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I. INTRODUCTION

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Physical Green's Functions In Quantum Gravity

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ABSTRACT: It is argued that the usual observables of quantum gravity are unphysical on account of a confusion which has been allowed to develop between gauge fixing and defining a realistic coordinate system in which to make observations. We avoid this problem by reformulating the theory on the Green's functions of a new interpolating field — the Mandelstam covariant — constructed solely from geodesics, parallel transport and the Riemann tensor. This interpolating field is nonlocal in all gauges and therefore may not give the usual S-matrix beyond tree order. It also seems to provide an operatorial, relative notion of time which agrees with the usual one for classical, asymptotically flat space but which *disagrees* on the quantum level and makes sense even when space is closed. We prove that field redefinitions of this general type — although not this one — can avoid ultraviolet divergences.

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Although it has long been realized that Einstein quantum gravity can not be perturbatively renormalizable in four dimensions [1], the suggestion was made in the late sixties that the on-shell S-matrix might be finite [2]. This hope was progressively disappointed as the understanding of perturbative quantum field theory grew. One loop calculations became possible in the early seventies with the confluence of the Faddeev-Popov prescription [2,3], the background field formalism [2] and dimensional regularization [4]. Although pure gravity was found to be finite at this order [5], most matter couplings were shown to engender nonrenormalizable divergences [5,6]. (The exception was supergravity [7].) Further, it was understood that the sourceless theory's one loop finiteness derived from an accident of four dimensions [5] and would not be likely to persist at two loops [8]. The confirmation of this expectation came in the mid eighties when the application of tensor manipulation programs made two loop calculations feasible [9]. While supergravity is known to evade these two loop divergences there is every indication that it succumbs at three loops [10]. Thus it seems that every theory whose gravitational self interaction is that of Einstein is bound to encounter fatal problems in the perturbative loop expansion.

Although it is conceivable that the problems of Einstein quantum gravity are figments of perturbation theory [11] the conventional approach is to take them seriously and consider remedial modifications of the gravitational self interaction. Such changes have taken the form of introducing higher derivatives into the kinetic operator or — in the case of string theory — of making the interactions nonlocal. Although these properties can endow models with renormalizability — or even finiteness — they forfeit stability by engendering new, negative energy degrees of freedom [12,13].

We wish here to explore the consequences of a different sort of remedy. Instead of changing the fundamental dynamical principle we propose to alter the way in which phys-

ical questions are asked. This idea is hardly new, nor is it unique to quantum gravity. The infrared catastrophe of QED affords a well known example. In testing QED early researchers modeled scattering experiments by exclusive S -matrix elements. As computations of these were extended beyond tree order it was found that radiative corrections between external legs introduced infrared divergences unrelated to the then equally disturbing ultraviolet divergences. The problem was resolved in 1937 when Bloch and Nordsieck made two important advances, one technical and the other philosophical [14]. The technical breakthrough was an approximation scheme which permitted the summation of the leading infrared divergent contributions to a given process. Application of this technique revealed that, far from diverging, exclusive cross sections actually vanish. The philosophical advance was the realization that any performable experiment is insensitive to absorption and emission of soft photons whose total energy is below the detector resolution. One must therefore sum over all such processes in order to represent a true scattering experiment. When this is done the resulting inclusive cross sections are found to be both nonzero and free of infrared divergences.

That the predictive mechanism is open to question is a generic problem in quantum mechanics. How an experiment is done generally affects the result, yet even the simplest real experiment is so complicated as to render hopeless the task of representing it in terms of field operators. Nor should this be necessary. If it is to have any meaning an observed quantity — say the mass of a particle — ought not to depend overmuch on what color eyes the principle investigator has. Clearly most experimental details can, for the purposes of theoretical treatment, be subsumed into initial conditions on the quantum state. However it can happen that idealized initial conditions are unattainable, that inevitable deviations from ideality critically affect the result, and that these deviations are determined by experimental details which have been neglected. The subtle and tantalizing trick is discovering where to draw the line. This work can be viewed as exploring the consequences of a slight

shift of that line in the direction of a more faithful representation of realistic observations.

The particular detail we propose to treat more carefully is the physical coordinate system. We shall do this by introducing a new interpolating field, the Mandelstam covariant, which is constructed solely from geodesics, parallel transport and the Riemann tensor. This field carries a quasi-local coordinate system which seems a much better representation of the way real experiments are calibrated than the sorts of gauge coordinate systems that characterize the usual observables. As we shall explain in Section III the Mandelstam covariant is not a local, invertible redefinition of the usual field, nor does it represent an eccentric choice of gauge. It follows that the Green's functions deriving from the Mandelstam covariant need not interpolate the usual S -matrix, nor contain the usual ultraviolet divergences. Indeed we will prove in Section IV that the usual result *does* change. Unfortunately it does so in the direction of greater divergence, however the reason for this seems to be correctable and we feel it is entirely possible that a modified covariant yields finite Green's functions.

Our program is interesting for three reasons. First, it is certainly possible to avoid ultraviolet divergences — or any other problem — by making a suitable nonlocal change in the interpolating field. The obstacle to this for most theories is the belief that the local field, or at least the S -matrix it interpolates, represents physically meaningful observations. Therefore nonrenormalizable divergences in the S -matrix are legitimate predictions of the theory whose conflict with experiment is thereby manifest. Which brings us to the second motivation: this isn't necessarily true for quantum gravity. The gravitational S -matrix doesn't strictly exist on account of the infrared problem and the off-shell Green's functions which interpolate it do not represent performable experiments. As we shall argue in Section II it is not even clear that the inclusive rates and cross sections of Weinberg [15] represent legitimate predictions of quantum gravity. While this does not empower us to

wantonly define the ultraviolet problem into oblivion by making a contrived choice for the interpolating field it does suggest that a study is in order of the ultraviolet properties of what seem to be physically meaningful Green's functions.

Finally, reformulating the theory around physically meaningful Green's functions would be necessary even if it had no effect on the ultraviolet problem. The usual inclusive rates and cross sections only exist when perturbing around an asymptotically flat background at zero temperature. Some of the most interesting questions in quantum gravity involve either nonzero temperature or backgrounds which are not asymptotically flat. Nor are the usual observables adequate to address the much-debated issue of gauge dependence in the effective action [16].

Our paper consists of five sections of which this introduction is the first. In Section II we make the case for changing variables generally. Section III defines and describes our specific proposal, the Mandelstam covariant. In Section IV we present the results of a perturbative computation of one loop corrections to the Mandelstam 2-point function. Section V comprises our conclusions.

In order to exploit dimensional regularization we will usually work in D spacetime dimensions. Coordinates are either represented abstractly or in $(D - 1) + 1$ notation — e.g. $x = (x^0, \vec{x})$. The dependence of field upon coordinates is indicated in the usual fashion by parentheses — e.g. $\phi(x)$; the dependence of functionals upon fields is denoted by brackets — e.g. $S[\phi]$. Both our Lorentz metric η_{ab} and Riemann metric $g_{\mu\nu}(x)$ have timelike signature. Our fundamental gravitational field is the vierbein $e_{\mu a}(x)$, from which the metric derives in the standard way:

$$g_{\mu\nu}(x) \equiv \eta^{ab} e_{\mu a}(x) e_{\nu b}(x) \quad (1.1)$$

Note that English letters stand for local Lorentz indices while Greek letters denote coordinate ones, and in both cases the Einstein summation convention is assumed. The two gravi-

tational symmetry groups are general coordinate invariance — which is parametrized by the invertible function $y^\mu \rightarrow y'^\mu(y)$ — and local Lorentz invariance — which is parametrized by a spacetime dependent transformation matrix Λ obeying $\eta_{ab} = \Lambda_a^c(x) \Lambda_b^d(x) \eta_{cd}$. The vierbein is defined to transform as a vector with respect to both symmetry groups:

$$e'_{ab}(x) \equiv \frac{\partial y'^\nu(x)}{\partial y^\mu(x)} \Lambda_a^b(y'^{-1}(x)) e_{\nu b}(y'^{-1}(x)) \quad (1.2)$$

Unless otherwise stated our pure gravitational action is that of Hilbert:

$$S_H[g] \equiv -\frac{\mu^{D-4}}{\kappa^2} \int d^D x \sqrt{-g(x)} R(x) \quad (1.3)$$

where g is the determinant of the metric, R is the Ricci scalar, $\kappa^2 \equiv 16\pi G$ and μ is the dimensional regularization mass. Our conventions for the Riemann tensor and the affine connection are:

$$R^\alpha_{\beta\gamma\delta} \equiv \Gamma^\alpha_{\delta\beta,\gamma} - \Gamma^\alpha_{\gamma\beta,\delta} + \Gamma^\alpha_{\gamma\rho} \Gamma^\rho_{\delta\beta} - \Gamma^\alpha_{\delta\rho} \Gamma^\rho_{\gamma\beta} \quad (1.4a)$$

$$\Gamma^\alpha_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\rho} (g_{\rho\beta,\gamma} + g_{\gamma\rho,\beta} - g_{\beta\gamma,\rho}) \quad (1.4b)$$

where the commas denote ordinary differentiation. The Ricci tensor is $R_{\mu\nu} \equiv R^\rho_{\mu\rho\nu}$ and the Ricci scalar is its trace. Flat space perturbation theory is implemented using the graviton field:

$$h_{\mu\nu}(x) \equiv \frac{1}{\kappa} (g_{\mu\nu}(x) - \eta_{\mu\nu}) \quad (1.5)$$

Note that by fixing the local Lorentz gauge it is entirely consistent to formulate perturbation theory in terms of $h_{\mu\nu}$ even when the vierbein is the fundamental field [17].

II. THE CASE FOR CHANGING VARIABLES

The purpose of this section is to argue that the usual observables of quantum gravity are unphysical and that different Green's functions do not necessarily harbor ultraviolet divergences. We begin by explaining what is wrong with Green's functions of the local field. We then discuss problems and potential problems with traditional asymptotic observables

such as the S-matrix. The section closes with a demonstration that nonlocal interpolating fields exist which have ultraviolet finite Green's functions — although we offer no proof that any of these fields is physically interesting.

II A — Problems with Green's functions of the local field

To our knowledge no one has ever suggested using local Green's functions as physical observables in quantum gravity. Our motive in criticizing them is to bring our doubts about the usual asymptotic variables into sharper focus. It turns out that the most important potential problems of the latter are best understood in terms of the undisputed problems of the former. Being clear about what renders local Green's functions unphysical will also be important in Section III when we attempt to design an improvement.

Much depends upon understanding the physical significance of gauge fixing so we will start by defining what is meant by a "complete general coordinate gauge." This is a set of conditions on the metric such that for every field configuration $g_{\mu\nu}(x)$ there is a coordinate redefinition, unique up to some finite dimensional group,* which makes the transformed field obey these conditions. To avoid confusion with generic coordinate transformations we will give this special transformation — or rather its inverse, since that is the more convenient quantity — the name $X^\mu|g, G|(x)$. Here the symbol "G" stands for the gauge conditions. The transformed metric:

$$g_{\mu\nu}|g, G|(x) \equiv \frac{\partial X^\rho|g, G|(x)}{\partial x^\mu} \frac{\partial X^\sigma|g, G|(x)}{\partial x^\nu} g_{\rho\sigma}(X|g, G|(x)) \quad (2.1)$$

obeys G-gauge and is invariant up to some finite dimensional, residual subgroup.

A complete gauge in D-space typically involves D local, "volume" conditions holding at each point in spacetime plus a set of "surface" conditions which provide the boundary

* In asymptotically flat perturbation theory this is usually the global Poincaré group. For the purposes of perturbation theory we can ignore the Gribov problem [18].

data necessary to make $X^\mu|g, G|(x)$ unique up to global transformations. To see the need for surface conditions let us consider the simplest and most popular volume condition, DeDonder gauge:

$$\partial_\nu(\sqrt{-g} g^{\nu\mu}) = 0 \quad (2.2)$$

It is easy to see that X^μ must obey:

$$\frac{\partial}{\partial X^\sigma} \left(\sqrt{-g(X)} g^{\rho\sigma}(X) \frac{\partial x^\mu}{\partial X^\sigma} \right) = 0 \quad (2.3a)$$

Whence it follows that:

$$x^\mu = X^\mu - \frac{1}{\partial_\rho \sqrt{-g} g^{\rho\sigma} \partial_\sigma} \left(\partial_\nu \sqrt{-g} g^{\nu\mu} \right) (X|g|(x)) \quad (2.3b)$$

is an exact solution — for any choice of the inverse of $\partial_\rho \sqrt{-g} g^{\rho\sigma} \partial_\sigma$. Surface conditions provide the boundary value data which eliminate the ambiguity in defining the inverse differential operator. In practice they are seldom explicitly stated. Their role in perturbation theory is to provide the boundary conditions which constrain the pure gauge part of the background configuration and determine the gauge dependent part of the "iε" terms in propagators [19]. The usual procedure is to put these things in by hand and ignore the surface gauge conditions which would have justified them.

Each complete gauge is associated with a set of Green's functions:

$$\mathcal{G}_{\mu_1 \dots \mu_N}^N[G](x_1, \dots, x_N) \equiv \int [dg] \delta[G] \det \left[\frac{\delta G}{\delta y} \right] h_{\mu_1 \nu_1}(x_1) \dots \dots h_{\mu_N \nu_N}(x_N) \exp(iS_H[g]) \quad (2.4)$$

Though $\mathcal{G}^N[G]$ appears to be the expectation value of a product of noninvariant, local operators this is not so. In fact it is the expectation value of a product of fully invariant,

* It is traditional to base perturbation theory in this gauge on the field $\phi^{\mu\nu} \equiv (\sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu})/\kappa$. Since $\phi^{\mu\nu}$ and $h_{\mu\nu}$ are related by a local, perturbatively invertible field redefinition, we are free to use $h_{\mu\nu}$ instead.

but gauge dependent and highly nonlocal operators [19]:

$$h_{\mu\nu}[g, G](x) \equiv \frac{1}{\kappa} \left(g_{\mu\nu}[g, G](x) - \eta_{\mu\nu} \right) \quad (2.5)$$

This becomes apparent upon expansion of the DeDonder gauge graviton:

$$h_{\mu\nu}[g, \text{DeDonder}] = h_{\mu\nu} + \frac{1}{\partial^2} \left(-h_{\mu\nu}{}^{\rho} + h^{\rho}{}_{\rho\mu\nu} - h_{\mu}{}^{\rho}{}_{\rho\nu} \right) + O(\kappa h^2) \quad (2.6a)$$

$$= \left(\text{harmonic terms} \right) - \frac{1}{\partial^2} \left(\frac{1}{2\kappa} R_{\mu\nu} \right) + O(\kappa h^2) \quad (2.6b)$$

The “harmonic terms” in (2.6b) are generally asymptotic surface integrals of the extrinsic curvature; their precise form depends upon the surface gauge condition. Higher order corrections serve to geometrically fix the various coordinates and indices in the DeDonder gauge coordinate system. Hence $\mathcal{G}^N[\text{DeDonder}]$ actually represents the observation of a complicated product of curvature invariants which just happens to simplify to N graviton fields when the metric obeys DeDonder gauge. Similar results obtain for other gauges.

There is a physical reason for this and a formal one. The formal reason is that only invariant operators exist in the canonical theory. Therefore to find the canonical operator whose expectation value a functional integral represents one must use the gauge condition to “invariantize” the functional argument:

$$A[g] \delta[G] = A[g, G] \delta[G] \quad (2.7a)$$

$$\int [dg] \delta[G] \det \left[\frac{\delta G}{\delta y} \right] A[g] \exp(iS_H[g]) = \left\langle T^* \left(A[g, G] \right) \right\rangle \quad (2.7b)$$

Various aspects of this procedure has been described at great length in a previous paper [19].

The physical reason is it has no operational significance to instruct an observer to “measure the $\mu\nu$ component of the graviton field at position x .” Rather one must ask for certain components of the tidal force in a geometrically meaningful frame and at a

geometrically meaningful location. In choosing $h_{\mu\nu}$ as our interpolating field we have not included any such information so the gauge condition provides it for us through the process of invariantization.

The choice of gauge can and does affect the ultraviolet divergence structure of local Green’s functions. For example Capper has shown that certain divergent tensor components of the graviton 2-point function can be gotten to change sign at one loop by altering the gauge [20]. In fact Kallosh, Tarasov and Tyutin have found gauges where there are no one loop divergences [21].^{*} There is nothing strange about this. Different gauges represent the observation of different components of the curvature in different coordinate systems. They should give different results.

The issue for us is not so much that local Green’s functions depend upon the gauge but rather that this gauge dependence makes them unwieldy experiments. In fact none of them is a reliable probe of quantum gravity because they call for measurements that can never actually be made. For example, (2.6) reveals that the DeDonder gauge graviton represents an average over spacetime of certain curvature measurements. Why should we have to do this in order to describe a process that occurs within a cubic Planck length and lasts the merest fraction of a second? More to the point, can we ever hope to perform the stated observation? And, if not, must we be very concerned over the fact that the theory predicts a divergent result for it?

The problem with local Green’s functions is that they obscure the distinction which ought to be made between gauge fixing and establishing the geometry of an experiment. The first is a mathematical device to facilitate perturbation theory; the second is the prelude to, and the consequence of, making any kind of measurement. Real observers do

* Of course the results of DeSer et. al. show that no such gauge exists when most matter is coupled to gravity [6]. Similarly the work of Goroff and Sagnotti demonstrates that no such gauge exists in pure gravity at two loops [9].

not establish global coordinate systems in which to take their data; they do not even fill a finite volume with coordinate labels. Yet the very act of measurement inevitably establishes a skeletal reference system in the lattice of measurement points. The geometrical relations between these may vary with the fields of the theory but such relations must exist because observations are always effected by some real process emanating, however indirectly, from a central control.

What we will propose in Section III is an interpolating field with such a coordinate system built into it. The product of N such fields would then be an invariant in its own right, without any need to refer to G .^{*} However we must first explain why we do not accept that the usual asymptotic observables correctly describe the physics of quantum gravity.

II.B — Problems with asymptotic observables

The traditional asymptotic observables of quantum gravity are the (infrared regulated) S-matrix and the inclusive rates and cross sections of Weinberg [15]. The former derives from LSZ reduction of local Green's functions; the latter are obtained by subjecting the S-matrix to the procedure of Block and Nordieck [14]. This defines the usual observables but it does not guarantee their physical interpretation. After all, what LSZ reduction *literally* gives is the inner product between a state whose wavefunctional is in the free Fock space at asymptotically early times and another state whose wavefunctional belongs to the free Fock space at asymptotically late times. The mass eigenstates of interacting quantum field theories are typically much different from the those of the corresponding free theories; how therefore is the above relation effected?

In an ordinary quantum field theory the answer to this question is supplied by the LSZ

^{*} Actually it is only covariant, depending upon a finite dimensional subgroup of coordinate transformations just as (2.1) and (2.5) do.

asymptotic condition. Briefly, it states that the field becomes weakly free at asymptotically early and late times. Therefore the free eigenstates are the interacting ones, for the purposes of certain inner products, in the asymptotically far past and future. The idea is that all normalizable states consist of combinations of particles with different momenta. As time evolves the particles separate and the interactions between them become negligible. Of course nothing can turn off self interactions but these are accounted for by mass and field strength renormalization; otherwise the field looks free to such states.

In fact this process doesn't quite work in quantum gravity owing to the masslessness of the graviton. This engenders a long range interaction which never really "turns off." For this reason the exclusive S-matrix vanishes, even as it does in QED. While it is certainly true that the procedure of Block and Nordieck [14] can be applied to give infrared finite, nonzero inclusive rates and cross sections in quantum gravity [15] we feel there are valid reasons for questioning their reliability as probes of the theory. Their physical interpretation still rests on the (infrared regulated) LSZ asymptotic condition and we suspect that it harbors unphysical coordinate dependence very similar to the sort we found for local Green's functions.

The usual argument is that LSZ reduction purges the local Green's functions of any such gauge dependence but a little thought reveals that this cannot be entirely true. LSZ reduction requires us to take something called "time" to infinity and this can only have meaning within the context of a coordinate system. In fact it is the time generated by the Hamiltonian of Arrowitt, Deser and Misner (ADM) [22] who were quite explicit about pointing out that their coordinate system is asymptotically flat. If one changes this — for example by varying the surface gauge condition or by altering the asymptotic volume gauge — then the S-matrix changes, and so too do the inclusive rates and cross sections which derive from it. This is only to be expected since the S-matrix comes from taking

an asymptotic limit of local Green's functions and by changing the asymptotic coordinate system we are changing what these Green's functions mean.

That admitted we must now ask whether there is anything unrealistic about this asymptotic coordinate system. For classical scattering we do not believe there is; once an asymptotically flat configuration is established causality forces it to stay that way. However quantization introduces an effect, local to asymptotic infinity, which disrupts asymptotic flatness. This is the Uncertainty Principle: if the wavefunctional is so constrained that all moments of the metric operator obey asymptotic flatness at a single time they will not do so an instant later.

It might be thought that we only need the asymptotic condition to hold in the weak sense but this is false. At many points in deriving the ADM Hamiltonian one must solve operator equations using the asymptotic condition.* If this condition is not true then the resulting Hamiltonian does not generate time evolution according to the Einstein equations. Therefore a conventional scattering amplitude would seem to represent an experiment which includes some sort of gigantic machine for preventing quantum oscillations in the asymptotic geometry.

This is precisely the sort of difficulty we require in order to have a hope of ascribing the ultraviolet problem to faulty measurement theory. On the classical level, where there is no problem, the machine does nothing. When quantum effects turn on so too does the machine, and it should make quite a difference. In a theory where geometry is a dynamical variable, in full interaction with the other fields, it has to be expected that whatever process is used to maintain the temporal and spatial relations between the various sensing devices will affect and be affected by whatever measurements are made. The natural question to

* Actually the analysis of ADM was entirely classical so they were correct in ignoring the quantum effect we are worried about. Our statements apply to the usual extrapolation of their analysis to the quantum theory.

ask is whether real experiments actually include such machines. We think they do not and we are therefore unwilling to accept the verdict of conventional scattering theory in quantum gravity.

The obvious way of seeing whether there is anything to this speculation is to define a different sort of observable and check its ultraviolet properties. The new interpolating field we shall propose in Section III carries an intrinsic, quasi-local definition of time which is free of any dependence upon the asymptotic coordinate system.* If the origin of this time is taken to asymptotic infinity then our variable seems to agree with that of ADM on the classical level — that is, where the asymptotic fields can be assumed to be nondynamical — but not quantum mechanically. This is promising in a qualitative sort of way. We close this section by explaining how a new interpolating field can eliminate ultraviolet divergences.

II.C — Existence of finite Green's functions

Central to our argument is the following theorem: *for every change induced in the local Green's functions by modifying the interaction it is possible to obtain the same result by using the Green's functions of a different interpolating field in the original theory.* The proof is general enough that we may as well work in the context of a scalar field $\phi(x)$ for simplicity. Let the action be:

$$S[\phi] \equiv \int d^D x \frac{1}{2} \{ \partial_\alpha \phi \partial^\alpha \phi - m^2 \phi^2 \} + g S_1[\phi] \quad (2.8)$$

where we will not give a precise form for the interaction S_1 . Although one could formulate all quantities in this quantum field theory using functional integrals the same information is

* We reiterate the point already made in the Introduction that such an observable is necessary to treat physically interesting situations even if it does nothing for the ultraviolet problem. There might, for example, be no asymptotic region. This is what happens in the case of a closed space or one that is full of matter. Present observation is entirely consistent with the possibility that the physical universe falls into this class, yet we are able to perform scattering experiments well enough. These experiments are obviously represented by observables which do not require a formally infinite spatial volume for their definition.

contained in the operator formalism. This is defined by using the Euler-Lagrange equation:

$$(\partial^2 + m^2)\phi(x) = g \frac{\delta S_1[\phi]}{\delta \phi(x)} \quad (2.9)$$

to solve for $\phi(t, \vec{x})$ in terms of the initial value field configurations. These last are the true operator degrees of freedom of the theory; their commutation algebra is determined by canonical quantization.

In perturbation theory this is always straightforward. The equations are solved by substituting the perturbative ansatz:

$$\phi(x) = \sum_{n=0}^{\infty} g^n \psi_n(x) \quad (2.10)$$

into (2.9) and segregating powers of g :

$$\begin{aligned} (\partial^2 + m^2)\psi_0(x) &= 0 \\ (\partial^2 + m^2)\psi_1(x) &= \frac{\delta S_1[\psi_0]}{\delta \phi(x)} \\ (\partial^2 + m^2)\psi_2(x) &= \int d^D y \frac{\delta^2 S_1[\psi_0]}{\delta \phi(x)\delta \phi(y)} \psi_1(y) \end{aligned} \quad (2.11)$$

and so on. The perturbative initial value data are $\phi_0(\vec{x}) \equiv \phi(0, \vec{x})$ and $\dot{\phi}_0 \equiv \dot{\phi}(0, \vec{x})$. The zeroth order solution is taken to agree with these operators at $t = 0$:

$$\psi_0[\phi_0, \dot{\phi}_0](t, \vec{x}) = \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \int d^{D-1}\vec{y} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \left\{ \phi(0, \vec{y}) \cos(\omega t) + \frac{\dot{\phi}(0, \vec{y})}{\omega} \sin(\omega t) \right\} \quad (2.12)$$

where $\omega^2 \equiv \vec{k} \cdot \vec{k} + m^2$. Corrections are found using the Green's function:

$$G(x, y) \equiv \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \frac{\exp[i\vec{k} \cdot (\vec{x} - \vec{y})]}{2\omega} \left\{ \sin[\omega|x^0 - y^0|] + \text{sgn}(y^0) \sin[\omega(x^0 - y^0)] \right\} \quad (2.13)$$

to enforce the initial conditions $\psi_n(0, \vec{x}) = 0 = \dot{\psi}_n(0, \vec{x})$. That is:

$$\psi_1[\phi_0, \dot{\phi}_0](x) = \int d^D y G(x, y) \frac{\delta S_1[\psi_0[\phi_0, \dot{\phi}_0]]}{\delta \phi(y)} \quad (2.14)$$

$$\psi_2[\phi_0, \dot{\phi}_0](x) = \int d^D y G(x, y) \int d^D z \frac{\delta^2 S_1[\psi_0[\phi_0, \dot{\phi}_0]]}{\delta \phi(y)\delta \phi(z)} \psi_1[\phi_0, \dot{\phi}_0](z)$$

and so on.

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The transformation to free fields proceeds in two steps. The first carries ϕ to ψ_0 on shell:

$$\bar{\phi}_1[\phi](x) \equiv \phi(x) - g \int d^D y G(x, y) \frac{\delta S_1[\phi]}{\delta \phi(y)} \quad (2.15)$$

Note that since $G(x, y)$ is not time translation invariant it is not generally true that the ϕ Hamiltonian generates time translations of $\bar{\phi}_1[t, \vec{x}]$. Further, even though $\bar{\phi}_1$ satisfies the free equation of motion it does not necessarily commute like a Klein-Gordon field because ϕ_0 and $\dot{\phi}_0$ may not do so. This typically arises when time derivatives are present in S_1 ; in fact for a temporally nonlocal interaction it is not even necessary that $[\phi_0(\vec{x}), \phi_0(\vec{y})] = 0$ [13].

The second step of our transformation restores conventional commutation relations. It begins by isolating the oscillators:

$$\bar{a}[\phi](t, \vec{k}) \equiv \int e^{-i\vec{k} \cdot \vec{x}} \left(\omega \bar{\phi}_1[\phi](t, \vec{x}) + i \dot{\bar{\phi}}_1[\phi](t, \vec{x}) \right) \quad (2.16)$$

We then transform the oscillators:

$$\begin{aligned} a[\phi](t, \vec{k}) &= \bar{a}(t, \vec{k}) + \sum_{n=1}^{\infty} \int \frac{d^{D-1}\vec{k}_1}{(2\pi)^{D-1}} \frac{1}{2\omega_1} \cdots \int \frac{d^{D-1}\vec{k}_{2n+1}}{(2\pi)^{D-1}} \frac{1}{2\omega_{2n+1}} \\ &\quad (2\pi)^D \delta^D(k + k_1 + \cdots + k_n - k_{n+1} - \cdots - k_{2n+1}) F_n(k_1, k_2, \dots, k_{2n-1}, k_{2n}) \\ &\quad \bar{a}^\dagger(t, \vec{k}_1) \cdots \bar{a}^\dagger(t, \vec{k}_n) a(t, \vec{k}_{n+1}) \cdots a(t, \vec{k}_{2n+1}) \end{aligned} \quad (2.17)$$

where $k_i^0 = \omega_i$. The functions F_n depend upon S_1 and are chosen to enforce the following operator algebra when ϕ obeys its Euler-Lagrange equation and commutation relations:

$$[a[\phi](t, \vec{k}), a[\phi](t', \vec{k}')] = 0 = [a^\dagger[\phi](t, \vec{k}), a^\dagger[\phi](t', \vec{k}')] \quad (2.18a)$$

$$[a[\phi](t, \vec{k}), a^\dagger[\phi](t', \vec{k}')] = 2\omega(2\pi)^{D-1} \delta^{D-1}(\vec{k} - \vec{k}') \quad (2.18b)$$

According to Pauli [23] this can always be done perturbatively in a Poincaré invariant

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theory of the sort we are considering. The final step:

$$\Phi_1[\phi](t, \vec{x}) \equiv \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \frac{1}{2\omega} \left\{ a[\phi](t, \vec{k}) e^{i\vec{k}\cdot\vec{x}} + a^\dagger[\phi](t, \vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right\} \quad (2.19)$$

produces a field which evolves and commutes like a free field when ϕ obeys its own field equation and commutation relations.

By means of the above transformation and its inverse we can relate any two theories which are built up perturbatively from the same free action. In particular this means that ultraviolet divergences can certainly be avoided by a suitable choice of interpolating field. If the fundamental theory with interaction S_1 is divergent while that with S_2 is not, merely take $\Psi[\phi] = \Phi_2^{-1}[\Phi_1[\phi]]$. Green's functions based on the field $\Psi[\phi]$ in the theory whose interaction is $S_1[\phi]$ will be identical to those based on ϕ in the theory whose interaction is $S_2[\phi]$. Therefore arbitrary perturbative dynamics can be subsumed into the choice of interpolating field; the dynamical principle has become an evanescent concept whose form changes as we alter the field basis in which we perceive it.

The remainder of the argument is trivial. There are actions with ultraviolet finite Green's functions — free theory for example — so it is certainly possible to achieve ultraviolet finiteness by changing the field variable.

II.D — An example from the literature

While correct the preceding result may seem contrived. It is therefore important to see that situations have already been noted in other theories where two different observables seem equally reasonable — and where they give dramatically different results. We have in mind the dual $SU(2)$ sigma model in two dimensions. The original model is based on a field $\theta^a(x)$ whose action is:

$$S[\theta] = \frac{1}{g^2} \int d^2x \left(U^{-1} \partial_\mu U U^{-1} \partial^\mu U \right) \quad (2.20a)$$

where:

$$U(x) \equiv \exp\left(\frac{ig}{2}\theta^i(x)\sigma^i\right) \quad (2.20b)$$

and the σ^i 's are the Pauli matrices. The model has a number of distinctive physical properties. Among them are asymptotic freedom [24] and the lack of particle creation [25], i.e. the scattering of n particles always gives precisely n particles. The equations of motion:

$$\partial^\mu \left(U^{-1} \partial_\mu U \right) = 0 \quad (2.21)$$

can be regarded as those of a current J_a^μ :

$$\partial_\mu J^{a\mu} = 0 \quad (2.22a)$$

subject to the constraints:

$$\frac{1}{2}\epsilon^{ab} \left(\partial_a J_b^i - \partial_b J_a^i + g\epsilon^{ijk} J_a^j J_b^k \right) = 0 \quad (2.22b)$$

The dual model is obtained by interchanging the role of dynamical equation and constraint.

The obvious way to do this was implemented by Nappi [26]. She enforced (2.22a) identically by defining the current in terms of a field ϕ^i :

$$J_a^i(x) = \epsilon_{ab} \partial^b \phi^i(x) \quad (2.23a)$$

Substitution into (2.22b) gives:

$$\partial^2 \phi^i - \frac{1}{2} g \epsilon^{ab} \epsilon^{ijk} \partial_a \phi^j \partial_b \phi^k \quad (2.23b)$$

Up to a constant the unique local action with this as its Euler-Lagrange equation is:

$$S[\phi] = \int d^2x \left\{ \frac{1}{2} \partial_a \phi^i \partial^a \phi^i - \frac{1}{2} g \epsilon^{ab} \epsilon^{ijk} \phi^i \partial_a \phi^j \partial_b \phi^k \right\} \quad (2.24)$$

It is simple to show that the energy of this theory:

$$H = \int \left\{ \frac{1}{2} \dot{\phi}^i \dot{\phi}^i + \frac{1}{2} \phi^{i'} \phi^{i'} \right\} \quad (2.25)$$

agrees with that of the original formulation; hence so too must the operator algebra implied by (2.23a). Yet Nappi's theory is neither asymptotically free nor does it conserve particle number.

An alternate dual representation was achieved by Fridling and Jevicki [27]. They started with a two-field action:

$$S[\psi, J] \equiv \frac{1}{2} \int d^2x \left\{ J_a^i J^{ia} + \psi^i \epsilon^{ab} (\partial_a J_b^i - \partial_b J_a^i + g \epsilon^{ijk} J_b^j J_c^k) \right\} \quad (2.26)$$

which reduces to (2.20) upon elimination of the Lagrange multiplier ψ^i and substitution of $J_a^i = Tr(\frac{1}{2} \sigma^i U^{-1} \partial_a U) / g$. The dual form is obtained by instead integrating out J_a^i . In a quadratic theory this is equivalent to substituting the classical solution:

$$J[\psi]_a^i = \frac{1}{1 + g^2 \psi^2} \left\{ \epsilon_{ab} \partial^b \psi^i + g^2 \psi^i \epsilon_{ab} \partial^b \psi^j \psi^j + g \epsilon^{ijk} \partial_a \psi^j \psi^k \right\} \quad (2.27)$$

where $\psi^2 \equiv \psi^i \psi^i$. The resulting action is:

$$S[\psi, J[\psi]] = \frac{1}{2} \int d^2x \frac{1}{1 + g^2 \psi^2} \left\{ \partial_a \psi^i \partial^a \psi^i + g^2 \partial_a \psi^i \psi^i \partial^a \psi^j \psi^j + g \epsilon^{ab} \epsilon^{ijk} \psi^i \partial_a \psi^j \partial_b \psi^k \right\} \quad (2.28)$$

from which Fridling and Jevicki were able to show that their formalism is asymptotically free and conserves particle number.

From (2.23a) and (2.27) it is apparent that ϕ and ψ are related on shell by:

$$\phi^i = \frac{1}{1 + g^2 \psi^2} \left\{ \psi^i + g^2 \psi^i \psi^j \psi^j - g \epsilon^{ijk} \psi^j \psi^k \right\} \quad (2.29a)$$

$$\phi^i = \frac{1}{1 + g^2 \psi^2} \left\{ \psi^i + g^2 \psi^i \psi^j \psi^j - g \epsilon^{ijk} \psi^j \psi^k \right\} \quad (2.29b)$$

Substitution into (2.25) and comparison with the Legendre transform of (2.28) reveals that the ψ Hamiltonian and the ϕ Hamiltonian agree. Since the ψ and ϕ field equations also imply one another it follows that the two operator algebras agree in the same sense. Therefore the formulation of Nappi really describes the same theory as that of Fridling and Jevicki; the apparent physical discrepancies between them arise from the use of different interpolating fields. We stress that they seem equally plausible in their own terms. Both have local actions. A ϕ observer would ascribe the fact that his ψ counterpart reports

both conservation of particle number and the high energy validity of perturbation theory to a tendentious choice of field basis. The ψ observer would advance the same explanation for the ϕ observer's failure to see these effects. Which — if either — of the two S-matrices actually describes the scattering of physical particles has yet to be established.*

III. THE MANDELSTAM COVARIANT

The purpose of this section is to define our new interpolating field and to describe its properties. We begin by reviewing the philosophical considerations which led us to it and state in words the observation it represents. We then express the field as an explicit functional of the vierbein. At this stage its relation is explained to the similar variable proposed long ago by Mandelstam [28]. We also derive its transformation properties and compare the notion of time it provides with the traditional one of ADM [22]. We then explain how our Mandelstam covariants are combined to form invariant Green's functions. The section closes with a discussion of ways in which our interpolating field may need to be modified in order to correctly probe both pure quantum gravity and gravity plus matter.

III.A — Philosophical considerations

We learned in Section II that the unphysical nature of local Green's functions — as well as the most important potential problem with the usual asymptotic observables — derives from a confusion that has been allowed to develop between gauge fixing and making geometrically well defined observations. Our proposal for avoiding this is to base the Green's functions upon a new interpolating field which carries its own physical coordinate system. Of course real coordinate systems involve material objects such as wires and beam-

* What seems to be going on here is that states composed of more than two particles must, in two dimensions, have particles with collinear momenta. If these particles are also massless — as they are, perturbatively, for this model — then no amount of evolution suffices to make them separate. Hence they are always interacting and the problem of creating exact physical states is as difficult in the asymptotic region as it is at finite times. Viewed from this perspective it would be miraculous if either of the two S-matrices were correct.

pipes in a complicated and quite experiment-specific fashion. It should not be necessary, nor is it even desirable, to incorporate such detail. What we need is a simple, universal model for a realistic observation in quantum gravity.

The physical coordinate system that it contains will render our interpolating field almost, but not quite, invariant. No object which possesses indices and depends upon spacetime variables — as any good interpolating field must — can be completely invariant. The best that can be hoped for is invariance up to some finite dimensional subgroup, as was the case for the invariantized graviton field (2.5). The idea is then that we achieve fully invariant Green's functions by doing things to products of the interpolating field. This, plus the fact that we want to avoid the necessity of making global measurements suggests that we take the interpolating fields to depend covariantly upon the transformation group at a single origin. We might think of this as the central control point of the experiment from which measurements are to be made in a fixed geometric relation to one another.

It is highly desirable that all indices be reported in the local Lorentz frame at the origin. In this way standard operations such as raising, lowering and contraction can be accomplished without the need to multiply by a dynamical field such as the metric. Similar considerations pertain to the variables which describe the geometric relations of the various measurement points. If they reside in the local Lorentz frame of the central origin then covariant differentiation can be accomplished without the need to multiply a dynamical connection.

In pure gravity there are three primitive geometrical concepts we can employ:

- (i) Free fall along a geodesic.
- (ii) Parallel transport.
- (iii) Curvature measurement.

An experiment of the type we seek can be put together from these in the following way. Set an observer adrift in free fall at the point x^μ in the direction $e^\mu_a(x)V^a$. The observer comes equipped with a clock, a set of gyroscopes for parallel transporting the local frame field from x , and an apparatus for measuring the tidal force. After he has drifted for a proper time $\sqrt{\eta_{ab}V^aV^b}$ the observer measures the $abcd$ component of the Riemann tensor with respect to the local frame field he has transported from x . If matter covariants are desired the experiment is the same except that the appropriate matter field — for example the field strength tensor in QED — is measured in place of the Riemann tensor.

III.B — The covariant and its properties

We refer to the operator which represents the experiment we have described above as the “Mandelstam Covariant” because a closely related variable was suggested by Mandelstam in the sixties [28]. Although it is a quantum operator we shall define it in the context of the functional formalism where the vierbein field is a C-number function. It is denoted by the symbol $\mathfrak{R}_{abcd}[e](V, x)$ and has the following definition:

$$\mathfrak{R}_{abcd}[e](V, x) \equiv \tilde{M}^k_a[e](V, x) \tilde{M}^l_b[e](V, x) \tilde{M}^m_c[e](V, x) \tilde{M}^n_d[e](V, x) \tilde{R}_{klmn}(\chi|e|(1, V, x)) \quad (3.1a)$$

where a bar over a familiar symbol denotes that its indices are local Lorentz:

$$\tilde{R}_{abcd}(z) \equiv e_{a\alpha}(z) e^\beta_b(z) e^\gamma_c(z) e^\delta_d(z) R^\alpha_{\beta\gamma\delta}(z) \quad (3.1b)$$

and we shall shortly explain what the other symbols mean.

By $\chi^\mu|e|(\tau, V, x)$ we mean the geodesic path which carries the Mandelstam observer from his origin x to the point at which he measures the curvature. As the notation suggests it takes values in the spacetime manifold, is a functional of the vierbein field and a function of the following variables: the geodesic parameter τ , the initial direction V^a and the origin x^μ . We will almost always suppress the functional dependence upon e and will

often suppress some of the arguments. When this last is done the argument list should be understood as having been truncated from right to left as follows:

$$(\tau, V) \equiv (\tau, V, -) \quad ; \quad (\tau) \equiv (\tau, -, -) \quad (3.2)$$

Missing arguments should always be clear from context. Dots over χ stand for differentiation with respect to τ . With this understood the geodesic is defined by the differential equation:

$$\ddot{\chi}^\mu(\tau) + \Gamma_{\rho\sigma}^\mu(\chi(\tau)) \dot{\chi}^\rho(\tau) \dot{\chi}^\sigma(\tau) = 0 \quad (3.3c)$$

and the initial conditions:

$$\chi^\mu(0) = x^\mu \quad (3.3b)$$

$$\dot{\chi}^\mu(0) = e_a^\mu(x) V^a \quad (3.3c)$$

Note that our initial conditions result in a unique geodesic, in contrast to the multivalued result which can emerge due to caustics when one attempts to fix the endpoints instead.

The invariant length along the geodesic is:

$$\begin{aligned} \ell(\tau, V) &\equiv \int_0^\tau dt' \sqrt{g_{\alpha\beta}(\chi(t')) \dot{\chi}^\alpha(t') \dot{\chi}^\beta(t')} \\ &= \tau \sqrt{\eta_{ab} V^a V^b} \end{aligned} \quad (3.4)$$

which is why we have evaluated the Riemann tensor in (3.1a) at $\chi(1)$.

By $\bar{M}_b^a[e](V, x)$ we mean the parallel transport matrix with which the Mandelstam observer carries the local frame field from his origin to the measurement point. It takes values in the tensor product of the local Lorentz vector spaces at $\chi^\mu(1)$ (first index) and $\chi^\mu(0) = x^\mu$ (second index). As the notation suggests it is a functional of the vierbein and a function of x^μ and V^a . The same argument suppression convention as we used for the

geodesic applies to $\bar{M}_b^a$ as well. It's defining relation is:

$$\bar{M}_b^a[e](V, x) \equiv e_a^\alpha(\chi(1)) M_{\beta|e}^a(V, x) e_b^\beta(x) \quad (3.5c)$$

$$M_{\beta|e}^a(V, x) \equiv P \exp \left(- \int_0^1 dt \dot{\chi}^\mu(\tau) \Gamma_{\mu\beta}^a(\chi(\tau)) \right) \quad (3.5b)$$

where the symbol " P " denotes path ordering according to the "later-on-the-left" convention.

One might wonder why we went to the trouble of constructing a variable based on the Riemann tensor instead of on the metric. The physical answer is that the curvature carries gauge invariant information about the local geometry whereas the metric does not. Further we see from (2.6) and its analogs in other gauges that measuring the metric in a gauge is effectively measuring the curvature. There is, however, a more trivial answer: as long as distant indices are related using a metric-compatible connection the metric transports onto itself. For example it is easy to show from (3.5b) that:

$$g_{\mu\nu}(x) = M_{\mu|e}^\rho(V, x) M_{\nu|e}^\sigma(V, x) g_{\rho\sigma}(\chi[e](1, V, x)) \quad (3.6)$$

Therefore we cannot really probe different points by using a "Mandelstam metric."

Though our interpolating field is quite similar to the one proposed by Mandelstam [28] four differences are worth noting:

- (1) Our fundamental field variable is the vierbein while Mandelstam states that his is the Riemann tensor.
- (2) We use geodesics to position the measurement point while Mandelstam uses unspecified but geometrically meaningful paths.
- (3) Our geodesic emanates from an arbitrary origin while Mandelstam's paths come in from spacelike infinity.

(4) The indices of our field are given in the local Lorentz frame of the arbitrary origin while Mandelstam's are reported in whatever coordinate basis pertains at spacelike infinity.

The motivation is also somewhat different. Writing before the advent of covariant quantization Mandelstam was trying to solve the problem of quantizing gravity (and Yang-Mills) in a unitary fashion. At the time it was not understood how to do this using noninvariant operators such as the metric (or the Yang-Mills connection) and Mandelstam's idea was to construct a canonical formalism based solely on invariant quantities. Today this problem would be solved by including the Faddeev-Popov determinant in the gauge-fixed functional formalism.

We come now to transformation properties. As always in field theory these are defined for composite objects such as $\chi^\mu[e](\tau, V, x)$, $\tilde{M}_b^a[e](V, x)$ and $\mathfrak{R}_{abcd}[e](V, x)$ by transforming the fundamental field, in this case vierbein. Under the combined general coordinate and local Lorentz transformation given in (1.2) it is not difficult to verify that the geodesic goes to:

$$\begin{aligned}\chi^\mu[e](\tau, V, x) &\equiv \chi^\mu[e'](\tau, V, x) \\ &= y^\mu \left(\chi[e](\tau, \Lambda^{-1}V, y'^{-1}(x)) \right)\end{aligned}\quad (3.7a)$$

where:

$$(\Lambda^{-1}V)^a \equiv \Lambda_b^a(x)V^b \quad (3.7b)$$

As the name suggests the parallel transport matrix carries frame field information from $\chi(1)$ to $\chi(0) = x$:

$$\begin{aligned}M_b^a[e](V, x) &\equiv M_b^a[e'](V, x) \\ &= \Lambda_b^a \left(\chi[e](1, \Lambda^{-1}V, y'^{-1}(x)) \right) M_d^c[e](\Lambda^{-1}V, y'^{-1}(x)) \Lambda_b^d(y'^{-1}(x))\end{aligned}\quad (3.8)$$

Whence it follows that the Mandelstam covariant transforms like a Lorentz tensor function

of V and a coordinate scalar function of x :

$$\begin{aligned}\mathfrak{P}_{abcd}'[e](V, x) &\equiv \mathfrak{P}_{abcd}[e'](V, x) \\ &= \Lambda_a^k(y'^{-1}(x)) \Lambda_b^l(y'^{-1}(x)) \Lambda_c^m(y'^{-1}(x)) \Lambda_d^n(y'^{-1}(x)) \mathfrak{P}_{klmn}[e](\Lambda^{-1}V, y'^{-1}(x))\end{aligned}\quad (3.9)$$

The set of all Mandelstam covariants taken from a given origin can serve as an operator basis for quantum gravity, at least perturbatively. That is, we can express the graviton field in any fixed gauge as a functional of $\mathfrak{P}_{abcd}[e](V, x)$ for fixed x . Thought of in this way the Mandelstam covariant is not just a Riemann tensor measured in a peculiar gauge. If it were then one could choose a gauge G (now thought of as including the local Lorentz group) such that:

$$\mathfrak{P}_{abcd}[e](V, x) \delta[G] = R_{abcd}(x + V) \delta[G] \quad (3.10)$$

for all values of V^a and all indices $abcd$. To see that this is not possible assume the converse. Inspection of (3.1) then implies:

$$\tilde{M}_b^a[e](V, x) \delta[G] = \delta_b^a \delta[G] \quad (3.11)$$

Whence it follows from (3.5) that:

$$\dot{\chi}^\mu[e](\tau, V, x) \Gamma_{\mu\beta}^a(\chi(\tau)) \delta[G] = 0 \quad (3.12a)$$

$$e_a^\mu(x) \delta[G] = \delta_a^\mu \delta[G] \quad (3.12b)$$

Combining these relations with (3.3) we obtain:

$$\chi^\mu[e](\tau, V, x) \delta[G] = (x^\mu + \tau V^\mu) \delta[G] \quad (3.13)$$

Substitution into (3.12a) and some trivial manipulation then gives:

$$g_{\alpha\beta,\mu}(x + \tau V) V^\mu \delta[G] = 0 \quad (3.14)$$

Hence we conclude that the V directional derivatives of the metric vanish at x , for all directions V^a . By combining the result for different V 's it follows that the mixed derivatives

vanish as well. But this implies that the curvature vanishes there — in fact it implies that all derivatives of the curvature vanish there. Such an invariant statement cannot be enforced by a choice of gauge so no such gauge exists, QED.*

Another way of seeing this is to note that condition (3.13) is, by itself, sufficient to establish a complete gauge. It gives what are usually known as “Riemann normal coordinates.” The invariantized Riemann tensor in this gauge is:

$$R_{abcd}[e](V, x) \equiv \frac{\partial \chi^\alpha(1)}{\partial V^\alpha} \frac{\partial \chi^\beta(1)}{\partial V^\beta} \frac{\partial \chi^\gamma(1)}{\partial V^\gamma} \frac{\partial \chi^\delta(1)}{\partial V^\delta} R_{\alpha\beta\gamma\delta}(\chi(1)) \quad (3.15a)$$

It relates to the Mandelstam covariant by the substitution:

$$\tilde{M}_b^a[e](V, x) \rightarrow e_\mu^a(\chi(1)) \frac{\partial \chi^\mu[e](1, V, x)}{\partial V^\beta} \quad (3.15b)$$

Since $\tilde{M}_b^a[e](V, x) \neq \delta_b^a$ in this gauge we know that $\mathcal{R}_{abcd}[e](V, x)$ cannot be local.

The fact that the Mandelstam covariant is not just a local function of the graviton field and its derivatives in some gauge has a profound significance for the meaning of time.

For example by taking the transformation parameters in (3.9) to be:

$$y^\mu(y) = y^\mu + \Delta t \delta_0^\mu \quad (3.16a)$$

$$\Delta_\alpha^b(z) = \delta_\alpha^b \quad (3.16b)$$

it is clear that ordinary time translation just shifts the Mandelstam origin x . This is what the ADM Hamiltonian generates. However the more natural quantity in terms of the Mandelstam operator basis we have just described is the “Mandelstam time,” i.e. V^0 .

* The above proof works for even for a finite number of directions V provided that number is greater than or equal to $D(D+1)/2$ for D spacetime dimensions. It is perhaps worth noting that the natural Yang-Mills gauge (but not general coordinate) covariant becomes local in radial gauge [29]. This is one of the differences between gravity and Yang-Mills and has to do with the fact that the symmetry group of gravity is not $GL(4)$.

They agree for matter covariants in flat space. For example the Mandelstam scalar:

$$\phi[e](V, x) \equiv \phi(\chi[e](1, V, x)) \quad (3.17)$$

degenerates in flat space to $\phi(x+V)$ and it is obviously irrelevant whether we say we are evolving x or V . The two notions of time also *probably* agree when the origin x is taken to spatial infinity and the asymptotic field is assumed to be non-dynamical.* However they are certainly different when the fields near x are dynamical — which is the case either when x is an interior point or when the asymptotic fields are allowed to undergo quantum oscillations. As we argued in II.B, this is precisely the sort of distinction we desire.

Using Mandelstam time amounts to adopting a sort of operator intrinsic evolution. It is important to realize that this sort of distinction can change physics. Indeed the transformations considered in II.C can be viewed as combinations of time independent field redefinitions and operatorial changes in the time coordinate.** The fact that this type of change alters the Hamiltonian was noted years previously by ADM [30]. They concluded that their Hamiltonian was unique provided the field variables were chosen so as to be time independent functionals of the metric and its first derivatives, and provided that the Hamiltonian density expressed in terms of these variables was free of explicit time dependence. In basing quantum gravity on a different Hamiltonian we are not questioning this result, but rather its assumptions. ADM were quite explicit about stating their belief that in “general relativity, the fundamental Heisenberg variables measured by rods and clocks

* The argument is easy to see perturbatively by expanding $\mathcal{R}$ in powers of the graviton field. Typical terms will involve products of h and its derivatives evaluated at $x+V$ multiplied by h 's at x and proper time integrals of h 's at $x+V$. If x is at asymptotic infinity and the field is non-dynamical there then $h(x) = 0$. To reach any finite point in spacetime from asymptotic infinity V must be infinite. Hence as long as r is any nonzero amount less than one the fields at $x+V$ are also in the non-dynamical region. Therefore only the fields at $x+V$ (and for r infinitesimally close to 1) contribute and these show no difference between translating x^0 and translating V^0 .

** To see this note that the equations of motion allow any field to be expressed in terms of some finite amount of initial value data. In comparing two field variables one ascribes differences in the initial value data to a time independent field redefinition. Any further differences that show up in evolution can then be subsumed into field-dependent redefinitions of the time coordinate.

are $g_{\mu\nu}$ and $\partial g_{\mu\nu}$." The reformulation we are proposing holds that it is the Mandelstam covariant which is the fundamental observable. It is not a time independent function of the metric and its first derivative, therefore we need not and do not agree with ADM time.

ADM time is the absolute coordinate time of a nondynamical observer at spatial infinity; Mandelstam time is the relative time of an inertial observer at x . Owing to this relative nature Mandelstam time makes sense even when ADM time does not. For example it can be applied when space is closed or not asymptotically empty. Note the distinction it makes from the usual resolution of the frozen formalism in which a component of the metric is chosen as "time" and the Wheeler-DeWitt equation is interpreted as the rule for evolving the wavefunction of the universe in terms of this variable [31]. Even though Mandelstam time differs from absolute time the component parts of the Mandelstam covariant are still created by evolving in absolute time — just as the fields of $\Psi[\phi] = \Phi_2^{-1}[\Phi_1[\phi]]$ in II.C evolve according to the S_1 equation of motion even though Ψ obeys the S_2 equation of motion. Therefore the functional formalism for Mandelstam Green's functions is the usual one.

III.C — Mandelstam Green's functions

We define the T^* -ordered Mandelstam N-point function as follows:

$$\mathcal{M}_{a_1 \dots a_N}^{*N} [\text{Gauge}](V_1, \dots, V_{N-1}; x) \equiv \int [dc] \left[\frac{\text{gauge fixing}}{\text{paraphernalia}} \right] \sqrt{-g(x)} \mathcal{R}_{a_1 b_1 c_1 d_1} [c](V_1, x) \mathcal{R}_{a_2 b_2 c_2 d_2} [c](V_2, x) \dots \mathcal{R}_{a_N b_N c_N d_N} [c](V_N, x) \exp(iS_H[g]) \quad (3.18a)$$

where the "gauge fixing paraphernalia" might impose an exact gauge (with a delta functional) or an average one (adding a gauge fixing term to the action) and includes both general coordinate and local Lorentz transformations. The V_i 's are assumed to sum to zero:

$$\sum_{i=1}^N V_i^a = 0 \quad (3.18b)$$

so that the N observations will really be centered on the Mandelstam origin x . The peculiar factor of $\sqrt{-g(x)}$ has been included to help purge the Green's function of gauge dependence. To obtain a fully gauge independent object the following steps are necessary:

- (1) Contract all indices. This can be done either into the existing indices or into derivatives with respect to the V_i^a 's.
- (2) Multiply against an invariant function of the V_i^a 's and integrate $\int d^D V_1 \dots \int d^D V_{N-1}$.
- (3) Integrate $\int d^D x$.

Our interpolating field has been crafted so that all of these operations can be carried out *after* the expectation value has been taken. As long as we work in a globally Poincaré invariant gauge they are not even necessary. There will be no dependence upon x^μ by virtue of translation invariance; and the result will contain no Lorentz vectors save the V_i^a 's as a consequence of Lorentz invariance.

Unfortunately $\mathcal{M}^{*N}$ lacks an important property: it does not permit a simple check of the equations of motion. When two indices on the same covariant are contracted it is easy to check that the parallel transport matrices just cancel and one is left with a Mandelstam Ricci scalar:

$$\eta^{ac} \mathcal{R}_{abcd} [c](V, x) = \bar{M}^d_b [c](V, x) \bar{M}^a_d [c](V, x) \bar{R}_{ln} (x|c|(1, V, x)) \quad (3.19)$$

This vanishes with the equations of motion, however contracting two indices within the same group in (3.18) — e.g. a_i and c_i — does not generally give zero. This derives from residual gauge dependence [19] and operator ordering problems. The canonical operator we are studying in the T^* -ordered product is obtained by expanding the functional argument — that is, the measure factor and the N covariants — in powers of the graviton field and its derivatives. One then time-orders the result with any derivatives taken *after* the time

ordering. Since the equations of motion contain derivatives it is necessary to operator order *after* differentiation if we want to see contraction (3.19) vanish in our Green's function.

However neither do we wish to perform simple time-ordering because this would lead to noncovariant factors of ϵ_0^0 . The scheme we have settled on is called "V-ordering." It makes use of the fact that every graviton field in the T^* -ordered product can be assigned a position on one of the background geodesics:

$$\chi^\mu |e_a^\nu = \delta_a^\nu |(\tau, V, x) = x^\mu + \tau V_a^\mu \quad (3.20)$$

V-ordering just arranges every graviton according to its value of τV_a^μ . It obviously reproduces T-ordering when no derivatives are present, but for derivatives it is the V direction which is special. This is because the ordering is according to time along the path, not simply time, and only differentiation along the path can change this.

As an example consider the correction to a two-graviton product when the gauge-fixed action is:

$$S_D[g] = S_H[g] - \frac{1}{2} \mu^{-4} \int d^D z \eta_{\mu\nu} G^\mu(z) G^\nu(z) \quad (3.21a)$$

$$G^\mu \equiv h^{\mu\nu}{}_{,\nu} - \frac{1}{2} h^{\nu\mu}{}_{,\nu} \quad (3.21b)$$

Using the associated canonical BRS theory it is easy to show that:

$$\begin{aligned} V(h_{\alpha\gamma}\beta\delta(x+V)h_{\alpha'\gamma'}\beta'\delta'(x-V)) - T^*(h_{\alpha\gamma}\beta\delta(x+V)h_{\alpha'\gamma'}\beta'\delta'(x-V)) \\ = i\mu^4 D P_{\alpha\gamma\alpha'\gamma'} D \beta\beta'\delta\delta' \delta^D(2V) + O(\kappa^2) \end{aligned} \quad (3.22a)$$

In this equation the propagator symbol is:

$$P_{\alpha\gamma\alpha'\gamma'} \equiv \eta_{\alpha\alpha'} \eta_{\gamma\gamma'} + \eta_{\alpha\gamma'} \eta_{\gamma\alpha'} - \bar{D}^2 \eta_{\alpha\gamma} \eta_{\alpha'\gamma'} \quad (3.22b)$$

* This is even true when perturbing around a nonflat background, although of course the expressions differ from (3.20).

and the differential operator $D\beta\beta'\delta\delta'$ is obtained by symmetrizing:

$$\begin{aligned} \frac{1}{4V^2} \left(V_\beta V_\delta \frac{\partial}{\partial V^{\beta'}} \frac{\partial}{\partial V^{\delta'}} + 4V_\beta V_{\beta'} \frac{\partial}{\partial V^\delta} \frac{\partial}{\partial V^{\delta'}} + V_{\beta'} V_{\delta'} \frac{\partial}{\partial V^\beta} \frac{\partial}{\partial V^\delta} \right) \\ - \frac{1}{V^4} \left(V_\beta V_\delta V^\mu V_{\beta'} \frac{\partial}{\partial V^{\delta'}} \frac{\partial}{\partial V^\mu} + V_{\beta'} V_{\delta'} V^\mu V_\beta \frac{\partial}{\partial V^\delta} \frac{\partial}{\partial V^\mu} \right) \\ + \frac{1}{4V^4} V_\beta V_\delta V_{\beta'} V_{\delta'} \left(-\eta^{\mu\nu} + \frac{4}{V^2} V^\mu V^\nu \right) \frac{\partial}{\partial V^\mu} \frac{\partial}{\partial V^\nu} \end{aligned} \quad (3.22c)$$

on the pairs $(\beta\delta)$ and $(\beta'\delta')$. Higher order corrections will involve operators, including the Faddeev-Popov ghost fields.

Even the V-ordered Mandelstam N-point function must still generally be subjected to the three operations described above in order to remove gauge dependence. However, as before, these operations are superfluous when using a globally Poincaré invariant gauge (GPIG) and background. We call this object the "invariant N-point function."

$$\begin{aligned} \mathcal{M}_{a_1 \dots a_N}^{N'}(V_1, \dots, V_{N-1}) \equiv \mathcal{M}_{a_1 \dots a_N}^{N'}[GPIG](V_1, \dots, V_{N-1}; 0) \\ - \left(\text{V-ordering corrections} \right)_{a_1 \dots a_N} [GPIG](V_1, \dots, V_{N-1}) \end{aligned} \quad (3.23)$$

Note that $\mathcal{M}^N$ does not depend upon the location of the Mandelstam origin. Since $\mathcal{M}^N$ represents the expectation value of an invariant operator contractions within the same index group really vanish by virtue of an argument we have given elsewhere [19]. This provided an important check on the calculation reported in Section IV. The "invariant S-matrix" would obtain from subjecting $\mathcal{M}^N$ to ordinary LSZ reduction just as though it were the expectation value of N linearized Riemann tensors at positions $x + V_i$.

III.D — Modifications and extensions

It should be noted that our geometrical insight may be faulty in translating the word-experiment of III.A into a quantum operator. Operators whose physical significance is clear on the classical level typically require corrections to mean the same thing in quantum field theory. Hence the Riemann tensor may not be at all what a quantum mechanical observer measures when he studies the tidal force. In fact these sorts of interpretational problems

can even occur classically. For example it is usual to assume that the metric measures the infinitesimal distance between two nearby points x and $x + dx$:

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \quad (3.24)$$

However if this is true in a conventional metric theory whose action is $S[g]$ then it is certainly not true in the related conformal scalar-metric theory whose action is:

$$S[\phi, g] \equiv S[\phi^2 g] \quad (3.25)$$

That role is played by the field $\phi^2(x)g_{\mu\nu}(x)$. Slavish insistence upon regarding $g_{\mu\nu}(x)$ as the physical metric contributed to the fallacious notion of a few years back that conformal scalar-metric theories might exhibit different physics from their metric parents [32]. And once the point is conceded in the conformal scalar-metric context it is a small step to questioning (3.24) in the original, pure metric formulations.

There is also the possibility that our word-experiment contains an idealization which precludes its physical realization. The feature that we are most suspicious of in this regard is the infinite sharpness with which the Mandelstam observer sticks to classical free fall. He is in fact following his classical geodesic, albeit with respect to the quantum metric. It turns out that this is equivalent to coupling the theory to a point particle of infinite mass and this sounds suspiciously unphysical.

To see the point note that the geodesic $X^\mu[e](\tau, V, x)$ obeys the classical equation of motion for a point particle coupled to gravity. Hence any functional of $X^\mu[e](\tau, V, x)$ can be obtained by taking the classical limit of the same functional of a quantum point particle $X^\mu(\tau)$:

$$F[X[e]] = \lim_{\hbar \rightarrow 0} \int_{\text{B.C.}} [dX] F[X] \exp \left[i \frac{1}{2\ell} \frac{m}{\hbar} S_X[X, g] \right] \quad (3.26a)$$

$$S_X[X, g] \equiv \int d\tau \left\{ g_{\alpha\beta}(X(\tau)) \dot{X}^\alpha(\tau) \dot{X}^\beta(\tau) - \ell^2(1, V) \right\} \quad (3.26b)$$

We must, of course, choose a state whose classical limit obeys initial conditions (3.3b)

and (3.3c). For the purposes of perturbation theory this is equivalent to specifying the propagator and the background — and is signified above by the notation “B.C.” The factor of $\frac{1}{2\ell}$ — where $\ell \equiv \ell(1, V)$ is defined in (3.4) — has been included to make the particle’s stress energy agree on shell with the usual one. Since the particle’s mass m and the quantization parameter only enter together in the ratio $\frac{m}{\hbar}$ we see that the classical limit ($\hbar \rightarrow 0$) agrees with the infinite mass one.

We believe that this is what causes the double poles in $(D-4)$ that we shall find in the Mandelstam 2-point function at one loop. If so the problem is easily fixed. One has merely to express the Mandelstam covariant using (3.26) and then not take the limit $\hbar \rightarrow 0$ (or $m \rightarrow \infty$). Physically this says that the Mandelstam observer only follows his classical path to within an uncertainty $\Delta x = \hbar/m$. We call this the “quantum observer” scenario for obvious reasons. The parameter m would thus play a role very similar to that of the detector efficiency in QED. Presumably it would furnish a sort of cutoff.

It is easy to contemplate other sorts of modifications. For instance, we could substitute some other geometrically meaningful path for the geodesic. The simplest way to do this would be to modify (3.3a) by introducing an arbitrary source $S^\mu(\tau)$:

$$\ddot{X}_S^\mu(\tau) + \Gamma^\mu_{\rho\sigma}(X_S(\tau)) \dot{X}_S^\rho(\tau) \dot{X}_S^\sigma(\tau) = S^\mu(\tau) \quad (3.27)$$

As long as $S^\mu(\tau)$ transforms like a vector at the point $X_S(\tau)$, the path will transform like (3.7) so the modified Mandelstam covariant would still obey (3.9). An example of such a source is:

$$S^\mu(\tau) = \kappa^2 R^\mu_{\alpha\beta\gamma} \dot{X}_S^\alpha(\tau) \dot{X}_S^\beta(\tau) \dot{X}_S^\gamma(\tau) \quad (3.28)$$

A physical reason for including it might be to account for the spin force the observer would feel due to the gyroscopes he is carrying. Such a source might also include matter fields in order to account for the renormalizations that matter loops induce in gravitational quantities.

Another possibility is to add an arbitrary tensor $S_{\mu\beta}^{\alpha}(\tau)$ to the affine connection:

$$\Gamma_{S\mu\beta}^{\alpha}(\tau) = \Gamma_{\mu\beta}^{\alpha}(\chi(\tau)) + S_{\mu\beta}^{\alpha}(\tau) \quad (3.29)$$

If $S_{\mu\beta}^{\alpha}(\tau)$ transforms like a tensor at $\chi(\tau)$ then $\Gamma_{S\mu\beta}^{\alpha}(\tau)$ will transform like a good connection and we can use it in place of $\Gamma_{\mu\beta}^{\alpha}$ in (3.5). The resulting modified parallel transport matrices will still obey (3.8) so the modified covariant will obey (3.9). An example would be:

$$S_{\mu\beta}^{\alpha}(\tau) = R_{\mu\beta\gamma}^{\alpha}(\chi(\tau)) \dot{\chi}^{\gamma}(\tau) \quad (3.30)$$

A physical reason for including it might be to more accurately model the parallel transport process. This might be very significant when matter couplings are added.

We view these sorts of modifications as a kind of "operator renormalization" in the struggle to obtain a perturbatively finite incarnation of quantum gravity. If one could be found that gave ultraviolet finite Green's functions, and for which there was any sort of plausible physical justification, we should feel quite unashamed in adopting it. The analysis of II.C suffices to show that this is possible. We know of no better way than to simply explore a few possibilities and see what happens. The rest of this paper is devoted to such an exploration for the simplest possibility.

IV. THE CALCULATION

The purpose of this section is to present our result for one loop corrections to the invariant 2-point function [33]. We begin by explaining what the notation means and roughly how the calculation was performed. One of the thirty principle component parts is presented in some detail as an example. It is also natural at this point to explain the curious "prefactors" -1 , 2^D , 2^{2D} and $-\frac{2^{D-1}}{D-1} 2F_1(D+1, 1-D; 2-D; \frac{1}{2})$ — which arise from the geodesic parameter integrals. The final answer is a linear combination of 8-index tensors whose coefficients are given in Tables iii-vi — one for each of the four prefactors.

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The section closes with a discussion of why we believe this computation — one of the most complex ever attempted in quantum gravity — is free of error.

IV.A — What we computed and how

We have computed the fourier transform of the invariant 2-point function:

$$\tilde{M}_{abcd\alpha'\epsilon'd'}^{2\alpha\beta\gamma\delta}(p) \equiv \int d^D(2V) \exp(i2V \cdot p) M_{abcd\alpha'\epsilon'd'}^{2\alpha\beta\gamma\delta}(V) \quad (4.1)$$

at order κ^2 (tree) and κ^4 (one loop). The basic method was straightforward if extremely tedious. We first expanded the functional $\sqrt{-g(0)} \mathcal{R}_{abcd}[c](V, 0) \mathcal{R}_{\alpha'\epsilon'\gamma\delta}[c](-V, 0)$ in powers of the graviton field. To the order we worked only 2, 3, and 4-graviton functions contribute. We organized the expectation values of these in a convenient way into thirty "k-point functions" which were then evaluated using dimensional regularization and the Feynman rules given in a previous paper [19]. The V -ordering corrections were then added and the final result was expressed as a linear combination of 8-index tensors constructed from η_{ab} and p_a . We begin by defining these "master tensor structures" and then sketch the graviton expansion. The former is of course necessary to make sense of the tabulated combination coefficients; the latter defines the thirty k-point functions whose values we have reported as a check on accuracy. The expansions would also be useful in solving for the Hamiltonian which generates Mandelstam time. We close by computing the V -ordering corrections.

By changing variables in (4.1) we conclude that the invariant 2-point function is symmetric under the interchange of index groups:

$$\tilde{M}_{abcd\alpha'\epsilon'd'}^{2\alpha\beta\gamma\delta}(p) = \tilde{M}_{\alpha'\epsilon'\gamma\delta\beta\delta\alpha\alpha'}^{2\alpha\beta\gamma\delta}(p) \quad (4.2a)$$

Since any 8-index object constructed from η_{ab} 's and p_a 's must contain an even number of p 's it is also invariant under $p \rightarrow -p$. The algebraic symmetries each Mandelstam covariant

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derives from the Riemann tensor imply three additional relations:

$$\tilde{M}_{ab\bar{c}d\bar{a}'b'c'd'}^2(p) = -\tilde{M}_{bac\bar{d}a'b'c'd'}^2(p) \quad (4.2b)$$

$$\tilde{M}_{ab\bar{c}d\bar{a}'b'c'd'}^2(p) = \tilde{M}_{cd\bar{a}b\bar{a}'b'c'd'}^2(p) \quad (4.2c)$$

$$\tilde{M}_{ab\bar{c}d\bar{a}'b'c'd'}^2(p) + \tilde{M}_{ac\bar{d}b\bar{a}'b'c'd'}^2(p) + \tilde{M}_{adb\bar{c}\bar{a}'b'c'd'}^2(p) = 0 \quad (4.2d)$$

Each of the thirty k-point functions into which we split $\tilde{M}^2$ were chosen so as to separately obey these relations.

Suppose that an 8-index tensor $T_{ab\bar{c}d\bar{a}'b'c'd'}$ is symmetric within the index pairs (ac) , (bd) , $(a'c')$ and $(b'd')$. It is easy to see that the following "standard permutation" yields a tensor $\{T_{ab\bar{c}d\bar{a}'b'c'd'}\}$ which obeys relations (4.2):

$$T_{ab\bar{c}d\bar{a}'b'c'd'}^i \equiv T_{ab\bar{c}d\bar{a}'b'c'd'} - T_{ab\bar{c}d\bar{c}'a'b'c'd'} + T_{ab\bar{c}d\bar{b}'a'c'b'c'd'} - T_{ab\bar{c}d\bar{c}'b'a'c'd'} \quad (4.3a)$$

$$T_{ab\bar{c}d\bar{a}'b'c'd'}^{ii} \equiv T_{ab\bar{c}d\bar{a}'b'c'd'}^i - T_{ab\bar{c}d\bar{a}'b'c'd'}^{i\bar{a}} + T_{ab\bar{c}d\bar{a}'b'c'd'}^{i\bar{b}} - T_{ab\bar{c}d\bar{a}'b'c'd'}^{i\bar{c}} \quad (4.3b)$$

$$\{T_{ab\bar{c}d\bar{a}'b'c'd'}\} \equiv T_{ab\bar{c}d\bar{a}'b'c'd'}^{ii} + T_{a'b'c'd'\bar{a}\bar{b}\bar{c}\bar{d}} \quad (4.3c)$$

The standard permutation cannot alter either the number of free indices residing on p 's versus the number on η 's, or how the indices are split between index groups. Therefore we span the space of tensors constructed from η_{ab} and p_a and satisfying (4.2) by merely listing the generic types according to the number of uncontracted p 's and how the indices are split. We call nine the elements of this tensor basis "master tensor structures" (MTS):

$$\hat{e}_i \equiv \left\{ \frac{1}{2} p_a p_c p_{a'} p_{c'} \eta_{bd} \eta_{b'd'} \right\} \quad (4.4a)$$

$$\hat{e}_{ii} \equiv \left\{ \frac{1}{2} p_a p_c p_{c'} p_{c'} (\eta_{bb'} \eta_{dd'} + \eta_{b'd'} \eta_{db'}) \right\} \quad (4.4b)$$

$$\hat{d}_i \equiv \left\{ \frac{1}{2} p^2 p_a p_c \eta_{a'c'} \eta_{bd} \eta_{b'd'} \right\} \quad (4.4c)$$

$$\hat{d}_{ii} \equiv \left\{ p^2 p_a p_c \eta_{a'c'} (\eta_{bb'} \eta_{dd'} + \eta_{b'd'} \eta_{db'}) \right\} \quad (4.4d)$$

$$\hat{d}_{iii} \equiv \left\{ \frac{1}{2} p^2 (p_a p_{a'} \eta_{cc'} + p_a p_{c'} \eta_{ca'} + p_c p_{a'} \eta_{ac'} + p_c p_{c'} \eta_{aa'}) \eta_{bd} \eta_{b'd'} \right\} \quad (4.4e)$$

$$\hat{d}_{iv} \equiv \left\{ \frac{1}{2} p^2 (p_a p_{a'} \eta_{cc'} + p_a p_{c'} \eta_{ca'} + p_c p_{a'} \eta_{ac'} + p_c p_{c'} \eta_{aa'}) (\eta_{bb'} \eta_{dd'} + \eta_{b'd'} \eta_{db'}) \right\} \quad (4.4f)$$

$$\hat{e}_i \equiv \left\{ \frac{1}{8} p^4 \eta_{ac} \eta_{bd} \eta_{a'c'} \eta_{b'd'} \right\} \quad (4.4g)$$

$$\hat{e}_{ii} \equiv \left\{ \frac{1}{2} p^4 \eta_{ac} \eta_{a'c'} (\eta_{bb'} \eta_{dd'} + \eta_{b'd'} \eta_{db'}) \right\} \quad (4.4h)$$

$$\hat{e}_{iii} \equiv \left\{ \frac{1}{4} p^4 (\eta_{ac} \eta_{cc'} + \eta_{ac'} \eta_{ca'}) (\eta_{bb'} \eta_{dd'} + \eta_{b'd'} \eta_{db'}) \right\} \quad (4.4i)$$

Note that we have suppressed the ubiquitous subscript list $abcd a' b' c' d'$.

Although all the MTS are needed in order to express our result for the thirty k-point functions the final answer for $\tilde{M}^2$ has a much more restricted form owing to the dynamical relations imposed by the equations of motion:

$$\eta^{ac} \tilde{M}_{ab\bar{c}d\bar{a}'b'c'd'}^2(p) = 0 \quad (4.5)$$

It turns out that only the following three linear combinations of MTS obey these relations:

$$\hat{e} \equiv \hat{e}_i - \frac{1}{2} (D-2) \hat{e}_{ii} - \frac{1}{4} \hat{d}_{iv} + \frac{1}{12} (D-2) \hat{d}_{iv} \quad (4.6a)$$

$$\hat{d} \equiv \hat{d}_i - \frac{1}{4} (D-2) \hat{d}_{ii} - \frac{1}{8} D \hat{d}_{iii} + \frac{1}{24} (D-2)^2 \hat{d}_{iv} - \frac{1}{4} \hat{e}_{ii} + \frac{1}{6} (D-2) \hat{e}_{iii} \quad (4.6b)$$

$$\hat{e} \equiv \hat{e}_i - \frac{1}{4} (D-1) \hat{e}_{ii} + \frac{1}{12} (D-2) (D-1) \hat{e}_{iii} \quad (4.6c)$$

We call these "restricted master tensor structures," or RMTS for short. They are linearly independent for all $D > 4$. At $D = 4$ it can be shown that $\hat{d} = \hat{e}$. For $D \leq 3$ all three RMTS vanish.

We turn now to the graviton expansions. We shall give these first for the building blocks — the density factor, the vierbein, the geodesic, the parallel transport matrix and the Riemann tensor. Then the Mandelstam covariant is expanded and the thirty k-point functions are defined. Because of dimensional regularization and the masslessness of the graviton we can ignore — to the order we are working — terms with more than two h 's evaluated at the same point.

The density factor is most easily obtained by first expanding the determinant:

$$\begin{aligned} -\det(g) &\equiv -\frac{1}{J} \epsilon^{\alpha_1 \dots \alpha_D} \epsilon^{\beta_1 \dots \beta_D} g_{\alpha_1 \beta_1} \dots g_{\alpha_D \beta_D} \\ &= 1 + \kappa h + \frac{1}{2} \kappa^2 (h^2 - h^{\rho\sigma} h_{\rho\sigma}) + O(\kappa^3 h^3) \end{aligned} \quad (4.7c)$$

Then one takes the square root:

$$\sqrt{-g} = 1 + \frac{1}{2} \kappa h + \frac{1}{8} \kappa^2 (h^2 - 2h^{\rho\sigma} h_{\rho\sigma}) + O(\kappa^3 h^3) \quad (4.7b)$$

As usual in perturbation theory graviton indices are raised and lowered with the Lorentz metric and $h \equiv \eta^{\mu\nu} h_{\mu\nu}$.

Although our fundamental field is the vierbein we can express it, for the purposes of perturbation theory, in terms of the graviton (1.5) by fixing to Lorentz symmetric gauge [17]:

$$e_{ab} - e_{ba} = 0 \quad (4.8a)$$

$$e_{\mu a} = \eta_{\mu a} + \frac{1}{2} \kappa h_{\mu a} - \frac{1}{8} \kappa^2 h_{\mu\nu} h_{\nu a} + O(\kappa^3 h^3) \quad (4.8b)$$

$$e^{\mu a} = \eta^{\mu a} - \frac{1}{2} \kappa h^{\mu a} + \frac{3}{8} \kappa^2 h^{\mu\nu} h_{\nu}^a + O(\kappa^3 h^3) \quad (4.8c)$$

In these and all subsequent expressions we shall make no distinction between local Lorentz and coordinate indices.

The geodesic expansion is defined as follows:

$$X^\mu(\tau, V, 0) \equiv A^\mu(\tau, V) + \kappa B^\mu(\tau, V) + \kappa^2 C^\mu(\tau, V) + O(\kappa^3) \quad (4.9a)$$

By solving (3.3) we see that the coefficient are:

$$A^\mu(\tau, V) = V^\mu_\tau \quad (4.9b)$$

$$B^\mu(\tau, V) = -\frac{1}{2} h^\mu_\nu(0) V^\nu_\tau - \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \delta^\mu_{\rho\sigma}(\tau_2 V) V^\rho V^\sigma \quad (4.9c)$$

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$$\begin{aligned} C^\mu(\tau, V) &= \frac{3}{8} h^{\mu\rho}(0) h_{\rho\sigma}(0) V^\sigma_\tau + \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 h^{\mu\nu}(\tau_2 V) h_{\nu\rho}(\tau_2 V) V^\rho V^\sigma \\ &\quad - \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \left(b^\mu_{\rho\sigma\nu}(\tau_2 V) V^\rho V^\sigma B^\nu(\tau_2, V) + 2b^\mu_{\rho\sigma}(\tau_2 V) V^\rho \tilde{B}^\sigma(\tau_2, V) \right) \end{aligned} \quad (4.9d)$$

Here the symbol " $b^\mu_{\rho\sigma}$ " denotes the order κ term in the expansion of the affine connection:

$$\Gamma^\alpha_{\beta\gamma} \equiv \kappa b^\alpha_{\beta\gamma} + \kappa^2 c^\alpha_{\beta\gamma} + O(\kappa^3) \quad (4.10a)$$

$$b^\alpha_{\beta\gamma} \equiv \frac{1}{2} (h^\alpha_{\beta,\gamma} + h^\alpha_{\gamma,\beta} - h_{\beta\gamma,\alpha}) \quad (4.10b)$$

We did not require the order κ^2 term.

The expansion for the parallel transport matrix is defined as follows:

$$\bar{M}^\alpha_b(V, 0) \equiv \delta^\alpha_b + \kappa \bar{b}^\alpha_b(V) + \kappa^2 \bar{c}^\alpha_b(V) + O(\kappa^3) \quad (4.11a)$$

Substitution of the previous expansions into (3.5) determines the coefficient functions to be:

$$\bar{b}^\alpha_b(V) = -\frac{1}{2} \int_0^1 d\tau \left(h^\alpha_{\mu,\beta}(\tau V) - h_{\beta\mu,\alpha}(\tau V) \right) V^\mu \quad (4.11b)$$

$$\begin{aligned} \bar{c}^\alpha_b(V) &= \frac{1}{4} \int_0^1 d\tau \left(\frac{1}{2} h^{\alpha\rho}(\tau V) h_{\rho b,\mu}(\tau V) - \frac{1}{2} h_{b\rho}(\tau V) h^{\rho\sigma}_{\mu}(\tau V) + h^{\alpha\rho}(\tau V) h_{\rho\mu b}(\tau V) \right. \\ &\quad \left. - h_b^\rho(\tau V) h_{\rho\mu,\alpha}(\tau V) - h^{\alpha\rho}(\tau V) h_{b\mu,\rho}(\tau V) + h_b^\rho(\tau V) h^{\alpha}_{\mu,\rho}(\tau V) \right) V^\mu \\ &\quad + \frac{1}{4} \int_0^1 d\tau \int_0^\tau d\tau' \left(h^\alpha_{\mu,\rho}(\tau V) - h_{\rho\mu,\alpha}(\tau V) \right) V^\mu \left(h^\rho_{\nu b}(\tau' V) - h_{b\nu,\rho}(\tau' V) \right) V^\nu \\ &\quad - \frac{1}{2} \int_0^1 d\tau \left(\left(h^\alpha_{\mu\nu b}(\tau V) - h_{b\mu\nu,\alpha}(\tau V) \right) V^\mu B^\nu(\tau, V) \right. \\ &\quad \left. + \left(h^\alpha_{\mu b}(\tau V) - h_{b\mu,\alpha}(\tau V) \right) \tilde{B}^\mu(\tau, V) \right) \end{aligned} \quad (4.11c)$$

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A good check on this last term is:

$$\varepsilon^{ab}(V) + \varepsilon^{ba}(V) + \bar{\varepsilon}_\rho^a(V) \bar{\varepsilon}^{b\rho}(V) = 0 \quad (4.11d)$$

which is the order κ^2 term in the expression that parallel transport to and from a point along the same path gives no change.

It is useful to expand $R_{\alpha\beta\gamma\delta}$ first, before attaching vierbeins to obtain $\bar{R}_{abcd}$. From (1.4) we find:

$$R_{\alpha\beta\gamma\delta} \equiv \kappa b_{\alpha\beta\gamma\delta} + \kappa^2 c_{\alpha\beta\gamma\delta} + O(\kappa^3) \quad (4.12a)$$

$$b_{\alpha\beta\gamma\delta} = -\frac{1}{2}(h_{\alpha\gamma,\beta\delta} - h_{\delta\alpha,\gamma\beta} + h_{\beta\delta,\alpha\gamma} - h_{\gamma\beta,\alpha\delta}) \quad (4.12b)$$

$$c_{\alpha\beta\gamma\delta} = -b^\rho \gamma_\alpha b_{\rho\delta}\beta + b^\rho \delta_\alpha b_{\rho\gamma}\beta \quad (4.12c)$$

There is no danger of confusion between these and the coefficients of the affine connection owing to the different number of indices. The problem of contracting an arbitrary, not necessarily symmetric, 2-index tensor r_b^a onto each index of an arbitrary 4-index tensor s_{abcd} occurs with such frequency that we have developed the following special notation for it:

$$[rs]_{abcd} \equiv r_a^p s_{pbcd} + r_b^p s_{apcd} + r_c^p s_{abpd} + r_d^p s_{abcp} \quad (4.13a)$$

$$[rrs]_{abcd} \equiv r_a^p r_b^q s_{pqcd} + r_a^p r_c^q s_{pbcd} + r_a^p r_d^q s_{pbcd} + r_b^p r_c^q s_{pabd} + r_b^p r_d^q s_{pabd} + r_c^p r_d^q s_{pabd} + r_c^p r_d^q s_{pabd} \quad (4.13b)$$

With this symbol the expansion for $\bar{R}_{abcd}$ becomes:

$$\bar{R}_{abcd} \equiv \kappa \bar{b}_{abcd} + \kappa^2 \bar{c}_{abcd} + O(\kappa^3) \quad (4.14a)$$

$$\bar{b}_{abcd} = b_{abcd} \quad (4.14b)$$

$$\bar{c}_{abcd} = c_{abcd} - \frac{1}{2}[h b]_{abcd} \quad (4.14c)$$

There should be no confusion between these and the coefficients of the parallel transport matrix owing to the different number of indices.

The expansion for the Mandelstam covariant is defined as follows:

$$\begin{aligned} \mathfrak{P}_{abcd}(V, 0) = & \kappa \beta_{abcd}^U(V) + \kappa^2 \left(\gamma_{abcd}^U(V) + \gamma_{abcd}^L(V') + \gamma_{abcd}^I(V') \right) \\ & + \kappa^3 \left(\delta_{abcd}^U(V) + \delta_{abcd}^L(V) + \delta_{abcd}^I(V) + \delta_{abcd}^{LI}(V) \right) + O(\kappa^4) \end{aligned} \quad (4.15)$$

We have found it convenient to break the result at each order up into “uncorrected” (superscript U) terms, “length corrections” (superscript L), “index corrections” (superscript I), and “length and index corrections” (superscript LI). The first derive solely from the Riemann tensor evaluated at V :

$$\beta_{abcd}^U(V) = \bar{b}_{abcd}(V) \quad (4.16a)$$

$$\gamma_{abcd}^U(V) = \bar{\varepsilon}_{abcd}(V) \quad (4.16b)$$

(δ_{abcd}^U is not needed because all three of its gravitons are evaluated at the same point.)

The length corrections account for the dynamical nature of the measurement point:

$$\gamma_{abcd}^L(V) = \bar{b}_{abcd,\mu}(V) B^\mu(1, V) \quad (4.17a)$$

$$\delta_{abcd}^L(V) = \bar{\varepsilon}_{abcd,\mu}(V) B^\mu(1, V) + \bar{b}_{abcd,\mu}(V) C^\mu(1, V) + \frac{1}{2} \bar{b}_{abcd,\mu\nu} B^\mu(1, V) B^\nu(1, V) \quad (4.17b)$$

The index corrections account for the dynamical nature of the measurement frame:

$$\gamma_{abcd}^I(V) = [\bar{b} \bar{b}]_{abcd}(V) \quad (4.18a)$$

$$\delta_{abcd}^I(V) = [\bar{b} \bar{\varepsilon}]_{abcd}(V) + [\bar{\varepsilon} \bar{b}]_{abcd}(V) + [\bar{b} \bar{b} \bar{\varepsilon}]_{abcd}(V) \quad (4.18b)$$

Note that the first tensors in the bracket symbols above must have two indices and hence from the parallel transport matrix while the final tensor must have four indices and is therefore from the Riemann tensor. The length and index corrections mingle the effect of the dynamical measurement point with that of the dynamical measurement frame:

$$\delta_{abcd}^{LI}(V) = [\bar{b} \bar{b}_{,\mu}] B^\mu(1, V) \quad (4.19)$$

These expansions could be used to make a perturbative determination of the Mandelstam Hamiltonian in flat space. The simplest gauge would seem to be that of (3.13), corre-

sponding to the Riemann normal coordinate system based on x . In these coordinates all the L and $L1$ corrections vanish.

The invariant 2-point function is broken down into k -point functions in a similar way. The symbol for each k -point function is a two or three letter code, followed by a superscript. The first one or two letters denote the number of gravitons — “ TW ” for 2, “ T ” for 3 and “ F ” for 4. The final one or two letters denote the type of correction. In addition to the previously considered “ U ”, “ L ”, “ T ” and “ LT ” designations we must add “ D ” for “density,” “ LD ” for “length and density” and “ UD ” for “index and density.” Keeping these straight will tell us many interesting things. For example $\tilde{M}^2$ would not be a gauge invariant and would not obey (4.5) but for the density factor, even though no integration is being performed over the Mandelstam origin.

Each of the thirty k -point functions is the fourier transform of a T^* -ordered Green's function evaluated in the gauge-fixed theory defined in (3.21). Although it abuses the notation somewhat we shall write them as expectation values of gauge-variant operators. We shall also suppress the ubiquitous index list $abcdab'c'd'$. First come the six terms which give the uncorrected Riemann tensor:

$$TWU^1 \equiv \kappa^2 \int d^D(2V) \exp(i2V \cdot p) \langle T^* (b_{abcd}(V) b_{a'b'c'd'}(-V)) \rangle \quad (4.20a)$$

$$TU^1 \equiv \kappa^3 \int d^D(2V) \exp(i2V \cdot p) \langle T^* (b_{abcd}(V) c_{a'b'c'd'}(-V) + c_{abcd}(V) b_{a'b'c'd'}(-V)) \rangle \quad (4.20b)$$

$$TU^2 \equiv -\frac{1}{2} \kappa^3 \int d^D(2V) \exp(i2V \cdot p) \langle T^* (b_{abcd}(V) [h]_{a'b'c'd'}(-V) + [h]_{abcd}(V) b_{a'b'c'd'}(-V)) \rangle \quad (4.20c)$$

$$FU^1 \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \langle T^* (c_{abcd}(V) c_{a'b'c'd'}(-V)) \rangle \quad (4.20d)$$

$$FU^2 \equiv -\frac{1}{2} \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \langle T^* (c_{abcd}(V) [h]_{a'b'c'd'}(-V) + [h]_{abcd}(V) c_{a'b'c'd'}(-V)) \rangle \quad (4.20e)$$

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$$FU^3 \equiv \frac{1}{4} \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \langle T^* ([h]_{abcd}(V) [h]_{a'b'c'd'}(-V)) \rangle \quad (4.20f)$$

The order κ^4 loop correction in (4.20a) was done using unpublished information provided by D. M. Capper [34] and partially reported in [20].

Next come the eight length corrections:

$$TL^1 \equiv \kappa^3 \int d^D(2V) \exp(i2V \cdot p) \langle T^* (b_{abcd,\mu}(V) B^\mu(1, V) b_{a'b'c'd'}(-V) + b_{abcd}(V) b_{a'b'c'd',\mu}(-V) B^\mu(1, -V)) \rangle \quad (4.21a)$$

$$FL^1 \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \langle T^* (c_{abcd,\mu}(V) B^\mu(1, V) c_{a'b'c'd'}(-V) + b_{abcd}(V) c_{a'b'c'd',\mu}(-V) B^\mu(1, -V)) \rangle \quad (4.21b)$$

$$FL^2 \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \langle T^* (c_{abcd}(V) b_{a'b'c'd',\mu}(-V) B^\mu(1, -V) + b_{abcd,\mu}(V) B^\mu(1, V) c_{a'b'c'd'}(-V)) \rangle \quad (4.21c)$$

$$FL^3 \equiv -\frac{1}{2} \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \langle T^* ([h]_{abcd,\mu}(V) B^\mu(1, V) \times b_{a'b'c'd'}(-V) + b_{abcd}(V) [h]_{a'b'c'd',\mu}(-V) B^\mu(1, -V)) \rangle \quad (4.21d)$$

$$FL^4 \equiv -\frac{1}{2} \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \langle T^* ([h]_{abcd}(V) b_{a'b'c'd',\mu}(-V) \times B^\mu(1, -V) + b_{abcd,\mu}(V) B^\mu(1, V) [h]_{a'b'c'd'}(-V)) \rangle \quad (4.21e)$$

$$FL^5 \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \langle T^* (b_{abcd,\mu}(V) B^\mu(1, V) \times b_{a'b'c'd',\mu}(-V) B^\mu(1, -V)) \rangle \quad (4.21f)$$

$$FL^6 \equiv \frac{1}{2} \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \langle T^* (b_{abcd,\mu\nu}(V) B^\mu(1, V) B^\nu(1, V) \times b_{a'b'c'd'}(-V) + b_{abcd}(V) b_{a'b'c'd',\mu\nu}(-V) B^\mu(1, -V) B^\nu(1, -V)) \rangle \quad (4.21g)$$

$$FL^7 \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \langle T^* (b_{abcd,\mu}(V) C^\mu(1, V) b_{a'b'c'd'}(-V) + b_{abcd}(V) b_{a'b'c'd',\mu}(V) C^\mu(1, -V)) \rangle \quad (4.21h)$$

We have reported TL^1 in two portions, TL^{1a} and TL^{1b} , according to the two terms in (4.9c). We also found it convenient to transfer some terms from FL^1 to FL^2 and from

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FL^3 to FL^4 . We report the modified 4-point functions as FL^{12} , FL^{21} , FL^{34} , and FL^{43} respectively.

There are also eight index functions:

$$TJ^1 \equiv \kappa^3 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left([\bar{b}b]_{abcd}(V) b_{a'b'c'd'}(-V) + b_{abcd}(V) [\bar{b}b]_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.22a)$$

$$FJ^1 \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left([\bar{b}c]_{abcd}(V) b_{a'b'c'd'}(-V) + b_{abcd}(V) [\bar{b}c]_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.22b)$$

$$FJ^2 \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left(c_{abcd}(V) [\bar{b}b]_{abcd}(-V) + [\bar{b}b]_{a'b'c'd'}(V) c_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.22c)$$

$$FJ^3 \equiv -\frac{1}{2}\kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left([\bar{b}|h b]_{abcd}(V) b_{a'b'c'd'}(-V) + b_{abcd}(V) [\bar{b}|h b]_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.22d)$$

$$FJ^4 \equiv -\frac{1}{2}\kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left([h b]_{abcd}(V) [\bar{b}b]_{a'b'c'd'}(-V) + [\bar{b}b]_{abcd}(V) [h b]_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.22e)$$

$$FJ^5 \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left([\bar{b}b]_{abcd}(V) [\bar{b}b]_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.22f)$$

$$FJ^6 \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left([\bar{b}\bar{b}b]_{abcd}(V) b_{a'b'c'd'}(-V) + b_{abcd}(V) [\bar{b}\bar{b}b]_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.22g)$$

$$FJ^7 \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left([\bar{c}b]_{abcd}(V) b_{a'b'c'd'}(-V) + b_{abcd}(V) [\bar{c}b]_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.22h)$$

For technical reasons we found it convenient to report FJ^5 , FJ^6 and FJ^7 each in two portions depending upon how the gravitons pair up into propagators. This will be explained in the next subsection.

The four density functions are much simpler:

$$TD^1 \equiv \frac{1}{2}\kappa^3 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left(h(0) b_{abcd}(V) b_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.23a)$$

$$FD^1 \equiv \frac{1}{2}\kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left(h(0) c_{abcd}(V) b_{a'b'c'd'}(-V) + h(0) b_{abcd}(V) c_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.23b)$$

$$FD^2 \equiv -\frac{1}{4}\kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left(h(0) [h b]_{abcd}(V) b_{a'b'c'd'}(-V) + h(0) b_{abcd}(V) [h b]_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.23c)$$

$$FD^3 \equiv \frac{1}{8}\kappa^8 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left((h(0) h(0) - 2 h^{\rho\sigma}(0) h_{\rho\sigma}(0)) \times b_{abcd}(V) b_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.23d)$$

So too are the four mixed functions:

$$FLJ^1 \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left([\bar{b}b_{\mu}]_{abcd}(V) B^{\mu}(1, V) b_{a'b'c'd'}(-V) + b_{abcd}(V) [\bar{b}b_{\mu}]_{a'b'c'd'}(-V) B^{\mu}(1, -V) \right) \right\rangle \quad (4.24a)$$

$$FLJ^2 \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left([\bar{b}b]_{abcd}(V) b_{a'b'c'd',\mu}(-V) B^{\mu}(1, -V) + b_{abcd,\mu}(V) B^{\mu}(1, V) [\bar{b}b]_{a'b'c'd'}(-V) \right) \right\rangle \quad (4.24b)$$

$$FLD^1 \equiv \frac{1}{2}\kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left(h(0) b_{abcd,\mu}(V) B^{\mu}(1, V) b_{a'b'c'd'}(-V) + h(0) b_{abcd}(V) b_{a'b'c'd',\mu}(-V) B^{\mu}(1, -V) \right) \right\rangle \quad (4.24c)$$

$$FID^1 \equiv \frac{1}{2}\kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left(h(0) [\bar{b}b]_{abcd} b_{a'b'c'd'}(-V) + h(0) b_{abcd}(V) [\bar{b}b]_{a'b'c'd',\mu}(-V) \right) \right\rangle \quad (4.24d)$$

So much for the k-point functions; we turn now to the question of V -ordering corrections. Since each of the graviton fields in the above T^* -ordered k-point functions are evaluated at multiples of the initial direction V^{μ} it is clear that all V -ordering corrections will be proportional to $\delta^D(V)$. A general k-graviton product contributes:

$$V(h(\tau_1 V) \dots h(\tau_k V)) - T^*(h(\tau_1 V) \dots h(\tau_k V)) = (h^{k-2}(0) + \kappa^2 h^k(0) + \dots) \delta^D(V) \quad (4.25)$$

where the fields on the right hand side may include BRS ghosts as well as gravitons and we have suppressed the indices. Because we are using dimensional regularization the

expectation values of products of coincident, massless fields can be ignored. Therefore all but the $k = 2$ and $k = 3$ contributions vanish. The fact that flat space is a solution makes the $k = 3$ terms vanish. Therefore only the 2-point function gives nonvanishing V -ordering contributions, and this only to lowest order. In fact they have already been largely worked out in expression (3.22). Taking the Fourier transform, permuting the indices and adding the result to $T w L^1$ we find:

$$\mu^{-D-4} \tilde{M}^2 = \frac{i\kappa^2}{8p^2} \left(\frac{-4}{D-2} \cdot \hat{e} + \frac{8}{(D-2)(D+4)} \cdot \hat{d} + \frac{-16}{(D-2)(D+2)(D+4)} \cdot \hat{e} \right) + O(\kappa^4) \quad (4.26)$$

It is worth noting that the V -ordering corrections were essential to reduce the tensor structure to RMTS — and hence to satisfy the equation of motion at this order.

IV.B — *A sample reduction: FL^4*

This was a fiendishly tedious calculation and we shall not attempt to cover all its details. However in order to give some idea of the procedure we will sketch the reduction of a representative 4-point function to MKS. This exercise also serves to explain the notation in which the final answer is expressed, namely “minimal prefactors” and “reduced standard K’s.”

The hardest part of this calculation was managing the tensor algebra. For example FL^4 consists of 128 different terms which are each contractions and permutations of the same generic object. Our solution was to express each k -point function as the standard permutation acting on a sum of “contraction types” of the relevant “generic k -point function.” For FL^4 the generic 4-point function turns out to be:

$$\hat{F}L^4_{abcd\rho\sigma\psi^{\prime}d^{\prime}} \equiv -\frac{1}{8}\kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left(h_{ab}(V) h_{cd\rho\sigma}(V) \right. \right. \\ \left. \left. B^{\mu}(1, -V) h_{\alpha\psi^{\prime}c^{\prime}d^{\prime}\mu}(-V) \right) \right\rangle \quad (4.27)$$

There are two contraction types:

$$\hat{F}L^4_{abcd\rho\sigma\psi^{\prime}c^{\prime}d^{\prime}} \equiv \hat{F}L^4_{\rho\sigma b d d a^{\prime} c^{\prime} b^{\prime} d^{\prime}} + \hat{F}L^4_{\rho\sigma a b b d a^{\prime} c^{\prime} b^{\prime} d^{\prime}} \quad (4.28a)$$

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$$\hat{F}L^4_{abcd\rho\sigma\psi^{\prime}c^{\prime}d^{\prime}} \equiv \hat{F}L^4_{\rho\sigma b a c d d a^{\prime} b^{\prime} c^{\prime} d^{\prime}} + \hat{F}L^4_{\rho\sigma d a c b b a^{\prime} c^{\prime} b^{\prime} d^{\prime}} \quad (4.28b)$$

The number of contraction types obviously depends upon the subfunction. There can be as few none — e.g. $\hat{F}L^1$ — or as many as six — e.g. $\hat{F}L^1$. We recover FL^4 by using the standard permutation defined in (4.3):

$$FL^4 = \{\hat{F}L^4_A\} + \{\hat{F}L^4_B\} \quad (4.29)$$

Because of the different graviton structures in $B^{\mu}(1, -V)$ it is useful to decompose $\hat{F}L^4$ into two “generic 4-point subfunctions:”

$$\hat{F}L^4_{abcd\rho\sigma\psi^{\prime}c^{\prime}d^{\prime}} \equiv -\frac{1}{16}\kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left(h_{ab}(V) h_{cd\rho\sigma}(V) \right. \right. \\ \left. \left. \times h^{\mu\nu}(0) V_{\nu} h_{\alpha\psi^{\prime}c^{\prime}d^{\prime}\mu}(-V) \right) \right\rangle \quad (4.30a)$$

$$\hat{F}L^4_{abcd\rho\sigma\psi^{\prime}c^{\prime}d^{\prime}} \equiv \frac{1}{8}\kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left\langle T^* \left(h_{ab}(V) h_{cd\rho\sigma}(V) \right. \right. \\ \left. \left. \times \int_0^1 d\tau \int_0^{\tau} d\tau' V^{\mu\theta\theta'} (-\tau' V) V_{\theta} V_{\theta'} h_{\alpha\psi^{\prime}c^{\prime}d^{\prime}\mu}(-V) \right) \right\rangle \quad (4.30b)$$

There could be as few as zero subfunctions — e.g. $\hat{F}L^1$ — or as many as six — e.g. $\hat{F}L^7$. Because it obviously makes sense to contract subfunctions as well as functions we often dealt with quantities such as:

$$FL^4_{abcd\rho\sigma\psi^{\prime}c^{\prime}d^{\prime}} \equiv \hat{F}L^4_{\rho\sigma a b c d d a^{\prime} c^{\prime} b^{\prime} d^{\prime}} + \hat{F}L^4_{\rho\sigma a b b d a^{\prime} c^{\prime} b^{\prime} d^{\prime}} \quad (4.31)$$

Hence the designation *contraction types*.

It is easy to write the expectation values in momentum space:

$$\hat{F}L^4_{abcd\rho\sigma\psi^{\prime}c^{\prime}d^{\prime}} = \frac{1}{32}\kappa^4 \frac{\partial}{\partial p^{\rho}} \int \frac{d^D k_1}{(2\pi)^D} \int \frac{d^D k_2}{(2\pi)^D} \int \frac{d^D k_3}{(2\pi)^D} \int \frac{d^D k_4}{(2\pi)^D} \\ \times (2\pi)^D \delta^D \left(p - \frac{1}{2}k_1 - \frac{1}{2}k_2 + \frac{1}{2}k_4 \right) k_{2\rho} k_{2\sigma} k_{4c^{\prime}} k_{4d^{\prime}} k_{4\mu} \\ \times \left\langle T^* \left(\tilde{h}_{ab}(k_1) \tilde{h}_{cd}(k_2) \tilde{h}^{\mu\nu}(k_3) \tilde{h}_{\alpha\psi^{\prime}}(k_4) \right) \right\rangle \quad (4.32a)$$

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$$\begin{aligned} \widehat{F}L^{4b}_{abcd\sigma\alpha'\nu'c'd'} &= \frac{1}{64}\kappa^4\frac{\partial}{\partial p^b}\frac{\partial}{\partial p^c}\frac{\partial}{\partial p^d} \int_0^\tau d\tau' \int_0^\tau d\tau' \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \int \frac{d^Dk_3}{(2\pi)^D} \int \frac{d^Dk_4}{(2\pi)^D} \\ &\quad \times (2\pi)^D \delta^D\left(p - \frac{1}{2}k_1 - \frac{1}{2}k_2 + \frac{1}{2}\tau'k_3 + \frac{1}{2}k_4\right) k_{2\rho}k_{2\sigma}k_{4c'}k_{4d'}k_{4\mu} \\ &\quad \times \left\langle T^*\left(\widetilde{h}_{ab}(k_1)\widetilde{h}_{cd}(k_2)\left(2\widetilde{h}^{\mu\delta}(k_3)k_\delta - \widetilde{h}^{\theta\theta}(k_3)k_\mu^\theta\right)\widetilde{h}_{\sigma'\nu'}(k_4)\right)\right\rangle \end{aligned} \quad (4.32b)$$

Now in the gauge we are using (3.21) the bare graviton propagator is:

$$i\Delta_{abcd}(p, q) \equiv \left\langle \Omega_0 \left| T^* \left(\tilde{h}_{ab}(p) \tilde{h}_{cd}(q) \right) \right| \Omega_0 \right\rangle \\ = \mu^{4-D} \frac{i P_{abcd}}{p^2 + i\epsilon} (2\pi)^D \delta^D(p+q) \quad (4.33)$$

where P_{abcd} is defined in (3.22b). To lowest order every 4-point function breaks up into three terms:

$$\begin{aligned} \left\langle T^a \left(\tilde{h}_{ab}(k_1) \tilde{h}_{cd}(k_2) \tilde{h}_{a'b'}(k_3) \tilde{h}_{c'd'}(k_4) \right) \right\rangle &= -\Delta_{abcd}(k_1 + k_2) \Delta_{a'b'c'd'}(k_3 + k_4) \\ &- \Delta_{aba'b'}(k_1 + k_3) \Delta_{cdcd'}(k_2 + k_4) - \Delta_{ab'cd'}(k_1 + k_4) \Delta_{cdab'}(k_2 + k_3) + \mathcal{O}(\kappa^2) \end{aligned} \quad (4.34)$$

We shall refer to these “propagator pairing types” by appending “i” (for 1 with 2), “ii” (for 1 with 3) or “iii” (for 1 with 4) to the subfunction designation superscript. For FL^4 all of the “i”-type pairings vanish because they contract two coincident, massless fields.

It is now a matter of sorting out delta functions. All of the resulting single-momentum integrals can be cast in the form:

$$J_n^{\mathcal{L}, m} \equiv \mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{k_{a_1} \dots k_{a_{\ell-n}} q_{b_1} \dots q_{b_n}}{(k^2 + i\epsilon) (q^2 + i\epsilon)} \quad (4.35)$$

where $q^\mu \equiv p^\mu - k^\mu$. Using this notation and changing variables judiciously we obtain:

$$\hat{F}L_{\text{4ali}}^{abcd\rho\sigma\alpha'b'c'd'} = \kappa^4 \mu^{4-D} 2^{D-7} P_{ab}^{\mu\nu} P_{cd\alpha'b'}^{\nu\sigma} \frac{\partial}{\partial x^{\mu}} \frac{\partial}{\partial x^{\nu}} \int_0^{y^{5,1}} c'd'\mu\sigma \quad (4.36a)$$

$$\hat{F}L^{4\text{iii}}_{abcd\sigma\sigma'a'b'c'd'} = \kappa^4 \mu^{4-D} 2^{D-5} P_{abc'b'} P_{cd}^{\mu\nu} \frac{\partial}{\partial \mu} \frac{\partial}{\partial \mu^2} j^{5,1}_{c'd'\mu\sigma} \quad (4.36b)$$

$$\hat{F}L_{abcdpqrs}^{4\text{ii}} = -\kappa^4 \mu^{4-D} 2^{D-7} \int_0^1 dt' (1+t')^{1-D} \quad (4.36c)$$

$$\times \frac{\partial}{\partial \rho^b} \frac{\partial}{\partial \rho^a} \left(2 P^{\mu b} P_{cda} v^{\mu,1}_{,1} c^d \mu \rho - P^{\theta \phi}_{ab} P_{cda} v^{\mu,1}_{,1} c^d \mu \rho \right)$$

The remaining term is reported in tables iii-vi as FL^3 . A similar transfer was made from FL^1 to FL^2 .

$$\widehat{F}L^{4\text{biii}}_{abcd\rho\sigma\tau'c'd'} = -\kappa^4\mu^{4-D}{}_2D^{-5}\int_0^1d\tau\int_0^1d\tau'(1+\tau')^{-1-D} \\ \times \frac{\partial}{\partial p^\mu}\frac{\partial}{\partial p^\nu}\left(2P_{ab\sigma'\tau'}P_{cd}{}^{\mu\phi}J_{cd}{}^{6,1}{}_\phi - P_{ab\sigma'\tau'}P_{cd}{}^{\mu\phi}J_{cd}{}^{6,1}{}_\mu\right) \quad (4.36d)$$

The parameter integrals are straightforward:

$$2^{D-7} \int_0^1 \int_0^{\tau} d\tau' (1+\tau')^{1-D} = \frac{(D-4)2^{D-3}+1}{16(D-3)(D-2)} \quad (4.37a)$$

$$2^{D-5} \int_0^1 \int_0^{\tau} d\tau' (1+\tau')^{-1-D} = \frac{(D-2)2^{D-1}+1}{16(D-1)D}. \quad (4.37b)$$

Note that the factors of 2^D invariably come from the limit where the integrated graviton is at 0.

Because FL^3 contains the derivative of a product of gravitons — $[h b]_{abcd;\mu}$ — it gives two sorts of momentum integrals. It turns out in the propagator pairings of the generic subfunctions that one of these terms has exactly the same index structure as the corresponding propagator pairing on $\widehat{FL}^4$. In the interest of simplifying the tensor algebra we have accordingly subsumed these terms into what we call $\widehat{FL}^{*43}$:

$$\hat{F}L^{43\text{ii}}_{abcd, p\sigma a'b'c'd'} = \kappa^4 \mu^{4-D} 2^{D-6} P^{\mu\nu}_{ab} P^{\mu\nu}_{cd a'b'} \frac{\partial}{\partial y^\nu} \frac{\partial}{\partial y^\nu} J^{5,1}_0 c'd'\mu\sigma \quad (4.38a)$$

$$\hat{F}L^{43\text{iii}}_{abcd\rho\sigma\alpha'\nu'c'd'} = \kappa^4 \mu^{4-D} 2^{D-4} P_{ab\alpha'\nu'} P^{\mu\nu}_{cd} \frac{\partial}{\partial p^\nu} J^{5,1}_{2\ c'd'\mu\rho\sigma} \quad (4.38b)$$

$$\widehat{F} L^{43\text{bii}}_{abcd\mu\sigma\alpha'b'c'd'} = -\kappa^4 \mu^4 \frac{(D-3)2^{D-2} + 1}{16(D-3)(D-2)} \times \frac{\partial}{\partial p^\beta} \frac{\partial}{\partial \bar{p}^\beta} \left(2 P^{\mu\beta}_{ab} P_{cd\alpha'b'} J^{6,1}_{c'd'\mu\rho\sigma} - P^{\theta\phi}_{ab} P_{cd\alpha'b'} J^{6,1}_{c'd'\mu\rho\sigma} \right) \quad (4.38c)$$

$$\hat{F}L^{43b\text{iii}}_{abcd\mu\sigma\sigma'\nu'c'd'} = -\kappa^4\mu^{4-D}\frac{(D-1)2^D+1}{16(D-1)D} \times \frac{\partial}{\partial\nu^\beta}\frac{\partial}{\partial\nu^\beta}\left(2P_{ab\sigma'\nu'}P_{cd}^{\mu\theta}J_{3c'd'}^{\mu\theta}J_{3c'd'\mu\sigma\sigma'}^\mu - P_{ab\sigma'\nu'}P_{cd}^{\mu\theta}J_{3c'd'}^{\mu\theta}J_{3c'd'\mu\sigma\sigma'}^\mu\right) \quad (4.38d)$$

These expressions have the form of products of two P 's and a J_n^m multiplied by a number which we call the "prefactor." These last can, in turn, be written as linear combinations — with rational functions of D as the combination coefficients — of four "minimal prefactors" (MPF): $1, 2^D, 2^{2D}$ and $\frac{2^{D-1}}{D-1} {}_2F_1(D+1, 1-D; 2-D; \frac{1}{2})$. Of course a traditional quantum gravity Greens function would only produce rational functions of D , i.e. the MPF 1. To understand where the other MPF's come from it is useful to consider the following generic 4-point function:

$$\tilde{F}_{a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4} \equiv \kappa^4 \int d^D(2V) \exp(i2V \cdot p) \left(T^* \left(h_{a_1 b_1}(V) h_{a_2 b_2}(\tau V) \right. \right. \\ \left. \left. \times h_{a_3 b_3}(\tau' V) h_{a_4 b_4}(-V) \right) \right) \quad (4.39)$$

As usual it gives three propagator pairing types:

$$\tilde{F}_{a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4}^{\text{I}} = -\kappa^4 \mu^{4-D} P_{a_1 b_1 a_2 b_2} P_{a_3 b_3 a_4 b_4} \left| \frac{2}{1-\tau} \right|^{D-2} \left| \frac{2}{1+\tau'} \right|^{D-2} J_0^{0,1} \quad (4.40a)$$

$$\tilde{F}_{a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4}^{\text{II}} = -\kappa^4 \mu^{4-D} P_{a_1 b_1 a_2 b_2} P_{a_2 b_2 a_4 b_4} \left| \frac{2}{1+\tau} \right|^{D-2} \left| \frac{2}{1-\tau'} \right|^{D-2} J_0^{0,1} \quad (4.40b)$$

$$\tilde{F}_{a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4}^{\text{III}} = -\kappa^4 \mu^{4-D} P_{a_1 b_1 a_4 b_4} P_{a_2 b_2 a_3 b_3} \left| \frac{2}{\tau-\tau'} \right|^{D-2} J_0^{0,1} \quad (4.40c)$$

Derivatives on the gravitons change the powers but not the fact that a factor of 2^D comes for each graviton evaluated at the origin. The hypergeometric function derives from integrating one of the parameters up to the other:

$$\int_0^1 d\tau \left| \frac{2}{1-\tau} \right|^\alpha \int_0^1 d\tau' \left| \frac{2}{1+\tau'} \right|^\beta \\ = \frac{2^\alpha}{(\alpha-1)(\beta-1)} \left(-2^\beta + 2F_1(\beta-1, 1-\alpha; 2-\alpha; \frac{1}{2}) \right) \quad (4.41)$$

It turns out that all the hypergeometric functions can be written in terms of our minimal one plus terms which involve 2^{2D} . Since no linear combination of the MPF's can cancel it follows that the coefficient of each in the final answer must separately obey the equations of motion (4.5). This means there are effectively $9(MTS) \times 4(MPF) = 36$ independent tensor-prefactor combinations, of which 24 are checked by the equations of motion.

While we are digressing we should also mention reduction strategies for the J_n^m 's. The integrals are simple enough to do in dimensional regularization but a few easily verified identities allow us to postpone this till after the tensor algebra is largely over. First there is the interchange identity for $m=1$:

$$J_n^{\ell,1}{}_{a_1 \dots a_{\ell-n} b_1 \dots b_n} = J_{\ell-n}^{\ell,1}{}_{b_1 \dots b_n a_1 \dots a_{\ell-n}} \quad (4.42a)$$

which has already been used to obtain (4.36). Then there are two important contraction identities:

$$\eta^{a_1 a_2} J_n^{\ell,m}{}_{a_1 \dots a_{\ell-n} b_1 \dots b_n} = 0 \quad (4.42b)$$

$$\eta^{a_{\ell-n} b_n} J_n^{\ell,1}{}_{a_1 \dots a_{\ell-n} b_1 \dots b_n} = \frac{1}{2} p^2 J_{n-1}^{\ell-2,1}{}_{a_1 \dots a_{\ell-n-1} b_1 \dots b_{n-1}} \quad (4.42c)$$

Finally we have the three differentiation identities:

$$\frac{\partial}{\partial p^c} J_n^{\ell,m}{}_{a_1 \dots a_{\ell-n} b_1 \dots b_n} = -2\ell J_{n+1}^{\ell+1,m+1}{}_{a_1 \dots a_{\ell-n} b_1 \dots b_n c} + J_{n-1}^{\ell-1,m}{}_{a_1 \dots a_{\ell-n} b_1 \dots b_{n-1}} \eta_{b_n c} \quad (4.42d)$$

$$\eta^{b_n} \frac{\partial}{\partial p^\mu} J_n^{\ell,m}{}_{a_1 \dots a_{\ell-n} b_1 \dots b_n} = (D+n-2m-1) J_{n-1}^{\ell-1,m}{}_{a_1 \dots a_{\ell-n} b_1 \dots b_{n-1}} \quad (4.42e)$$

$$\eta^{\mu a_{\ell-n}} \frac{\partial}{\partial p^\mu} J_n^{\ell,m}{}_{a_1 \dots a_{\ell-n} b_1 \dots b_n} = (D+\ell-n-3) J_{n-1}^{\ell-1,m}{}_{a_1 \dots a_{\ell-n-1} b_1 \dots b_n} \quad (4.42f)$$

where parenthesizing indices denotes symmetrization.

By using these identities to perform the derivatives and by decomposing the propagator symbols according to (3.22b) we can reduce $\hat{F}_{L^{4Aii}}$ and its relatives to linear combinations of prefactors times a single $J_n^{\ell,m}$ and one, two or three η 's. These last are called "kernel integrals." Under the standard permutation they give what we call a "K." For example:

$$K2A1 \equiv \{ \eta_{bd} \eta_{b'd'} \eta_{ac} \eta_{a'c'} J_1^{4,1}{}_{ac\rho} \} \\ = I(4D(D+1) \cdot \hat{d}_1 - 16(D+1) \cdot \hat{e}_1) \quad (4.43)$$

where:

$$I \equiv \frac{J_0^{0,1}}{16(D-1)(D+1)} \quad (4.44a)$$

$$J_0^{0,1} = \frac{i}{2D\pi^{D/2}} \frac{\Gamma(2-\frac{D}{2})\Gamma^2(\frac{D}{2}-1)}{\Gamma(D-2)} \left(-\frac{p^2}{\mu^2} \right)^{D/2-2} \quad (4.44b)$$

By using (4.42) and a few other simple tricks all possible K's can be reduced to linear

combinations of at set of 29 "lowest-m K's" (LMK). These are obviously a more convenient way to express the answer than the MTS, unfortunately they are overcomplete. What we did was to select a basis of nine "standard K's" (SK). They are $K2A1$ and the following:

$$K4A1 \equiv \{\eta_{bd} \eta_{bd'} J_0^{4,1}{}_{acc'e'}\} \\ = I(2(D+2)(D+4) \cdot \hat{e}_i - 4(D+2) \cdot \hat{d}_i - 2(D+2) \cdot \hat{d}_{ii} + 8\hat{e}_i + 2 \cdot \hat{e}_{ii}) \quad (4.45a)$$

$$K4B1 \equiv \{\{\eta_{bb'} \eta_{dd'} + \eta_{bd'} \eta_{db'}\} J_0^{4,1}{}_{acc'e'}\} \\ = I(2(D+2)(D+4) \cdot \hat{e}_{ii} - 2(D+2) \cdot \hat{d}_{ii} - 2(D+2) \cdot \hat{d}_{iv} + 2\hat{e}_{ii} + 4 \cdot \hat{e}_{iii}) \quad (4.45b)$$

$$K4A2 \equiv \frac{1}{2} \{ \eta_{bd} \eta_{bd'} (J_1^{4,1}{}_{acc'e'} + J_1^{4,1}{}_{acc'e'}) \} \\ = I(2(D+2)(D+4) \cdot \hat{e}_i + 4 \cdot \hat{d}_i + 2 \cdot \hat{d}_{iii} - 8\hat{e}_i - 2 \cdot \hat{e}_{ii}) \quad (4.45c)$$

$$K4B2 \equiv \frac{1}{2} \{ (\eta_{bb'} \eta_{dd'} + \eta_{bd'} \eta_{db'}) (J_1^{4,1}{}_{acc'e'} + J_1^{4,1}{}_{acc'e'}) \} \\ = I(2(D+2)(D+4) \cdot \hat{e}_{ii} + 2 \cdot \hat{d}_{ii} + 2 \cdot \hat{d}_{iv} - 2\hat{e}_{ii} - 4 \cdot \hat{e}_{iii}) \quad (4.45d)$$

$$K4A7 \equiv \{\eta_{bd} \eta_{bd'} J_2^{4,1}{}_{acc'e'}\} \\ = I(2(D-2)D \cdot \hat{e}_i - 4(D+2) \cdot \hat{d}_i + 2D \cdot \hat{d}_{iii} + 8\hat{e}_i + 2 \cdot \hat{e}_{ii}) \quad (4.45e)$$

$$K4B7 \equiv \{(\eta_{bb'} \eta_{dd'} + \eta_{bd'} \eta_{db'}) J_2^{4,1}{}_{acc'e'}\} \\ = I(2(D-2)D \cdot \hat{e}_{ii} - 2(D+2) \cdot \hat{d}_{ii} + 2D \cdot \hat{d}_{iv} + 2\hat{e}_{ii} + 4 \cdot \hat{e}_{iii}) \\ K4P1 \equiv \frac{1}{2} \left\{ \eta_{ac} \left(\eta_{bb'} (J_1^{4,1}{}_{a'c'dd'} + \eta_{bd'} J_1^{4,1}{}_{a'c'bd'}) \right. \right. \\ \left. \left. + \eta_{db'} J_1^{4,1}{}_{a'c'bd'} + \eta_{dd'} J_1^{4,1}{}_{a'c'bd'} \right) \right\} \\ = I(2(D+1) \cdot \hat{d}_{ii} - 2(D+1) \cdot \hat{d}_{iii}) \quad (4.45f)$$

$$K4P6 \equiv \frac{1}{2} \left\{ \eta_{ac} \left(\eta_{bb'} (J_2^{4,1}{}_{a'c'dd'} + \eta_{bd'} J_2^{4,1}{}_{a'c'bd'}) \right. \right. \\ \left. \left. + \eta_{db'} J_2^{4,1}{}_{a'c'bd'} + \eta_{dd'} J_2^{4,1}{}_{a'c'bd'} \right) \right\} \\ = I(-2(D+1) \cdot \hat{d}_{ii} - 2(D+1) \cdot \hat{d}_{iii}) \quad (4.45g)$$

The 29 LMK were then tabulated as linear combinations of these SK though we will not trouble to report the result — or even the definitions of the other 20 LMK [33].

The rest is algebra. For example:

$$FL^{43}A\ddot{a}\ddot{b} = \mu^{4-D} \kappa^4 2^{D-6} \left\{ (p_0^{\rho\mu\nu} P_{pca'e'} + p_\epsilon^{\rho\mu\nu} P_{pca'e'}) \frac{\partial}{\partial p^\nu} J_0^{5,1}{}_{bd't'\mu} \right\} \\ - \mu^{4-D} \kappa^4 \frac{2^{D-4}}{D-2} \left\{ P_{aca'e'} \frac{\partial}{\partial p^\mu} J_0^{5,1}{}_{bd't'\mu} \right\} \\ = -\mu^{4-D} \kappa^4 2^{D-4} \left(\frac{D+2}{D-2} \right) \left\{ P_{aca'e'} J_0^{4,1}{}_{bd't'd'} \right\} \\ = \mu^{4-D} \kappa^4 \cdot \frac{(D+2)}{8(D-2)^2} \cdot 2^D \cdot K4A1 - \mu^{4-D} \kappa^4 \cdot \frac{(D+2)}{16(D-2)} \cdot 2^D \cdot K4B1 \quad (4.46)$$

We report the results — without the factors of $\mu^{4-D} \kappa^4$ — by subfunction for MPF 1 in table i and for MPF 2^D in table ii.

IV.C — The final answer and checks on its accuracy

The order $\mu^{4-D} \kappa^4$ contribution of each k-point function are reported by minimal prefactor in tables iii-vi. For each MPF the result is expressed as a sum of the nine SK's defined in (4.43) and (4.45), except that for MPF's 1 and 2^D we have added $-2^{D-2} \cdot K4A1$ to each $K4B1$ and reported the remaining $K4A1$'s under the heading "residual." The reason for this is that the combination

$$K4B1 \equiv -\frac{2^2}{D-2} \cdot K4A1 + K4B1 \\ = -\frac{4(D+2)(D+4)}{D-2} \cdot \hat{e} + \frac{4(D+2)}{D-2} \cdot \hat{d} - \frac{16}{D-2} \cdot \hat{e} \quad (4.47a)$$

occurs naturally in isolated subfunctions — for example it appears in (4.46) — and independently obeys our principal check (4.5). The other "restricted standard K's" (RSK):

$$K4B2 \equiv \frac{(D+1)}{(D-2)^2} \cdot K2A1 + \left(\frac{D-1}{D-2} \right) \cdot K4A1 - \frac{(D^2-5D+8)}{(D-2)^2} \cdot K4A2 \\ + K4B2 - \frac{D}{D-2} \cdot K4A7 \\ = -4(D+2) \cdot \hat{e} + \frac{8(D^2+2)}{(D-2)^2} \cdot \hat{d} - \frac{16(D+4)}{(D-2)^2} \cdot \hat{e} \quad (4.47b)$$

$$\begin{aligned}
\mathbf{K4B7} \equiv & \frac{2(D+1)}{(D-2)^2(D-1)} \cdot K_{2A1} + \frac{(3D-4)}{(D-2)(D-1)} \cdot K_{4A1} - \frac{2(D^2+5D-8)}{(D-2)^2(D-1)} \cdot K_{4A2} \\
& - \frac{3D}{(D-2)(D-1)} \cdot K_{4A7} + K_{4B7} - \frac{(3D-2)}{(D-2)(D-1)} \cdot K_{4P1} + \frac{1}{D-1} \cdot K_{4P6} \\
& = -4D \cdot \dot{\epsilon} + \frac{8D(D+4)}{(D-2)^2} \cdot \dot{d} - \frac{16(D^2+D+6)}{(D-2)^2(D-1)} \cdot \dot{\epsilon} \quad (4.47c)
\end{aligned}$$

occur only through superposing many k -point functions, which are themselves sums over contraction types of many subfunctions. The possibility for error in $\mathbf{K4B1}$ term is consequently much greater and we wished to devote special attention to them. We have also isolated the troublesome “iii” pairing type (interior with interior) for $F^j F^5$, $F^j F^6$ and $F^j F^7$ for MPF's 1 and 2^D .

The final answer is most simply expressed in terms of the RSK's:

$$\begin{aligned}
\mu^{D-4} \tilde{\mathcal{M}}^2 = & (4.26) + \kappa^4 \left(\alpha_1 \cdot \mathbf{K4B1} + \alpha_2 \cdot \mathbf{K4B2} + \alpha_3 \cdot \mathbf{K4B7} \right) \\
& + \kappa^4 H \left(\beta_1 \cdot \mathbf{K4B1} + \beta_2 \cdot \mathbf{K2B2} + \beta_3 \cdot \mathbf{K4B7} \right) + \mathcal{O}(\kappa^6) \quad (4.48a)
\end{aligned}$$

where H is a linear combination of two MPF's:

$$H \equiv \frac{2^{2D-1}}{D} + \frac{2^{D-1}}{D-1} {}_2F_1(D+1, 1-D; 2-D; \frac{1}{2}) \quad (4.48b)$$

and the various coefficient functions are:

$$\alpha_1 \equiv -\frac{(D^6 + 8D^5 - 11D^4 - 82D^3 - 8D^2 + 112D - 64)}{64(D-4)(D-2)D(D+1)} \quad (4.48c)$$

$$\alpha_2 \equiv \frac{(D^2 - D + 2)(D^4 + 2D^3 - 19D^2 + 20D - 8)}{32(D-2)(D-1)D(D+1)} \quad (4.48d)$$

$$\alpha_3 \equiv -\frac{(D-1)(D^4 - 6D^3 + 9D^2 - 8)}{64D(D+1)^2} \quad (4.48e)$$

$$\beta_1 \equiv \frac{(D-1)}{4(D+1)} \quad (4.48f)$$

$$\beta_2 \equiv \frac{(D-2)(D+1)}{4(D-1)} \quad (4.48g)$$

$$\beta_3 \equiv -\frac{(D-1)(D^2 - D - 1)}{4(D+1)^2} \quad (4.48h)$$

Before discussing accuracy we wish to draw attention to four points concerning divergences:

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(1) For $D = 3$ and $p^2 \neq 0$ $\tilde{\mathcal{M}}^2$ is finite but not zero.

(2) For $p^2 \neq 0$ $\tilde{\mathcal{M}}^2$ has double poles in $D - 4$ at order κ^4 .

(3) For $p^2 \neq 0$ $\tilde{\mathcal{M}}^2$ has single poles in $D - N$ at order κ^4 for every odd integer $N > 3$.

(4) For $p^2 \neq 0$ $\tilde{\mathcal{M}}^2$ has double poles in $D - N$ at order κ^4 for every even integer $N > 4$.

The first result is surprising because all the RMTS — and hence also the RSK — vanish for $D = 3$. However it is easy to show that certain contractions of them only contain a single power of $(D - 3)$. Since the hypergeometric MPF:

$$\frac{2^{D-1}}{D-1} {}_2F_1(D+1, 1-D; 2-D; \frac{1}{2}) = \sum_{k=0}^{\infty} \frac{\Gamma(D+1+k)}{\Gamma(D+1)} \frac{1}{k+1-D} \frac{2^{D-k-1}}{k!} \quad (4.49)$$

contains a pole in $(D - 3)$ — as it does for all the positive integers — the result is finite and nonzero as we have claimed.

This is not inconsistent with the fact that classical sourceless gravity has zero curvature in three dimensions. *Quantum* gravity possesses divergences — even for $D = 3$ and cannot be discussed meaningfully until it has been regulated. Dimensional regularization does this by varying the dimension of spacetime. Therefore it does not automatically preserve results specific to a particular dimension. Although the pole *does* derive from the peculiar coordinate system in which we have measured the curvature it is worth pointing out that the situation for $\tilde{\mathcal{M}}^2$ is no worse than for the uncorrected product of Riemann tensors we can obtain from (4.20). As table iii shows this also fails to vanish for $D = 3$.

The divergences in (4.49) — and those in the other MPF's — originate from parameter integrals in which two interior gravitons approach one another. It is this extra source of divergence which is responsible for the fact that our results contain one more pole than usual. The special relation that exists between the RMTS at $D = 4$ — namely $\dot{d} = \dot{\epsilon}$ — is

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not enough to prevent this. In fact it is easy to show that near $D = 4$:

$$\mu^{D-4} \tilde{M}^2 \sim \frac{i\kappa^4}{\pi^2(D-4)^2} \left(\frac{7}{2^4 \cdot 3^2} \cdot \hat{\epsilon} + \frac{7^4}{2^6 \cdot 3 \cdot 5^2} \cdot \hat{d} \right) \quad (4.50)$$

In higher dimensions there are no special relations so the usual poles from the momentum integrals combine with those from (4.49) — for $D > 4$ none of the other MPF's contain poles — to give results (3) and (4).

If this is correct then the Mandelstam covariant does no good for ultraviolet finiteness; in fact Mandelstam Green's functions are *more* divergent than the usual ones. The question is, can the result of such an intricate calculation be trusted? We feel there is some reason to doubt the *method*, i.e. flat space perturbation theory. Our qualms here are really the subject of another paper. Briefly, the idea is that the Mandelstam covariant is nonlocal and of exponential order in the metric. Some of the coordinate corrections, for example (3.26), can be written in a way as to be indistinguishable from the action. They ought thus to affect the stationary field configuration around which perturbation theory is developed. If the new extremal point is nonperturbatively far from flat space in classical perturbation theory — and it is — then quantum perturbation theory around flat space does not give the correct asymptotic series expansion for the Green's function. We shall discuss this at greater length elsewhere [35].

Within the realm of flat space perturbation theory we feel the result is quite reliable. As mentioned before our basic check was provided by the equations of motion (4.5). This fixes 24 of the 36 MPF-SK coefficients we computed. Since computing prefactors is simple and since very few give pure MPF's — for example those of $\hat{F}L^{4b}$ contribute to both 1 and 2^D — the check tends to be much better. Watching thousands of independently computed terms sum to verify (4.5) is a compelling assurance of accuracy. It should be noted in this regard that the loop in TwU^1 was performed not even performed by us [20,34].

An important supplement to (4.5) was the cancellation of all the 2^D terms and the fact that the 2^{2D} coefficients turned out to be proportional to those of the hypergeometric function. We did not even anticipate this beforehand, although the reason for it is clear enough now in a qualitative sort of way. The fact that the effect occurs independently among the following groups of k-point functions is suggestive:

- (1) TL^1 , TI^1 , and TD^1 .
- (2) FL^{5-7} , FJ^{5-7} , FD^3 , FLJ^{1-2} , FLD^1 and FID^1 .
- (3) FL^{1-2} , FJ^{1-2} and FD^1 .
- (4) FL^{3-4} , FJ^{3-4} and FD^2 .

The 3-point functions in (1) are all corrections to the same term, $b_{abcd}(V) b_{a'b'c'd'}(-V)$, as are the 4-point functions of (2). Similarly the 4-point functions of (3) all correct $c_{abcd}(V) b_{a'b'c'd'}(-V) + b_{abcd}(V) c_{a'b'c'd'}(-V)$ and the 4-point functions of (4) all correct $-\frac{1}{2}[h b]_{abcd}(V) b_{a'b'c'd'}(-V) - \frac{1}{2}b_{abcd}(V) [h b]_{a'b'c'd'}(-V)$. Recall that one gets a factor of 2^D for each graviton sitting at the origin. If the parameter integrals could be continued through the origin so that they ran from $\tau = +1$ to $\tau = -1$ then there wouldn't be any such terms.* Expression (3.26) shows that the geodesic parameters can be written in this way. It turns out that a similar trick works for the parallel transport matrices [35].

One still sees the hypergeometric MPF because nothing prevents interior gravitons from being integrated up to one another. The reason the 2^{2D} joins it in the way that it

* This of course ignores the gravitons which aren't integrated to the origin but rather reside there permanently. An example would be the first term in $B^\mu(\tau, V)$ of (4.9c). We think these drop out because the graviton at any one point can be gauged to zero by a global transformation.

does seems to be that precisely this combination gives a symmetric integrand:

$$\begin{aligned}
H &\equiv \frac{2^{2D}}{2D} + \frac{2^{D-1}}{D-1} {}_2F_1(D+1, 1-D; 2-D; \frac{1}{2}) \\
&= \frac{2^{2D}}{2D} - 2 \int_0^1 dt \left(\frac{2}{1-\tau} \right)^D \left(\frac{2}{1+\tau} \right)^{D+1} \\
&= -\frac{1}{2} \int_{-1}^1 dt \left(\frac{2}{1-\tau} \right)^{D+1} \left(\frac{2}{1+\tau} \right)^{D+1} \\
&= \frac{2^{D-1}}{D} {}_2F_1(D+1, -D; 1-D; \frac{1}{2})
\end{aligned} \tag{4.51}$$

Had we seen that this would happen we would, of course, have taken the final hypergeometric function as one of our MPFs. As it was the surprise was a pleasant one since it provided many new checks. Especially welcome were those on the troublesome K4B1 terms which (4.5) does not probe well. Whereas a single subfunction could yield a K4B1 — and hence satisfy (4.5) in toto — it still required the collaboration of many to suppress the 2^D terms.

Finally it should be mentioned that we only noticed the 2^D cancellation because (4.5) had culled most of the errors. In fact we found only one additional error — it was in a $2^D \cdot \text{K4B1}$ term — after we began to suspect that the 2^D cancellation was more than an accident. If the final answer we have given still harbors errors one would rather have seen a gradual decrease in the observed error rate as the additional checks were used up. Instead it looks as if we found the final error and the curve (of errors discovered versus checks enforced) went flat.

V. CONCLUSIONS

Though the results of the previous section make it clear — barring the possible problem with using flat space perturbation theory [35] — that the Mandelstam covariant does not resolve the ultraviolet finiteness problem of quantum gravity we are not discouraged.

The double poles we encountered are almost certainly the result of constraining the Mandelstam observer to follow a classical geodesic. As explained at the end of Section III this is equivalent to coupling an infinite mass point particle into the problem, and it is not surprising that this should lead to new divergences. The more realistic “quantum observer” scenario — in which the Mandelstam observer is given a finite, nonzero mass m — ought to avoid the problem. This parameter would then play a role very much akin to the detector efficiency in the QED infrared problem.

Far more significant from our point of view is the fact that *we got the usual result to change by making observations with respect to a different coordinate system*. A particularly elegant way of viewing the field redefinitions of II.C — by which, we remind the reader, any two theories with the same free limit can be related in perturbation theory — is as a combination of a transformation of the initial value configurations and an operator (and spacetime) dependent redefinition of time. Changes of this sort seem very artificial in matter theories but the same can not be said in gravity where geometry is a dynamical variable. In fact we showed at the end of III.B that adopting the Mandelstam covariant is equivalent to just such a transformation. And, as we saw in III.D, it is only the simplest in a long series of such variables. The result of II.C *proves* that some of these give completely finite Green’s functions in perturbation theory. If one can be found which is also physically reasonable then we should be quite happy to reformulate quantum gravity around it.

The result would be a classically irrelevant reshuffling of the dynamical degrees of freedom into special combinations chosen to survive quantization without developing ultraviolet divergences.* We emphasize that there is little reason to accept the verdict of the usual variables of quantum gravity. As we saw in II.A Green’s functions of the local

* At least as long as the new fields are noncoincident. Since the old fields would have to be expressible in terms of the new ones it follows that divergent quantities could be formed using the new variables — even as they can be in free scalar field theory. The point is that these divergences would be ascribed to divergent operations such as taking a coincidence limit.

field correspond, in traditional gauges, to observations which can never be made. This derives from the failure to specify a physical coordinate system and the consequent use of the unphysical, gauge one. There is no doubt that changing these coordinate systems can affect the divergence structure [20-21]. Though the traditional asymptotic variables depend much more weakly upon gauge coordinate systems they are certainly affected by gauge changes in the asymptotic region. And as we argued in II.B they may be unphysical in depending upon the absence of quantum oscillations in the asymptotic geometry. We have just seen what imposing a similarly unphysical condition does to Mandelstam Green's functions; it does not seem too much to suggest that the same sort of thing is responsible for the divergences in asymptotic observables.

The picture that emerges is one of nongravitational theories required to be perturbatively renormalizable while quantum gravity, because it is a theory of geometry, can take fuller advantage of the sort of transformations described in II.C. The procedure could be thought of as an operator renormalization in which the physical geometry is adjusted to cancel divergences. The Mandelstam covariant is certainly too simple to incorporate these changes. In particular it is independent of matter whereas we know that the lowest order effect of adding matter loops is to make previously finite, one loop gravitational amplitudes diverge. Perhaps the answer lies in causing source terms described in (3.27) and (3.29) to depend upon matter couplings. As previously mentioned such modifications might be justified as the effect the Mandelstam observer notices due to matter.

We should not close without mentioning the relevance of our work to situations which are not amenable to the usual asymptotic analysis — such as when space is closed or else not asymptotically empty. The Mandelstam covariant provides a quasi-local definition of time that makes sense even in the absence of an asymptotic region. The same is true for all the modifications that we have been discussing. Since these are relative times the operators

that generate them need not vanish on physical states as does the constrained, closed space Hamiltonian. Yet because the usual Hamiltonian is used to evolve the unconstrained, non-invariant component parts of the Mandelstam covariant and its relatives their expectation values are still obtained from the usual functional formalism. Such variables should be useful in quantum cosmology. We regard their elaboration as sufficient justification for our labors, even if no amelioration of the ultraviolet problem should emerge.

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Table i - The coefficient of $\mu^{4-D} k^4 - 1$ in FL^{43}

Subfunction	K2A1	K4A1	K4B1	K4A2	K4B2	K4A7	K4B7	K4P1	K4P6
$FL^{43A} b_{ii}$	0	$\frac{-(D+2)}{2(D-3)(D-2)^2}$	$\frac{D+2}{4(D-3)(D-2)}$	$\frac{D+2}{2(D-3)(D-2)^2}$	$\frac{-(D+2)}{4(D-3)(D-2)}$	0	0	$\frac{-D(D+2)}{4(D-3)(D-2)^2}$	0
$FL^{43B} b_{ii}$	0	$\frac{D+2}{2(D-2)^2}$	$\frac{-(D+2)}{4(D-2)}$	$\frac{-(D+1)(D+2)}{2(D-3)(D-2)^2}$	$\frac{(D+1)(D+2)}{4(D-3)(D-2)}$	0	0	0	0
$FL^{43A} b_{iii}$	$\frac{D}{4(D-2)^2(D-1)}$	$\frac{1}{4(D-1)}$	0	$\frac{-(D^2+6D-12)}{4(D-2)^2(D-1)}$	$\frac{1}{2(D-2)(D-1)}$	0	$\frac{1}{4(D-1)D}$	$\frac{-(D^2+D+2)}{4(D-2)(D-1)D}$	$\frac{-1}{4(D-1)D}$
$FL^{43B} b_{iii}$	$\frac{D}{4(D-2)^2(D-1)}$	$\frac{1}{4(D-1)}$	0	$\frac{-(D^2+6D-12)}{4(D-2)^2(D-1)}$	$\frac{1}{2(D-1)}$	0	$\frac{-1}{4(D-1)D}$	$\frac{-1}{2(D-2)D}$	$\frac{1}{4(D-1)D}$
Total	$\frac{D}{2(D-2)^2(D-1)}$	$\frac{D^2+5D+5D-3}{2(D-3)(D-2)^2(D-1)}$	$\frac{-(D+1)(D+2)}{4(D-3)(D-2)}$	$\frac{D^2+2D^2-16D+18}{(D-3)(D-2)^2(D-1)}$	$\frac{D^2+4D-6}{4(D-3)(D-2)}$	0	0	$\frac{-(2D^2+3D^2-D+6)}{4(D-3)(D-2)^2(D-1)}$	0

Subfunction	K2A1	K4A1	K4B1	K4A2	K4B2	K4A7	K4B7	K4A8	K4P6
F _{43A} ...	0	$\frac{D+Z}{8(D-Z)^2}$	$\frac{16(D-Z)}{32(D-Z)}$	0	0	0	0	0	0
F _{43B} ...	0	$\frac{-(D-4)(D+Z)}{16(D-Z)^2}$	$\frac{10(-4)(D+Z)}{32(D-Z)}$	0	0	0	0	0	0
F _{43A} ...	0	0	$\frac{1}{8(D-Z)}$	0	0	$\frac{D}{8(D-Z)^2}$	$\frac{-(D+Z)}{8(D-Z)}$	0	0
F _{43B} ...	0	0	0	0	0	$\frac{D}{8(D-Z)^2}$	$-\frac{8}{8(D-Z)}$	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	$\frac{8(D-Z)^2}{-(D+Z)}$	$\frac{16(D-Z)^2}{10(-3)(D+Z)}$	$\frac{8(D-Z)^2}{-(D+1)(D+Z)}$	$\frac{16(D-Z)^2}{-(D+1)(D+Z)}$	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0
F _{43A} ...	0	0	0	0	0	0	0	0	0
F _{43B} ...	0	0	0	0	0	0	0	0	0

Table 11 - The coefficient of $x^{4-D} y^D$ in F_{L+3}

Table III - The Coefficient of $\mu^{4-D} K^4 \cdot 1$

Function	$K_2 A_1$	$K_4 A_1$ (Residual)	$-\frac{D^2}{2} * K_4 A_1 + K_4 B_1$
T_{u11}	0	$\frac{-(10^5 - 350^2 + 140D^2 - 244D + 250)}{4(10 - 2)^2 D(D+1)}$	$\frac{-(140^5 + 170^2 - 9D^2 - 34D + 44)}{32(10 - 2) D(D+1)}$
T_{u1}	0	$\frac{D^4 - D^2 - 4D^2 - 8D - 64}{32(10 - 2)^2 D(D+1)}$	$\frac{D^2 - 8D + 8}{16(10 - 2)}$
T_{u2}	0	$\frac{-(13D^4 - 333D^3 + 448D^2 - 243D - 40)}{4(10 - 2)^2 D(D+1)}$	$\frac{D^3 + 30D^2 - 24D + 80}{8(10 - 2) D(D+1)}$
F_{u1}	$\frac{(10 - 7) D^2}{32(10 - 2)^2}$	$\frac{D^4 - 10D^3 + 16D^2 - 14D + 16}{32(10 - 2) D}$	$\frac{-10D^3 + D^2 - 16D - 8}{64 D}$
F_{u2}	$\frac{D^2}{4(10 - 2)^2}$	$\frac{1}{4} (D + 1)$	$-\frac{1}{8} (D - 2)$
F_{u3}	0	$\frac{-(10^3 - 2D^2 - 4D + 4)}{4(10 - 2)^2 D}$	$\frac{-(10^3 - 2D^2 - 22D + 6)}{16(10 - 2) D}$
T_{L14}	0	$\frac{D^3 - 3D^2 + 10D - 10}{4(10 - 2) (D - 1) (D + 1)}$	$\frac{3D^5 - 18D^4 + 11D^3 + 14D^2 - 5D - 12}{8(10 - 2) (D - 2) (D - 1) D(D + 1)}$
T_{L1b}	$-\frac{D}{4(10 - 2) (D - 1)}$	$\frac{-(12D^2 - D + 2)}{8(10 - 2) (D - 1) (D + 1)}$	$\frac{-(12D^4 - 15D^3 + 35D^2 - 54D - 52)}{8(10 - 4) (D - 3) (D - 2) (D - 1) D}$
F_{L24}	0	$-\frac{1}{2(10 - 2) (D - 1)}$	$\frac{1}{2(10 - 2) (D - 1)}$
F_{L23}	$\frac{D}{2(10 - 2) (D - 1)}$	$\frac{1}{2(10 - 2) D}$	$\frac{-(D - 4) (D + 2)}{4(10 - 2) (D - 2)}$
F_{L5}	$\frac{1}{4(10 - 2) (D + 1)}$	$\frac{-(D^2 - 5D + 8)}{16(10 - 2) (D + 1)^2}$	$\frac{(10 - 2) (D - 3) (D^2 - 2D + 4)}{16(10 - 4) (D - 3) (D - 2) (D - 1) D}$
T_{I1}	$\frac{D}{4(10 - 2) (D - 1)}$	$\frac{2D^2 - 5D + 10D - 4}{8(10 - 2) D(D + 1)}$	$\frac{-10 - 2(17D^3 + 7D^2 - 5D + 3)}{8(10 - 2) (D - 1) D(D + 1)}$
F_{I2}	$-\frac{D^2}{2(10 - 2) (D - 1)}$	$-\frac{(D - 1)}{2(10 - 2)}$	$\frac{1}{2(10 - 2)}$
F_{I4}	$\frac{D(10 + 5)}{8(10 - 2)^2}$	$\frac{D^2 + 2D - 4}{8(10 - 2)}$	$-\frac{1}{8} D$
$F_{I5(11)}$	$\frac{-(10 + 1)}{8(10 - 2)^2}$	$-\frac{D}{16(10 - 2)}$	0
$F_{I5(11)}$	$\frac{-D(10 + 1) (D^2 - 3D + 4)}{16(10 - 2)^2 (D - 1)}$	$\frac{-D(10^2 - 23D + 4)}{16(10 - 2)^2}$	$\frac{D^3 - D^2 - 6D + 16}{16(10 - 2) (D - 1)}$
F_{L12}	0	$-\frac{1}{2(10 - 2) (D + 1)}$	$\frac{(10 - 2) (D^2 - 3D - 15)}{4(10 - 2) (D - 2) (D - 1) D(D + 1)}$
Total	$\frac{(10 - 2) (10^4 + 10^3 - 3D + 2)}{32(10 - 2)^2 (D + 1)}$	$\frac{2D^5 + 9D^4 + 6D^3 - 23D^2 + 4D - 16}{64(10 - 2) D(D + 1)^2}$	$\frac{(10^4 - 110D^5 + 107D^4 - 100D^3 - 50D^2 - 120D + 4)}{64(10 - 4) (D - 2) (D - 1) D(D + 1)}$

Table iii (continued)

Func _{in}	K4A2	K4B2	K4A7
TWU^1	0	$\frac{-(4D^9 + 17D^8 + 9D^7 - 24D + 64)}{16(D-2)(D+1)}$	$\frac{D^9 + 39D^3 + 108D^2 + 4D + 128}{64(D-2)^2(D+1)}$
TU^1	$\frac{2D^2 - 5D - 10}{2(D-2)^2(D+1)}$	$\frac{D^2 - 8D + 8}{8(D-2)}$	$\frac{-(D^3 + 6D^2 + 8D + 64)}{32(D-2)^2}$
TU^2	$\frac{D^3 + 3D^2 + 6D + 16}{4(D-2)^2(D+1)}$	$\frac{D^3 + 3D^2 - 24D + 20}{4(D-2)D}$	$\frac{7D^2 - 26D + 40}{8(D-2)^2}$
FU^1	$\frac{-10^5 + 3D^3 + 90D^2 + 16D - 108}{32(D-2)^2}$	$\frac{D^4 - 5D^3 + 12D^2 - 34D - 16}{32(D-2)D}$	$\frac{2D^3 - 10D^2 + 23D + 124}{32(D-2)^2}$
FU^2	$\frac{-(D^3 + 5D^2 - 10D + 16)}{4(D-2)^2}$	$\frac{D + 10}{4(D-2)}$	0
FU^3	0	$\frac{1}{2D}$	$\frac{-(D^2 - 4D + 2C)}{8(D-2)^2}$
TL^4	$\frac{-(3D^4 - 4D^3 + 4D^2 + 10D - 24C)}{2(D-2)(D-1)(D+1)}$	$\frac{(3D^4 - 16D^3 + 12D^2 - 4)}{4(D-2)^2(D-1)(D+1)}$	$\frac{-D^3(2D-7)}{2(D-2)^2(D-1)(D+1)}$
TL^6	$\frac{7D^5 - 20D^4 + 25D^3 + 8D^2 + 40D + 84}{4(D-2)(D-2)(D-1)(D+1)}$	$\frac{-(3D^5 - 17D^4 + 34D^3 - 24D + 24)}{4(D-2)(D-2)(D-1)}$	$\frac{D(7D^2 - 15D - 4)}{8(D-2)^2(D-1)(D+1)}$
FL^2A	$\frac{-3}{(D-2)^2(D-1)}$	$\frac{D+2}{4(D-2)(D-1)}$	$\frac{-D(D+1)}{4(D-2)^2(D-1)}$
FL^3	$\frac{-(D^3 + 2D^2 - 16D + 18)}{(D-2)(D-2)(D-1)}$	0	0
FL^5	$\frac{-(D^2 + 5D - 8)}{4(D-2)^2(D+1)}$	$\frac{D^2 - D + 4}{16(D-2)(D+1)^2}$	$\frac{-1}{4(D+1)^2}$
TI^1	$\frac{7D^2 + 7D - 6}{4(D-2)(D-1)(D+1)}$	$\frac{-10D^3 + 15D^2 + 4D - 4}{8(D-2)(D-1)(D+1)}$	$\frac{-15D^3 - 15D^2 + 4D - 4}{8(D-2)(D-1)(D+1)}$
FI^2	$\frac{D(D^2 + 2D - 2)}{2(D-2)^2(D-1)}$	$\frac{-D}{(D-2)(D-1)}$	$\frac{1}{2}$
FI^4	$\frac{-(D^3 + 4D^2 + 14D - 24)}{8(D-2)^2}$	$\frac{D(D+6)}{8(D-2)^2}$	$\frac{-10(-1)}{4(D-2)}$
$FI^{5,ii}$	$\frac{D(D+1)}{8(D-2)^2}$	$\frac{-D(D+2)}{16(D-2)^2}$	0
$FI^{5,iii}$	$\frac{D(D^2 + 3D + 1)(D^2 - 5D - 5)}{16(D-2)^2(D-1)}$	$\frac{-D(D^2 + 3D + 4)}{16(D-2)(D-1)}$	0
FLI^2	$\frac{-1}{2(D-2)(D+1)}$	0	0
Total except $FI^{5,iii}$	$\frac{-(D^2)(D^5 + 5D^4 + D^3 + 4D^2 - 20D + 2)}{32(D-2)^2(D+1)^2}$	$\frac{D^4 + 6D^3 - 7D^2 - 10D - 8}{32(D-2)}$	$\frac{3(D^4 - 6D^3 + 9D^2 - 8)}{64(D-2)(D+1)^2}$

Table iii (continued)

Func _{in}	K4B7	K4P1	K4P6
TWU^1	$\frac{-(4D^9 + 17D^8 + 9D^7 - 24D + 64)}{16(D-2)(D+1)}$	$\frac{-17D^5 - 57D^4 - 31D^3 + 14D^2 + 12D - 25C}{64(D-2)^2(D+1)}$	$\frac{9D^5 - 21D^4 - 18D^3 - 33D^2 + 18D - 25C}{64(D-2)^2(D+1)}$
TU^1	$\frac{D^2 - 5D + 8}{16(D-2)}$	$\frac{7D^4 - 25D^3 - 34D^2 - 25D + 8C}{32(D-2)^2(D+1)}$	$\frac{-(19D^3 - 44D^2 - 44D + 43)}{32(D-2)^2}$
TU^2	$\frac{D^5 + 3D^2 - 14D + 20}{4(D-2)D}$	$\frac{15D^3 - 27D^2 - 14D + 20}{8(D-2)D(D+1)}$	$\frac{-(13D^2 - 10D + 2C)}{8(D-2)D}$
FU^1	$\frac{-(D^4 - 11D^3 + 12D^2 + 4D + 16)}{64(D-2)D}$	$\frac{-1D^5 - 7D^4 + 11D^3 - 36D^2 + 4C(D-16)}{32(D-2)^2D}$	$\frac{4D^4 - 17D^3 + 8D^2 + 44D - 16}{32(D-2)^2D}$
FU^2	0	$\frac{-D^2}{4(D-2)}$	0
FU^3	$\frac{-(D^3 - 3D^2 - 18D + 8)}{16(D-2)D}$	$\frac{-(D-4)}{16D}$	$\frac{3D^2 + 6D + 8}{16(D-2)D}$
TL^4	$\frac{3D^3 - 11D^2 + 2D - 8}{8(D-2)(D-1)(D+1)}$	$\frac{7D^2 - 10D + 4}{4(D-2)^2(D-1)(D+1)}$	$\frac{-1}{2(D-1)(D+1)}$
TL^6	$\frac{-(3D^5 - 7D^4 - 6D^3 - 8)}{8(D-2)(D-1)(D+1)}$	$\frac{D^4 + 10D^3 - 17D^2 - 2D + 48}{9(D-2)(D-2)(D-1)(D+1)}$	$\frac{-(10^5 + 33D + 8)}{8(D-2)(D-1)(D+1)}$
FL^2A	$\frac{D+1}{8(D-1)}$	$\frac{-3}{4(D-2)(D-1)}$	$\frac{D+1}{4(D-2)(D-1)}$
FL^3	0	$\frac{-(12D^3 - 3D^2 - D + 4)}{4(D-2)(D-2)^2(D-1)}$	0
FL^5	$\frac{-1}{4(D+1)^2}$	$\frac{D^2 - 7D + 2}{16(D-2)(D+1)^2}$	$\frac{D+3}{16(D+1)^2}$
TI^1	$\frac{-(15D^3 - 15D^2 + 4D - 4)}{8(D-2)(D+1)}$	$\frac{-(4D^5 - 27D^4 + 3D^3 - 20D^2 + 8D - 8)}{8(D-2)(D-1)(D+1)}$	$\frac{-(15D^5 - 15D^4 - 46D - 4)}{8(D-2)(D-1)(D+1)}$
FI^2	$\frac{1}{2}$	$\frac{2D^2 - 3D + 4}{4(D-2)(D-1)}$	$\frac{1}{2(D-1)}$
FI^4	$\frac{-(D+1)}{4(D-2)}$	$\frac{-D(D+2)}{8(D-2)^2}$	$\frac{-1}{4(D-2)}$
$FI^{5,ii}$	0	0	0
$FI^{5,iii}$	0	$\frac{D^2(D^2 - 2D + 4)}{16(D-2)^2(D-1)}$	0
FLI^2	0	$\frac{-(D-1)}{4(D-2)(D-1)}$	0
Total except $FI^{5,iii}$	$\frac{-(D^2)(D^4 - 6D^3 + 9D^2 - 8)}{64(D+1)^2}$	$\frac{-(10^5 + 11D^5 - 15D^4 - 24D^3 + 46D^2 - 16)}{64(D-2)(D+1)^2}$	$\frac{-(1^4 - 6D^3 + 9D^2 - 8)}{64(D+1)^2}$

Table iv - The Coefficient of $\mu^4 \nu^2 \kappa^4 \alpha^2 \nu^2$

Funcn	$K3A1$	$K4A1$ (Reduced)	$\frac{-2}{D^2} * K4A1 + K4B1$
TL^{14}	0	$\frac{D^3 - D^2 + 3D + 2}{4(D-2)(D+1)}$	$\frac{7D^5 + 4D^4 + 5D^3 - 7D^2 - 13D + 4}{64(D-2)(D+1)(D+1)}$
TL^{16}	$\frac{-D}{4(D-2)^2}$	$\frac{-D^2}{2(D-1)(D+1)}$	$\frac{-17D^5 + 4D^4 + 10D^3 + 4D^2 + 15D + 8}{64(D-2)(D+1)(D+1)}$
FL^{12}	$\frac{D(D+1)}{16(D-2)^2}$	$\frac{1}{16(D-2)(D+1)}$	0
FL^{21}	$\frac{-D(D+1)}{16(D-2)^2}$	$\frac{-10D^2 + 2D + 6}{16(D-2)(D+1)}$	$\frac{1}{4(D-1)}$
FL^{34}	0	0	$\frac{D}{4(D-2)}$
FL^{43}	$\frac{D}{2(D-2)^2}$	$\frac{1}{2}$	$\frac{-1}{2} = (D+2)$
FL^{55}	$\frac{D-2}{4(D-2)(D+1)}$	$\frac{2D^2 - 13D + 12}{8(D-2)(D+1)^2}$	$\frac{(D-2)(D+1)(D^2 + 4D + 4)}{128(D-2)(D+1)(D+1)}$
FL^6	$\frac{D+3}{2(D-2)(D+1)}$	0	$\frac{(D+2)(D+3)(D+5)(D+6)}{128(D-2)(D+1)(D+1)}$
FL^7	$\frac{-10D^2 + 14D + 1}{4(D-2)(D+1)}$	$\frac{-13D^2 + 5D + 20}{8(D-2)(D+1)^2}$	$\frac{-64(D-2)}{(D-1)(D+2)}$
TI^1	$\frac{D}{4(D-2)^2}$	$\frac{D^3 - 3D^2 + 3D + 2}{4(D-2)(D+1)(D+1)}$	$\frac{5D^3 + 4D^2 + 10D + 4}{16(D-2)(D+1)(D+1)}$
FI^1	$\frac{-D(D+1)}{4(D-2)(D+1)}$	$\frac{-D(D+1)}{8(D+1)}$	0
FI^2	$\frac{D^2}{4(D-2)^2(D+1)}$	$\frac{D-1}{4(D-2)}$	$\frac{-1}{4(D-1)}$
FI^3	$\frac{D(D+1)}{32(D-2)^2}$	$\frac{D^2 - 5D + 4}{32(D-2)}$	$\frac{1}{16}$
FI^4	$\frac{-D(D-1)^2}{32(D-2)^2}$	$\frac{-10D^2 + 17D + 2}{32(D-2)}$	$\frac{D}{32}$
$FI^{5,6}$	$\frac{D+1}{4(D-2)^2}$	$\frac{D}{4(D-2)}$	0
$FI^{6,6}$	$\frac{-D(D+1)}{4(D-2)^2}$	0	0
$FI^{7,6}$	$\frac{-10D^2 + 17D + 2}{96(D+1)}$	$\frac{-10D^2 + 2D}{96(D+1)}$	0
$FI^{5,7,6}$	0	0	$\frac{-10D^2 + 10D - 5D + 6}{32(D-2)(D+1)}$
TD^1	0	$\frac{-10D^2 + 2D}{4(D-2)(D+1)}$	$\frac{32(D-2)(D+1)}{32(D-2)(D+1)}$
FD^1	$\frac{-D}{32(D-2)^2}$	$\frac{1}{2}$	0
FD^2	0	0	$\frac{-1}{4(D-2)}$
FLI^1	$\frac{2+1}{4(D-2)(D+1)}$	0	$\frac{10D^2 + 10D - 5D + 6}{64(D-2)(D+1)}$
FLI^2	$\frac{-D}{32(D-2)^2}$	$\frac{-10D^2 + 2D}{512(D+1)}$	$\frac{10D^2 + 10D - 5D + 6}{64(D-2)(D+1)}$
FLD^1	$\frac{-1}{4(D-2)(D+1)}$	0	$\frac{-10D^2 + 2D}{32(D-2)(D+1)}$
FID^1	$\frac{-1}{32(D-2)^2}$	0	0
Total	0	0	0

Table iv (continued)

Funcn	$K4A2$	$K4B2$	$K4H7$
TL^{14}	$\frac{-17D^2 + 16D^2 + 3D + 2}{4(D-2)(D+1)}$	$\frac{7D^4 + 2D^3 + 3D^2 + 5D + 4}{16(D-2)(D+1)}$	$\frac{-18D^5 + 36D^4 - 5D^3 + 4D^2 + 15D + 8}{64(D-2)(D+1)(D+1)}$
TL^{16}	$\frac{9D^2 + 10D^2 + 3D + 2}{512(D-2)(D+1)}$	$\frac{-17D^5 + 37D^4 + 36D^3 + 4D^2 + 15D + 8}{16(D-2)(D+1)}$	$\frac{D(9D^3 + 36D^2 + 10D + 10)}{64(D-2)(D+1)(D+1)}$
FL^{12}	$\frac{-10D^2 + 10D^2 + 3D + 2}{16(D-2)(D+1)}$	$\frac{D+1}{4(D-2)}$	$\frac{2(D-2)(D+1)}{32(D-2)(D+1)}$
FL^{21}	$\frac{D^2 + 10D^2 + 3D + 2}{16(D-2)(D+1)}$	$\frac{-D(D+2)}{4(D-2)(D+1)}$	$\frac{3(D-2)(D+1)}{32(D-2)(D+1)}$
FL^{34}	$\frac{D^2}{32(D-2)^2}$	$\frac{-D^2}{16(D-2)}$	$\frac{10D^2 + 2D}{16(D-2)}$
FL^{43}	$\frac{-15D^2 + 15D - 32}{8(D-2)^2}$	$\frac{D(D+6)}{16(D-2)}$	$\frac{D}{10D^2 + 2D}$
FL^{55}	$\frac{-10D^2 + 10D^2 + 3D + 2}{4(D-2)(D+1)^2}$	0	$\frac{-10D^2 + 10D + 4}{64(D-2)(D+1)^2}$
FL^6	$\frac{-10D^2 + 10D^2 + 3D + 2}{16(D-2)^2}$	0	0
FL^7	$\frac{D^3 + 10D^2 + 10D + 4}{4(D-2)(D+1)^2}$	0	$\frac{D^2 + 7D + 4}{32(D-2)(D+1)^2}$
TI^1	$\frac{-10D^2 + 10D^2 + 3D + 2}{4(D-2)(D+1)(D+1)}$	$\frac{5D^3 + 4D^2 + 10D + 4}{32(D-2)(D+1)(D+1)}$	0
FI^1	$\frac{D^2 + 10D^2 + 10D + 4}{4(D-2)(D+1)}$	$\frac{-10D^2 + 10D + 4}{4(D-2)(D+1)}$	$\frac{-D}{32(D-2)(D+1)}$
FI^2	$\frac{-10D^2 + 10D^2 + 3D + 2}{4(D-2)(D+1)}$	$\frac{D^2}{32(D-2)}$	$\frac{D}{32(D-2)}$
FI^3	$\frac{D^3 + 10D^2 + 10D + 4}{32(D-2)^2}$	$\frac{-10D^2 + 10D + 4}{32(D-2)}$	$\frac{D}{32(D-2)}$
FI^4	$\frac{-D(D+1)}{32(D-2)^2}$	0	$\frac{D(D+2)}{32(D-2)}$
$FI^{5,6}$	$\frac{-D(D+1)}{4(D-2)^2}$	0	$\frac{D(D+2)}{64(D-2)}$
$FI^{6,6}$	$\frac{D(D+1)}{4(D-2)^2}$	0	$\frac{D(D+2)}{64(D-2)}$
$FI^{7,6}$	$\frac{2D^2 + 10D^2 + 10D + 4}{64(D-2)(D+1)}$	0	$\frac{D(D+2)}{64(D-2)}$
$FI^{5,7,6}$	0	0	0
TD^1	$\frac{D-6}{4(D-2)^2}$	$\frac{-10D^2 + 10D^2 + 3D + 2}{32(D-2)(D+1)}$	$\frac{D-6}{4(D-2)^2}$
FD^1	$\frac{D^2 + 10D^2 + 10D + 4}{4(D-2)^2}$	$\frac{-1}{32(D-2)}$	0
FD^2	0	0	$\frac{D}{10D^2 + 2D}$
FLI^1	$\frac{-D}{4(D-2)}$	0	$\frac{D(D+2)}{64(D-2)}$
FLI^2	$\frac{D^3 + 10D^2 + 10D + 4}{32(D-2)(D+1)}$	0	$\frac{D^2}{4(D-2)^2}$
FLD^1	$\frac{D}{16(D-2)^2}$	0	0
FID^1	$\frac{D}{32(D-2)^2}$	0	$\frac{-D}{32(D-2)^2}$
Total	0	0	0

Table v - The Coefficient of $M^{+D} \times 4 \cdot 2^{2D}$

Function	K2A1	K4A1	K4B1	K4A2	K4B2
TL ^{1a}	0	$\frac{-(D^2-2D+2)}{4(D-2)D}$	$\frac{1}{8(D+1)}$	$\frac{1}{2(D-2)}$	$\frac{D^2-2D-4}{8D(D+1)}$
TL ^{1b}	0	$\frac{-1}{8(D+1)}$	0	$\frac{1}{2(D-2)(D+1)}$	0
FL ⁵	$\frac{-(D-1)}{2(D-2)^2(D+1)}$	$\frac{D^3-10D^2+25D-20}{16(D-2)(D+1)^2}$	0	$\frac{D^2+6D^2-21D+22}{4(D-2)^2(D+1)^2}$	0
FL ⁶	$\frac{D+3}{4(D-2)(D+1)}$	0	0	$\frac{-(D+3)}{4(D-2)}$	0
FL ⁷	$\frac{-(D^3-2D^2+4D-5)}{4(D-2)^2(D+1)(D+1)}$	$\frac{7D^3-19D^2+16}{8(D-2)(D+1)^2}$	0	$\frac{D^4-D^3-7D^2+35D-32}{4(D-2)^2(D+1)^2}$	0
TI ¹	0	$\frac{5D^2-10D+8}{8(D-2)(D+1)}$	$\frac{-1}{4(D+1)}$	$\frac{-3}{2(D-2)(D+1)}$	$\frac{-(D-2)}{4D(D+1)}$
FI ⁵	$\frac{-(D+1)}{8(D-2)^2}$	$\frac{-D}{16(D-2)}$	0	$\frac{D(D+1)}{8(D-2)^2}$	0
FI ⁶	$\frac{D+1}{8(D-2)^2}$	0	0	$\frac{-D(D+1)}{8(D-2)^2}$	0
FI ⁷	$\frac{D+1}{8(D-2)^2}$	$\frac{1}{4(D-2)(D+1)}$	0	$\frac{-(D^3+D+8)}{8(D-2)^2(D+1)}$	0
TD ¹	0	0	$\frac{1}{8D(D+1)}$	0	$\frac{D+2}{8D(D+1)}$
FD ³	0	0	0	0	0
FLI ¹	$\frac{-(2D^2-D-2)}{4(D-2)^2(D+1)}$	0	0	$\frac{D^2}{2(D-2)^2}$	0
FLI ²	$\frac{D}{4(D-2)^2}$	$\frac{1}{4(D+1)}$	0	$\frac{-(D^2+4D-4)}{4(D-2)^2(D+1)}$	0
FLD ¹	$\frac{-1}{2(D-2)(D+1)}$	0	0	$\frac{1}{2(D-2)}$	0
FID ¹	$\frac{1}{2(D-2)^2}$	0	0	$\frac{-D}{2(D-2)^2}$	0
Total	$\frac{-(D-1)^2}{8(D-2)^2(D+1)}$	$\frac{-(D-1)(2D^3-5D^2+4D+4)}{8(D-2)D(D+1)^2}$	$\frac{D-1}{8D(D+1)}$	$\frac{(D-1)^2(D^2+5D-8)}{8(D-2)^2(D+1)^2}$	$\frac{(D-2)(D+1)}{8D(D+1)}$

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Table v (continued)

Function	K4B7	K4P1	K4P6
TL ^{1a}	$\frac{7D^3+10D^2+3D+5}{16(D-2)(D+1)(D+1)}$	$\frac{-(13D^3-10D^2+24D-16)}{16(D-2)^2(D+1)}$	$\frac{D^3-2D^2+3D+2}{8(D-2)(D+1)}$
TL ^{1b}	$\frac{7D^3+10D^2+3D+5}{16(D-2)(D+1)(D+1)}$	$\frac{3D^3+10D^2+4D+12}{16(D-2)(D+1)(D+1)}$	$\frac{-(D^2+2)}{5D(D+1)}$
FL ^{1a}	$\frac{-(D-2)(D+1)}{16(D-2)(D+1)}$	$\frac{-1}{16(D-2)(D+1)}$	$\frac{-(D+1)}{5D(D+1)}$
FL ^{2a}	$\frac{D+2}{16(D-2)}$	$\frac{D^2-2D^2-3D+10}{16(D-2)(D+1)}$	$\frac{D+1}{5D(D+1)}$
FL ^{3a}	$\frac{D+2}{4(D-2)}$	$\frac{D^2}{16(D-2)^2}$	$\frac{D+2}{4(D-2)}$
FL ^{4a}	$\frac{-D}{4(D-2)}$	$\frac{-D(7D+10)}{16(D-2)^2}$	$\frac{-D}{4(D-2)}$
FL ⁵	$\frac{-(D-2)}{4(D-2)^2}$	$\frac{-(15D^2+10D+2)}{8(D-2)(D+1)^2}$	$\frac{-(D+5)}{8(D+1)^2}$
FL ⁶	0	0	0
FL ⁷	$\frac{D-3}{4(D-2)^2}$	$\frac{D^2+10D+8}{8(D-2)(D+1)^2}$	$\frac{2D^2}{4(D+1)^2}$
TI ¹	0	$\frac{-(9D^3-2D^2+4D+2)}{8(D-2)(D+1)(D+1)}$	0
FI ¹	$\frac{D-2}{8(D-2)}$	$\frac{D-2}{8(D-2)}$	$\frac{1}{4(D-1)}$
FI ²	$\frac{D-2}{8(D-2)}$	$\frac{-(D^2-3D+4)}{8(D-2)(D+1)}$	$\frac{-1}{4(D-1)}$
FI ³	$\frac{-(D-1)}{4(D-2)}$	$\frac{9D^2-(D+6)}{32(D-2)^2}$	$\frac{-1}{4(D-2)}$
FI ⁴	$\frac{D-1}{4(D-2)}$	$\frac{D(D^2-6D+10)}{32(D-2)^2}$	$\frac{1}{4(D-2)}$
FI ^{5a}	0	0	0
FI ^{6a}	0	0	0
FI ^{7a}	0	$\frac{D-1}{16(D+1)}$	0
FI ^{5b}	0	0	0
TD ¹	$\frac{-(D^2+D-4)}{9(D-2)D}$	$\frac{-D+2}{8D(D+1)}$	$\frac{1}{4D}$
FD ¹	0	$\frac{1}{8}$	0
FD ²	$\frac{-1}{2(D-1)}$	0	$\frac{-1}{2(D-1)}$
FLI ¹	0	0	0
FLI ²	0	$\frac{-(D-1)(D-10)}{16(D-2)(D+1)}$	0
FLD ¹	0	0	0
FID ¹	0	0	0
Total	0	0	0

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Table vi (continued)

Functn	K4A7	K4B7	K4P1	K4P6
TL ^{1a}	$\frac{-D(3D^2-5D+10)}{4(D-2)^2(D+1)}$	$\frac{D^2-5D+2}{8(D+1)}$	$\frac{-(2D^2-3D+2)}{4(D-2)(D+1)}$	$\frac{D-1}{4(D+1)}$
TL ^{1b}	$\frac{-D^2(3D^2-2D+4)}{4(D-2)^2(D+1)}$	$\frac{D^2(D-3)}{8(D+1)}$	$\frac{-D^3}{4(D-2)(D+1)}$	$\frac{D^2}{4(D+1)}$
FL ⁷	$\frac{D(30^3+14D^2-3D^2+30D+8)}{8(D-2)^2(D+1)^2}$	$\frac{D^2}{(D+1)^2}$	$\frac{-D(D^3-10D^2+7D-2)}{8(D-2)(D+1)^2}$	$\frac{-D(D^2+4D-1)}{8(D+1)^2}$
TI ¹	$\frac{D(11D^2-13D+16)}{4(D-2)^2(D+1)}$	$\frac{-(D-1)(3D-2)}{4(D+1)}$	$\frac{-(D^3-7D^2+8D-4)}{4(D-2)(D+1)}$	$\frac{-(3D-2)}{4(D+1)}$
FI ⁷	$\frac{-D(3D^2-2D+8)}{8(D-2)^2}$	$\frac{(D-1)D}{8}$	$\frac{D(D^2-5D+2)}{8(D-2)(D+1)}$	$\frac{1}{8}D$
Total	$\frac{-3(D-1)D(D^2-D-1)}{4(D-2)(D+1)^2}$	$\frac{(D-1)^2(D^2-D-1)}{4(D+1)^2}$	$\frac{-(D-1)(2D^2-4D^2+D+2)}{4(D-2)(D+1)^2}$	$\frac{(D-1)(D^2-D-1)}{4(D+1)^2}$