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IMPROVED BOUNDS ON MANY-BODY HAMILTONIANS

I) - SELF-GRAVITATING BOSONS

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A B S T R A C T

We propose a new method to obtain lower bounds on the ground state energy of an N-body Hamiltonian considered as a sum of two-body Hamiltonians. When this is applied to a system of non-relativistic gravitating bosons, the difference between variational upper bounds and the new lower bound never exceeds 17% or, if we accept to take a refined numerical calculation for the three-body case as input, 11%. The variational upper bound is used to get a semi-relativistic upper limit on the mass of a miniboson star.

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It has long been recognized that the derivation of absolute lower bounds for the ground-state energies of many-body systems has a fundamental importance. In this context, the general criteria established by Fischer and Ruelle [1] in order to prove the existence of the thermodynamic limit, and the proof given by Dyson and Lenard [2] of the stability of usual matter (i.e., screened Coulomb forces) constitute fundamental and pioneering achievements.

To these works, one must add the remarkable analysis of Lévy-Leblond [3] on self-gravitating systems and the non-saturation of gravitational forces.

The purpose of this series of notes is to show that these rigorous results not only constitute absolute bounds, but can also be cast in the form of quite remarkable estimates of the actual ground-state energies under consideration. This is achieved, in particular under a simple but noticeably efficient improvement of the Fisher-Ruelle-Dyson-Lévy-Leblond (FRDL³) technique.

We concentrate in this paper on the problem of a self-gravitating system of N particles. Other applications will be given in further publications.

We first recall the (FRDL³) technique in the case of N particles of mass m interacting pairwise via the Newton potential $V(\vec{r}_i - \vec{r}_j) = -Gm^2/|\vec{r}_i - \vec{r}_j|$. The total Hamiltonian is:

$$H = \sum_{i=1}^N \frac{p_i^2}{2m} - \sum_{1 \leq i < j \leq N} \frac{g^2}{r_{ij}} \quad (1)$$

where we have set $g^2 = Gm^2$, and $r_{ij} = |\vec{r}_i - \vec{r}_j|$, which can identically be written as a sum of $N(N-1)/2$ (non-commuting) two-body Hamiltonians h_{ij}

$$H = \sum_{i > j} h_{ij} \quad (2)$$

with

$$h_{ij} = \frac{p_i^2}{2m(N-1)} + \frac{p_j^2}{2m(N-1)} - \frac{g^2}{r_{ij}} \quad (3)$$

We assume that the N particles are identical bosons. Since h_{ij} appears as a hydrogenic-type two-body Hamiltonian with mass $m(N-1)$ instead of m , its expectation value in the ground state satisfies the inequality

$$\langle h_{ij} \rangle > -1/4 \, m(N-1) \frac{g^4}{\hbar^2} \quad (4)$$

and, therefore, the ground-state energy E_0 of this self-gravitating boson system is bounded from below by [3]

$$E_0 = -1/8 \, m \, N(N-1) \frac{g^4}{\hbar^2} \quad (5)$$

which, of course, reduces to the exact result (e.g., positronium) for $N = 2$ as can easily be checked.

It is clear, however, that in the above derivation, the energy has been overestimated. Indeed, the total centre-of-mass kinetic energy of the system (which corresponds to the global motion and is of no interest) has not been separated out, and the two-body subsystems are not at rest. This can be achieved by using the identity:

$$\sum_{i=1}^N p_i^2 = 1/N \left[\left(\sum_{i=1}^N \vec{p}_i \right)^2 + \sum_{i < j} \left(\vec{p}_i - \vec{p}_j \right)^2 \right] \quad (6)$$

$\sum_{i=1}^N \vec{p}_i = \vec{P}$ being the total momentum of the system. Hence the relative motion of the N particles in their centre-of-mass system ($\vec{P} = 0$) is described by the Hamiltonian

$$H_{\text{rel}} = \sum_{i < j} \tilde{h}_{ij} \quad (7)$$

with

$$\tilde{h}_{ij} = \frac{(\vec{p}_i - \vec{p}_j)^2}{2mN} - \frac{g^2}{r_{ij}} \quad (8)$$

Notice, however, that the canonical momentum conjugate to $\vec{r}_i - \vec{r}_j$ is not $\vec{p}_i - \vec{p}_j$ but $\vec{\pi}_{ij} = (\vec{p}_i - \vec{p}_j)/2$, therefore

$$\tilde{h}_{ij} = \frac{4 \pi_{ij}^2}{2mN} - \frac{g^2}{r_{ij}} \quad (9)$$

where π_{ij} and $\vec{r}_{ij}$ obey canonical commutation relations.

Hence, we obtain

$$\langle \tilde{H}_{1j} \rangle > - \frac{mN}{8} \frac{g^4}{\hbar^2} \quad (10)$$

and

$$E_0 > - \frac{m}{16} \frac{N^2(N-1)}{16} \frac{g^4}{\hbar^2} \quad (11)$$

which agrees, as it should, with the exact result for $N = 2$, but improves Lévy-Leblond's result by a factor two for large N ($1/16$ instead of $1/8$), where one obtains

$$E_0 > -0.0625 \frac{N^2(N-1)}{16} \frac{G^2 m^5 \hbar^2}{16} \quad (12)$$

(instead of -0.125).

The following comments are in order at this point.

A) Aside from improving previous results by an unexpected amount, the result is interesting in that it disproves the numerical result of Ruffini and Bonazzola [4]:

$$E_0 = -0.1626 \frac{N^3}{16} \frac{G^2}{m^5} \hbar^{-2}$$

(notice that Lévy-Leblond's result also disproved it).

B) More interesting is that an upper bound can be calculated for this energy by standard variational techniques.

Using a trial wave function of the form

$$\psi = \prod_{i=1}^N \phi(\vec{r}_i) \quad (13)$$

one is led to minimizing the expression

$$\frac{1/2m}{\int |\phi|^2 d^3x} - \frac{Gm^2}{\int |\phi|^2 d^3x} \frac{\int |\phi(x)|^2 |x-y|^{-1} |\phi(y)|^2 d^3x d^3y}{(\int |\phi|^2 d^3x)^2} \quad (14)$$

which leads to what is called the Choquard equation [5] but which appears also in the work of Lévy-Leblond [3] and of Ruffini and Bonazzola [4]. If we use a trial function

$$\phi(\vec{r}_i) = \exp - \mu/\lambda^2 r_i^2 + 1 \quad (15)$$

we get the upper bound

$$E_0 < -0.0542 \frac{N(N-1)}{16} \frac{G^2 m^5}{16} \quad (16)$$

Therefore, comparing Eqs. (12) and (16), one approaches the exact result within less than 14% in the limit of very large N . For small N , however, the discrepancy between the upper bound (14) and the lower bound (12) gets worse. This is presumably because the trial function (15) is not translation invariant. For the three-body case, an upper bound from a Gaussian trial function

$$\exp -\lambda[(r_1-r_2)^2 + (r_2-r_3)^2 + (r_1-r_3)^2]$$

and an "exact" value from a calculation using hyperspherical expansion up to a generalized angular momentum $L = 8$ have been obtained. They are, respectively,

$$E_0 < -0.95492 \frac{Gm^5}{16} \quad (17)$$

$$E_0 = -1.067 \frac{Gm^5}{16}$$

while the lower bound (12) gives

$$E_0 > -1.125 \frac{Gm^5}{16} \quad (18)$$

Therefore the lower bound deviates again by only 6% from the exact value.

For general finite N , we have obtained an upper bound on E_0 in the hypercentral approximation, which is equivalent to taking a variational trial function f depending only on the variable $\sqrt{\sum_{ij}(\vec{r}_i - \vec{r}_j)^2}$. Then the trial function satisfies a Schrödinger equation with a Coulomb potential and a fictitious orbital angular momentum depending on N :

$$-\frac{\hbar^2}{2m} \frac{d^2}{d\xi^2} - \frac{\lambda_N(I+1)}{\xi^2} \} u(\xi) - \frac{Gm^2 \lambda_N}{\xi} u(\xi) = E u(\xi) \quad (19)$$

where

$$u(\xi) \sim \xi^{(3N-4)/2} f(\xi) \quad (20)$$

$$\lambda_N = \frac{3N-6}{2} \quad (21)$$

$$\lambda_N = \frac{\Gamma(1)}{\Gamma(3/2)} \frac{\Gamma(3/2 + (3N-6/2))}{\Gamma(1 + (3N-6/2))} \frac{N(N-1)}{2/2} \quad (22)$$

Hence the upper bound on the energy is

$$E_0 < -\frac{G^2 m^5}{4\pi} \frac{\Gamma(\frac{3N-3}{2})}{\Gamma(\frac{3N-2}{2})} \left[\frac{\Gamma(\frac{3N-2}{2})}{\Gamma(\frac{3N-3}{2})} \right]^2 N^2 (N-1)^2 \quad (23)$$

For $N = 3$ this gives

$$E_0 < -G^2 m^5 1.0375$$

very close to the "exact" answer. For arbitrary N , the ratio between the absolute value of the lower bound (14) and the absolute value of the upper bound (23) is

$$R(N) = \frac{\pi}{4} (N-1) \frac{\Gamma(\frac{3N-2}{2})}{\Gamma(\frac{3N-3}{2})} \left[\frac{\Gamma(\frac{3N-2}{2})}{\Gamma(\frac{3N-3}{2})} \right]^2 \quad (24)$$

It is easy to see that $R(N)$ increases with N . Indeed,

$$\frac{R(N+2)}{R(N)} = \frac{N^2}{N^2-1} \frac{N^2 - 4/9}{N^2 - 1/9} \quad (25)$$

can be shown to be larger than unity by elementary means. We conclude

$$R(N) < R(\infty) < 1.1781 \quad (26)$$

In fact, the variational bound (16) supersedes the hypercentral bound for $N > 65$ and finally the ratio between the lower bound and the best upper bound is

$$R(N) < 1.172 \quad \forall N \quad (27)$$

C) By generalizing our method it is possible to narrow down further the gap between the upper bound and the lower bound. Notice first that H_{rel} defined by Eq. (7)

represents the N body Hamiltonian with the centre-of-mass kinetic energy removed. Its lower bound is therefore the N body ground state energy at rest. Now instead of writing H_{rel} for N particles as a sum of two-body relative Hamiltonians we can write it, for instance, as a sum of three-body Hamiltonians:

$$H_{rel} = \frac{1}{N-2} \sum_{i,j,k} (\tilde{h}_{ij} + \tilde{h}_{jk} + \tilde{h}_{ik}),$$

but, precisely $\tilde{h}_{ij} + \tilde{h}_{jk} + \tilde{h}_{ik}$ represents the relative Hamiltonian for a three-body system with the mass $Nm/3$ and the same g^2 . Hence we get

$$E_0 > \frac{1}{N-2} \frac{N(N-1)(N-2)}{6} \times \frac{N}{3} E_0 \text{ (three-body)}$$

and if we trust the three-body calculation (17),

$$E_0 > N^2(N-1) 0.0593 G^2 m^5$$

Then the ratio between the best lower bound and the best upper bound is at most 1.11 for any N , and at most 1.09 for very large N .

D) We recall that such investigations are not purely academic. The critical mass of such an object - a so-called mini-boson star - can be estimated by various methods, the crudest one being to say that

$$E_0 = -N m c^2,$$

where c is the velocity of light.

We know from B) that the variational energy given by (16) is a good estimate of the true non-relativistic binding energy of the system. This gives (here we take $\hbar = c = 1$):

$$N = \left(\frac{1}{0.0542 G^2 m^4} \right)^{1/2} = 4.2 \left(\frac{M_{Planck}}{m} \right)^2 \quad (28)$$

and defining $M_c = Nm$

$$M_c = 4.2 M_{Planck}^2 / m \quad (29)$$

However, more refined semi-relativistic estimates can be made following the method used by one of us (A.M.) [6]. The semi-relativistic Hamiltonian of the N bosons can be written

$$H = \sum_i p_i^2 + m^2 - Gm^2 \sum_{i>j} \frac{1}{r_{ij}} \quad (30)$$

We have

$$H < N \left[\frac{m^2}{2\mu} + \frac{1}{2} \right] + \left[\sum_{i>j} \frac{1}{2\mu} - Gm^2 \sum_{i>j} \frac{1}{r_{ij}} \right]$$

where μ is an arbitrary positive parameter. In the second bracket, we recognize a non-relativistic Hamiltonian. Hence the mass of the ground state of N particles is given by

$$M_N < N \left[\frac{m^2}{2\mu} + \frac{1}{2} \right] - 0.0542 N(N-1)^2 G^2 m^4 \mu \quad (31)$$

Minimizing with respect to μ , we get

$$M_N = -\infty \quad \text{if} \quad N-1 > \frac{3.037}{Gm^2} \quad (32)$$

i.e., the collapse of the system for

$$N > \left(\frac{M_{\text{Planck}}}{m} \right)^2 \quad 3.037 \quad \text{and, if} \quad N-1 < \frac{3.037}{Gm^2}, \quad (33)$$

$$M_N < Nm \sqrt{1 - 0.1084 G^2 m^4 (N-1)^2} \quad (34)$$

Maximizing this expression with respect to N we get

$$M_{\text{max}} < \frac{1.51}{Gm} \quad (35)$$

approximately a factor of two above the general relativistic calculation [4,7]:

$$M_{\text{max}} = \frac{0.633}{Gm}$$

To give an order of magnitude, let us indicate that if the lightest super-symmetric particle is a sneutrino of 10 GeV, this could result in miniboson stars of a few million tons.

Our further and more important remarks are the following.

i) In principle, our method applies to any type of two-body potential $V(r_{ij})$. Equation (9) transforms into

$$\tilde{H}_{ij} = \frac{4}{2mN} \pi_{ij}^2 + V(r_{ij}) \quad (36)$$

the major problem being to calculate the energy levels E_n of the two-body problem as a function of the mass. For power law potentials, this can be done analytically, by using scaling properties, and we shall return to this problem in the next paper [8].

ii) Notice that, as the number of particles goes to infinity, the "effective" mass entering Eq. (8) also increases indefinitely. Therefore, if the potential is bounded from below, $V > V_0$, one tends to a "classical situation" in the boson case:

$$E_0 > \frac{N^2}{2} V_0 + O(N^{3/2}),$$

if the potential is twice differentiable near its minimum.

iii) We hope that the fermion case can be handled along the same lines as in the work of Lévy-Leblond [3]. We shall also return to this problem in future work. The main problem is there a counting problem, which is trivial in the Coulomb (or Newton) and harmonic oscillator cases owing to the symmetry groups involved, but becomes trickier even for power-law potentials.

iv) Nevertheless, even when a numerical approach is unavoidable, the present method gives remarkably accurate and quick estimates of ground state energies of many-body systems as we shall see in the next paper.

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