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Pseudoscalars in the Instanton Liquid Model[†]

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ABSTRACT

The masses and decay constants of the nonet of pseudoscalar mesons are investigated in the instanton liquid model of the QCD vacuum using the long wavelength approximation. The instanton induced interactions between the light quarks yield conservation laws for the flavor currents that are consistent with QCD including anomalies. We show that the induced anomaly in the flavor singlet current gives rise to an η' mass that is in agreement with the Witten-Veneziano relation. The $\eta - \eta'$ mixing matrix coincides with the result derived previously by Veneziano.

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Recently there has been a considerable interest in the instanton picture of the QCD vacuum. Due to quantum disorder interacting instantons and anti-instantons in the vacuum stabilize in a dilute liquid state free of the long standing infrared problem [1,2]. The liquid state consists of an amorphous network of instantons and anti-instantons constantly absorbing and emitting light quarks of different flavors. The amorphous structure leads to a delocalization of the fermionic zero modes and to the spontaneous breakdown of chiral symmetry.

Detailed numerical investigations have been carried out to assess quantitatively these effects [3,4]. For a vacuum state with a realistic number of flavors and quark masses the instantons are in a liquid state [5] for which the chiral symmetry is spontaneously broken. For decreasing instanton density (*e.g.* by increasing the temperature [6,7]) the amorphous structure breaks apart in favor of sparse molecular configurations. This state lacks flavor coherence and preserves the chiral symmetry. The spontaneous breakdown of chiral symmetry in the instanton liquid phase is reminiscent to the metal-insulator transition (Mott-transition) in solids.

In his pioneering work on instantons 't Hooft [8] pointed out that at scales larger than the instanton size, the effects of a single instanton on the light quarks can be well described in terms of an effective *albeit* nonlocal interaction. Following this suggestion we were able to derive an effective action that describes light quarks in the instanton liquid picture [4]. Our construction was based on a generalization of the one instanton partition function, and at first we ignored fluctuations in the instanton density.

Recently, by including bulk correlations in the liquid state and proper coarse graining, we have shown that in the long wavelength limit our effective action is consistent with generic effective actions obtained from large N_c arguments [9]. In our case, the scalar glueball field is identified with the local variations in the density of instantons in the vacuum, and the pseudoscalar glueball field is generated by allowing for local variations in the topological susceptibility. Our effective action satisfies both the chiral and scale anomaly. The latter is directly related to the

compressibility of the instanton liquid state.

In this letter we will derive the masses and decay constants of the lowest pseudoscalar excitations using our effective action. In particular, we will show that in the chiral limit the π , K and η -particles are Goldstone modes in the liquid state, as first noted by Diakonov and Petrov [10], whereas the η' -particles are heavy due to the explicit breaking of the $U_A(1)$ anomaly by instantons. In the presence of current masses, the bulk parameters of the liquid state yield a pseudoscalar nonet spectrum in good agreement with the data.

A fermion propagating in the liquid state can be either trapped in a zero mode state (ϕ_I) of a given instanton (anti-instanton) or undergo background field scattering to the next anti-instanton (instanton). Since the instanton field is short ranged the scattering part can be approximated by a free propagator (S_0). If we were to assume that instantons and anti-instantons in the liquid state are distributed according to their invariant measure¹ then the Euclidean vacuum partition function $Z[0]$ associated to the propagation of a fermion can be written as follows

$$Z[0] = \int d\psi d\psi^\dagger \exp\left(-\int \psi^\dagger S_0^{-1} \psi\right) < \prod_{I'} \left(1 - \frac{1}{2im} \int \psi^\dagger S_0^{-1} \phi_I \int \phi_I^\dagger S_0^{-1} \psi\right) >, \quad (1)$$

where m is the current quark mass matrix, and the average is meant to be over the collective coordinates (position, color and size) of the instantons and anti-instantons. By using a proper coarse graining for the product over the pseudoparticles and by invoking a cumulant expansion for the averages, the fermionic action can be written as

$$S = \int d^4z (\psi^\dagger S_0^{-1} \psi - n^+(z) \log \det_+(z) - n^-(z) \log \det_-(z)), \quad (2)$$

¹Ignoring induced fermionic correlations and pseudoparticle interactions.

where the smeared 't Hooft determinants are given by ²

$$\det_{\pm} = \frac{1}{N_f!} \det_{I\theta} \left(m_{\bar{\psi}} - \frac{\bar{\rho}}{2i} < \int \psi^\dagger S_0^{-1} \phi_{\pm} \int \phi_{\pm}^\dagger S_0^{-1} \psi^{\theta} >_{U_{\theta}} \right). \quad (3)$$

The brackets $< \dots >_{U_{\theta}}$ denote averaging over the gauge group and the normalized size distribution. Since the zero modes have definite chirality, the determinantal factors break explicitly the $U_A(1)$ symmetry (leaving invariant the remaining flavor transformations) and induce nonlocal interactions between fermions on the scale of the size of the instantons. At the mean-field level they also induce the spontaneous breakdown of chiral symmetry. Notice that for an equal density $n_+(z)$ of instantons and density $n_-(z)$ of anti-instantons the partition function (1) involves exclusively the $U_L(3) \times U_R(3)$ invariant combination $\det_+ \det_-$. By allowing for fluctuations in the number of pseudoparticles (as in a grand canonical description for instance) the $U_A(1)$ breaking combination $\det_+ / \det_-$ appears leading to the explicit breaking of the symmetry in the vacuum.

To describe the long wavelength pseudoscalar excitations, we can either study source-source correlators in the vacuum described by (1), or equivalently bosonize the fermionic degrees of freedom in (1). To achieve this we use the completeness relation

$$1 = \int d\pi^{\pm} \int dP^{\pm} \exp -i \int P^{\pm} (\pi^{\pm} - < \psi^\dagger S_0^{-1} \phi_{\pm} \phi_{\pm}^\dagger S_0^{-1} \psi >), \quad (4)$$

where $P^{\pm}$ are the auxiliary fields associated to the pseudoscalar fields $\pi^{\pm}$. Notice that $\psi^\dagger$ and ψ are independent grassmannian fields. Therefore the $P^{\pm}$ need not to be hermitean and in general they are $N_f \times N_f$ complex matrices. The fields $\pi^{\pm}$ are eliminated with the help of the mean field equations for fixed values of $P^{\pm}$. In this way the gaussian fluctuations in $\pi^{\pm}$ are included in the partition function.

After bosonization the fermionic part of the effective action takes the following

form

²Note that there is a printing error in eq. (24) of ref. [9]. The averaging brackets should be around the second term inside the determinant only.

$$S_F = -N_c \sum_f \text{Tr} \log \left(\hat{k} + im + iP^+(k - l) \left(1 + \frac{im\hat{k}}{k^2} \right) \gamma_5^+ \left(1 + \frac{im\hat{l}}{l^2} \right) \sqrt{M_k M_l} \right. \\ \left. + iP^-(k - l) \left(1 + \frac{im\hat{k}}{k^2} \right) \gamma_5^- \left(1 + \frac{im\hat{l}}{l^2} \right) \sqrt{M_k M_l} \right). \quad (5)$$

The trace is over the Dirac indices and the four momenta. In order to distinguish the Goldstone modes from the massive modes we consider eq. (5) for zero quark masses. Then the action is invariant under the global axial (at this point we do not consider the invariance under global vector transformations) transformation³

$$P^+ \rightarrow e^{+\frac{i\kappa}{2}} P^+ e^{+\frac{i\kappa}{2}}, \\ P^- \rightarrow e^{-\frac{i\kappa}{2}} P^- e^{-\frac{i\kappa}{2}}. \quad (6)$$

This suggests the parametrization

$$P^{\pm} = e^{\pm \frac{i\kappa}{2}} \sigma e^{\pm \frac{i\kappa}{2}}, \quad (7)$$

where κ and σ are $N_f \times N_f$ hermitean matrices. The κ -variables can be identified as the pseudoscalar Goldstone modes. The σ -variables are massive and can be eliminated with the help of the saddle point equations. To lowest order the saddle point solution $\bar{\sigma}$ does not depend on the quark masses. We can set $\bar{\sigma} = 1$ after a suitable redefinition of the constituent mass M_k in (5). To extract the mass term in (5) we choose to redefine $P^{\pm} \rightarrow e^{\pm \frac{i\kappa}{2}} P^{\pm} e^{\pm \frac{i\kappa}{2}}$. As a result the contribution to (5) to order κ^2 and m depends solely on the contribution:

$$\text{Tr} \left(m \left(1 - \frac{1}{2} \kappa^2 \right) \right). \quad (8)$$

This expression can be rewritten as

³No anomaly is expected in the absence of gauge couplings.

where the quark condensate and the pseudoscalar decay constant f

$$f^2 = \lim_{q^2 \rightarrow 0} \frac{1}{q^2} 4N_c \int \frac{d^4 k}{(2\pi)^4} \frac{(M(k_+)k_- - M(k_-)k_+)^2}{(k_+^2 + M^2(k_+))(k_-^2 + M^2(k_-))} \quad (13)$$

$$k_{\pm} = k \pm q/2$$

are evaluated in the chiral limit. A similar result has been derived in [10]. For π and K excitations we have the following

$$m_{\pi}^2 = -2m < \bar{\psi}\psi > / f^2 \quad (14)$$

$$m_K^2 = -(m + m_s) < \bar{\psi}\psi > / f^2 \quad (15)$$

We have explicitly used the relation $< \bar{\psi}\psi > = -i < \psi^{\dagger}\psi >_E$ between Euclidean and Minkowski condensates. To account for the mass of the η' we have to add to (12) the gluonic part due to the fluctuations in the topological susceptibility. In Ref. [9] we have shown that the terms proportional to the difference $n^+(z) - n^-(z)$ in the pseudoparticle densities are given by

$$S_G = \frac{V}{2N} \int d^4 z (n^+(z) - n^-(z))^2 - i\sqrt{2N_f} \int d^4 z (n^+(z) - n^-(z))\eta'. \quad (16)$$

This gives rise to a gluonic contribution to be added to the fermionic contribution (5). Integrating out the pseudoparticle difference gives the term

$$N_f \frac{N}{V} \eta'^2. \quad (17)$$

After adding this to (12) and substituting (14) and (17) into (12), we obtain the following result for the second order expansion in the η, η' -fields of our effective action (momentum integration understood)

$$\begin{aligned} & \frac{1}{2} f^2 p^2 (\eta^2 + \eta'^2) + \frac{1}{2} 2N_f \frac{N}{V} \eta'^2 + \\ & \frac{1}{2} f^2 \eta^2 \left(\frac{4}{3} m_K^2 - \frac{1}{3} m_{\pi}^2 \right) + \frac{1}{2} f^2 \eta'^2 \left(\frac{2}{3} m_K^2 + \frac{1}{3} m_{\pi}^2 \right) + \frac{1}{2} f^2 \eta \eta' \frac{4\sqrt{2}}{3} (m_{\pi}^2 - m_K^2). \end{aligned} \quad (18)$$

$$\begin{aligned} & 2m + m_s \\ & - \frac{1}{2} \left(\frac{4m}{3} (\kappa_0 + \frac{1}{\sqrt{2}} \kappa_8)^2 + \frac{2m_s}{3} (\kappa_0 - \sqrt{2} \kappa_8)^2 \right) \\ & + 2m \sum_{k=1,2,3} \kappa_k^2 + (m + m_s) \sum_{k=4,5,6,7} \kappa_k^2, \end{aligned} \quad (9)$$

where we have used the decomposition $\kappa = \sum_{k=0}^8 \kappa_k \lambda_k$ ($\text{Tr} \lambda_a \lambda_b = 2\delta_{ab}$).

For zero momentum pseudoscalar excitations (constant κ_I) we can immediately write down the relations (to order m and κ^2)

$$\begin{aligned} \frac{\partial^2 S_F}{\partial \kappa_k^2} &= -m \frac{\partial S_F}{\partial m}, \quad k = 1, 2, 3 \\ \frac{\partial^2 S_F}{\partial \kappa_k^2} &= -\frac{(m + m_s)}{2} \frac{\partial S_F}{\partial m}, \quad k = 4, 5, 6, 7, \end{aligned} \quad (10)$$

and similar relations involving the κ_0 and κ_8 fields. The derivatives with respect to m are related to the Euclidean condensate by

$$\frac{\partial S_F}{\partial m} = -i < \psi^{\dagger}\psi > \quad (11)$$

from which we immediately obtain the PCAC relation. Note that to order m the chiral condensate is flavor independent.

To investigate the momentum content of the pseudoscalars consider first the κ_0 and κ_8 excitations, which can be identified as the η' and the η field, respectively. Inserting (7) in (5) and expanding around zero momentum gives (momentum integration understood)

$$\begin{aligned} & \frac{1}{2} f^2 p^2 (\eta^2 + \eta'^2) \\ & + i \frac{< \psi^{\dagger}\psi >_E}{2} \left(\eta^2 \left(\frac{2}{3} m + \frac{1}{3} m_s \right) + \eta'^2 \left(\frac{1}{3} m + \frac{2}{3} m_s \right) + \eta \eta' \frac{2\sqrt{2}}{3} (m - m_s) \right). \end{aligned} \quad (12)$$

⁵When we differentiate with respect to m , we take into account that m is the average of m_u and m_d . By the derivative with respect to m we mean the derivative either with respect to m_u or with respect to m_d .

This result coincides with the result obtained by Veneziano [11]. In particular, for current zero quark masses, we obtain the Witten-Veneziano [11,12] formula. At this point, we disagree with the analysis of [10] concerning the $U(1)$ problem.⁶ The present analysis shows the importance of the fluctuations in the difference between the number of instantons and anti-instantons (topological susceptibility) in generating the mass of the η' and thus solving the $U(1)$ problem.

Using the standard instanton liquid model parameters $N/V = 1\text{fm}^{-4}$ and $\bar{\rho} = \frac{1}{3}\text{fm}$ we obtain for the constituent mass $M(0) = 345\text{ MeV}$, for the condensate $\langle \bar{\psi}\psi \rangle = (255\text{ MeV})^3$ and for the decay constant $f = 91\text{ MeV}$. The current masses $m = 5\text{ MeV}$ and $m_\pi = 120\text{ MeV}$ reproduce the experimental values of m_π and m_K . Diagonalization of the mass matrix given in (18) yields $m_\eta = 527\text{ MeV}$ (exp.: 549 MeV) and $m_{\eta'} = 1172\text{ MeV}$ (exp.: 958 MeV). The corresponding mixing angle comes out to be $\theta = -11.5^\circ$ (exp.: $\approx -20^\circ$ [13]).

We have shown that the effective action for light quarks derived from the instanton liquid picture of the QCD vacuum leads to the proper description of low momentum pseudoscalar excitations. In particular, the spontaneously broken vacuum state yields the correct PCAC relations to leading order in the current quark masses. The gaussian fluctuations in the number difference between instantons and anti-instantons lead to an η' mass in agreement with the Witten-Veneziano relation. In the long wavelength limit and to leading order in the current mass our effective action reduces to the effective action used by Veneziano. The bulk instanton liquid parameters yield masses and decay constants that are in good agreement with the data.⁷ It all seems to indicate that an instanton liquid description of the QCD vacuum provides a fair account of low momentum pseudoscalar excitations.

⁶Indeed, for $N_f > 1$ their result leads to tachyonic singlet excitation. They circumvented the problem by taking $N_f \rightarrow 0$.

⁷We have used here the value of the decay constant in the chiral limit. Taking into account the finite current masses for the η' decay constant would improve the agreement with the data. Details will be given elsewhere.

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