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Simulating the Quark-Gluon Plasma on the Lattice[†]

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1. INTRODUCTION

The idea of using a discrete space-time lattice to regularize Euclidean quantum field theories¹⁾ opened the entire repertory of statistical physics for the analysis of non-perturbative properties of these theories. In particular the application of Monte Carlo simulation techniques²⁾ turned out to be a powerful approach, which for the first time allowed to perform a quantitative study of non-perturbative aspects of Quantum Chromodynamics (QCD) on the basis of first principles, i.e. by simulating the complete lattice regularized QCD Lagrangian³⁾.

For a long time it had been expected that the properties of strongly interacting matter change drastically at high temperature and/or densities and it even has been speculated that strong interactions with their copious production of resonances provide a mechanism that leads to the existence of a limiting temperature at which the density of states diverges⁴⁾. However, such considerations assumed hadrons as being elementary and ignored their internal structure. At temperatures of the order of the pion mass, $T \sim 140 \text{ MeV}$, or densities a few times that of ordinary nuclear matter, $n \sim (3-5)n_0$, with $n_0 = 0.15 \text{ fm}^{-3}$, conditions are reached where hadrons would start overlapping and the interaction among the internal partonic degrees of freedom, described by QCD, becomes important. The asymptotic freedom property of QCD⁵⁾ led to the expectation that nuclear matter at such high temperatures or densities would behave as an asymptotically free gas of quarks and gluons⁶⁾. The qualitative differences between this perturbative high temperature (density) regime and the non-perturbative confinement regime at low temperature (density) led to the speculation that these two regimes may in fact be separated by a phase transition.

The lattice regularized formulation of QCD made it possible to approximate the lattice action by simpler statistical systems, whose phase structure is easier to analyze. In this way it could first be shown that in the strong coupling limit of lattice QCD a phase transition exists^{7,8)}. Although hitherto still not proved rigorously it became a firm belief that QCD predicts a phase transition from a confined low temperature phase with broken chiral symmetry to a deconfined, chiral symmetric high temperature phase, the quark-gluon plasma. The properties of hot (dense) hadronic matter have then been exploited for a long time using various approximations for the low and high temperature (density) region^{9,10)}, which usually assume the existence of a phase transition. Monte Carlo simulations of lattice QCD aim at replacing this approach by a systematic, non-perturbative analysis of the QCD thermodynamics, based on the complete lattice regularized QCD Lagrangian. They can provide us with quantitative results for the transition temperature, order of the phase transition, equation of state and other quantities which are of immediate interest for the experimental search for the quark-gluon plasma in heavy ion collisions¹¹⁾ as well as for our understanding of the early stages in the expansion of the universe¹²⁾. Already the first Monte Carlo simulations provided us with quantitative results about the phase structure of QCD at finite temperature as well as the thermodynamic properties of the high temperature phase¹³⁻¹⁵⁾. At present large effort is put into the refinement of these early quantitative results for pure $SU(N)$ gauge theories. This includes the attempt to eliminate finite lattice artifacts by simulating QCD on larger and larger lattices, also the problem of introducing dynamical fermions in the lattice simulations and the whole problem of performing simulations of QCD at non-zero baryon number density.

In this review we want to discuss the lattice approach to the thermodynamics of QCD and summarize its present status. We start in section 2 with a discussion of the continuum formulation of finite temperature QCD and summarize some perturbative results. In section 3 we describe the lattice formulation of QCD thermodynamics. Section 4 is devoted to a discussion of the phase structure of QCD as it emerges from lattice simulations. In section 5 we present Monte Carlo results for the equation of state and analyze the thermodynamic properties of the high temperature phase. Section 6 gives a discussion of screening lengths in the plasma phase, in particular we analyze the influence of screening on the heavy quark potential. The problem of lattice QCD at non-zero chemical potential or

baryon number is discussed in section 7. Finally we give some conclusions and an outlook in section 8.

2. PERTURBATIVE QCD AT FINITE TEMPERATURE

The partition function of a $SU(N)$ gauge theory coupled to n_f flavours of fermions with masses m_i , $i = 1, \dots, n_f$, is given by

$$Z(T, V) = \int DAD\bar{\psi} D\psi \exp[-S(T, V)] . \quad (1)$$

Here we have suppressed the explicit dependence of the partition function on the quark masses (m_i), as well as colour (N) and flavour (n_f) degrees of freedom. The dependence on temperature T and volume V enters in the action $S(T, V)$ through the integration limits for the four-dimensional space-time integral over the QCD Lagrangian

$$S(T, V) = \int_0^{1/T} dx_0 \int_V d^3x \mathcal{L}(\mathcal{A}(x), \bar{\psi}(x), \psi(x)) , \quad (2)$$

with

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \sum_{i=1}^{n_f} \bar{\psi}_i(D(\mathcal{A}) + m_i)\psi_i . \quad (3)$$

Here $D(\mathcal{A}) = \gamma_\mu [i\partial_\mu + \frac{1}{2}gA_\mu^a\lambda^a]$ denotes the covariant derivative and $F_{\mu,\nu}^a$ is the gluonic field strength tensor, $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$. The fact that in the Hamiltonian formulation the partition function is a trace over all possible states of the system is in the Euclidean formulation reflected in the temporal boundary conditions for the fields. While the bosonic fields $\mathcal{A}_\mu(x_0, \vec{x})$ obey periodic boundary conditions in Euclidean time, $\mathcal{A}_\mu(1/T, \vec{x}) = \mathcal{A}_\mu(0, \vec{x})$, the fermionic fields $\bar{\psi}(x), \psi(x)$ obey anti-periodic boundary conditions in this direction of space-time.

Starting from the partition function, eq.(1), thermodynamic observables can be obtained from derivatives of $\ln Z$ with respect to T and/or V . In particular the energy density and pressure are given by

$$\begin{aligned} \epsilon &= \frac{T^2}{V} \frac{\partial}{\partial T} \ln Z , \\ p &= T \frac{\partial}{\partial V} \ln Z . \end{aligned} \quad (4)$$

Let us define the thermodynamic potential $\Omega = -\frac{1}{VT^3} \ln Z$, which is directly related to the free energy density, $F/T^3 = \Omega$. Its perturbative expansion is known

up to $O(g^4 \ln g^2)^{16-19}$. For massless QCD, $m_i = 0$, one obtains

$$\Omega = -\frac{1}{VT^3} \ln Z = -[a_0 + a_2 g^2 + a_3 g^3 + a_4 g^4 \ln g^2] + O(g^4) , \quad (5)$$

with the expansion coefficients a_i given by

$$\begin{aligned} a_0 &= \frac{\pi^2}{45} (N^2 - 1 + \frac{7}{4} N n_f) \\ a_2 &= -\frac{1}{144} (N^2 - 1) (N + \frac{5}{4} n_f) \\ a_3 &= \frac{1}{12\pi} (N^2 - 1) \left[\frac{N}{3} + \frac{n_f}{6} \right]^{3/2} \\ a_4 &= \frac{1}{32\pi^2} N (N^2 - 1) \left[\frac{N}{3} + \frac{n_f}{6} \right] . \end{aligned} \quad (6)$$

We note that Ω depends on T only implicitly through the renormalization of the bar coupling constant g , which at high temperature is governed by the renormalization group equation,

$$T \frac{\partial g(T)}{\partial T} = -b_0 g^3 - b_1 g^5 + O(g^7) . \quad (7)$$

Here b_0 and b_1 are the first two universal coefficients of the $SU(N)$ β -function:

$$\begin{aligned} b_0 &= \frac{1}{16\pi^2} \left[11 - \frac{2}{3} n_f \right] \\ b_1 &= \left(\frac{1}{16\pi^2} \right)^2 \left[\frac{34}{3} N^2 - \left(\frac{10}{3} N + \frac{N^2 - 1}{N} \right) n_f \right] . \end{aligned} \quad (8)$$

Eq.(7) reflects the fact that QCD is asymptotically free; the running coupling constant, $g(T)$, approaches zero at infinite temperature. It is then obvious from eqs.(4) and (5) that in the limit $T \rightarrow \infty$ the equation of state approaches that of an ideal gas of quarks and gluons, $\epsilon = 3p$, with $p/T^4 = a_0$, and a_0 denoting the Stefan-Boltzmann constant for an ideal quark-gluon gas.

Starting with eq.(5), perturbative results for other thermodynamic quantities can be obtained. In particular one finds for the pressure

$$p/T^4 = -\Omega , \quad (9)$$

the energy density

$$\epsilon/T^4 = -3\Omega + T \frac{\partial g(T)}{\partial T} \frac{\partial \Omega}{\partial g} , \quad (10)$$

and the entropy density $S = (\epsilon + P)/T$

$$S/T^3 = -4\Omega + T \frac{\partial g(T)}{\partial T} \frac{\partial \Omega}{\partial g} . \quad (11)$$

The perturbative approach suggests, that QCD at high temperatures closely resembles the behaviour of a free quark-gluon gas. As can be seen from eqs. 9 and 10 any deviations from the ideal gas equation of state, $\epsilon = 3p$, can only arise from the temperature dependence of the running coupling constant $g(T)$, i.e. they are controlled by the QCD β -function, eq.(7). However, already the presence of the $O(g^3)$ term (plasmon term) in the perturbative expansion of Ω signals the importance of non-perturbative effects in the high temperature phase. In fact, this term is obtained from a summation of an infinite set of diagrams. This became necessary, in order to eliminate infra-red divergences that occur at $O(g^4)$ in the perturbative expansion of the thermodynamic potential Ω . To this order the infra-red problems could be cured because the summation of an infinite set of ring-diagrams generates a mass term in the electric part of the gluon propagator,

$$m_E(T)^2 = \left(\frac{N}{3} + \frac{n_f}{6} \right) g^2(T) T^2 , \quad (12)$$

which is of $O(g)$. In fact, the $O(g^3)$ term in the expansion of Ω results from a contribution that is proportional to $(m_E/T)^3$.

More severe infra-red problems arise in higher orders of the perturbative expansion of Ω from the magnetic sector of the gluon propagator. A magnetic mass,

$$m_M \sim g^2(T) T , \quad (13)$$

is needed²⁰⁾ in order to control the resulting divergences and to get a finite contribution to Ω at $O(g^6)$. Similar problems arise in all higher orders and, assuming the appearance of a magnetic mass of the form given in eq.(13), a diagram with n 4-gluon vertices will lead to contributions of the form $g^6(g^2 T/m_M)^{n-3}$ and thus will contribute to the $O(g^6)$ term in Ω . The coefficient of the g^6 -term receives therefore contributions from all higher orders in perturbation theory and hence is in calculable within the context of perturbation theory. These infra-red problems of QCD at high temperatures have been taken as an indication for the instability of the classical vacuum around which the perturbative expansion is performed. It has been argued, that the time component of the gauge fields, A_0 , may acquire

a non-vanishing vacuum expectation value²¹⁻²⁴⁾. If true, this would improve the infra-red properties of the theory. A perturbative expansion around the "false", classical vacuum, however, would be completely unreliable and would cast doubt even on lowest order perturbative calculations for some finite temperature observables like, for instance, the electric screening mass given in eq.(12).

The above discussion shows that perturbation theory cannot provide us with a satisfactory description of the high temperature phase. Its validity regime is uncertain and the inherent infra-red problems indicate that in addition to the characteristic perturbative length scale, $l_P \sim 1/T$, a hierarchy of non-perturbative length scales like the electric (l_E) and magnetic (l_M) mass gap are generated in the plasma phase:

$$\begin{aligned} l_P &< l_E < l_M \\ 1/T &< 1/gT < 1/g^2T \end{aligned} \quad (14)$$

A perturbative treatment of the long-distance properties of QCD plasma phase thus seem to be impossible. Moreover, it is obvious that the perturbative approach is not suitable for studying such inherently non-perturbative effects as the appearance of a phase transition that separates the high temperature plasma phase from the low temperature hadronic phase. A non-perturbative approach is thus called for.

Monte Carlo simulations of lattice regularized QCD allow a non-perturbative analysis of the phase structure of QCD, its critical properties at the transition point as well as the thermodynamics of the high temperature phase. In the following we will present results from this non-perturbative approach to QCD thermodynamics. We will, however, extensively compare the numerical results with perturbation theory in order to test the validity regime of the perturbative approach and the importance of non-perturbative effects in the plasma phase.

3. FINITE TEMPERATURE QCD ON THE LATTICE

In this section we will discuss the formulation of finite temperature QCD on the lattice. We start in section 3.1 with a brief introduction into the lattice formulation of QCD, which mainly serves to fix our basic notations. For a more detailed discussion of basic aspects of lattice QCD and results obtained from simulations at $T = 0$ the reader is referred to recent reviews on these topics^{25,26)}. In section 3.2 we concentrate on aspects more specific to the finite temperature formulation

of QCD on the lattice.

We would, of course, also like to extend these considerations to the thermodynamics at non-zero chemical potential μ (non-zero baryon number density). However, although the general framework for doing this on the lattice has been developed^{27–30}, one encounters severe technical problems in a quantitative numerical analysis. We thus will restrict our discussion here to the finite temperature, zero chemical potential case and come back to the problem of non-zero baryon number density in section 7.

3.1. QCD on the lattice

Let us start our discussion of lattice QCD by defining the basic degrees of freedom of QCD on a four-dimensional discrete space-time lattice. First of all we replace the integration over the four-volume $T^{-1}V$, appearing in the definition of the action in eq.(2), by a sum over sites of a four-dimensional hypercubic lattice[†]. On this lattice we define bosonic (fermionic) fields $U_{n,\mu}$ ($\bar{\chi}_n, \chi_n$) which replace the continuum fields $A_\mu(x)$ ($\psi(x)$, $\bar{\psi}(x)$). Here $n = (n_0, n_1, n_2, n_3)$ labels a site on the four-dimensional hypercubic lattice, which is related to the continuous space-time coordinate x through $x = n \cdot a$, with a denoting the lattice spacing. Gluonic fields are represented by elements of the colour gauge group $SU(N)$, $U_{n,\mu} \in SU(N)$. They are defined on links (n, μ) of the lattice that originate at a site n and end at the neighbouring site, $n + \hat{\mu}$, in μ -direction, $\mu = 0, 1, 2$ or 3 . The link variables $U_{n,\mu}$ are related to the gauge fields $A_\mu(x)$ through

$$U_{n,\mu} = \exp \left[-iga \sum_{i=1}^{N^2-1} \lambda^i A_\mu^i(x) \right], \quad (15)$$

where g is the bare coupling constant of QCD. Using this relation one can easily verify that the single plaquette action proposed by Wilson¹⁾

$$S_G = \frac{2N}{g^2} \sum_{n; 0 \leq \mu < \nu \leq 3} P_{n,\mu\nu}, \quad (16)$$

with

$$P_{n,\mu\nu} = 1 - \frac{1}{N} \text{ReTr} U_{n,\mu} U_{n+\hat{\mu},\nu} U_{n+\hat{\mu}+\hat{\nu},\mu}^\dagger U_{n,\nu}^\dagger, \quad (17)$$

[†] Other lattice structures are, of course possible, and have been used to achieve a better approximation of the continuum theory. In particular the use of a random lattice has been suggested³¹⁾ which has the advantage that it preserves the rotational symmetry of the continuum theory. However, a cubic symmetry has the advantage of being easy to handle and thus is commonly used.

approximates the gluonic part of the continuum action up to terms of $O(a^6)$, $P_{n,\mu\nu} = \frac{1}{4} a^4 F_{\mu\nu}^a F_{\mu\nu}^a + O(a^6)$. In the continuum limit, $a \rightarrow 0$, these higher order corrections become irrelevant.

The discretization of the fermionic part of the action leads to some difficulties. The fermion action contains only first derivatives of the fields. As a consequence of this one finds that a naive discretization of the action leads to additional poles in the lattice fermion propagator. In the continuum limit these additional poles would give rise to 16 additional, unwanted fermion species rather than only the one we started with. By distributing the four components of the continuum spinor over different sites of the lattice it is possible to reduce somewhat the number of additional species³²⁾. This problem of species doubling can be avoided entirely by giving a large mass to the additional species¹⁾. However, this is possible only for the price of explicitly breaking the chiral invariance of the continuum action. In fact, if we insist in introducing only short range interactions among the fermionic fields on the lattice it is in general impossible to avoid species doubling and at the same time preserve chiral symmetry³³⁾.

For the purpose of finite temperature calculations on the lattice it is convenient to work with a lattice action which preserves at least a part of the $SU(n_f) \times SU(n_f)$ chiral symmetry of the continuum action as phase transitions at finite temperature are generally characterized by the spontaneous breaking/restoration of global symmetries of the QCD Lagrangian. In the following we thus will restrict our discussion to the staggered fermion formulation³²⁾. If we introduce $\bar{f}$ different fermion species on the lattice the staggered lattice action will lead to $n_f = 4\bar{f}$ species of fermions in the continuum limit. It preserves, however, a global $U(\bar{f}) \times U(\bar{f})$ chiral symmetry, i.e. an abelian subgroup of the continuum chiral symmetry.

The staggered fermion action is given by

$$S_F = \sum_{i=1}^{\bar{f}} \bar{\chi}_{n,i} Q_{n,m}^i \chi_{m,i}, \quad (18)$$

Here there fermion fields, χ ($\bar{\chi}$), are anticommuting Grassmann variables defined on the sites of the lattice and the fermion matrix $Q_{n,m}^i$ is defined as

$$Q_{n,m}^i = \sum_{\mu=0}^3 D_{n,m;\mu} + m_i a \delta_{n,m}. \quad (19)$$

The hopping matrices $D_{n,m;\mu}$ have non-zero elements only for $m = n \pm \hat{\mu}$. They are given by

$$D_{n,m;\mu} = \frac{1}{2} \eta_{\mu}(n) [U_{n,\mu} \delta_{n,m-\hat{\mu}} - U_{n,\mu}^{\dagger} \delta_{n,m+\hat{\mu}}] , \quad (20)$$

with phase factors $\eta_{\mu}(n) = (-1)^{\eta_0 + \dots + \eta_{\mu-1}}$ for $\mu > 0$ and $\eta_0(n) = 1$. Finally the partition function takes on the form

$$Z = \int \prod_{n,\mu} dU_{n,\mu} \prod_{n,i} d\chi_{n,i} d\bar{\chi}_{n,i} e^{[-S_G - S_F]} . \quad (21)$$

As the fermionic part of the action is bilinear in the fields $\bar{\chi}_{n,i}$, $\chi_{n,i}$, these can be integrated out easily and the partition function can be represented in terms of gluonic degrees of freedom only,

$$Z = \int \prod_{n,\mu} dU_{n,\mu} \prod_{i=1}^{\bar{I}} \det Q^i e^{-S_G} . \quad (22)$$

Using this partition function expectation values of any observable X can be calculated in the usual way,

$$\langle X \rangle = Z^{-1} \int \prod_{n,\mu} dU_{n,\mu} X \prod_{i=1}^{\bar{I}} \det Q^i e^{-S_G} . \quad (23)$$

Finally we have to discuss how results of a lattice calculation can be converted into physical units and how the lattice cut-off can be removed in order to get results relevant for the continuum limit. Physical quantities are calculated on the lattice in units of the lattice spacing a ; a mass will be given in units of a^{-1} , the energy density in units of a^{-4} , etc.. The lattice action, however, does not contain any dimensionful parameters. The lattice cut-off a^{-1} enters through the renormalization of the bare coupling g^2 (and the quark masses $m_i a$). In the continuum limit ($g^2 \rightarrow 0$) g^2 and a are related through the renormalization group equation

$$a \Lambda_L = (b_0 g^2)^{-b_1/2b_0^2} e^{-1/2b_0 g^2} , \quad (24)$$

with b_0 , b_1 given by eq.(8). This can then be used to eliminate the cut-off from measured quantities in favour of the scale parameter Λ_L . If the lattice simulations

have been performed in the continuum regime, results should be cut-off independent, i.e. physical quantities expressed in units of Λ_L should be independent of the coupling g^2 used in the actual calculation. Numerical studies of the QCD β -function using Monte Carlo renormalization group techniques gave evidence that the asymptotic scaling behaviour, described by eq.(24), is approximately valid for $g^2 < 1$ ³⁴⁾. For $g^2 \leq 1$, however, strong violations of asymptotic scaling have been found.

3.2. Finite temperature QCD on the lattice

For simplicity lattice simulations are generally performed on isotropic lattices, i.e. lattices with identical lattice spacings a_r and a in temporal and spatial directions. However, in order to discuss the finite temperature formalism on the lattice it is convenient to work on an anisotropic lattice ($a_r \neq a$) and introduce the anisotropy parameter³⁵⁾

$$\xi = \frac{a}{a_r} . \quad (25)$$

This allows to keep track of the temperature dependence, which enters in a complicated way the lattice action. While T appears in the continuum formulation explicitly only as an integration limit for the integration over the time components of the fields, on the lattice it appears through the finite number of lattice sites in the time direction of the lattice as well as through the lattice cut-off ($1/a_r$) in that direction. The continuum four-volume $T^{-1}V$ is replaced by a lattice of size $N_r \times N_g^3$ with lattice spacings a_r (a) in the temporal (spatial) directions such that

$$\begin{aligned} 1/T &= N_r a_r , \\ V^{1/3} &= N_g a . \end{aligned} \quad (26)$$

As we have seen in the previous section, the lattice spacings (a_r , a) do not enter the action explicitly. They can only be tuned through the bare couplings. An anisotropy can be introduced on the lattice by choosing different couplings for space-space and space-time plaquettes in the gluonic action and similarly by introducing a new coupling in the 0-th component of the Dirac operator. The gluonic action, eq.(16), then becomes³⁵⁾

$$S_G = \beta_G \left[\sum_{n; 0 < \mu < \nu \leq 3} P_{n,\mu\nu} + \gamma_G^2 \sum_{n; 0 < \mu \leq 3} P_{n,0\mu} \right] \quad (27)$$

and the fermion matrix, eq.(19), now reads

$$Q_{n,m}^i = \sum_{j=1}^3 D_{n,m;j} + \gamma_F D_{n,m;0} + m_i a \delta_{n,m} . \quad (28)$$

Let us look at these expressions in somewhat more detail. The QCD action on anisotropic lattices depends on three dimensionless couplings β_σ , γ_G and γ_F , as well as the bare quark masses $m_i a$. The lattice spacings a , a_τ enter only through the renormalization of these couplings in the continuum limit. All three couplings may then be considered as functions of g^2 , defined on isotropic lattices ($\xi = 1$), and the anisotropy parameter ξ . The dependence on the latter can be established perturbatively by demanding rotational invariance in the continuum limit, even though this limit is performed on an anisotropic lattice³⁶⁻⁴¹. In the naive classical continuum limit rotational invariance is guaranteed if we choose $\gamma_G = \gamma_F = \xi$ and $\beta_\sigma = \xi^{-1}\beta$, with $\beta = 2N/g^2$ denoting the bare coupling on an isotropic Euclidean lattice. The latter is related to the lattice cut-off $1/a$ through the QCD renormalization group equation, eq.(24). Quantum corrections to this leading order relations have been calculated for β_σ and γ_G in the pure gauge theory^{36,37} as well as for QCD⁴¹ by demanding rotational invariance of the effective action. Similarly $O(g^2)$ corrections have been calculated for γ_F by demanding rotational invariance for the fermion propagator³⁸. One finds then

$$\begin{aligned} \beta_\sigma &= \xi^{-1}\beta [1 + c_\sigma(\xi)g^2 + O(g^4)] , \\ \gamma_G &= \xi [1 + c_G(\xi)g^2 + O(g^4)] , \\ \gamma_F &= \xi [1 + c_F(\xi)g^2 + O(g^4)] . \end{aligned} \quad (29)$$

In the definition of the gluonic part of the action, eq.(27), we have introduced the couplings β_σ and γ_G in order to stress the similarity of the modifications in the gluonic and fermionic part of the action required on anisotropic lattices. The couplings γ_G and γ_F determine the relative weights of space- and time-like components in the gluonic and fermionic action. Quite often one uses instead the pair β_σ and $\beta_\tau = \beta_\sigma \gamma_G^2$, where β_τ now denotes the coupling in front of the temporal plaquettes. Its weak coupling expansion may be written as $\beta_\tau = \xi\beta(1 + c_\tau(\xi)g^2 + O(g^4))$, with c_τ being related to the functions c_σ and c_G through

$$c_\tau(\xi) = \frac{1}{2}(c_\tau(\xi) - c_\sigma(\xi)) . \quad (30)$$

The functions $c_\sigma(\xi)$, $c_\tau(\xi)$ and $c_F(\xi)$ depend on the number of colours and flavours. Their explicit form^{36-38,41} is not of importance here, as we are finally interested in the isotropic limit $\xi \equiv 1$ where these functions vanish. The derivatives of c_σ , c_τ and c_F with respect to ξ , however, do not vanish for $\xi = 1$. This is of importance as thermodynamic observables are obtained from the partition function by taking suitable derivatives with respect to V and/or T . On a lattice of fixed size $N_r \times N_\sigma^3$ the temperature derivative can be written as

$$\frac{\partial}{\partial(1/T)} = -\frac{\xi^2}{N_r a} \frac{\partial}{\partial \xi} . \quad (31)$$

Derivatives of c_σ , c_τ and c_F thus will enter thermodynamic expressions even in the isotropic limit. These derivatives for $\xi = 1$ are given by^{36-38,41}

$$\begin{aligned} c'_\sigma &= \frac{dc_\sigma}{d\xi} \Big|_{\xi=1} = 4N \left[\frac{N^2-1}{32N^2} 0.58644 + 0.000499 \right] - \frac{1}{2} n_f 0.00062 \\ c'_\tau &= \frac{dc_\tau}{d\xi} \Big|_{\xi=1} = 4N \left[-\frac{N^2-1}{32N^2} 0.58644 + 0.005306 \right] - \frac{1}{2} n_f 0.00782 , \\ c'_F &= \frac{dc_F}{d\xi} \Big|_{\xi=1} = -\frac{N^2-1}{2N} 0.1599 \end{aligned} \quad (32)$$

Using these expressions one obtains for instance for the energy density

$$\epsilon/T^4 = (\epsilon_G + \epsilon_F)/T^4 = \left(\frac{N_r}{N_\sigma} \right)^3 \frac{\partial}{\partial \xi} \ln Z \Big|_{\xi=1} , \quad (33)$$

with the gluonic contribution ϵ_G given by

$$\epsilon_G/T^4 = -3N_r^4 \frac{d\beta_\sigma}{d\xi} \Big|_{\xi=1} [\langle P_\sigma \rangle + \langle P_\tau \rangle] - 6\beta N_r^4 \frac{d\gamma_G}{d\xi} \Big|_{\xi=1} \langle P_\tau \rangle , \quad (34)$$

and the corresponding fermionic part ϵ_F

$$\epsilon_F/T^4 = N_r^4 \frac{n_f}{4} \left(\frac{d\gamma_F}{d\xi} \Big|_{\xi=1} \langle \text{Tr} D_0 Q^{-1} \rangle + \frac{d m a}{d\xi} \Big|_{\xi=1} \langle \text{Tr} Q^{-1} \rangle \right) . \quad (35)$$

Unless explicitly stated we will assume equal masses for all fermionic species and suppress the flavour index i in the following. Furthermore we have used the abbreviation P_τ for space-time plaquettes $P_{n,0\mu}$, $\mu = 1, 2, 3$ and P_σ for space-space plaquettes $P_{n,\mu\nu}$, $\mu, \nu = 1, 2, 3$. In eqs.(34) and (35) we have introduced the normalized expectation values $\langle X \rangle$, which are defined by

$$\langle X \rangle = \frac{1}{Z} \text{Tr} X = \langle X \rangle_{N_r, N_\sigma} - \langle X \rangle_{N_r, N_\sigma} , \quad (36)$$

This takes care of the subtraction of vacuum contributions to thermodynamic quantities and thus normalizes the gluonic and fermionic parts of the energy density to zero at $T = 0$.

The pressure can be obtained in a similar way directly from eq.(4). However, it is instructive to write it in terms of the energy density as

$$\frac{\epsilon - 3p}{T^4} = N_r^4 \frac{d\beta}{d \ln a} \left(\overline{\langle P_\sigma \rangle} + \overline{\langle P_r \rangle} \right) - N_r^4 \frac{n_f}{4} \frac{dma}{d\xi} \overline{\langle \text{Tr} Q^{-1} \rangle} \quad (37)$$

This shows that also on the lattice deviations from the ideal gas equation of state are proportional to the QCD renormalization group equations for the bare couplings g^2 and ma . Note also that for massless QCD, $ma \equiv 0$, we find that $\epsilon - 3p$ is given entirely in terms of gluonic operators.

Using eqs.(33) and (37) we obtain for the entropy density

$$\begin{aligned} \frac{S}{T^3} &= \frac{\epsilon + p}{T^4} \\ &= \frac{4}{3} \frac{\epsilon/T^4}{T^4} - \frac{1}{3} N_r^4 \left[\frac{d\beta}{d \ln a} \left(\overline{\langle P_\sigma \rangle} + \overline{\langle P_r \rangle} \right) - \frac{n_f}{4} \frac{dma}{d\xi} \overline{\langle \text{Tr} Q^{-1} \rangle} \right] \quad (38) \end{aligned}$$

We note that the analysis of the thermodynamic observables discussed above involves the calculation of two conceptually different quantities. First of all one has to evaluate expectation values of certain gluonic and fermionic operators. This can be done in a Monte Carlo simulation. In addition one also needs to know the various derivatives of couplings entering the expressions given in eqs.(34-38). Using eqs. (29) and (33) the derivatives of β_σ , γ_G and γ_F with respect to ξ can be replaced by the leading order perturbative results[†]. In particular in the fermionic sector these relations may not be accurate enough to get a correct description of all non-perturbative aspects of the thermodynamics of the high temperature phase³⁸⁾ and higher order corrections may be needed.

An entirely non-perturbative treatment of the thermodynamics of QCD on the lattice would also require a non-perturbative calculation of the derivatives of the couplings discussed above. This can be achieved in principle³⁹⁾, in practice it is, however, rather time consuming to obtain accurate results for these quantities. At present one thus usually uses the perturbative relations given in eq.(32).

[†] Quantum corrections for the remaining derivative, $dma/d\xi$, have not been calculated so far. This derivative is, however, important only at finite quark mass, it vanishes in the chiral limit.

A detailed discussion of the numerical analysis of the thermodynamic quantities defined above will be given in section 5. However, before coming to that we start in the next section our discussion of Monte Carlo simulations of finite temperature QCD with a detailed analysis of the finite temperature phase structure of QCD as it emerges from these simulations.

4. DECONFINEMENT AND CHIRAL SYMMETRY RESTORATION

The low temperature phase of QCD is characterized by the confinement of quarks and gluons as well as the spontaneous breaking of chiral symmetry. The asymptotic freedom of QCD at high temperatures suggests that these non-perturbative features of QCD are lost in the high temperature phase. The linearly rising $q\bar{q}$ potential is expected to be replaced by a Debye screened Coulomb potential and chiral symmetry is expected to be restored in the plasma phase.

The mechanisms leading to confinement and chiral symmetry breaking are seemingly unrelated and it thus has been speculated that QCD may undergo two separate phase transitions⁴²⁾. Deconfinement could set in at some critical temperature T_d giving rise to free partons which still have an effective mass. Chiral symmetry thus would still be broken and would get restored during a second phase transition at $T_{ch} \geq T_d$. As will become clear from the following discussion this scenario does not seem to be realized for QCD at finite temperature, i.e. the Monte Carlo data suggest $T_d = T_{ch}$. Two separate transitions have, however, been found in gauge theories with fermions in higher representations⁴³⁾. It may also be possible that two separate transitions occur in a cold QCD plasma, i.e. at low temperature as a function of baryon number density.

4.1. Global symmetries of the lattice action

Phase transitions are generally characterized by the spontaneous breaking/restoration of global symmetries of the system under consideration. The QCD Lagrangian possesses exact global symmetries only in the limiting cases of vanishing (chiral limit) or infinite (pure gauge theory limit) quark masses. In the latter case the QCD action reduces to the gluonic action, S_G , defined in eqs.(16) and (17). It has a global $Z(3)$ symmetry, the breaking of which can be related to the deconfinement phase transition. If we perform a global $Z(3)$ rotation of all timelike gauge fields $U'_{n,0}$ originating at sites $n = (n_0, \vec{n})$ of a given temporal

hyperplane (fixed n_0) of the lattice,

$$U_{(n_0, \vec{n}), 0} \rightarrow U'_{(n_0, \vec{n}), 0} = z U_{(n_0, \vec{n}), 0} \quad , \quad z \in Z(3), \quad n_0 \text{ fixed} \quad , \quad (39)$$

the action remains unchanged, i.e. $S_G(\{U'_{n,\mu}\}) = S_G(\{U_{n,\mu}\})$, while for instance the Polyakov loop,

$$L_{\vec{n}} = \prod_{n_0=1}^{N_\tau} U_{(n_0, \vec{n}), 0} \quad , \quad (40)$$

transforms non-trivially under this transformation,

$$L_{\vec{n}} \rightarrow z L_{\vec{n}} \quad , \quad z \in Z(3) \quad . \quad (41)$$

The thermal expectation value of the Polyakov loop

$$\langle L \rangle = \frac{1}{N_\tau^3} \langle \sum_{\vec{n}} \text{Tr} L_{\vec{n}} \rangle \quad , \quad (42)$$

will thus vanish as long as the global $Z(3)$ symmetry of the action is preserved but will acquire a non-vanishing value if this symmetry is broken spontaneously. The Polyakov loop can be thought of as representing a static fermionic test charge which probes the screening properties of the surrounding gluonic medium^{13,44}. Its expectation value is related to the excess free energy, $F_q(T)$,

$$e^{-F_q(T)/T} \simeq \langle L \rangle \quad , \quad (43)$$

induced by the presence of this source in the gluonic heat bath. In the absence of dynamical quarks ($ma \rightarrow \infty$) a single coloured charge cannot be screened in the confined phase, its free energy, F_q , is infinite and $\langle L \rangle = 0$. In the deconfined phase, however, F_q is finite and $\langle L \rangle \neq 0$. The expectation value $\langle L \rangle$ is thus an order parameter for the deconfinement phase transition in the pure $SU(3)$ gauge theory, which is related to the spontaneous breaking of a global $Z(3)$ symmetry of the gluonic action S_G .

In the massless limit, $ma \rightarrow 0$, the QCD Lagrangian has a global chiral symmetry. Using staggered fermions on the lattice the symmetry group is $U(\vec{f}) \times U(\vec{f})$, i.e. a subgroup of the $SU(n_f) \times SU(n_f)$ chiral symmetry group of the continuum theory. In the low temperature phase this symmetry is spontaneously broken and leads to a non-vanishing chiral condensate,

$$\langle \bar{\chi} \chi \rangle = \frac{1}{N_\tau N_\sigma^3} \frac{\partial}{\partial ma} \ln Z \quad . \quad (44)$$

Figure 1: The chiral order parameter $\langle \bar{\chi} \chi \rangle$ (triangles) and the Polyakov loop $\langle L \rangle$ (dots) versus $\beta = 6/g^2$ for QCD with $n_f = 4$ and quarks of mass $m/T = 0.1$ on a 4×8^3 lattice. Data are taken from ref.45.

This symmetry is expected to be restored at high temperature.

In fig.1 we show results for the temperature dependence of the deconfinement order parameter $\langle L \rangle$ and the chiral condensate $\langle \bar{\chi} \chi \rangle$ obtained in a simulation of QCD with 4 light quark flavours ($n_f = 4$) of mass $m/T = 0.1^{45}$. This shows that the restoration of the chiral symmetry is accompanied by a large change in the excess free energy of a heavy quark source.

The numerical analysis of the deconfinement/chiral phase transition concentrates around two questions - the order of the phase transition and a quantitative determination of the transition temperature. As can be seen from fig.1 the transition is rather abrupt. Whether it is continuous or whether there is a discontinuity in the order parameter which would signal the presence of a first order transition is a subtle question. To answer it a careful analysis of the transition region on various lattice sizes is needed. As we will see, the problem is not completely settled even for the simplest case - the pure $SU(3)$ gauge theory. Similarly a quantitative determination of the transition temperature requires good control over the approach to the continuum limit in order to be able to translate the critical couplings β_c , determined in lattice simulations, into physical values for the transition

temperature. These two problems will be discussed in the next two subsections.

4.2. Order of the phase transition

It has been pointed out first by Svetitsky and Yaffe that the critical properties of four-dimensional $SU(N)$ gauge theories at finite temperature can be related to that of a 3-d effective spin theory. Universality arguments can then be used to classify the order of the phase transition in the pure gauge sector⁴⁶⁾. A similar argument has later on been used by Pisarski and Wilczek to classify the phase transitions in the chiral limit⁴⁷⁾.

Svetitsky and Yaffe argue that the effective theory for the (3+1)-dimensional finite temperature $SU(N)$ gauge theory is a 3-d, $Z(N)$ symmetric spin model with short-range interactions, dominated by a ferromagnetic nearest neighbour coupling.

In the case of $SU(2)$ the validity of these arguments has been verified in great detail. A Monte Carlo renormalization group analysis has shown that the interactions in the effective theory are indeed short-ranged and ferromagnetic⁴⁸⁾. One would then expect that the critical properties are the same as those of the 3-d Ising model. Indeed, excellent agreement has been found between the critical exponents of the $SU(2)$ gauge theory and those of the 3-d Ising model⁴⁹⁾.

In the case of $Z(3)$ symmetric models it is of particular importance to verify that the effective theory is dominated by ferromagnetic interactions as it is known that anti-ferromagnetic interactions can give rise to second order phase transitions⁵⁰⁾ in this case. In the class of globally $Z(3)$ symmetric models with ferromagnetic short-range interactions, on the other hand, no fixed point is known, and one thus expects that these models have first order phase transitions. In the strong coupling limit one can explicitly derive the effective 3-d spin model from the QCD partition function. One can carry out the integration over all spatial degrees of freedom, $U_{\vec{n},i}$, $i = 1, 2, 3$. This leads to an effective theory in terms of the Polyakov loop operator^{46,51,52)}

$$Z_{eff} = \int \prod_{\vec{n}} dL_{\vec{n}} \exp[\beta' \sum_{\vec{n}} \sum_{i=1}^3 \text{Tr} L_{\vec{n}} \text{Tr} L_{\vec{n}+\vec{i}}^\dagger + c.c.] , \quad (45)$$

with $\beta' = \beta^{N_c} + \dots$. This is a 3-d, $Z(3)$ spin model with ferromagnetic coupling, $\beta' > 0$. It can be shown that also in higher orders of the strong coupling expansion the interactions are ferromagnetic. Also at larger values of β the effective

spin model characterizing the deconfinement phase transition in pure $SU(3)$ gauge theory has been constructed recently⁵³⁾. On a 4×24^3 lattice near T_c ($\beta = 5.6925$) it has been found that the effective spin model has only short ranged ferromagnetic interactions. This gives strong support for the applicability of the universality arguments given by Svetitsky and Yaffe for the $SU(3)$ gauge theory and it therefore is expected that the $SU(3)$ gauge theory indeed exhibits a first order deconfining phase transition.

The numerical analysis of the effective model⁵⁴⁾ defined by eq.(45) as well as the 3-d, 3-state Potts model^{55,56,57)} shows that these $Z(3)$ symmetric spin system with ferromagnetic couplings indeed undergo first order phase transition. Thus, at least for QCD motivated 3-d, $Z(3)$ symmetric models the first order nature of the phase transition seems to be established. The early numerical investigations of $SU(3)$ pure gauge theory, performed on rather small lattices, gave evidence that also in this case the $Z(3)$ symmetry breaking deconfinement phase transition is first order. Rather large discontinuities in the order parameter $< L >$ and the internal energy of the system have been observed in these simulations^{58,59)}. Later systematic studies⁶⁰⁻⁶³⁾ of the transition showed that at the critical point the system flipped between ordered and disordered phases giving rise to a double peak structure in the probability distribution of the Polyakov loop that is characteristic for a first order phase transition. The time history of such a Monte Carlo simulation for the order parameter $< L >$ and the entropy density is shown in fig.2.

Although these results are very suggestive of a first order phase transition, some care is in order. Strictly speaking there are no discontinuities in the order parameter or other physical quantities on finite lattices. When averages are calculated over the complete time history of a Monte Carlo simulation the frequent flips between different metastable states will only lead to a rapid, but smooth, changes in all thermodynamic observables. Discontinuities will develop only in the infinite volume limit. One thus has to study the finite size dependence of thermodynamic quantities in order to judge whether the transition is first or second order in the infinite volume limit. The finite size scaling behaviour is expected to show clear differences in these cases. For instance the critical coupling determined on finite lattices from the relative population of the peaks in the probability distribution of

Figure 2: Time history of the phase, $\Theta \equiv \arg\{<P>\}$, of the Polyakov loop operator, $L \equiv |L|\exp(i\Theta)$, for $SU(3)$ gauge theory on a 4×24^3 lattice at $\beta = 5.6925$ (fig.2b). Noisy regions indicate the time intervals during which the system is in the symmetric phase, the remaining time it spent in one of the 3 degenerate vacua with broken $Z(3)$ symmetry where $\Theta \simeq 0, \pm 2\pi/3$. Fig.2a shows the corresponding abrupt changes in the entropy density when the system flips from the broken to the symmetric phase and back. Data are taken from ref.63.

the order parameter, L , or the peak in the specific heat, is expected to scale like

$$\beta_c = \beta_c(V) + \frac{A}{V^{1/\nu d}}, \quad (46)$$

with $\nu d = 1$ for a first order transition and $\nu d < 1$ for a second order transition. Such an analysis has been attempted for the $SU(3)$ gauge theory⁶⁰. In fig. 3 we show results for lattices with $N_\tau = 4$ and compare with a similar analysis performed for the 3-d, 3-state Potts model⁵⁵. As can be seen in the latter case the determination of ν is statistically significant and clearly supports the first order nature of the transition whereas in the case of the $SU(3)$ gauge theory at best consistency with a first order transition can be deduced from such an analysis.

Additional evidence for the first order nature of the transition comes from a finite size scaling analysis of response functions like the specific heat or susceptibility. Let us, for instance, consider the susceptibility of the order parameter defined as

$$\chi = V \langle 4bs(L)^2 \rangle > - \langle 4bs(L) \rangle^2 \rangle. \quad (47)$$

On a finite lattice this quantity reaches a maximum value, χ^{peak} , at β_c . With increasing lattice size χ^{peak} increases and will diverge in the infinite volume limit,

Figure 3: Finite size analysis of the critical couplings in the 3-d, 3-state Potts model (a) and the $SU(3)$ gauge theory for $N_\tau = 4$ (b).

irrespective of the order of the transition, i. e. $\chi^{peak} \sim V^\alpha$. For a first order phase transition $\alpha = 1$, while $\alpha < 1$ for a continuous transition. The finite size scaling behaviour of χ^{peak} has been analyzed in $Z(3)$ symmetric spin models and supports the first order nature of the transition^{55,54}. A similar analysis has recently been performed for the $SU(3)$ gauge theory on lattices of size $4 \times N_\sigma^3$, $N_\sigma = 8 - 32$, which also suggests that $\chi^{peak} \sim V^{53}$. These results are shown in fig.4. It is interesting to note that in both cases the results for the ratio χ^{peak}/V are in good agreement with the value expected from the discontinuity of the Polyakov loop expectation value⁵⁴

$$\chi^{peak}/V = \left(\frac{1}{2}\Delta L\right)^2. \quad (48)$$

This supports the expectation that the discontinuity in $\langle L \rangle$, extracted on finite lattices from the two separate peaks in the probability distribution of L , will survive the infinite volume limit.

The analysis of the finite size scaling behaviour of the critical coupling and the peaks in the specific heat or susceptibility allows a clear distinction between first and second order phase transitions. The situation is somewhat more subtle for correlation lengths near a first or second order phase. This is partly due to the fact that the correlation length cannot be obtained by a simple measurement of a lattice operator but has to be extracted from the long distance behaviour of a two-point correlation function. Moreover, one has to be careful in disentangling the physical mass gap (inverse correlation length) from the tunnelling correlation

Figure 4: Finite size analysis of the susceptibility peak in the $Z(3)$ spin model defined in eq.(45) (a) and the $SU(3)$ gauge theory for $N_\tau = 4$ (b). The band shows the value expected for χ^{P44}/V on the basis of eq.(48).

length, which diverges at β_c irrespective of the order of the transition. Unfortunately the divergence of the tunnelling correlation length itself cannot be used to classify the order of the transition. In a recent large scale analysis of the deconfinement transition the first order nature of the $SU(3)$ deconfinement transition has been questioned^[64–66] on the basis of a finite size analysis of the correlation length at the phase transition. The analysis of the Polyakov loop correlations as well as plaquette-plaquette correlations close to T_c indicated that for these quantities the correlation lengths increased at T_c with increasing spatial volume of the lattice. In fact the correlation lengths at T_c turned out to rise proportionally to the linear size of the lattice thus suggesting that they actually diverge in the infinite volume limit. This led to a detailed study of correlation lengths in the vicinity of first and second order phase transitions^[54,55,57,53,67,68]. In particular the analysis of the correlation length of both the 3-d, 3-state Potts model^[55] and the 3-d effective spin model^[54], defined by eq.(45), has shown that the correlation length in these models becomes large close to the critical point. However, in order to judge whether the physical correlation length (inverse mass gap) diverges at β_c one has to be careful in separating its contribution to the correlation functions from that of the tunnelling correlation length^[55]. The latter diverges for all $\beta > \beta_c$ in the infinite volume limit, irrespective of the order of the transition. Qualitatively the behaviour of the correlation length found in the $Z(3)$ symmetric spin models seems to be very similar to that observed in the $SU(3)$ gauge theory, although the transition clearly

is first order in the former case. It thus seems that it is particularly difficult to deduce the order of the phase transition from the scaling behaviour of the correlation length alone. We will give a more detailed discussion of the behaviour of the $SU(3)$ screening length in section 6.2.

Although there seems to be strong evidence that the $SU(3)$ deconfinement transition is first order, the above discussion shows that a detailed finite size analysis of the pure gauge theory on even larger lattices is still required to establish the order of the deconfinement transition beyond any doubt. In the presence of dynamical fermions the situation is by far more complicated as it is difficult to perform high statistics analyses of full QCD on large lattices that would allow a detailed finite size study as it has now been done in the pure gauge sector. The qualitative changes due to dynamical fermions are, however, theoretically well understood and reproduced by Monte Carlo simulations. The presence of dynamical fermions leads to an explicit breaking of the global $Z(3)$ symmetry of the pure gauge action. In terms of a $1/m$ expansion it can be seen that fermions act like an additional magnetic field in the effective spin model^[51,52,69]. This leads to a weakening of the 1st-order transition. It is thus expected that the transition seen in the pure gauge sector weakens for finite quark masses and it has been speculated that in an intermediate mass regime there would be no trace of the deconfinement transition left and only in the chiral limit would the restoration of the chiral symmetry lead to a new phase transition.

The order of the chiral transition has been analyzed by Pisarski and Wilczek^[47] using universality arguments for 3-d chiral models with the general Lagrangian

$$L_\Phi = -\frac{1}{2}T\tau(\partial_\mu\Phi^\dagger\partial_\mu\Phi) - \frac{1}{2}m^2T\tau(\Phi^\dagger\Phi) + \frac{\pi^2}{3}g_1(T\tau(\Phi^\dagger\Phi))^2 + \frac{\pi^2}{3}g_2T\tau((\Phi^\dagger\Phi))^2 + c(\det\Phi + \det\Phi^\dagger) \quad (49)$$

for a field $\Phi_{i,j}$, $i, j = 1, \dots, n_f$, which is invariant under the chiral symmetry group $Z(n_f) \times SU(n_f) \times SU(n_f)$. A renormalization group (RG) analysis of this Lagrangian shows that the nature of the transition is strongly dependent on the number of flavours. For $n_f \geq \sqrt{3}$ the RG-equations have no infra-red stable fixed point. The transition is thus expected to be first order in the chiral limit for $n_f = 3$ and larger. A non-zero mass term weakens this transitions and it will disappear completely for masses larger than a certain critical mass. For $n_f = 2$ the term proportional to $\det\Phi + \det\Phi^\dagger$ acts itself like a mass term and thus may change

the order of the transition, depending on the magnitude of the coefficient c . The RG analysis thus does not lead to a prediction of the order of the transition for $n_f = 2$. Finally for $n_f = 1$ it predicts that there is no chiral transition at all, as $\det\Phi + \det\Phi^\dagger$ acts like an external magnetic field in this case.

The above scenario for the n_f -dependence of the chiral transition has been found to be correct in the instanton liquid approximation of QCD^{70–73}. A mean-field analysis of this model suggests that there is no chiral phase transition for $n_f = 1$. However, it predicts a second order transition for $n_f = 2$ and a first order transition for all $n_f > 2$ ^{72,73}.

Monte Carlo simulations for full QCD with dynamical quarks have by now been performed for a variety of different quark masses and flavours (A detailed summary can be found in refs. 26 and 74.). Indeed the transition has been found to be rather sensitive to the choice of quark masses and flavours. It seems that for $n_f > 4$ the transition is first order for all quark masses m . For $n_f \leq 4$ the situation is still to some extent uncertain. Nonetheless, there are strong indications that for sufficiently small quark masses ($m/T \lesssim 0.1$) the transition is first order for all $2 \leq n_f \leq 4$. In fig.1 we have shown results of a simulation for $n_f = 4$ with light quarks of mass $m/T = 0.1$ on a 4×8^3 lattice, which strongly suggests a first order transition. At intermediate quark masses there seems to be a region where the transition is likely to be continuous, although still accompanied by a very rapid change in the order parameters $\langle L \rangle$ and $\langle \bar{\chi}\chi \rangle$.

However, one should once again stress the uncertainties in the determination of the order of the chiral transition in Monte Carlo simulations. At present most finite temperature simulations of QCD with light quarks are performed on rather small lattices, $N_\sigma \simeq 4 - 12$, $N_\tau = 4, 6$, and it seems that the dependence of the chiral transition on the spatial lattice size is not fully understood. At intermediate quark masses ($0.4 \lesssim m/T \lesssim 1.0$) the volume dependence has been studied^{75–78}. This showed that metastabilities, which could be considered as being indicative for a first order transition, tend to disappear with increasing lattice size. In fig.5 we present results from a finite size analysis of the phase transition at $m/T = 0.4$ and $n_f = 2$. As can be seen the distribution of the Polyakov line is broad on a small lattice and gives a clear two-state signal, which has been taken as indication of a 1st order transition. However, on the larger lattice the distribution becomes narrower for all values of β and there remains little evidence for a first order transition. If this behaviour persists at smaller quark masses and on larger lattices a

Figure 5: Probability distribution of $|L|$ for $n_f = 2$ and $m/T = 0.4$ on a 4×8^3 lattice at $\beta = 5.38$ (a) and a 4×12^3 lattice at $\beta = 5.375$ (b). These β values are close to the critical couplings on these lattices which have been estimated to be 5.3825(50) and 5.376(3), respectively. Data are taken from ref.78.

second order transition, particularly for $n_f = 2$, cannot be ruled out.

For the actual case of QCD it is also important to understand the influence of an intermediately heavy strange quark ($m_s/T \sim O(1)$) in addition to a light isodoublet ($m_u = m_d \simeq 0$). Simulations which try to approximate this situation^{79,80} indicate that the influence of a strange quark with $m_s/T = 1$ on the order of the transition is small. The critical behaviour of QCD thus seems to be controlled by the $n_f = 2$ chiral theory.

3.3. The critical temperature

For all $SU(N)$ gauge groups it has been rigorously established that the global $Z(N)$ centre symmetry is spontaneously broken on the lattice for couplings β larger than a certain critical coupling $\beta_c = 2N/g_c^2$ ⁸¹. A similar existence proof has been given for the chiral phase transition in the case of $SU(2)$ lattice gauge theory⁸². However, in both cases the bounds on β_c are weak and, in particular, they do not rule out that the transition temperature shifts to infinity in the continuum limit.

A quantitative determination of the critical temperature from lattice simulations involves two steps. First of all one has to show that the transition temperature is independent of the lattice cut-off, i.e. when increasing the temporal lattice size N_τ the transition has to occur at a smaller lattice spacing, $a(g_c)$. The critical coupling, g_c , has to scale such that the product $N_\tau a(g_c) = 1/T_c$ stays constant. In the continuum limit the dependence of g_c on N_τ is given by the renormalization

Figure 6: $T_c/\Lambda_{\overline{MS}}$ for pure gauge theory, $n_f = 0$, (dots) and massless QCD with $n_f = 2$ (triangles) and $n_f = 4$ (squares) flavours versus temporal extent of the lattice. The solid line gives the critical temperature for the pure gauge theory, $T_c\Lambda_{\overline{MS}} = 1.78 \pm 0.03$, estimated on the basis of the results obtained for $N_t = 10, 12$ and 14 .

group equation,

$$T_c/\Lambda_L = N_t^{-1} (b_0 g_c^2)^{b_1/2b_0} e^{1/2b_0 g_c^2} \quad (50)$$

Monte Carlo renormalization group studies³⁴⁾ suggest that this asymptotic scaling relation holds approximately for $g_c^2 < 1$. In order to achieve that the critical coupling is smaller than 1 large temporal lattices are necessary. Using nonetheless the asymptotic scaling relation, eq.(50), also on smaller lattices to extract T_c/Λ_L leads to results which will still depend on N_t (or g^2), indicating that the validity regime of the asymptotic scaling relation has not been reached. Results for the $SU(3)$ gauge theory are shown in fig.6. (Rather than using Λ_L to fix the scale we converted this to the continuum scale parameter $\Lambda_{\overline{MS}}$ which is less sensitive to the number of flavours.) As can be seen $T_c/\Lambda_{\overline{MS}}$ is roughly N_t independent on lattices with $N_t \geq 10$, or equivalently for $g_c^2 \leq 1$.

In order to extract the critical temperature in physical units from these measurements one has to eliminate the scale parameter, Λ_L , by measuring a second physical quantity in the continuum regime. Using measurements of the string

tension⁸³⁾ or hadron masses in the quenched approximation⁸⁴⁾ one obtains

$$T_c = \begin{cases} 0.58(4)\sqrt{\sigma} & = 254(18)\text{MeV} \\ 0.30(5)m_\rho & = 231(38)\text{MeV} \\ 0.21(2)m_N & = 197(20)\text{MeV} \end{cases} \quad (51)$$

Here we have used $\sigma = 0.192\text{GeV}^2$ for the string tension⁸⁵⁾ and $m_\rho = 770\text{MeV}$, $m_N = 940\text{MeV}$ for the rho and nucleon masses respectively. The difference in T_c obtained by either using m_ρ or m_N to fix the lattice scale $a\Lambda_L$ reflects the difficulties one has in reproducing the experimental mass spectrum in lattice simulations. The ratio m_N/m_ρ calculated on the lattice does not agree with the experimental result so far, it comes out too high (see ref. 26 for a detailed discussions of hadron mass calculations on the lattice).

A similar detailed analysis of the transition temperature for QCD with light quarks does not yet exist. So far only rather small lattices with $N_t \leq 6$ could be analyzed. Most detailed results exist for $n_f = 2$ and 4 on lattices with $N_t = 4$ and 6. Rather surprisingly it has been found that the violations of asymptotic scaling seen in the measurement of the critical temperature when going from $N_t = 4$ to $N_t = 6$ are in good quantitative agreement with those present already in the pure gauge sector. In order to convert the critical couplings into a critical temperature, measured in units of $\Lambda_{\overline{MS}}$ using the asymptotic scaling relation, eq.(50), an additional extrapolation to zero quark mass is required. A comparison of these critical temperatures for $n_f = 2$ and 4 with the corresponding results for $n_f = 0$ is also shown in fig.6. A further conversion of the critical temperature into units of MeV again requires the measurement of a second physical quantity, for instance m_N or m_ρ , for the same value of the lattice cut-off (same coupling β). For full QCD this leads to considerably smaller values for the critical temperature than in the pure gauge sector. For $n_f = 2$ and quark masses $m/T = 0.15$ one finds⁸⁶⁾

$$T_c = \begin{cases} 0.19(3)m_\rho & = 146(23)\text{MeV} \\ 0.12(1)m_N & = 113(9)\text{MeV} \end{cases}, \quad (52)$$

on lattices with $N_t = 6$. Results obtained for $N_t = 4$ lead to about 10% smaller values for T_c ^{86,87)}. A similar analysis of the transition temperature for $n_f = 4$ and $N_t = 4$ and 6⁸⁸⁾ leads to values for T_c measured in units of hadron masses that are about 15% smaller than those obtained for $n_f = 2$. However, one should keep in mind that at present the calculations with dynamical fermions are by far not as accurate as corresponding simulations in the pure gauge sector. The finite

size effects still present in the determination of both T_c as well as m_ρ or m_N have the tendency to lead to larger $T_c/m_{\rho(N)}$ ratios when the spatial lattice size is increased. On larger lattices β_c will shift to larger values and $m_{\rho(N)}$ will become smaller. Although the first effect is expected to lead only to small modifications the effect of the latter is difficult to judge. Also the changes in the $T_c/m_{\rho(N)}$ ratio with increasing N_τ are not studied systematically.

5. THERMODYNAMICS OF THE PLASMA PHASE

5.1. Equation of state

Here we want to discuss Monte Carlo results for the equation of state (e.o.s.) in the high temperature phase. We will compare these results with perturbation theory to $O(g^2)$. Deviations from this will give some indication of the importance of non-perturbative effects in the high temperature plasma phase. In order to perform this comparison we have to take into account that Monte Carlo simulations are still performed on rather small lattices. In particular the small temporal lattice size, needed to simulate finite temperature effects, leads to rather severe finite size effects. This prohibits a direct comparison of the Monte Carlo data with continuum perturbation theory. As an intermediate step one thus calculates the lowest order perturbative expressions on finite lattices⁸⁹⁻⁹¹⁾ and compares these with the Monte Carlo data.

Let us start with a discussion of the entropy density defined in eq.(38). This expression still involves the derivative of the couplings with respect to the anisotropy parameter. We will approximate these derivatives by their lowest order weak coupling expressions^{36,38,41)}. The entropy density can then be written as

$$\begin{aligned} \frac{S}{T^3} = & \frac{8N}{g^2} N_\tau^4 \left(1 - \frac{c'_\sigma - c'_\tau}{2} g^2\right) (\langle P_\sigma \rangle - \langle P_\tau \rangle) \\ & + \frac{n_f}{3} N_\tau^4 (1 + c'_F g^2) \left(\langle T_\tau D_0 Q^{-1} \rangle - \frac{1}{4} N \right) \end{aligned} \quad (53)$$

where $c'_{\sigma(\tau)}$ and c'_F are given by eq.(32) and we have ignored an additional term which is proportional to the bare quark mass (see eq.(38)). We will first consider the pure gauge sector only. The entropy density involves then only differences of space-space (P_σ) and space-time (P_τ) plaquettes.

In fig.7 we show results for S/T^3 obtained on lattices of size $4 \times N_\sigma$, $N_\sigma = 12^{92)}$, $16^{63)}$ and $24^{63)}$. The numerical results are compared with perturbative

Figure 7: Entropy density in units of T^3 versus β on lattices of size $4 \times N_\sigma$, with $N_\sigma = 12$ (dots)⁹²⁾, 16 (triangles)⁶³⁾ and 24 (squares)⁶³⁾. The data on the 4×16^3 lattice have been shifted by $\Delta\beta = -0.02$ (see discussion in ref.63). The dashed and dashed-dotted curves give the perturbative results on a 4×12^3 lattice. Perturbative results for larger N_σ differ only little from this. Lines are drawn to guide the eye.

Figure 8: Energy density and pressure in units of the ideal gas values on finite lattices versus T/T_c . Data on the 4×12^3 lattice (dots and triangles) are taken from ref.92 and include a subtraction of the vacuum contributions. The data on the 4×24^3 (squares) are taken from ref.63 and do not include a subtraction of the vacuum part. Lines are drawn to guide the eye.

behaviour up to $O(g^2)$ obtained on a 4×12^3 lattice⁹⁰). It can be seen that the entropy density approaches ideal gas behaviour at temperatures $T \sim (2 - 3)T_c$. Close to T_c ($\beta_c \simeq 5.692$), however, the entropy drops to about 30% of the corresponding ideal gas value.

It is instructive to analyze what causes these large deviations from ideal gas behaviour close to T_c . To this extent we look separately at the contribution of the energy density and the pressure to S/T^3 . These quantities are displayed in fig.8. A comparison with the perturbative results shows that the deviations from ideal gas behaviour seen in S/T^3 close to T_c are mainly due to the non-perturbative behaviour of the pressure. The energy density on the other hand is quite close to the Stefan-Boltzmann limit already at $T \gtrsim 1.2T_c$.

At least for $T \gtrsim 1.2T_c$ the good agreement between the 4×12^3 and 4×24^3 data in fig.7 and 8 shows that finite size effects are small for this size lattices. There are, however, large discrepancies between the results for the latent heat obtained in earlier simulations^{93,94} on smaller lattices and the recent analysis on a 4×24^3 lattice⁶³) close to T_c . Simulations on a 4×10^3 lattice gave for the latent heat of the transition $\Delta\epsilon/T_c^4 \simeq 5.53 \pm 0.37^{94}$). This is substantially larger than what has been found from the simulation on a $N_r \times 24^3$ lattice⁶³),

$$\frac{\Delta\epsilon}{T_c^4} = \begin{cases} 2.54 \pm 0.12 & , \text{ for } N_r = 4 \\ 2.48 \pm 0.24 & , \text{ for } N_r = 6 \end{cases} \quad (54)$$

The origin of this discrepancy is not completely clarified. One would expect any discontinuity to be smoothed out on smaller lattices and becoming more pronounced on larger lattices. In view of the large finite size dependence of the latent heat measurements found for lattices with only 4 sites in temporal direction ($N_\tau = 4$) one has to be sceptical about the accuracy obtained so far for $\Delta\epsilon$ on lattices with larger N_τ ⁹⁵) (the signal for $\Delta\epsilon$ decreases as N_τ^{-4} !).

The result for the latent heat, given in eq.(54), should be compared with the energy density of an ideal gluon gas calculated on a lattice of same size. For $N_\tau = 4$ (6) one finds $\epsilon_G/T^4 = 7.85(6.16)$, i.e. the latent heat is about $1/3$ of the energy density of an ideal gas at the same temperature. Notice, however, that even for $N_\sigma \rightarrow \infty$ the lattice result for ϵ/T^4 is larger by a factor $1.49(1.18)$ than the continuum Stefan-Boltzmann value $\epsilon/T^4 = \frac{8}{15}\pi^2$ for a free gas of 8 massless gluons. Only for $N_\tau \rightarrow \infty$ does the lattice ideal gas approach the continuum value. It thus is to be expected that the present estimates for $\Delta\epsilon$ are still affected

by finite lattice size effects and it seems that a more detailed finite size analysis is needed in order to establish an accurate value for the latent heat.

Qualitatively similar results for the entropy density as well as for energy density and pressure have been obtained for the $SU(2)$ gauge theory^{35,96-98}) and also for QCD with light quarks^{99,100}). In fig.9 we show results for the entropy density, s/T^3 , obtained for two flavour QCD ($n_f = 2$) with light quarks of mass $m/T = 0.1^{38}$). The corresponding results for energy density and pressure are given in fig.10⁹⁹). As can be seen also in this case ϵ/T^4 rises rapidly at T_c and stays approximately constant already close to T_c . The pressure, on the other hand, rises only slowly and the ideal gas relation $\epsilon = 3p$ is satisfied only for $T \sim 2T_c$. Also the entropy density approaches quickly the value expected for an ideal quark-gluon gas. This is similar to the pure gauge sector. However, in the case of QCD the ideal gas value seems to be approached from above, whereas in the pure gauge sector the approach is from below and even seems to be in agreement with the $O(g^2)$ perturbative corrections calculated on the lattice. Whether this overshooting of the perturbative value observed for QCD is a lattice artifact or will persist in the continuum limit is at present difficult to judge. At this point it is crucial to keep in mind that the thermodynamic observables calculated on the lattice are not exact. We have used approximate perturbative expressions for the derivatives of the couplings with respect to the anisotropy (see discussion in section 3.2). It has been demonstrated recently that it is important to include the $O(g^2)$ corrections $c'_{\sigma(\tau,F)}$ in the expressions used on the lattice to calculate the entropy density, eq.(53), and other observables. This insures that their perturbative evaluation on the lattice is consistent with continuum perturbation theory. Leaving them out leads to an even larger overshooting of the perturbative limit¹⁰¹). It is thus conceivable that also the presently used expressions for the thermodynamic quantities are not accurate enough in the presence of light quarks and a non-perturbative determination of the derivatives appearing in the definitions of thermodynamic quantities given in eq.(33-38) might be necessary^{35,39}).

In the pure gauge sector as well as for full QCD a large difference in behaviour is observed for the pressure and the energy density in the plasma phase close to T_c . The equation of state in this region is obviously highly non-perturbative. At least for the pure gauge sector the deviations from the ideal gas behaviour seen in the entropy density close to T_c also rules out the validity of a simple bag equation of state in this region, which would lead to $S = 4a_0 T^3$, with a_0 denoting

Figure 9: Entropy density in units of T^3 versus β for two flavour QCD with quarks of mass $m/T = 0.1$. Shown are results obtained on 4×8^3 (triangles) and 4×12^3 (dots) lattices¹⁰⁰. The full line is the lowest order perturbative result on a lattice of size 8×12^3 , the dashed line is the corresponding $O(g^2)$ correction.

Figure 10: Energy density and pressure in units of T^4 versus β for two flavour QCD with quarks of mass $m/T = 0.1$ on a 4×8^3 lattice. Data are taken from ref.99, they, however, do not include contributions from $O(g^2)$ corrections to $d\gamma_F/d\xi$ given in eq.(53). Lines are drawn to guide the eye.

the Stefan-Boltzmann constant for an ideal quark-gluon gas given in eq.(6). The present analysis of the bulk thermodynamic quantities suggests that at least for temperatures $T \gtrsim 1.2T_c$ the plasma phase can well be described by an e.o.s which is of the form⁹⁹

$$\begin{aligned} \epsilon &= 3a_0 T^4 + O(g^2) \\ P &= \frac{1}{3} \epsilon - bT \end{aligned} \quad (55)$$

with $b \simeq 1.5T_c^3$.

The different behaviour of energy density and pressure is not unexpected: in the case of a second order phase transition the singular part of the partition function is expected to be of the form $\ln Z \sim VT^3(1-t)^{2-\alpha}$, $t = T_c/T \leq 1$. Here α is the thermal exponent that governs the singular behaviour of the specific heat, $c_V = (\partial\epsilon/\partial T)_V \sim (1-t)^{-\alpha}$, at T_c . From this ansatz it follows that close to T_c the energy density ϵ and the pressure P are given by

$$\begin{aligned} \epsilon &\sim (T - T_c)^{1-\alpha} \\ P &\sim (T - T_c)^{2-\alpha} \end{aligned} \quad (56)$$

P thus will drop faster close to T_c than ϵ ¹⁰². For a weak first order phase transition a remnant of this behaviour may be visible as P is continuous through the transition and has to connect smoothly to the low pressure in the hadronic phase whereas the energy density jumps at T_c .

Independent of these general considerations, which are valid only close to T_c , the quasi ideal gas behaviour of the energy density and the non-perturbative behaviour of the pressure has been taken as indication for the importance of massive excitations^{97,103,104} in the plasma phase. These are expected to affect the pressure more than the energy density in the plasma phase. The basic idea is the observation that in the plasma phase, heavy colour singlet excitations will contribute dominantly in the low momentum regime to the thermodynamic potential, while for large momenta a free massless gluon gas gives the dominant contribution. A simple model calculation based on this idea shows that the introduction of heavy low frequency modes in the gluon propagator affects predominantly the pressure whereas the energy density remains basically unaffected¹⁰¹. In the simplest version one may think of a single mass scale which defines the relevant momentum cut-off in the dispersion relation

$$E(\vec{k}) = \begin{cases} |\vec{k}| & , |\vec{k}| > m_c \\ \sqrt{\vec{k}^2 + m_c^2} & , |\vec{k}| < m_c \end{cases} \quad (57)$$

The partition function reads then

$$\ln Z = dV T^3 \frac{\pi^2}{90} - dV h(T, m_c), \quad (58)$$

where the first term is just the contribution of an ideal gas with d degrees of freedom and the second term parametrizes the deviations due to massive excitations,

$$h(T, m_c) = -\frac{T^3}{2\pi^2} \int_0^{m_c/T} dx x^2 \ln \frac{1 - e^{-\sqrt{x^2 + (m_c/T)^2}}}{1 - e^{-x}}. \quad (59)$$

For the energy and pressure one obtains from this ansatz

$$\begin{aligned} p &= d \frac{\pi^2}{90} T^4 - dT h(T, m_c) \\ \epsilon &= 3d \frac{\pi^2}{90} T^4 - dT^2 \frac{\partial}{\partial T} h(T, m_c) \end{aligned} \quad (60)$$

For $m_c \sim T$ it turns out that $h(m_c, T)$ is only weakly dependent on T and the derivative term contributing to ϵ in eq.(60) is thus negligible, whereas the contribution in p coming from $h(m_c, T)$ is linear in T and thus leads to a correction term for the pressure as given in eq.(55). The above ansatz can be further refined to get a good phenomenological description of the e.o.s. extracted from lattice simulations of QCD. An explicit comparison of the above ideas with the high statistics data available for the $SU(2)$ thermodynamics shows that a good quantitative description of the behaviour of energy density and pressure can be obtained this way⁹⁸⁾.

5.2. Strangeness content of the plasma

A realistic simulation of the QCD e.o.s. has to take into account the contribution of strange quarks to the partition function. Their mass is of the order of the phase transition temperature and thermal production of strange quarks is thus expected to become relevant for $T \sim T_c$. They may influence the chiral phase transition itself^{79,80)} as well as the dynamics of the high temperature phase. As the strangeness content of the low temperature phase is small it has been argued that strangeness production in the plasma phase can yield genuine signals of plasma formation in heavy ion experiments^{105,106)}.

On the basis of ideal gas relations for massive fermion gases one would expect a large contribution of the strange quark sector to the energy density. The energy

Figure 11: Fermionic energy density of light u- and d-quarks and heavy s-quark in units of T^4 versus β on a 4×12^3 lattice. Data are taken from ref.80. The dashed line gives the lowest order ideal gas behaviour on a finite lattice.

density in a massive fermion gas is given by

$$\epsilon_F(m/T) = dT^4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \left[15 \left(\frac{mn}{T^4} \right)^2 K_2(mn/T) + 3 \left(\frac{mn}{T} \right)^3 K_1(mn/T) \right], \quad (61)$$

with d counting the number of degrees of freedom and K_i , $i = 1, 2$ denoting Bessel-functions of second kind. From eq.(61) one finds for the energy density of quarks with masses which are of the order of the temperature of the system

$$\begin{aligned} \epsilon_F(m/T=1) &= 0.88 \\ \epsilon_F(m/T=0) & \end{aligned} \quad (62)$$

In a non-interacting gas of massive fermions with $m \sim T$ the energy density at a given temperature is thus only little less than the corresponding energy density in a gas of massless fermions.

The contribution of strange quarks to the energy density of QCD has been studied recently⁸⁰⁾ in a simulation with a light isodoublet of mass $m_u/T = m_d/T = 0.05$ and a heavy quark of mass $m_s/T = 1.0$. Results for the fermionic part of the energy density are shown in fig.11. They indicate that only at rather high temperatures is the perturbative result, eq.(62), approached. Close to T_c one finds $\epsilon_F(m_s/T)/\epsilon_F(m_u/T) \simeq 0.5$.

However, looking only at the fermionic part of the energy density is not completely correct, as the presence of fermions also influences the gluonic contribution ϵ_G to the total energy density. Furthermore, the derivative $d\gamma_F/d\xi$ appearing in the definition of the energy density has been approximated by 1 in the above analysis. It is known, that the $O(g^2)$ corrections to this will lead to a reduction of the fermionic contribution to the total energy density and this reduction will depend on the quark mass³⁸⁾. These effects have not been studied so far. The present result for $\epsilon_F(m_s/T)/\epsilon_F(m_u/T)$ thus should only be taken as a first attempt to analyze the influence of an intermediately heavy strange quark on the e.o.s.. However, if verified on larger lattices it suggests that the thermal production of strange quarks close to T_c is rather small. In any case, as should have become clear from our discussion of the QCD e.o.s. near T_c , it is certainly not to be expected that ideal gas relations are valid for the strangeness sector in the plasma phase in this temperature regime.

5.3. Other observables

We have seen that the analysis of bulk thermodynamic quantities in pure $SU(3)$ gauge theory as well as QCD indicate that for $T \gtrsim 1.2T_c$ the thermodynamic behaviour is close to what one would expect for an ideal gas of quarks and gluons. This is also reflected in the behaviour of other observables. For instance the quark number susceptibility¹⁰⁷⁾ shows that it is easy to create baryonic excitations in the plasma phase, suggesting that the relevant degrees of freedom can be associated with light quarks. Also the analysis of transport coefficients in the high temperature phase suggests that the heat conductivity is small as one would expect for a weakly interacting gas¹⁰⁸⁾.

Close to T_c one observes significant non-perturbative effects. There are large deviations from the ideal gas equation of state, $\epsilon = 3p$. This is also reflected in a calculation of the velocity of sound, $c_s^2 = dp/d\epsilon$ in pure $SU(3)$ gauge theory^{109,110)}. While it is compatible with the ideal gas value, $c_s = \sqrt{1/3}$, for $T \gtrsim 1.2T_c$, it drops suddenly close to T_c .

The discontinuity in the entropy density (latent heat) at T_c seems to be small. At present the numerical data are indeed not accurate enough in order to rule out completely the possibility that the transition for pure $SU(3)$ gauge theory is continuous⁶⁴⁻⁶⁶⁾. A careful finite size analysis will be needed to clarify this point. If the transition turns out to be first order different phases can coex-

ist at T_c . The interface between these phases will have a non-vanishing surface energy. The thermodynamics of such interfaces has been studied in mean-field approximation¹¹¹⁾ and simple 2-dimensional models^{112,113)}. A first analysis of the interface energy density for the $SU(3)$ gauge theory has been performed recently on a rather small lattice with only two sites in temporal direction ($N_\tau = 2$)^{114,115)}. In this case the deconfinement transition is clearly first order and a stable interface between the confined and deconfined phases can be produced. This exploratory study yields for the surface energy, α , in units of the deconfinement temperature, $\alpha/T_c^3 = 0.24 \pm 0.06$.

6. SCREENING LENGTHS AND THE HEAVY QUARK POTENTIAL

The non-perturbative effects seen in the analysis of bulk quantities in the plasma phase seem to be related to the complicated infra-red structure of finite temperature QCD. This should show up also in various screening lengths that can be studied in the plasma phase. As discussed in section 2 it would be of particular importance to understand in more detail the behaviour of magnetic screening lengths in the plasma phase. This, however, is also difficult to obtain from numerical simulations, and only some exploratory studies of the magnetic screening length have been performed so far¹¹⁶⁾. Work has concentrated instead on the analysis of the electric screening length, which itself is not well understood perturbatively²²⁻²⁴⁾, and also is difficult to extract on the lattice.

6.1 The heavy quark potential

The heavy quark potential plays a central role in our attempts to understand the phase structure of QCD. A quantitative analysis of its short and intermediate distance properties is important for the understanding of the heavy quark bound state spectrum. Its temperature dependence recently found much attention and is expected to lead to characteristic signatures for plasma formation in heavy ion collisions^{117,118)}.

Besides the interest in the quantitative properties of the potential at intermediate distances an understanding of its asymptotic large distance behaviour is of fundamental importance. At low temperatures the potential rises linearly at large distances leading to confinement. In the deconfined phase, however, it is screened due to the presence of nearly free colour sources. We thus expect a character-

istic change in the functional form of the potential as we cross the QCD phase transition:

$$V_{\bar{q}q}(r, T) = \begin{cases} \frac{\alpha(T)}{\alpha(T_c)} + \sigma(T)r, & T \leq T_c \\ \frac{\alpha(T)}{\alpha(T_c)} e^{\mu(T)r}, & T \geq T_c \end{cases} \quad (63)$$

Here σ denotes the string tension and μ the screening mass in the high temperature phase. Their temperature dependence is of fundamental interest and recently led to conflicting statements on the order of the phase transition.

On the lattice the potential for a static $q\bar{q}$ pair separated by a distance $R \equiv r/a = |\vec{r}|$ can be obtained from the correlation function $< \text{Tr} L_0 \text{Tr} L_n^\dagger >$ for two Polyakov loops. We define the heavy quark potential by

$$\exp(-V_{\bar{q}q}(r, T)/T) = \frac{< \text{Tr} L_0 \text{Tr} L_n^\dagger >}{< |L| >^2} \quad (64)$$

The normalization of the loop-loop correlation function with $< |L| >^2$ is used to eliminate divergent self-energy terms in the potential. Eq.(64) then provides a well-defined potential, free of self-energy divergences. The heavy quark potential defined this way gives a thermal average of colour singlet and octet quark-antiquark potentials⁴⁴⁾. In perturbation theory the singlet potential is attractive, whereas the octet potential is repulsive. Their relative strength is such that it leads to a cancellation of the leading, $O(g^2)$, contributions to the colour averaged potential defined in eq.(64). The leading perturbative contribution thus is $O(g^4)$,

$$V_{\bar{q}q}(r, T)/T = -\frac{1}{16} \frac{V_1(r, T)^2}{T^2}, \quad (65)$$

with the singlet potential V_1 given by

$$V_1(r, T) = -g^2(T) \frac{N^2 - 1}{8\pi N} \frac{1}{r} e^{-m_E(T)r}. \quad (66)$$

In leading order perturbation theory the screening mass, m_E , of course vanishes. However, including self energy terms in the bare gluon propagator, leads to eq.(66) with m_E given by eq.(12). Whether this leading order result for the heavy quark potential is of any significance at all, or gets completely invalidated due to the infra-red problems that occur at higher orders is unclear²²⁻²⁴⁾. A detailed non-perturbative analysis of the functional form of the heavy quark potential thus is of particular importance for understanding more about the infra-red properties of

high temperature QCD. In the following we will discuss some Monte Carlo results for the heavy quark potential and compare these with leading order perturbation theory. This will give some indication about the applicability of the leading order perturbative result.

Let us first discuss the relation between the potential calculated on the lattice and the corresponding continuum potential. Due to the periodic spatial boundary conditions the lattice potential is periodic around $N_\sigma/2$ and the quark-antiquark separation can only vary between $R = 1, 2, \dots, N_\sigma/2$. On lattices with fixed asymmetry N_σ/N_τ there is thus only a limited range of distances measured in units of T accessible,

$$\frac{1}{N_\tau} \leq rT \leq \frac{1}{2} \frac{N_\sigma}{N_\tau}. \quad (67)$$

no matter how large N_σ and N_τ are chosen. For instance, on a lattice with $N_\sigma/N_\tau = 2$ the potential will be periodic around $rT = 1$ and one may expect that the potential will be strongly distorted for $rT \sim 1$ due to the periodicity of the lattice. The behaviour of the heavy quark potential in the continuum limit can then be determined by comparing calculations of the lattice potential on different size lattices performed at the same temperature $T = 1/N_\tau a$. In fig.11 we show results for the $SU(2)$ heavy quark potential at a high temperature, $T = 250\Lambda_L$ ($T \simeq 6T_c$), obtained on lattices of various sizes and with varying asymmetry between spatial and temporal lattice extent, $N_\sigma/N_\tau = 1, 2$ and 3 . We note that results from different size lattices follow a unique curve as long as the ratio N_σ/N_τ is the same. This suggests that we are indeed calculating the continuum heavy quark potential in a finite physical volume. In addition it is obvious that the potential becomes steeper with increasing N_σ/N_τ and that the distortion of the infinite volume potential is largest around $rT \sim N_\sigma/2N_\tau$.

A detailed analysis of the finite size dependence of the $SU(2)$ potential¹¹⁹⁾ shows that this can be parametrized as

$$\ln \left[\frac{V_\infty(r, T)}{T} \right] = \ln \left[\frac{V(r, T)}{T} \right] - \frac{c(r, T)}{VT^3}. \quad (68)$$

An extrapolation to infinite volume based on this ansatz is also shown in fig.12.

In order to compare the Monte Carlo results for the heavy quark potential in the plasma phase with perturbation theory it seems to be appropriate to use a general ansatz of the form

$$\frac{V(r, T)}{T} = \frac{\alpha(T)}{(rT)^d} e^{-\mu r} \quad (69)$$

Figure 12: The heavy quark potential $-V(r, T)/T$ versus rT at fixed temperature $T = 250 \Lambda_L$. The couplings used in the simulation are $\beta = 2.988$ ($N_r = 4$), $\beta = 3.147$ ($N_r = 6$) and $\beta = 3.26$ ($N_r = 8$). The dashed line gives the potential extrapolated to infinite volume.

for fits to the numerical data. Parameters obtained from fits to the potentials shown in fig.12 using this ansatz (with periodic boundary conditions taken into account) are given in Table 1. This shows that finite size effects tend to lower the screening mass in a finite volume. For $N_c/N_r = 3$ the screening mass is still about 20% lower than the asymptotic infinite volume result. This volume dependence of the screening mass is in qualitative agreement with perturbative calculations on finite lattices¹²⁰⁾. However, we also see that the power of the subleading $1/rT$ -term is $d \simeq 1.6$, i.e. smaller than the perturbative value, $d = 2$. As the temperature is lowered one finds that d decreases further and for $T \simeq T_c$ is compatible with $d = 1$ ¹¹⁹⁾.

Like in the discussion of bulk thermodynamic quantities we thus find also here

N_c/N_r	α	d	μ/T
1	0.0386	1.30(1)	0.13(2)
2	0.0085	1.72(1)	0.81(3)
3	0.0127	1.58(1)	2.30(8)
∞	0.0119	1.61(1)	2.87(20)

Table 1: Parameters for the fits shown in fig.12.

that deviations from perturbative predictions are large close to T_c . Similar results have been obtained for the functional form of the $SU(3)$ potential¹²¹⁾. Also in that case one finds indications that the perturbative form of the potential with $d = 2$ yields a better description of the Monte Carlo data at high temperature.

Further information about the validity regime of perturbative predictions for the heavy quark potential can be obtained from a direct comparison of the functional form of the potentials in the singlet and octet channels¹²¹⁾. These potentials are given by²¹⁾

$$e^{-V_1(r, T)/T} = 3 \frac{\langle \text{Tr} L_0 L_{\pi}^{\dagger} \rangle}{\langle |L| \rangle^2}, \quad (70)$$

$$e^{-V_8(r, T)/T} = \frac{9 \langle \text{Tr} L_0 \text{Tr} L_{\pi}^{\dagger} \rangle}{8 \langle |L| \rangle^2} - \frac{3 \langle \text{Tr} L_0 L_{\pi}^{\dagger} \rangle}{8 \langle |L| \rangle^2}. \quad (71)$$

As The observables $\langle \text{Tr} L_0 L_{\pi}^{\dagger} \rangle$, entering the definitions of V_1 and V_8 , are not gauge invariant. One thus has to analyse them in a fixed gauge. A suitable choice is the Landau gauge¹²²⁾ as this preserves the rotational symmetry of the potential. In perturbation theory the singlet and octet potentials are related by

$$\frac{V_1(r, T)}{V_8(r, T)} = -8 + O(g^4). \quad (72)$$

Note that the $O(g^2)$ corrections cancel here, while that is not the case for the relation between the singlet and colour-averaged potentials

$$\frac{V_1(r, T)^2}{T V(r, T)} = -16 + O(g^2). \quad (73)$$

Figure 13: The ratios $-V_1(r, T)/V_0(r, T)$ (a) and $-V_1(r, T)^2/TV(r, T)$ (b) versus R for $\beta = 5.75$ (dots), 6.1 (squares) and 8.00 (triangles). The dashed lines correspond to the perturbative results.

These relations have been studied at three different couplings, $\beta = 5.75, 6.10$ and 8.0 on a $12^3 \times 4$ lattice. The β -values used correspond to temperatures $T/T_c = 1.16, 2.3$ and 13.7^{121} . Results are shown in fig.13. Clearly the agreement gets worse at large distances. We note, however, that the perturbative relations are approached with increasing temperature. For the largest β value one indeed finds the expected ratio between the singlet and octet potentials, while the relation between colour-averaged and singlet potentials is not quite fulfilled, even at this high temperature. This shows the importance of $O(g^2)$ corrections in these relations.

6.2. Debye screening lengths

The screening length extracted from the asymptotic large distance behaviour of the heavy quark potential has been studied recently for pure $SU(3)$ gauge theory on rather large lattices^{53,63,64}. In particular its behaviour close to T_c has been analyzed in detail. From the experimental point of view the behaviour of the potential close to T_c , i.e. $T \simeq (1 - 1.5)T_c$ is most interesting as this is the temperature regime which at best can be hoped to be reached in heavy ion experiments and in which the melting of heavy quark bound states may provide a signal for the formation of the quark-gluon plasma^{117,118}.

Figure 14: Screening masses in units of the lattice spacing for the pure $SU(3)$ gauge theory obtained on lattices of size 4×24^3 (ref.53, ref.63) and $4 \times 8^2 \times 32$ (ref. 64).

The analysis of the $SU(2)$ as well as the $SU(3)$ potentials close to T_c has shown that the potential is well described by a simple Debye-screened Coulomb potential

$$V(r, T) = \frac{\alpha(T)}{r} e^{-\mu r}, \quad T \geq T_c. \quad (74)$$

The perturbative behaviour, eq.(73), is found to set in only at very high temperatures. The screening length can be extracted from the large distance behaviour of the Polyakov-loop correlation function

$$\mu/T = -N_r \lim_{|\vec{r}| \rightarrow \infty} \ln \frac{1}{|\vec{r}|} \left[\langle \text{Tr} L_0 \text{Tr} L_{\vec{r}}^\dagger \rangle - \langle L \rangle^2 \right]. \quad (75)$$

In practice, however, it is difficult to extract information about the asymptotic behaviour of the correlation functions, is rather difficult to measure correlations at large values of $rT = |\vec{r}|N_r$. In the first numerical determinations of the screening mass¹²³⁻¹²⁵ the potential could only be analyzed up to distances $rT \sim 1$. In this intermediate distance regime the subleading power behaviour of the potential is still very important and, as we have seen in the previous section, the exponent of the subleading $1/r$ is essentially unknown. Assuming nonetheless a potential of the form given by eq.(74), i.e. assuming $d = 1$, will lead to an overestimate

μ/T β	ref.53	ref.63	ref.64	ref.121
5.695	0.84(13)	0.70(29)	0.66(16)	
5.70	1.15(16)	1.36(8)	1.0(5)	
5.71		1.44(5)	1.7(3)	
5.72		1.68(5)	1.9(3)	
5.73			1.6(3)	
5.75		2.56(19)		2.44(28)
6.1				2.20(28)
8.0				2.32(24)

Table 2: Screening masses in units of the temperature for $T > T_c$ for the pure $SU(3)$ gauge theory obtained on lattices of size $4 \times N_s^3$ with $N_s = 12$ (ref.121), 24 (ref.53, ref.63). In ref.64 a cylindrical lattice of size $4 \times 8^2 \times 32$ has been used.

of the screening mass. Recent high statistics simulations on large lattices^{63–64} allowed a determination of the screening mass μ from the heavy quark potential for $1 < rT < 5$. These calculations lead to substantially smaller values for μ/T than those obtained in earlier simulations^{123,124}, where the screening mass has been extracted from the region $rT \sim 1$. Some results for μ/T for pure $SU(3)$ gauge theory are shown in fig.14. Results for $T \geq T_c$ are also given in table 2. It can be seen that the screening mass decreases strongly close to T_c ($\beta_c \simeq 5.692$). In fact, it has been suggested that $\mu(T_c) \rightarrow 0$ if the spatial lattice size moves to infinity^{64–65}. However, this does not seem to be supported by a recent study of the screening mass on a large, 4×32^3 , lattice⁵³. There are thus still uncertainties concerning the exact behaviour of the mass gap at T_c . However, should also be noted that the regime of couplings shown in fig.14 corresponds to a rather small temperature interval around T_c ($\Delta T/T_c \lesssim 0.2$). Already at $T \simeq 1.2T_c$ ($\beta = 5.75$) one finds $\mu/T \simeq 2.5$, and this ratio stays constant for larger temperatures^{121,126}. We note that the above result is already close to the lowest order perturbative expression, $\mu = 2m_D \sim 2T$, with m_D given by eq.(12) if we assume $g(T) \sim 1$.

Simulations with light quarks^{109,127} are at present in a stage similar to the early $SU(3)$ calculations. The statistics collected in these simulations is not sufficient to go to large distances or analyse the potential with an arbitrary power for

the $1/rT$ -term. Nonetheless, as we have seen, eq.(74) seems to give a satisfactory description of the intermediate-distance part of the potential close to T_c , and can be used to analyse the Monte Carlo data in this regime. One should, however, interpret the screening length extracted from these data as an effective mass, μ_{eff} , that characterizes the behaviour of the potential for $rT \leq 1$. Present calculations for QCD give $\mu_{eff}/T \simeq 3.5$ for $T \simeq 1.1T_c$ which rises up to $\mu_{eff}/T \simeq 5.0$ for $T \gtrsim 2T_c$ ^{109,127}. The behaviour is thus similar to what has been found in the pure gauge sector. The asymptotic value at large temperatures, however, seems to be larger than for pure $SU(3)$ gauge theory, which is in accordance with the expectation that the presence of light quarks leads to a more efficient screening of colour charges.

As mentioned above, the quantitative results obtained for the heavy quark potential close to T_c are of immediate interest for the discussion of bound state formation in the quark-gluon plasma. A screening mass of about 700MeV seems to be large enough to prohibit bound state formation with charmed quarks in a Debye-screened Coulomb potential¹¹⁸. The numerical simulations thus suggest that the screening mass at $T \sim (1.3 - 1.4)T_c$ is large enough to melt heavy quark bound states like the J/ψ in a thermalized quark-gluon plasma. Whether this is indeed the correct interpretation of the experimentally observed J/ψ suppression in heavy ion experiments²⁸ is still an open question and will require also a detailed analysis of possible other suppression mechanisms.

6.3. Hadronic screening lengths

The change in the heavy quark potential and the temperature dependence of the screening length extracted from the Polyakov loop correlation functions reflects the change in the confining properties of QCD when crossing the transition temperature to the high temperature phase. The results are very much in agreement with the intuitive picture of a confinement-deconfined transition for QCD. For QCD with massless quarks, however, the only true order parameter related to a global symmetry of the QCD Lagrangian is the chiral condensate. The different chiral properties of the low and high temperature phase are reflected in the behaviour of hadronic correlation functions. In the chirally symmetric phase there is no difference between scalar (vector) and pseudoscalar (pseudovector) states. Spatial correlation functions of operators with the corresponding quantum numbers should thus show the same exponential fall-off at large distances, i.e. the screen-

ing masses in the scalar/pseudoscalar and vector/pseudovector channels should be degenerate in the chiral symmetric phase whereas they are splitted in the broken phase. Moreover, the pion is no longer a Goldstone boson in the chiral symmetric phase, one thus may expect that the corresponding screening length in the scalar/pseudoscalar channel is non-zero. This picture is supported by simulations of QCD with light quarks¹²⁹⁾. In particular a rather large value for the screening mass in the pionic sector has been found above T_c , i.e. $m_\pi/T \simeq 2.5$.

7. QCD AT FINITE BARYON NUMBER DENSITY

Up to now we have discussed the thermodynamics of QCD at finite temperature and vanishing chemical potential, μ . This is sufficient for the description of the situation met in the early universe or the central region of heavy ion collisions. To describe the equation of state in the baryon rich fragmentation region of heavy ion collisions or the interior of neutron stars we would, however, very much like to understand the phase structure of QCD at non-vanishing chemical potential, i.e. at finite baryon number density. The introduction of a finite chemical potential on the lattice is straightforward²⁷⁻³⁰⁾. It is achieved by a modification of the 0-th component of the lattice Dirac operator. The fermion matrix $Q_{n,m}$, defined in eqs.(19) and (20), then becomes

$$Q_{n,m}(\mu) = \sum_{\nu=0}^3 D_{n,m;\nu} + m\delta_{n,m}, \quad (76)$$

where $D_{n,m;\nu}$ is not affected by the chemical potential for $\nu = 1, 2$, and 3 and is given by eq.(20). The 0th component, however, becomes

$$D_{n,m;0} = \frac{1}{2} [e^{\mu a} U_{n,0} \delta_{n,m-\hat{0}} - e^{-\mu a} U_{m,0}^\dagger \delta_{m,n+\hat{0}}]. \quad (77)$$

The consequences of this modification of the fermion matrix for any numerical simulation of the theory is far reaching. For all $SU(N)$ groups with $N \geq 3$ the matrix Q is no longer positive definite[†], i.e. its determinant becomes complex and in fact even for small values of μ is a rapidly oscillating function; its phase is proportional

[†] The case of $SU(2)$ symmetry groups is special. The fermion determinant is still positive definite and Monte Carlo simulations have been performed in this case¹³⁰⁾. However, as the lowest baryonic state is massless in this theory no obvious generalization to $SU(3)$ is possible from these results.

to the lattice size. This unpleasant behaviour makes a direct application of Monte Carlo simulation techniques impossible. Unlike in the case of hadron spectroscopy quenched simulations where one approximates the fermion determinant by unity and thus calculates fermionic operators on gluonic background configurations, are inconsistent in the case of non-vanishing chemical potential^{131,132)}. One thus has to consider the case of full QCD with dynamical fermions right from the beginning.

Due to the difficulties one faces at $\mu \neq 0$ the phase diagram has been studied so far mainly in the mean-field approximation using either the above Euclidean formulation^{133,134)} or a Hamiltonian approach¹³⁵⁾. These calculations suggest the existence of a first order chiral phase transition at finite chemical potential.

First numerical results for the phase structure of QCD with non-vanishing baryon number density have been obtained in the strong coupling limit¹³⁴⁻¹³⁶⁾. QCD in the strong coupling limit simplifies considerably. The gauge action vanishes in this limit and all gluonic degrees of freedom factorize. They thus can be integrated out and a simple fermionic model remains, which can be mapped exactly to a monomer-dimer system¹³⁹⁾. For $SU(N)$ theories with even N this statistical system has a partition function with positive weights only. It is thus suitable for Monte Carlo simulations at non-zero μ . Simulations for $SU(4)$ clearly show the presence of a chiral symmetry restoring transition in the strong coupling limit of QCD¹³⁴⁾.

For $SU(3)$ the monomer-dimer partition function still contains some terms with negative weights. These sign factors can in principle be added to the expectation values one wants to calculate. In general these sign factors will lead to large cancellations between contributions with positive and negative weights. However, as all gluonic degrees of freedom have been integrated out in the strong coupling limit, the fluctuations in the phase factors are small and can be taken into account properly. A simulation of the strong coupling $SU(3)$ theory with the monomer-dimer algorithm is then still possible¹³⁷⁾. The disadvantage of this approach is that it is restricted to the extreme strong coupling ($\beta = 0$) limit. There have been recently attempts to develop algorithms for simulating QCD also at $\mu \neq 0$. At present these techniques are, however, extremely time-consuming and thus restricted to very small lattices^{138,136)}.

All the above-mentioned simulations agree about the properties of QCD at finite density in the strong coupling limit. One finds in this limit that QCD undergoes a chiral symmetry restoring phase transition at $\mu_c a = 0.63$. Expressed

in units of the strong coupling nucleon mass this corresponds to

$$\mu_c = 0.21 m_N \simeq 200 \text{ MeV} \quad (78)$$

Similarly the critical baryon number density has been found to be¹³⁷⁾

$$n_c = (0.0017 \pm 0.0002) m_N^3 = (0.22 \pm 0.02) f m^{-3}, \quad (79)$$

which is only slightly larger than that of ordinary nuclear matter, $n_0 = 0.17/fm^3$. This, however, can only be taken as first indication for the critical parameters of the chiral transition at finite baryon number density. Better simulation techniques have to be developed, which allow an analysis of this problem at sufficiently large β in order to extract results, which are indicative for the physics in the continuum limit.

8. CONCLUSIONS

Monte Carlo simulations of QCD thermodynamics are with us now already for nearly 10 years. Still we did not reach the final goal of a realistic simulation of QCD with light, nearly massless, quarks on lattices large enough for finite lattice artifacts to be ignored safely. However, we already learned a lot about the phase structure of QCD at finite temperature and there is steady progress with new simulation techniques and faster computer generations coming up.

We have discussed the information gathered so far about the phase structure of QCD. The order of the deconfinement phase transition in purely gluonic QCD is now systematically analyzed on large lattices and, although still under debate, there is growing evidence that the transition is first order. Also the analysis of the physically most interesting case of two-flavour QCD with light quarks seems to indicate that the chiral phase transition is first order.

Bulk thermodynamic quantities agree well with the perturbative predictions for $T \gtrsim (2 - 3)T_c$. For lower temperatures strong deviations from perturbative predictions have been found, which lead to deviations from the ideal gas equation of state. Non-perturbative effects are clearly visible in the behaviour of the pressure and the entropy density for $T_c \leq T \lesssim (2 - 3)T_c$, while the energy density is in good agreement with perturbative results already for $T \gtrsim 1.2T_c$. This can be understood as the energy density is less sensitive to the short distance properties of the plasma phase.

The heavy quark potential seems to deviate clearly from perturbative predictions close to T_c . In particular the analysis of the singlet and octet potentials shows that perturbative predictions can be bad for certain observables even at temperatures as high as $T \sim 10T_c$. The comparison between Monte Carlo data and perturbation theory becomes worse at larger distances. This stresses the non-perturbative nature of the QCD plasma at large distances even at rather high temperature. There are, however, indications that the potential approaches the expected perturbative form for high temperatures and that the screening mass is close to the lowest order perturbative value already for $T \sim 2T_c$. Given the poor understanding of the screening mechanism gained from perturbative analyses so far, a further detailed numerical analysis of the large-distance part of the potential at high temperatures is certainly needed. This also includes the entire problem of magnetic screening, which has been studied only little with Monte Carlo techniques and is fundamental to our understanding of non-perturbative features of the high temperature phase.

Present simulations of full QCD with light quarks suffer from large finite size effects. Although technically feasible simulations on large enough lattices are extremely time consuming. While in the gluonic sector the transition temperature has been determined on lattices with temporal sizes up to $N_\tau = 14$, the corresponding results for full QCD exist only for $N_\tau = 4$ and 6. At present simulations for $N_\tau = 8$ are being performed. If the similarity in the scaling violations found for QCD and the pure gauge sector at $N_\tau = 4$ and 6 persist for larger N_τ , this should bring us close to the validity regime of asymptotic scaling relations, i.e. to the continuum limit. It has been estimated that the computer time required for the determination of T_c/m_ρ on a 8×16^3 lattice is of the order of 30.000 hours on a Cray-YMP. Such a computation thus brings us to the limit of what can be done with present day computers and simulation techniques.

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