$. Accordingly there is a decomposition of the spinor character, which after some trivial rearrangement can be written as

$$\text{Ch}_s^{D_N} = \sum_{r=0, \text{ even}}^{N-1} \mathcal{Y}_r (\mathcal{W}_{N-2r}^{2N} + \mathcal{W}_{N+2r}^{2N}). \quad (6.9)$$

This decomposition is straightforward, because it is simply a matter of rewriting a lattice partition function in terms of a sublattice of the root lattice. In particular there are no fixed point problems to worry about.

Now compare (6.9) with the definition of the branching functions,

$$\mathcal{X}_s^{(1)} \mathcal{X}_s^{(1)} = b_0^{ss} \mathcal{X}_0^{(2)} + b_T^{ss} \mathcal{X}_T^{(2)} + \sum_r b_r^{ss} \mathcal{X}_r^{(2)}$$

(the only highest weight states of $SO(N)_2$ are 0, T , r plus spinor representations that cannot appear in $s \times s$). Because of the field identification we know that $b_0^{ss} = b_T^{ss}$. Using (6.8) we

conclude then *

$$b_0^{ss} + b_T^{ss} = 2\mathcal{W}_0^{2N}$$

$$b_r^{ss} = \mathcal{W}_{N-2r}^{2N} + \mathcal{W}_{N+2r}^{2N}$$

The fixed point problem is manifest: the identity character has an overall factor 2, but the other characters do not. Fortunately, the solution is now also obvious. As explained above, the fixed point characters can be modified by the characters of the non-unitary theory corresponding to each fixed point. From (6.7) we get for these characters

$$\mathcal{X}_{r,1} = \mathcal{W}_{N-2r}^{2N} - \mathcal{W}_{N+2r}^{2N} ,$$

so that by taking sums and differences with b_r^{ss} and dividing by two we get well-defined characters, which are in fact nothing but torus characters.

Since the coset theories under considerations have $c = 1$, one expects to find either a torus or an orbifold partition function (or possibly a combination of both). Which one do we actually get? To answer this question we have listed all non-vanishing branching functions in table I.

All primary fields except one have already been defined above. The extra one is denoted “ $(vs)^\eta$ ”, and has Dynkin-labels $(1, 0, \dots, 0, 1)$. Although the integer part of the h -values are not easy to determine from those of $\mathcal{G}$ and $\mathcal{H}$, one can benefit from the identification with torus states to fix h completely. In the table only one field out of each set of identified fields is listed. The last two entries correspond to the fixed point field discussed above. The torus primary fields are the identity, the operator ∂X plus a field for every integer charge $-2N + 1 \leq q \leq 2N$. Only about half of these charges actually appear as primary fields in table I. One can in fact identify the coset fields precisely with the absolute values of all the charges, for $|q| > 2N$. Note that on the other hand there are two fields with charge $2N$.

After having identified the torus charges, one is left with a set of fields with $h = \frac{1}{16}$ and $h = \frac{9}{16}$. These are precisely the well-know twist field and excited twist field of the $c = 1$ orbifold. Thus we conclude that the diagonal partition function that one obtains from the coset $SO(N) \times SO(N)/SO(N)$ must be the orbifold partition function.

* One cannot really draw this conclusion on the basis of the q -dependence alone, but one has to take into account the $SO(N)$ transformations of the fields.

$SO(N)_1 \times SO(N)_1$	$SO(N)_2$	h -value	interpretation
$(0,0)$	(0)	0	vacuum (NS)
$(0,0)$	(T)	1	∂X (NS)
$(0,0)$	(r)	$-\frac{4rN-4r^2}{8N} + \text{integer}$	$q \equiv 0 \pmod{4}$ (NS)
$(0,v)$	(0)	$\frac{N}{2}$	fermionic current ($q = 2N$; NS)
$(v,0)$	(0)	$\frac{N}{2}$	fermionic current ($q = 2N$; NS)
$(0,v)$	(r)	$-\frac{4rN-4r^2-4N}{8N} + \text{integer}$	$q \equiv 2 \pmod{4}$ (NS)
$(s,0)$	(s)	$\frac{1}{16}$	orbifold twist field (NS)
$(s,0)$	(vs)	$\frac{9}{16}$	excited twist field (NS)
$(0,s)$	(s)	$\frac{1}{16}$	orbifold twist field (R)
$(0,s)$	(vs)	$\frac{9}{16}$	excited twist field (R)
(s,s)	(0)	$\frac{N^2}{8N}$	$q = N$ (R)
(s,s)	(r)	$\frac{(N-2r)^2}{8N}$	$q = N - 2r$ (R)
(s,s)	(r)	$\frac{(N+2r)^2}{8N}$	$q = N + 2r$ (R)

Table I. Branching functions of the coset $SO(N)_1 \times SO(N)_1 / SO(N)_2$ and their interpretation

Among the orbifold primary fields there are several simple currents, built out of simple currents of $\mathcal{G}$ and $\mathcal{H}$. First of all there are two fields of spin $\frac{N}{2}$ (i.e. half-integer spin for N odd). The other fields can be subdivided into Neveu-Schwarz and Ramond fields with respect to each of these currents, and we have indicated this division for the current $(0, v; 0)$. In addition the field $(0, 0; T)$ is a simple current of spin 1, which can be used to build an integer spin modular invariant partition function.

The effect of the latter current is as follows. The orbifold fields are projected out, and the identity is combined with the current into a bigger module. The two fermionic currents are also combined into one representation. All other torus states are fixed points, although for somewhat different reasons: the states of the form $(*, *, r)$ are fixed points because r is a fixed point of T , whereas $(s, s; 0)$ is a fixed point because it is mapped to $(s, s; T)$, with which it is identified.

If our conjecture is correct, the fixed points must be identifiable with some conformal field theory, and indeed they are. Consider the shifted lattice $(\frac{q}{2N} + 2m)\sqrt{N}$, $m \in \mathbb{Z}$, where N need not be odd and may even be half-integer. The values of h for the charges $q = 1, \dots, 2N - 1$ are $h_q = \frac{q^2}{8N}$. These values are equal up to a constant to those of $SU(2)$, level $2(N - 1)$:

$$h_q - \frac{1}{24} = \frac{j(j+2)}{4(k+2)} - \frac{k}{8(k+2)} + \frac{1}{12},$$

with $j = q - 1$. Thus all $SU(2)$ Kac-Moody algebras can be identified with the fixed points of a $c = 1$ theory. It is noteworthy that $SU(2)$ level 1 does not appear for $N = \frac{1}{2}$ (the $SU(2)$ torus), but for $N = \frac{3}{2}$.

The fixed point resolution replaces each orbifold primary field by two fields with the same value of h . These two fields are distinguished by the chiral algebra (generated by ∂X) by having opposite charges. Clearly the partition function we have obtained is simply the standard one for the torus.

The foregoing discussion has a very elegant formulation in terms of cosets. The orbifold simple current $(0, 0; T)$ is identified with $(s, s; 0)$. Thus the action of this current has the effect of simultaneously changing $SO(N)$ to $SU(N)$ and $SO(N) \times SO(N)$ to $SO(2N)$. Thereby the coset $SO(N) \times SO(N) / SO(N)$ (the orbifold) is transformed into $SO(2N) / SU(N)$, which is manifestly a torus.

7. Conclusions

In this paper we have proposed solutions to two problems in conformal field theory, namely the problem of resolving fixed points in integer spin modular invariants, and the problem of fixed points in the field identification of coset models. These problems are closely related, since the problem of field identification in coset models $G/\mathcal{H}$ is equivalent to that of understanding a particular integer spin modular invariant in the "tensor product" $\mathcal{G} \times \mathcal{H}^c$, where $\mathcal{H}^c$ is the complement of $\mathcal{H}$. We believe that we have found the general solution to both problems, but at present this remains to be proved rigorously.

Many very interesting and important questions remain. It would clearly be very important to find a general proof (or a counter-example...) to our main conjecture, which says that fixed points of (half)-integer spin currents of prime order are related to some conformal

field theory in a very definite way (see section 4). One could in principle try to prove this conjecture by exhaustion, by first checking it for all Kac-Moody algebras, and then for all coset theories, thus covering most, if not all, rational conformal field theories. This might well be possible (we have completed this for $SU(N)$ Kac-Moody algebras in Appendix A), but it is not a very satisfactory approach to the problem. Although we have made extensive use of the simple current formalism developed in [3], (which is all that is needed for the application to coset models) it appears that our results apply to exceptional invariants as well.

If our conjecture is correct, an extremely interesting question is which conformal field theories one gets in this way. Apparently the connection between a CFT and the CFT associated with its fixed points is not very straightforward: although the results for $SU(N)$ suggest a simple answer, it is rather surprising to see non-unitary CFT's appear for $SO(N)$ Kac-Moody algebras. Equally unexpected to us was the appearance of the $SU(2)$ Kac-Moody algebra in relation to the fixed points of the current ∂X in the $c = 1$ orbifold. In addition there appear to be CFT's associated with the set of non-self-conjugate representations of some Kac-Moody algebras (or all Kac-Moody algebras, or perhaps even all conformal field theories).

If the aforementioned conjecture is proved, one still has to show that the matrix γ associated with the fixed point CFT does indeed lead to the correct resolution of the fixed points. It is not obvious that the new S -matrix will lead to positive integer fusion coefficients; even if it does, it is not obvious that there could not be additional solutions.

Finally, it should be proved that the characters of the fixed point CFT can always be used to modify the branching functions of a coset theory in precisely the right way to allow a proper normalization of the partition function. This argument can be turned around to conclude that the strongest argument in favor of the general validity of our conjectures is the likely existence of coset models and their characters. Since character modification seems to be the only way out of the fixed point dilemma in coset models, it must be true that there is a fixed point CFT with proper characters.

Our belief that all or most of the above is true is largely based on examples. However, the set of examples we have investigated so far is sufficiently large and sufficiently non-trivial to demonstrate that our ideas are on the right track, and to stimulate further work on these problems. We hope and expect to report further progress on at least some of them in the

future.

ACKNOWLEDGEMENTS

We are deeply indebted to Wolfgang Lerche, who drew our attention to the fixed point problem in coset theories and also pointed out a possible relation with simple currents. We also thank D. Nemeschansky for very helpful explanations of branching functions, and S. Panda and B. Blok for discussions.

APPENDIX A

$SU(N)$ fixed points

In this appendix we will derive the fixed point CFT's for all simple currents of $SU(N)_k$. The simple currents of $SU(N)_k$ form a $\mathbb{Z}_N$ orbit generated by $J = (k, 0, \dots, 0)$. Powers $(J)^l \equiv J_l$ of the generating current J may generate the full $\mathbb{Z}_N$ or a finite subgroup. Since any two currents that generate the same orbit will obviously have the same fixed points, we can without loss of generality concentrate on generators of the form J^l , with l a divisor of N .

The fusion rule of J with any other primary field Φ is most easily described in terms of Young tableaux. One simply adds one box to the first k columns of the Young-tableau representing Φ . If after doing this the first column has length N it can be removed, exactly as one would do for Lie-algebra tensor products. To compute the fusion rules with J_l one simply iterates this l times.

We will begin by determining the fixed points of J . The current J obviously modifies any Young tableau Y with less than $N - 1$ boxes in the first column. If, however, the first column has precisely $N - 1$ boxes it will be removed, and the second column takes the place of the first. There are now three possibilities: (a) The second column had less than $N - 2$ boxes. Then the Young tableau does not describe a fixed point. (b) The second column has $N - 2$ boxes (c) The second column has $N - 1$ boxes. If (c) is true the second column is removed as well, and then the third column takes the place of the first. Then we have again options (a), (b) and (c) for the third column, etc.

Suppose now that the first p columns have length $N - 1$, and the $(p + 1)^{\text{th}}$ length $N - 2$. Since the first p columns are cancelled and replaced by columns $p + 1, \dots, 2p$ after adding J , it follows that columns $p + 1, \dots, 2p$ must have length $N - 2$ if Y describes a fixed point. Then columns $2p + 1, \dots, 3p$ end up replacing the $p + 1, \dots, 2p$ after adding J , and hence these columns must have $N - 3$ boxes. Hence the Young tableau must have the shape of an (inverse) staircase with steps of length p and height 1. The last p columns of Y must obviously have length 0, which is only possible if $pN = k$. Hence the current J has a fixed point if and only if k is a multiple of N , and then there is only one such fixed point. When expressed in terms of Dynkin labels, the fixed point we just found is $\vec{p\delta} = (p, p, \dots, p)$, where $\vec{\delta}$ is half the sum of the positive roots.

The discussion is just slightly more complicated for J_l . Clearly a Young tableau must have at least $N - l$ boxes in the first column to describe a fixed point. All columns with at least $N - l$ boxes will be cancelled after J has been added l times. Assume that column $p + 1$ is the first one with less than $N - l$ boxes. Then this column replaces the first one, and hence its length must be equal to that of the first one, minus l . Similarly column $p + 2$ must have the length of column 2 minus l . In general we find that the length of column $jp + i$ must be equal to the length of column i minus jl .

The resulting pattern is most easily described as follows. Consider a staircase diagram, with steps of length p and height l , and a first column of length $N - l$. This is obviously a fixed point for J_l if $k = p\frac{N}{l}$. However this is not the only fixed point. We can add to every step of the staircase any valid Young tableau with at most $l - 1$ rows and p columns. So if the column lengths of the staircase diagram are

$$\underbrace{(N - l, \dots, N - l, N - 2l, \dots, N - 2l, \dots, 0, \dots, 0)}_p, \quad \underbrace{\dots, 0, \dots, 0}_p,$$

then we can choose some Young tableau of $SU(l)_p$ with column lengths $(m_1, \dots, m_p)$ and add it to each step to obtain

$$(N - l + m_1, \dots, N - l + m_p, N - 2l + m_1, \dots, N - 2l + m_p, \dots, m_1, \dots, m_p).$$

Thus we see that the fixed point of J_l are in one-to-one correspondence with the primary fields of $SU(l)_p$.

We would now like to check that the conformal weights are also related. For this purpose we use the following formula for the quadratic Casimir eigenvalue of an $SU(N)$ representation. Let R be a representation described by a Young tableau with row lengths f_i , $i = 1, l - 1$ and column lengths g_i . Then

$$C(R) = \frac{1}{2} \left[r \left(N - \frac{r}{N} \right) + \sum_i f_i^2 - \sum_i g_i^2 \right],$$

where r is the total number of boxes, i.e. $r = \sum f_i = \sum g_i$. It is not difficult to use this formula to express the Casimir eigenvalue of the fixed point field in terms of the Casimir eigenvalue of the $SU(l)_p$ Young tableau that was stacked on top of the staircase diagram. After some straightforward computations one finds that the conformal weights of the $SU(N)$ fixed points and those of the corresponding primary fields of $SU(l)_p$ are related as follows

$$\left(h - \frac{c}{24} \right)_{\text{fixed point of } J_i} = \left(h - \frac{c}{24} \right)_{SU(l)_p} + \frac{k(N^2 - l^2)}{24N}. \quad (\text{A1})$$

It is easy to check that the last terms is equal to an even (odd) integer over 24 if the correct J_l has (half)-integer conformal spin.

Note that if J^l has integer spin and is used to build a modular invariant partition function one will not only encounter short orbits of length 1 due to fixed points of J^l itself, but also short orbits that are due to fixed points of powers of J_l . Only if the order of J_l is prime (i.e. if $\frac{N}{l}$ is prime) such intermediate length orbits do not occur.

Note also that the fixed point CFT's $SU(l)_p$ have simple current themselves, which may have fixed points. In terms of the $SU(N)$ Young tableau, these $SU(l)_p$ fixed points correspond in an obvious way to fixed points of currents J_m , where m is a divisor of l .

APPENDIX B

Examples of fixed point resolution

The appearance of a multiplicity larger than 1 in a modular invariant partition function often implies that there is more than one distinct field for a given character. However, this is not always true. In this appendix we give some instructive examples where the interpretation of such a modular invariant partition function is different. Nevertheless, in all cases there is an interpretation in terms of a diagonal invariant of an extended chiral algebra, in agreement with the proof of [4].

The first example occurs for $SU(4)$ level 4. The following combination of characters is modular invariant

$$|\chi_{000} + \chi_{012} + \chi_{040} + \chi_{210}|^2 + |\chi_{004} + \chi_{400} + \chi_{101} + \chi_{121}|^2 + 4|\chi_{111}|^2, \quad (\text{B1})$$

where the subscripts are Dynkin labels. The factor 4 might suggest four distinct fields, but this interpretation is not correct. The easiest way to see that it cannot be correct is to compute the matrix elements S_{00} and S_{0a_i} , where 0 denotes the identity, and $a_i, i = 1, \dots, 4$ one of the four distinct fields. It turns out that $S_{00} = \frac{1}{2}$, and $S_{0a_i} = \frac{1}{4}\sqrt{2} < S_{00}$. However in a unitary CFT S_{0j} must be *larger* than S_{00} [13]. In addition the fixed point violates our conjecture, since $h - \frac{c}{24} = \frac{5}{8}$.

The correct answer is not hard to guess: instead of resolving the fixed point, one should absorb the factor of 4 into the character. Thus there is just one primary field for this fixed point, and its character is $2\chi_{111}$. since there is no need to resolve the fixed point, there is no conformal field theory (not even a trivial one) associated with it, and therefore our conjecture does not apply.

The new extended algebra is in fact nothing but $SO(15)$, level 1, and the example corresponds to the conformal sub-algebra $SO(15) \supset SU(4)$. It is easy to check that our previous interpretation is indeed the correct one, since the $SO(15)$ spinor has dimension 128, whereas the representation $(1, 1, 1)$ of $SU(4)$ has dimension 64.

Another way to arrive at the correct answer is to observe that the modular invariant under consideration can be thought of as a “secondary” invariant, built on top of the one generated by the simple current $(0, 4, 0)$ (note that $(2, 1, 0)$ is not a simple current, so that

(B1) is an exceptional invariant). The simple current invariant has the form

$$\begin{aligned} & |\chi_{000} + \chi_{040}|^2 + |\chi_{210} + \chi_{012}|^2 + |\chi_{002} + \chi_{220}|^2 + |\chi_{200} + \chi_{022}|^2 \\ & + |\chi_{400} + \chi_{004}|^2 + |\chi_{011} + \chi_{121}|^2 + |\chi_{010} + \chi_{030}|^2 + |\chi_{103} + \chi_{301}|^2 \\ & + 2|\chi_{020}|^2 + 2|\chi_{202}|^2 + 2|\chi_{111}|^2. \end{aligned}$$

The three fixed points in this expression have to be resolved, and this can be done without problems. The resulting new S -matrix admits a modular invariant partition function corresponding to (B1). It turns out that after resolving the primary fixed points the two fields corresponding to χ_{111} belong to the same multiplet of the secondary extended algebra. Hence all multiplets consist now of two characters of the intermediate theory, and a fixed point problem never arises. This also demonstrates the advantage of extending the chiral algebra step-by-step, rather than looking directly at the final result.

A more complicated example occurs for $SU(5)$, level 5. Here we expect to find at least two modular invariant partition functions, namely one generated by the simple current $(5, 0, 0, 0)$, and an exceptional invariant corresponding to the conformal embedding $SU(5) \subset SO(24)$. The former is easy to find, and is built out of the character combinations

$$\begin{aligned} \chi_1 &= \chi_{0000} + \chi_{5000} + \chi_{0500} + \chi_{0050} + \chi_{0005} \\ \chi_2 &= \chi_{0102} + \chi_{2010} + \chi_{0220} + \chi_{1022} + \chi_{2201} \\ \chi_3 &= \chi_{0013} + \chi_{3100} + \chi_{0131} + \chi_{1310} + \chi_{1001} \\ \chi_4 &= \chi_{0021} + \chi_{2100} + \chi_{0212} + \chi_{2120} + \chi_{2002} \\ \chi_5 &= \chi_{0110} + \chi_{0301} + \chi_{1030} + \chi_{1103} + \chi_{3011} \\ \chi_6 &= \chi_{1111} \end{aligned}$$

These orbits can easily be derived using the fusion rules of the simple current $(5, 0, 0, 0)$, and include all primary fields with charge ("pentiality") equal to zero. The modular invariant partition function is

$$\sum_{i=1}^5 |\chi_i|^2 + 5|\chi_6|^2. \quad (\text{B2})$$

With some more work one can obtain a second modular invariant partition function which

looks like a simple sum of squares, namely

$$|\chi_1 + \chi_2|^2 + 2|\chi_3|^2 + 10|\chi_6|^2. \quad (\text{B3})$$

Although this expression looks quite reasonable, it is not the partition function corresponding to the conformal embedding, and presumably not the partition function of *any* conformal field theory!

To understand what is going on we begin by resolving the fixed point of the primary modular invariant (B2). This allows us to compute the fusion rules of this theory, and we find that these fusion rules have an automorphism. This implies [13][4] that there exists yet another modular invariant partition function. In fact there are two new ones, since both (B2) and the secondary invariant (B3) can be twisted. These partition functions are

$$|\chi_1|^2 + |\chi_2|^2 + |\chi_4|^2 + |\chi_5|^2 + (\chi_3\chi_6^* + \text{c.c.}) + 4|\chi_6|^2 \quad (\text{B4})$$

and

$$|\chi_1 + \chi_2|^2 + 2|\chi_3\chi_6|^2 + 8|\chi_6|^2. \quad (\text{B5})$$

The latter is obtained by multiplying the matrices M representing (B3) and (B4). Partition function (B4) is part of a short sequence that also includes the E_7 invariant of $SU(2)$, level 16, and the automorphism invariant of $SU(3)$ level 9 found in [4]. The common feature of these three invariants is that the value of $h - \frac{c}{24}$ of the fixed point field is equal to one. Indeed, from (A1) one finds that for $SU(N)$ Kac-Moody algebras with single fixed points (*i.e.* N prime, $l=1$) the value of $h - \frac{c}{24}$ for the fixed point is $k(N^2 - 1)/24N$, which is equal to one only if $N = 2, k = 16$ or $N = 3, k = 9$ or $N = 5, k = 5$. The significance of this is not clear to us.

Although we now have four modular invariant partition functions, it turns out that still none of them corresponds to the conformal embedding. To get it, one has to add (B3) and (B5) and divide the result by two. The answer can be written as

$$\begin{aligned} & |\chi_1 + \chi_2|^2 + |\chi_3|^2 + |\chi_3\chi_6|^2 + 9|\chi_6|^2 \\ & = |\chi_1 + \chi_2|^2 + |\chi_5 + \chi_6|^2 + 2|2\chi_6|^2 \end{aligned} \quad (\text{B6})$$

As this way of writing the result suggests, the partition function does indeed have an interpretation in terms of 4 primary fields, with h -values $0, \frac{1}{2}$ and twice $\frac{1}{2}$. The ground

states have dimension 1, 24 and twice 2048 respectively. Thus this must be the partition function corresponding to the conformal embedding $SU(5) \subset SO(24)$.

Again this interpretation becomes more sensible when one considers the result as a secondary invariant. After resolving the fixed point of (B2) in the standard way one obtains first a theory with ten primary fields, with characters $\mathcal{X}_i, i = 1, \dots, 5$ and 5 copies of $\mathcal{X}_6$, denoted $\mathcal{X}_6^j, j = 1, \dots, 5$. This theory has a secondary modular invariant

$$|\mathcal{X}_1 + \mathcal{X}_2|^2 + |\mathcal{X}_3 + \mathcal{X}_6^1|^2 + |\mathcal{X}_6^2 + \mathcal{X}_6^3|^2 + |\mathcal{X}_6^4 + \mathcal{X}_6^5|^2.$$

Of course all permutations of the labels of $\mathcal{X}_6$ are allowed as well. Note that as in the previous example, there is nothing unusual about this invariant. It is simply an integer spin modular invariant without a fixed point. Thus in this way one immediately gets the $SO(24)$ partition function in the correct form.

It is easy to check that the intermediate theory has a second modular invariant namely

$$|\mathcal{X}_1 + \mathcal{X}_2|^2 + 2|\mathcal{X}_3|^2 + 2 \sum_j |\mathcal{X}_6^j|^2. \quad (B7)$$

Since the fixed point multiplicities are 2, there seems to be no other choice but to resolve each one into two fields. This is easy, since the matrix ρ (T restricted to the fixed points) is 1, so that $S = 1$ (note that this S is not the S -matrix of a conformal field theory, although one could interpret it as the direct sum of six trivial S -matrices). However, one readily discovers that this answer leads to non-integer fusion rules, and that furthermore the inequality $S_0 > S_{00}$ is violated. Our conclusion is therefore that (B7) (or equivalently (B3)) does not correspond to a conformal field theory.

This still leaves (B5) unexplained, but this partition function has a simple interpretation as a tertiary modular invariant, built (therefore) out of the $SO(24)$ characters $\mathcal{X}_0 = \mathcal{X}_1 + \mathcal{X}_2$, $\mathcal{X}_v = \mathcal{X}_3 + \mathcal{X}_6^1$, $\mathcal{X}_s = \mathcal{X}_6^2 + \mathcal{X}_6^3$ and $\mathcal{X}_c = \mathcal{X}_6^4 + \mathcal{X}_6^5$. The primary fields corresponding to the last three characters are half-integer spin simple currents, and can be used to build new modular invariants. In particular, using the $SO(24)$ spinors one gets a modular invariant of the form $(0, 0) + (v, s) + (s, v) + (c, c)$. When written back in terms of $SU(5)$ characters this is precisely (B5). So here we have a half-integer spin simple current invariant built on top of an exceptional invariant, which in its turn is built on top of an integer spin simple current invariant.

Note that the modular invariant (B3) is a linear combination of (B5) and (B6), both of which have a sensible interpretation. This is probably the only correct way of interpreting (B3) itself. This example serves thus as an explicit reminder that not everything that looks like a regular modular partition function corresponds to a conformal field theory. It is perhaps rather surprising that a linear combination of two positive modular invariants M_i with $M_{00} = 1$ can be another invariant with those properties, but obviously this possibility exists.

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