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QUATERNIONIC MANIFOLDS FOR TYPE II-SUPERSTRING VACUA
OF CALABI-YAU SPACES

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ABSTRACT. In type II theories compactified on Calabi-Yau manifolds the moduli fields are accompanied by (Ramond-Ramond) scalar superpartners. $N = 2$ space-time supersymmetry requires the low energy effective mutual interaction of these excitations be described by a quaternionic σ -model in target space. In this paper we explicitly construct these manifolds from the restricted Kähler geometry of the moduli space of arbitrary $(2,2)$ superconformal systems.

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1. Introduction.

Recently it has been shown^{1,2,3,4} that the metric of the moduli space of superstring compactified on Calabi-Yau vacua⁵ has a restricted Kähler structure as implied by $N = 2$ space-time supersymmetry^{6,7} when these manifolds are thought of as vacua of type II superstrings.^{8,9} These restrictions have also been recently derived² (for superstrings compactified on arbitrary internal $(2,2)$ superconformal field theories¹⁰) by exploiting the Ward identities of $N = 2$ world-sheet supersymmetry on the four-point function of the moduli fields which is related to the curvature tensor of the moduli space.

$N = 2$ space-time supersymmetry requires the moduli space, related to deformations of the Kähler class $\{(1,1)$ harmonic forms $\}$ and of the complex structure $\{(2,1)$ form $\}$ to be a product space^{3,1,9}

$$M = M_A \times M_B$$

with restricted Kähler potentials of the form¹

$$K_{A(B)} = -\log Y_{A(B)} \quad \left\{ \begin{array}{l} A = (1,1) \text{ moduli} \\ B = (2,1) \text{ moduli} \end{array} \right.$$

with $Y_{A(B)}$ given, in a certain choice of coordinates for the moduli, by^{6,7}

$$Y_{A(B)} = f_{A(B)} + \bar{f}_{A(B)} - \frac{1}{2} \left(\frac{\partial f_{A(B)}}{\partial \phi_{A(B)}} - \frac{\partial \bar{f}_{A(B)}}{\partial \bar{\phi}_{A(B)}} \right) \cdot \left(\phi_{A(B)} - \bar{\phi}_{A(B)} \right).$$

*In this paper we will (somewhat improperly) use the word Calabi-Yau and their moduli space for generic $(2,2)$ internal superconformal field theories.

$f_A(B)$ ($\Phi_A(B)$) are two holomorphic functions of the complex moduli fields $\Phi_A(B)$ which determine all low-energy couplings of heterotic¹¹ strings as well as type II strings¹² compactified on an arbitrary (2,2) systems.

In the field theory limit the functions $Y_A(B)$ are given by^{1,3,4}

$$Y_A = V = \int J \wedge J \wedge J; \quad Y_B = i \int \Omega \wedge \bar{\Omega},$$

where V is the volume of the Calabi-Yau manifold, J is the Kahler form as a function of the (1,1) moduli and Ω is the holomorphic (3,0) form as a function of the (2,1) moduli.

In type II theories, due to the occurrence of extra massless scalar fields (Ramond-Ramond scalars), the Kahler spaces of the moduli are enlarged to quaternionic spaces,^{8,1} as required by $N = 2$ space-time supersymmetry.^{6,7,13}

This phenomenon occurs when the moduli fields are members of $N = 2$ hypermultiplets, which is the case for the (2,1) moduli in type IIA strings and for the (1,1) moduli in type IIB strings.

If one also includes in the sector of massless excitations the dilaton and the space-time axion (four-dimensional antisymmetric tensor), the complete manifolds are^{8,1}

$$M = \frac{SU(1,1)}{U(1)} \times M_A \times M_B$$

in heterotic strings,

$$M' = \tilde{M}_A \times Q_B$$

in type IIA strings,

$$M'' = Q_A \times M_B$$

in type IIB strings, where M_A and M_B are restricted Kahler manifolds of complex dimension $h_{(1,1)}$, $h_{(2,1)}$ respectively and Q_A , Q_B are quaternionic manifolds of (real) dimension $4(h_{(2,1)} + 1)$, $4(h_{(1,1)} + 1)$ which contain, as submanifolds, the Kahler manifolds $\frac{SU(1,1)}{U(1)} \times M_B$ and $\frac{SU(1,1)}{U(1)} \times M_A$ respectively. The $\frac{SU(1,1)}{U(1)}$ part refers to the dilaton, axion sector. The relation between these manifolds are called c and s -map in Ref. 1. These quaternionic manifolds were called dual quaternionic manifolds in Ref. 1 where it was pointed out that their metric should be completely determined in terms of the same holomorphic functions $f_A(B)$ which determine their Kahler submanifolds $M_A(B)$ (s -map). Particular cases of such quaternionic manifolds which correspond to symmetric or homogeneous Kahler manifolds have been discussed in the literature.^{14,15}

Although the existence of such manifolds was established, no explicit construction of these spaces have been carried out yet. In Ref. 1 the simpler problem of determining the s -map for rigid supersymmetry was solved. In that case the dual hyper-Kahler manifolds of dimension $4n$, from complex (rigid) Kahler manifolds of dimension n were obtained.

In the present paper we solve the problem of the s -map in local supersymmetry. The result is not only of physical but also of mathematical interest in view of the fact that it allows the construction of continuous families of quaternionic manifolds, allowed by $N = 2$ supergravity (Einstein spaces of negative curvature with a specific value), whose geometry in a certain coordinate system, depends entirely on the very same holomorphic function of the restricted Kahler manifolds of the moduli space of (2,2) superconformal theories.¹⁻⁴

We anticipate some of the properties shared by all these dual manifolds.

Let us call the preferred coordinate systems for these manifolds (z^a, C_i, S) where z^a ($a = 1, \dots, n$) are the original coordinates of the restricted Kahler manifold, (moduli fields) C_i ($i = 1, \dots, n+1$) are the Ramond-Ramond scalars and S is a complex field related to the dilaton and the (space-time) axion.

- (a) At each point of the moduli space $z^a = z^{a^0}$ ($\partial_\mu z^{a^0} = 0$), the Ramond-Ramond scalar parametrizes a $\frac{SU(1, n+1)}{U(1) \times SU(n+1)}$ manifold.
- (b) At each point ($\text{Re } C_i = 0$, $\text{Im } C_i = \text{Im } C_i^0$) ($\partial_\mu C_i = 0$) the (z^a, S) fields parametrize a $\frac{SU(1, 1)}{U(1)} \times M_n$ manifold when M_n is the original Kahler manifold. This is the Kahler manifold of the heterotic string when the charged matter fields are set to zero.
- (c) The quaternionic manifold $Q_4(n+1)$ has at least $2n+4$ isometries acting on all coordinates but the z^a coordinates.
- (d) The dual quaternionic manifolds are Einstein spaces with negative curvature $R = -8(n+3)(n+1)$.
- (e) Those moduli which have vanishing Yukawa couplings¹⁶ together with their (Ramond) $N = 2$ partners form a Kahler quaternionic submanifold $SU(2, n')/SU(2) \times SU(n') \times U(1)$ of the original (non-Kahler) quaternionic manifold.

2. Properties of Dual Quaternionic Spaces.

Let us anticipate a particular result stated in Ref. 1. In the case of symmetric quaternionic spaces, all of them are s-maps of symmetric Kahler spaces with the exception of the $Sp(n+1)/Sp(1) \times Sp(n)$ series. All symmetric¹⁵ restricted (or homogeneous)¹⁹⁻²¹ Kahler manifolds correspond to a holomorphic function f which is a polynomial of degree two or degree three in the z .¹⁵ These particular polynomials lead to a vanishing (for quadratic functions) or constant (for cubic functions) Yukawa coupling¹⁶ in heterotic strings.^{1,2,9} In particular, vanishing Yukawa coupling means that the f function is quadratic and in that case the restricted Kahler manifold is always $SU(1, n)/U(1) \times SU(n)$ and the dual quaternionic manifold $SU(2, n+1)/SU(2) \times SU(n+1) \times U(1)$ which is also Kahler.¹ These results have been confirmed in special cases in Refs. 17, 1, and 7, where the moduli metric was computed for orbifolds and for Calabi-Yau spaces obtained by tensoring several copies of the $N = 2$ minimal series.²

Now let us consider the general case in which, for arbitrary choice of the holomorphic function $f(z)$ the restricted Kahler manifold is not symmetric nor homogeneous. This is of course the case of interest for generic Calabi-Yau spaces.

The metric of the dual quaternionic manifold is derived^{1,14} by performing a dimensional reduction from $D = 4$ to $D = 3$ dimensions of $N = 2$ supergravity coupled to n -vector multiplets with holomorphic function $f(z^a)$ ($a = 1, \dots, n$).

The $N = 2$ Lagrangian for vector multiplets is^{6,7} (bosonic part)

$$e^{-1} \mathcal{L} = \frac{1}{2} R - \kappa_{ab} \partial_\mu z^a \partial^\mu z^b + \frac{1}{4} \text{Re } N_{ij} F_{\mu\nu}^i F^{\mu\nu j} + \frac{1}{4} \text{Im } N_{ij} F_{\mu\nu}^i F^{\mu\nu j} \quad (2.1)$$

with

$$N_{ij} = \frac{1}{4} \bar{F}_{ij} - \frac{(NZ)_i (NZ)_j}{(ZNZ)} \quad (2.2)$$

$$N_{ij} = \frac{1}{4} (F_{ij} + \bar{F}_{ij}) \quad (2.3)$$

$$K = -\ell n \bar{Z}NZ = -\ell n \left[f + f^* - \frac{1}{2} (Z^a - Z^{*a}) (f_a - f_a^*) \right] \quad (2.4)$$

F is a homogeneous holomorphic function of degree two in $n+1$ variables x^i ($Z^i = x^i/x^0$).

$$F(\lambda x) = \lambda^2 F(x) \quad \text{and} \quad f(z) = x^{0-2} F(x) \quad (2.5)$$

Dimensional reduction from $D = 4$ ($N = 2$) to $D = 3$ ($N = 4$) is obtained using a triangular gauge for the vierbein

$$e_{\mu}^{\hat{a}} = \begin{bmatrix} e_{\mu}^{\hat{a}} & 0 \\ 1/2 B_{\mu} & 1/2 \end{bmatrix}, \quad \hat{a}, \hat{\mu} = 1, \dots, 4; \quad a, \mu = 1, \dots, 3 \quad (2.6)$$

and for four-vectors we have

$$\Lambda_{\mu}^{\hat{a}} = (\Lambda_{\mu}^{\hat{a}} + B_{\mu}^{\hat{a}}, \zeta^{\hat{a}}) \quad (\zeta^{\hat{a}} = \Lambda_4^{\hat{a}}) \quad (2.7)$$

The Lagrangian (2.1) reduced to three dimensions, after a Weyl rescaling $e_{\mu}^{\hat{a}} \rightarrow \phi^{-1/2} e_{\mu}^{\hat{a}}$, becomes

$$e^{-1} \mathcal{L} = \frac{1}{2} R - \frac{1}{4\phi^2} (\partial_{\mu} \phi)^2 + \frac{1}{4} \phi^2 H_{\mu}^2 +$$

$$\begin{aligned} & -K_{ab} \partial_{\mu}^2 \partial_{\nu}^b - \frac{\phi}{2} R_{ij} (F_{\mu}^{i+H} \zeta^j) (F_{\mu}^{j+H} \zeta^i) + \frac{1}{2\phi} R_{ij} \partial_{\mu}^i \partial_{\nu}^j \zeta^j \\ & - \text{Im } N_{ij} (F_{\mu}^{i+H} \zeta^j) \partial_{\mu} \zeta^j, \end{aligned} \quad (2.8)$$

where $H_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}$, $H_{\mu} = -\frac{1}{2} \epsilon_{\mu\nu\rho} H_{\nu\rho}$ and $R_{ij} = \text{Re } N_{ij}$. We now use 3D-duality in order to convert 3D-vector fields into scalar fields. For this purpose we add the Lagrange multipliers

$$-F_{\mu}^i \partial_{\nu} \zeta_i + \frac{1}{2} H_{\mu} \partial_{\nu} (\tilde{\phi} - \zeta^i \zeta_i) \quad (2.9)$$

and integrate over the F_{μ}^i , H_{μ} fields to get

$$\begin{aligned} e^{-1} \mathcal{L} = & \frac{1}{2} R - K_{ab} \partial_{\mu}^a \partial_{\nu}^b + \frac{1}{4\phi^2} (\partial_{\mu} \phi)^2 + \frac{1}{4\phi^2} (\partial_{\mu} \tilde{\phi} + \zeta_{\mu}^i \partial_{\nu}^i \zeta_i)^2 \\ & - \left(\frac{1}{2\phi} \right) R_{ij} \partial_{\mu}^i \zeta^j \partial_{\nu}^j \zeta^i \\ & - \frac{1}{2\phi} (\partial_{\mu} \zeta_i + \text{Im } N_{ik} \partial_{\mu}^k \zeta^k) (R^{-1})^{ij} (\partial_{\mu} \zeta_j + \text{Im } N_{jl} \partial_{\mu}^l \zeta^l). \end{aligned} \quad (2.10)$$

Finally, let us define the fields

$$\begin{aligned} C_i &= N_{ij} \zeta^j + i \zeta_i \\ S &= \phi + i \tilde{\phi} - \frac{1}{2} (C+\bar{C})_i (R^{-1})^{ij} (C+\bar{C})_j, \end{aligned} \quad (2.11)$$

then the Lagrangian describing the scalar manifold is

$$e^{-1} \mathcal{L} = -K_{ab} \partial_{\mu}^a \partial_{\nu}^b - \frac{1}{\left[S + \bar{S} + \frac{1}{2} (C+\bar{C}) R^{-1} (C+\bar{C}) \right]^2} x$$

$$\begin{aligned}
& \times \left| \partial_\mu^S + (C+\bar{C}) R^{-1} \partial_\mu^C - \frac{1}{4} (C+\bar{C}) R^{-1} \partial_\mu^N R^{-1} (C+\bar{C}) \right|^2 \\
& + \frac{1}{\left[S + \bar{S} + \frac{1}{2} (C+\bar{C}) R^{-1} (C+\bar{C}) \right]^2} \left(\partial_\mu^C - \frac{1}{2} \partial_\mu^N R^{-1} (C+\bar{C}) \right) \\
& \times R^{-1} \left(\partial_\mu^{\bar{C}} - \frac{1}{2} \partial_\mu^{\bar{N}} R^{-1} (C+\bar{C}) \right). \quad (2.12)
\end{aligned}$$

positivity of the kinetic energy (in Eq. (2.10) required K_{ab}^- and $-R_{ij}$ to be positive definite matrices.

Eq. (2.12) defines a manifold for the $2(n+1)$ complex scalar fields $S, \bar{S}^a$, C_i that according to Ref. 1, should be the dual quaternionic manifold Q_{4n+4} obtained by the s map from the restricted Kähler manifold K_n (specified by the holomorphic function $f(z)$). More specifically, as mentioned in the introduction, the Lagrangian given by Eq. (2.12) (in $D = 4$ dimensions) describes the (low-energy) interactions of moduli fields (z^a), dilaton and axion (S) and the Ramond scalars C_i present in type II superstrings. For a generic compactification on a $(2,2)$ system the z^a moduli refer (in the language of Calabi-Yau manifolds) to deformation of the Kähler class in type IIB strings and to deformation of the complex structure in type IIA strings. The dual quaternionic manifolds have therefore (quaternionic) dimensions $h(2,1)+1$ in type IIA and $h(1,1)+1$ in type IIB, where $h(1,1)$ and $h(2,1)$ are the Hodge numbers of the manifold. It is our aim to prove that the metric given by Eq. (2.12) is quaternionic and discuss its properties.

Let us first discuss properties (a), (b), (c), and (e). An alternative form for the metric given by Eq. (2.12) is

$$e^{-1} \mathcal{L} = -K_{ab}^{-1} \partial_\mu^a \partial_\mu^b \bar{z}^b - \bar{K}_{SS}^{-1} \partial_\mu^S \partial_\mu^{\bar{S}} - \bar{K}_{SC_i}^{-1} \partial_\mu^S \partial_\mu^{\bar{C}_i} - \bar{K}_{C_i S}^{-1} \partial_\mu^{\bar{C}_i} \partial_\mu^{\bar{S}} +$$

$$- \bar{K}_{C_i C_j}^{-1} \partial_\mu^{\bar{C}_i} \partial_\mu^{\bar{C}_j}, \quad (2.13)$$

where

$$D_\mu^C = \partial_\mu^C - \frac{1}{2} \partial_\mu^N R^{-1} (C+\bar{C})$$

$$D_\mu^{\bar{C}} = \partial_\mu^{\bar{C}} + \frac{1}{4} (C+\bar{C}) R^{-1} \partial_\mu^N R^{-1} (C+\bar{C})$$

and

$$\bar{K} = -\ell n \left(S + \bar{S} + \frac{1}{2} (C+\bar{C}) R^{-1} (C+\bar{C}) \right). \quad (2.14)$$

The above equations show that for fixed C ($\partial_\mu^C = 0$) and $C+\bar{C} = 0$ the manifold Q contains the Kähler submanifold $\frac{SU(1,1)}{U(1)} \times K_n$ with coordinates $(S, \bar{S}^a)$, while for fixed $\bar{S}^a$ it contains the Kähler submanifold $\frac{SU(1,n+2)}{U(1) \times SU(n+2)}$ with coordinates (S, C) . * The first manifold is the Kähler manifold for heterotic strings which contains the dilaton, the axion and the moduli fields, the second manifold is the manifold of (Ramond-Ramond) scalars for fixed values of the moduli.

We can also remark that if the matrix $H(Z, \bar{Z})$ is holomorphic, i.e. does not depend on $\bar{Z}$ then the manifold Q is Kähler with Kähler potential $K+\bar{K}$.** In view of Eq. (2.2) this is the case if the function $f(z)$ is a quadratic polynomial^{7,15} and the Kähler quaternionic manifold turns out to be $^{1,13} SU(2,n+1)/SU(2) \times SU(n+1) \times U(1)$.

*The standard metric for the manifold is best seen by making the following (holomorphic for fixed z) field redefinition: $S \rightarrow S - \frac{1}{2} C R^{-1} C$.

**Note that the covariant derivative in Eq. (2.13) just reproduces the terms in the Kähler metric proportional to $\bar{K}_{S\bar{Z}}, \bar{K}_{C\bar{Z}}$ and their hermitian conjugates.

This corresponds to the statement (e) in the Introduction.

Let us now consider another important property of the Q manifold, i.e. its isometries. We may note from Eq. (2.10) that among the $4(n+1)$ scalar fields only $2n+1$ may appear with non-polynomial interactions. These are the true moduli (Z^a , $\text{Re } S$) which correspond to the deformation of the manifolds and the dilaton. These are actually at least $2n+3$ isometries associated with the axion and the Ramond fields

$$S \rightarrow S + i\alpha - 2C\gamma - \gamma\# \gamma$$

$$C \rightarrow C + i\beta + \# \gamma, \quad (2.15)$$

where α, β, γ^i are $2n+3$ real parameters. An additional isometry is the scale-transformation $S \rightarrow \lambda S$, $C \rightarrow \lambda^{1/2} C$. Therefore the Q manifold has at least $2n+4$ isometries.

We may compare these results with the simpler case of rigid supersymmetry described in Appendix B of Ref. 1. In rigid supersymmetry, the $N = 2$ gravity sector (graviton and graviphoton) was missing (this sector contains 4 degrees of freedom) so we had a correspondence between a (rigid) restricted Kahler manifold of complex dimension n with a hyper-Kahler (rather than quaternionic) manifold¹⁸ of (quaternionic) dimension n . Since a hyper-Kahler manifold can be regarded as a limiting case (for vanishing $\text{Sp}(1)$ connection) of a quaternionic manifold¹³ (as in local supersymmetry) the Einstein space structure of the latter is replaced by the Ricci flatness of the former.^{15,1}

3. The Geometry of the Q-Manifolds.

In this section we will prove that the manifold defined by Eq. (2.10) is indeed a quaternionic manifold.

Let us recall that for quaternionic (4d-dimension) manifolds^{13,19-21} there are three locally defined (1,1) tensors $(J^u)^\mu_\nu$ which satisfy the quaternionic algebra

$$J^u \times J^v = -\delta^{uv} + \varepsilon^{uvw} J^w. \quad (3.1)$$

Moreover, the three two-forms*

$$J^u = \frac{1}{2} J^\mu_\nu dx^\mu dx^\nu, \quad ((J^u)^\mu_\nu = g_{\mu\rho} (J^u)^\rho_\nu) \quad (3.2)$$

are covariantly constant with respect to a $\text{Sp}(1)$ connection ω

$$dJ + \omega J - J\omega = 0, \quad J = J^u \sigma^u.$$

The $\text{Sp}(1)$ curvature is proportional to the J two-forms

$$d\omega + \omega\omega = i\lambda J \quad (3.3)$$

for some constant λ .

The holonomy group of a quaternionic manifold is a subgroup of $\text{Sp}(1) \times \text{Sp}(d)$. In addition quaternionic manifolds are Einstein spaces with $R_{\mu\nu} =$

*Wedge product of forms $dx \wedge dy$ will be denoted by $dx dy$.

$2\lambda(d+2)g_{\mu\nu}$. Consistent coupling to supergravity requires¹³ λ to be negative and fixed to -1. So only quaternionic manifolds with negative curvature can be coupled to supergravity. We will see later that this property is automatically satisfied for the Q-manifolds irrespective of the holomorphic function $f(z)$.^{*}

Let us consider the original Kahler manifold K_n with Kahler (closed) two-form given by

$$J = ie^A \bar{e}^{\bar{A}}, \quad (e^A = e_a^A dz^a) \quad (3.4)$$

The Kahler metric is

$$K_{a\bar{b}} = e_a^A (e_{\bar{b}}^{\bar{A}})^* \quad (3.5)$$

It is very convenient to define a $n \times (n+1)$ matrix P_i^A as follows:

$$P_a^A = e_a^A, \quad P_0^A = -e_a^A z_a^A \quad (3.6)$$

Note that P satisfies

$$P \cdot Z = 0 \quad (Z^0 = 1) \quad (3.7)$$

$$P^+ \cdot P = -\frac{1}{2N2} \left(N - \frac{(N2)(\bar{N}2)}{2N2} \right) \quad (3.8)$$

^{*}We will use capital letters for flat indices, small letters for curved indices, initial letters of the alphabet a, A run from 1 up to n while middle letters i, I run from 1 up to n+1.

$$PN^{-1}P^+ = -\frac{1}{2N2} \quad (3.9)$$

The vierbein one forms are then given by

$$e = PdZ$$

$$E = e^{(\tilde{K}-K)/2} PN^{-1} \left(dC - \frac{1}{2} d\bar{H}R^{-1}(C+\bar{C}) \right)$$

$$u = 2e^{(\tilde{K}+K)/2} z \left(dC - \frac{1}{2} d\bar{H}R^{-1}(C+\bar{C}) \right)$$

$$v = e^{\tilde{K}} \left(dS + (C+\bar{C}) R^{-1} dC - \frac{1}{4} (C+\bar{C}) R^{-1} d\bar{H}R^{-1}(C+\bar{C}) \right) \quad (3.10)$$

The Lagrangian for the quaternionic manifold takes the form^{*}

$$-e^{-1} \mathcal{L} = e \otimes \bar{e} + E \otimes \bar{E} + u \otimes \bar{u} + v \otimes \bar{v}$$

$$= \sum_{\alpha=1,2;I=1,\dots,n+1} e^{\alpha I} (e^{\alpha I})^* \quad (3.11)$$

in terms of the $2(n+1)$ component vierbeins

$$e^{\alpha I} = (e^{+I}, e^{-I})$$

$$e^{+I} = \begin{pmatrix} u \\ e^A \end{pmatrix}, \quad e^{-I} = \begin{pmatrix} v \\ E^A \end{pmatrix} \quad (3.12)$$

^{*}The $\otimes$ symbol denotes the sum of the product of components of two one-forms.

To find the connections we compute the exterior derivatives of the vierbein one-forms

$$de = -\omega e$$

$$dv = v\bar{v} + u\bar{u} + E\bar{E}$$

$$du = \left(-\frac{1}{2} (v+\bar{v}) + \frac{\bar{Z}NdZ - ZN\bar{d}\bar{Z}}{2ZNZ} \right) u - \bar{E}e$$

$$dE = \left(-\omega - \frac{1}{2} (v+\bar{v}) - \frac{\bar{Z}NdZ - ZN\bar{d}\bar{Z}}{2ZNZ} \right) E$$

$$- \bar{u}e - \frac{ZNZ}{2} PN^{-1} d(N-iY) N^{-1} P^T E.$$

Here ω is the connection for the original Kähler manifold K_n

$$\omega = -\frac{\bar{Z}NdZ - ZN\bar{d}\bar{Z}}{2ZNZ} + \frac{ZNZ}{2} \{ dPN^{-1}P^+ - PN^{-1}dP^+ - iPN^{-1}dYN^{-1}P^+ \}$$

and

$$N_{ij} + iY_{ij} = \frac{1}{2} F_{ij}.$$

The curvature two-form for K_n is

$$(d\omega + \omega\omega)^A_B = -\delta^A_B e^C e^D - e^A e^C e^D - \frac{f_{ACE} f_{EDB}}{16(ZNZ)^2} e^C e^D$$

with $f_{ABC} = f_{abc} e^a_A e^b_B e^c_C$, $f_{abc} = \frac{\partial}{\partial z^a} \frac{\partial}{\partial z^b} \frac{\partial}{\partial z^c} f(z)$ and $e^a_A = (e^A_a)^{-1}$. Eq. (3.16) is in agreement with Ref. 7.

The connections for the Q manifold are given by

$$de^{\alpha I} + p^\alpha_\beta e^{\beta I} + q^I_J e^{\alpha J} + t^I_J e^{\alpha\alpha'} \overline{e^{\alpha' J}} = 0 \quad (3.17)$$

where

$$p = \begin{bmatrix} \frac{1}{4} (v-\bar{v}) - \frac{1}{4} \frac{\bar{Z}NdZ - ZN\bar{d}\bar{Z}}{ZNZ} & -u \\ -\frac{1}{4} (v-\bar{v}) + \frac{1}{4} \frac{\bar{Z}NdZ - ZN\bar{d}\bar{Z}}{ZNZ} & -\bar{u} \end{bmatrix} \quad (3.18)$$

(p is the Sp(1) connection)

$$q = \begin{bmatrix} -\frac{3}{4} (v-\bar{v}) - \frac{1}{4} \frac{\bar{Z}NdZ - ZN\bar{d}\bar{Z}}{ZNZ} & \bar{E} \\ -E & \omega - \frac{1}{4} (v-\bar{v}) + \frac{1}{4} \frac{\bar{Z}NdZ - ZN\bar{d}\bar{Z}}{ZNZ} \end{bmatrix} \quad (3.19)$$

$$t = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{f_{ABC} E}{4ZNZ} \end{bmatrix}. \quad (3.20)$$

The Sp(1) x Sp(n+1) connection is most easily seen by defining a 4(n+1) component vierbein*

$$v^{\alpha I} = \begin{bmatrix} e^{\alpha I} \\ e^{\alpha\beta} (e^{\beta I})^* \end{bmatrix}. \quad (3.21)$$

The flat space metric is $\frac{1}{2} e_{\alpha\alpha'} \rho_{II'}$, with $\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $v^{\alpha I}$ is covariantly constant

* $v^{\alpha I}$ correspond to $\gamma_i^{AZ} d\phi^i$ in the notation of Ref. 13. They satisfy the reality condition $v^{\alpha I} = \frac{1}{2} (v^{\alpha I})^*$.

$$(d+\Omega) v = 0 \quad (3.22)$$

with connection

$$\Omega = p \times 1_{2(n+1)} + 1_2 \times \begin{pmatrix} q & t \\ -t^+ & -q^+ \end{pmatrix},$$

$$p^+ = -p, \quad q^+ = -q, \quad t^+ = +t,$$

$$\Omega^+ = -\Omega, \quad (3.23)$$

where the two terms respectively correspond to the $Sp(1)$ and $Sp(n+1)$ connections. The curvature two-form for the Q-manifold is

$$R = d\Omega + \Omega \Omega = -i J \times 1_{2(n+1)} + 1_2 \times \tilde{R} \quad (3.24)$$

where $\tilde{R}$ is the $Sp(n+1)$ curvature.

We are mainly interested in the $Sp(1)$ curvature

$$-iJ = dp + p p = \frac{1}{2} \begin{pmatrix} \bar{e}^+ \\ \bar{e}^- \end{pmatrix} \sigma^u \begin{pmatrix} e^+ \\ e^- \end{pmatrix} \cdot \sigma^u$$

or

$$(-iJ)^\alpha_\beta = -v d\Gamma^\alpha_\beta \Gamma^\beta_\gamma \quad (3.25)$$

J^u define the three "complex structures" satisfying the quaternionic algebra given by Eq. (3.1). The construction of the three two-forms given by Eq. (3.25) is the final proof that Q is a quaternionic manifold of quaternionic

dimension $n+1$.

It is of interest to give the expression for the $Sp(n+1)$ curvature as well. This is a $(2(n+1) \times 2(n+1))$ matrix valued two-form

$$\tilde{R} = \begin{pmatrix} r & r' \\ -r' & -r \end{pmatrix}$$

in which r, r' are two $((n+1) \times (n+1))$ matrix valued two forms. Their expression is

$$r^0_0 = -\frac{3}{2} (u\bar{u} + v\bar{v}) - \frac{1}{2} (\bar{e}\bar{e} + \bar{e}\bar{e})$$

$$r^A_0 = -(r^0_A)^* = \bar{u}e^A + \bar{v}e^A$$

$$r^A_B = -\frac{1}{2} \delta^A_B (\bar{e}\bar{e} + \bar{e}\bar{e} + u\bar{u} + v\bar{v}) - e^A_{\bar{e}} \bar{e}^B_{\bar{e}} - e^A_{\bar{e}} \bar{e}^B_{\bar{e}} +$$

$$-\frac{\bar{f}_{ACE} \bar{f}_{EDB}}{16(2NZ)^2} (e^C_{\bar{e}} e^D_{\bar{e}} + \bar{e}^C_{\bar{e}} \bar{e}^D_{\bar{e}})$$

$$r^A_B = \frac{1}{4NZ} \bar{f}_{ABC} (\bar{e}^C_{\bar{e}} + v\bar{e}^C_{\bar{e}})$$

$$+ \frac{1}{16(2NZ)^2} \bar{f}_{ABC} \bar{f}_{CDE} e^D_{\bar{e}} e^E_{\bar{e}} - \frac{1}{4NZ} \frac{1}{4} \bar{X}^A_{BCD} e^C_{\bar{e}} \bar{e}^D_{\bar{e}}$$

$$X^{\bar{a}\bar{b}\bar{c}\bar{d}}_{abcd} = \bar{f}_{abcd} + \frac{(NZ) \bar{f}_{bcd} + (NZ) \bar{f}_{acd} + (NZ) \bar{f}_{abd} + (NZ) \bar{f}_{abc}}{2NZ}$$

$$- \frac{1}{4} (\bar{f}_{abe} \bar{f}_{bcd} + \bar{f}_{ace} \bar{f}_{fdb} + \bar{f}_{ade} \bar{f}_{fbc}) (N^{-1})^{\bar{e}\bar{f}} \quad (3.26)$$

The $Sp(n+1)$ curvature can alternately be written as ¹³

$$(\tilde{R})_{\Gamma}^{\Gamma} = -v^{\alpha\Gamma} v_{\alpha\Gamma'} + \rho^{\Gamma\alpha} \Omega_{\Gamma\Gamma'} v^{\alpha\Gamma''} v_{\alpha\Gamma''} \quad (3.27)$$

where $\Omega_{\Gamma\Gamma'} v^{\alpha\Gamma''} v_{\alpha\Gamma''}$ is completely symmetric due to the Bianchi identity $R^{\Gamma}{}_{\Gamma} = 0$. The quaternionic manifold, which is always an Einstein space, has scalar curvature given by

$$R = -8(n+1)(n+3) \quad (3.28)$$

in agreement with Ref. 13. We remark that the Q-curvature, unlike the K_n curvature (given by Eq. (3.15)), depends, through r' , up to the fourth derivative of the holomorphic function $f(z)$.

4. Some Examples.

In Section 2 we have seen that for a quadratic holomorphic function $F(x)$ the Q manifolds become Kahler. This result can be obtained in a more general way by observing that the two-form

$$e \bar{e} + E \bar{E} + u \bar{u} + v \bar{v} \quad (4.1)$$

is always closed. However, this does not mean in general that the manifold is Kahler unless the vierbeins are holomorphic. The vierbeins are holomorphic if and only if dF is holomorphic which is actually the case only for quadratic F functions. On the other hand the non-holomorphic part of the Q-connection given by t in Eq. (3.24) is proportional to the third derivative of f which consistently shows that for quadratic f , the connection becomes a $SU(2) \times SU(n+1) \times U(1)$ connection, as appropriate for the Kahler quaternionic manifold $\frac{SU(2, n+1)}{SU(2) \times SU(n+1) \times U(1)}$. In view of the relation between the third derivative of f and the Yukawa couplings in heterotic superstrings the previous result implies that those moduli fields which correspond to vanishing Yukawa coupling correspond to a $SU(1, n_0)/U(1) \times SU(n_0)$ restricted Kahler sub-manifold whose s -map is the quaternionic Kahler manifold

$$\frac{SU(2, n_0+1)}{SU(2) \times SU(n_0+1) \times U(1)} \quad (4.2)$$

The general form of the $Sp(n+1)$ connection in Eq. (3.24) actually tells us that the above manifolds are the only Q manifolds which are also Kahler.

The Kahler potential of these manifolds is (see Section 2)

$$K_Q = K + \bar{K} = -\ell n 2(1-z^2\bar{z}^2) - \ell n \left(s + \bar{s} + \frac{1}{2} (C+\bar{C}) R^{-1} (C+\bar{C}) \right) \quad (4.3)$$

5. Conclusions.

In this paper we have worked out the geometry of the dual quaternionic manifolds which describe the effective interactions of the moduli fields of (2,2) superstring vacua and of their (Ramond-Ramond) scalar superpartners in type II strings. These results have been obtained by space-time supersymmetry arguments but it is likely that they can be reproduced with a S-matrix approach as discussed recently by Dixon, Kaplunovsky and Louis.²

Quaternionic spaces which correspond to non-vanishing Yukawa couplings have a complicated structure unless the Yukawa couplings are independent of the moduli. In the latter case, for untwisted sectors of orbifold compactifications, one may even get symmetric spaces of the type discussed in Refs. 1 and 8. However, when twisted fields are involved or general Calabi-Yau compactification are considered, these spaces are not symmetric nor seem to be homogeneous. For those moduli fields which correspond to vanishing Yukawa couplings, the quaternionic sub-spaces are both Kahler and symmetric. More generally, at each point of the moduli space, the pure Ramond-Ramond sector is a $SU(1,n+2)/U(1) \times SU(n+2)$ sub-manifold. The results we described can alternatively be obtained by a Kaluza-Klein compactification of the 10D $N = 2$ supergravity on Calabi-Yau spaces. In that case the extra Ramond-Ramond scalars come, in the case of type IIA supergravity, from the (2,1) and (3,0) internal components of the three form A_{ijk} . The third and fourth derivative of the F holomorphic function, which appear in the Q-curvature (Eqs. (3.24) and (3.26)), are related to the following overlapping integrals on Calabi-Yau spaces^{3,4}

$$\int_K \Omega \wedge \frac{\partial \Omega}{\partial z_i \partial z_j \partial z_k} \quad \text{and} \quad \int_K \Omega \wedge \frac{\partial \Omega}{\partial z_i \partial z_j \partial z_k \partial z_l}$$

when Ω is a three-form as a function of the deformation of the complex structure. It is of interest to investigate whether the field theory formulae, the formulae obtained by 3D duality and the ones obtained by a S-matrix approach manifold give the same answer. Besides the physical motivation related to the point-field limit of superstring theories, the interest on the construction of the dual quaternionic manifolds is that they offer examples of continuous families of manifolds whose geometry depends on a holomorphic function. These manifolds are the generalization of the class of manifolds studied in Refs. 19-21, which were related to symmetric or homogeneous (restricted) Kahler manifolds. We have given here a component description of these manifolds. From the paper of Ref. 13, this is sufficient to write the entire $N = 2$ supergravity action. However, if one wants to describe arbitrary invariants and coupling to other multiplets, a general tensor calculus is required. The tensor calculus developed in Ref. 6 is not at present suitable to describe general quaternionic manifolds. The superfield formulation based on a harmonic superspace²² seems more adequate. A formulation of the c-map in superspace would be a very useful simplifying tool.

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