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Chern-Simons 3-Dimensional
And Yang-Mills-Higgs 2-Dimensional Systems
As 4-Dimensional Topological Quantum Field Theories.

Laurent Baulieu

CERN, Geneva*

ABSTRACT

We show that the second Chern class considered as a classical topological action $\int_{M_4} \text{Tr} F F$ over the four-dimensional manifold $M_3 \times [0, \infty]$ can be gauge fixed into a supersymmetric action defined over a three-dimensional submanifold M_3 and containing the Chern Simons action $\int_{M_3} \text{Tr}(A dA + \frac{2}{3} A A A)$ plus other metric dependent terms including a regularizing ordinary Yang-Mills action $\int_{M_3} \text{Tr} F^* F$. We use a temporal gauge in M_4 and choose a gauge in the three-dimensional quantum field theory to probe the moduli space of solutions to the equation $*F = 0$ in M_3 . Ward identities should explain the metric independence of expectation values of Wilson loops computed from this action. We also show that another gauge fixing of the second Chern class gives a two dimensional Yang-Mills-Higgs quantum field theory.

* Permanent address: Laboratoire de Physique Théorique et Hautes Energies de l' Université Pierre et Marie Curie, Paris, France.

1. Introduction

In [1], Witten introduced the idea of topological quantum field theory and showed how to obtain the Donaldson polynomials from an action depending on four dimensional vector potentials. In [2] he showed that Jones polynomials and some generalizations are related to expectation values of Wilson loops computed in a quantum field theory based on the three-dimensional Chern-Simons action. In [3], we showed how to interpret Witten's result on Donaldson polynomials by starting with a purely topological action $I_{top} = \frac{1}{8\pi^2} \int_{M_4} \text{Tr} F_\Lambda F$, i.e. the second Chern class. Since the action is a constant function on vector potentials, the path integral $\int [DA] \exp - I_{top}$ needs interpretation. We did so by choosing appropriate gauge functions and applying the BRST formalism. The topological invariance followed from the BRST symmetry. This symmetry and the ghost fields introduced by gauge fixing have a geometric interpretation on $M_4 \times A/G$, where M_4 is the 4-manifold and A/G is the orbit space of vector potentials equivalent under gauge transformations.

In this paper, inspired by Witten, we define topological invariants of a compact manifold M_3 via a quantum field theory. Rather than starting directly from the Chern-Simons action defined over M_3 as Witten did, we shall start from the purely topological action $I_{top} = \frac{1}{8\pi^2} \int_{M_4} \text{Tr} F_\Lambda F$ where $M_4 = M_3 \times [0, \infty]$. We will gauge fix this action as we did in the case of Donaldson polynomials but with different gauge functions adapted to the moduli space that we want to probe. We will end up with a supersymmetric gauge fixed action defined on M_3 and containing as a metric independent part the Chern-Simons action, and as a metric dependent part the usual Yang-Mills action $\int_{M_3} \text{Tr}^* F_\Lambda F$ among other terms. The presence of the metric dependent term $\int_{M_3} \text{Tr}^* F_\Lambda F$ is important, because it is a second-order action which regularizes the Chern-Simons action [4]. On the other hand, expectation values of Wilson loops and cocycles computed from our gauge fixed action should not depend on our choice of metric, since we introduce the metric through a BRST exact action in M_4 . As a result of our analysis the Donaldson and Jones invariants are linked in the sense of being computable by a quantum field theory stemming from the same classical action, namely the second Chern class, but gauge-fixed with different gauge functions.

By using other gauge functions, namely ones to probe the solutions to the Bogomolny equations for a Yang-Mills-Higgs system in two-dimensions, we show how to gauge-fix the second Chern class down to a two dimensional quantum field theory. Technically speaking, the topological ghosts of ghosts are identified with the Higgs fields. We find this kind of

transmutation between gauge field components and ghosts interesting because it jumps two-dimensions. Clearly topological actions in higher dimensions may be reducible to other effective supersymmetric actions in four and two-dimensions, but we shall not investigate this possibility here.

Witten's work on the quantization of the pure Chern-Simons action has stimulated other papers on the subject. Most adopt a different point of view from ours since our principle is to gauge fix and quantize the second Chern class and not the Chern-Simons action. However we wish to call attention to the work of Chapline and Grossman and of Schaposnik and Thompson [5], since one of the gauge-fixed actions that we obtain can be identified (up to details) with the topological actions that they have considered for describing the moduli space of Bogomolny equations in two-dimensions. The link between the four-dimensional Yang-Mills topological quantum field theory and the three dimensional Chern-Simons theory was also indicated in [6], by using the formalism of stochastic quantization. (There is in fact a general correspondence between the supersymmetric representation of a stochastically quantized D-dimensional gauge theory and a covariant $(D+1)$ -dimensional topological quantum field theory.[7])

2 . From the second Chern class to the regularized Chern-Simons action

Let M_3 be a given three-dimensional manifold and A a G -valued Yang-Mills field defined on M_3 . The Chern Simons action $\int_{M_3} \text{Tr}(A_\Lambda dA + \frac{2}{3} A_\Lambda A_\Lambda A)$ action is reparametrization invariant without being metric dependent and seems thus a good candidate for generating a topological field theory in three-dimensional space, as the second Chern class is in four dimensions. Witten has performed the canonical quantization of this action and reduced it to a bidimensional conformal theory with topological observables, namely the Jones polynomials [2]. On the other hand, the Chern-Simons action is first order : the momenta of the fields are functions on the fields and the corresponding Hamiltonian vanishes modulo the classical constraints. Therefore, the definition of the Chern-Simons theory in a path integral formulation is conceptually difficult to understand.

Here we shall construct a topological quantum field theory for M_3 by introducing a manifold $M_4 = M_3 \times [0, \infty]$. We assume given the principal bundle $B = M_4 \times G$. From the definition of Chern-Simons' and Stokes' theorem, we have :

$$I_{top} = \int_{M_4} \text{Tr} F_\Lambda F = \int_{M_3} \text{Tr}(A_\Lambda dA + \frac{2}{3} A_\Lambda A_\Lambda A). \quad (1)$$

The difference between the antighost spectrum in equation (3) and the one used in [3] for building the action generating Donaldson invariants is that we have replaced the G -valued self-dual 2-form antighost $\chi_{\alpha\beta}$, which counts in four dimensions for -3 degrees of freedom, by the set made from antighost Ψ_μ and the mixed ghost-antighost L . $\bar{\Psi}_\mu$ counts for -4 degrees of freedom and L for +1 degree of freedom so that these two fields count as a whole for -4+1=-3 degrees of freedom, which make the substitution a consistent one. Analogously the G -valued Stueckelberg fields β_μ and η (4-1=3 degrees of freedom) replace the auxiliary self-dual 2-form $b_{\alpha\beta}$ used in [3] as a Lagrange multiplier for imposing on the Yang-Mills field the self-duality gauge condition.

We define the partition function as :

$$\int [dA_\mu][d\bar{c}][d\bar{c}][db][d\bar{\Psi}_\mu][d\bar{\Psi}_\mu][d\bar{\beta}_\mu][d\bar{\Phi}][dL][d\bar{\Phi}][d\eta][d\bar{\eta}] \exp - I_{GF} \quad (5)$$

with

$$I_{GF} = \int_{M_4} Tr F_\wedge F + s \{ \dots \} \quad (6)$$

where a metric dependence is introduced through the term $s \{ \dots \}$. Since $s^2 = 0$ this action is obviously BRST invariant for all possible choices of $\{ \dots \}$ in (6).

We want to define gauge functions which concentrate the functional integral around the moduli space of flat vector potentials on M_3 . In what follows latin indices $i, j, k, \dots$ stand for three-dimensional indices of M_3 and the index 0 refers to the fourth index in M_4 . (One has, for instance, $\Psi_\mu = (\Psi_0, \Psi_i)$.) Using $M_4 = M_3 \times [0, \infty]$, we choose as gauge functions $A_0, \star \tilde{F} + DL, D^\star \tilde{\Psi}, D^\star \tilde{A}$ where $\tilde{\Psi}$ and $\tilde{A}$ mean the restrictions of Ψ and A to M_3 . These gauge functions are not BRST invariant. The Donaldson invariants case breaks the symmetry between ghosts and antighosts and there is no anti-BRST invariance in this case. The BRST and anti-BRST equations can be written in a most compact form :

$$(d + s + \bar{s})(A + c + \bar{c}, A + c + \bar{c}) = F + \Psi + \bar{\Psi} + \Phi + L + \bar{\Phi} \\ (d + s + \bar{s})(F + \Psi + \bar{\Psi} + \Phi + L + \bar{\Phi}) = -(A + c + \bar{c}, F + \Psi + \bar{\Psi} + \Phi + L + \bar{\Phi})$$

The BRST and anti-BRST equations are recovered by expansion in form and ghost degrees. The nilpotency property $s^2 = \bar{s}s + s\bar{s} = 0$ follows from the fact that the last equation is the Bianchi identity of the former one.

functions can be written as follows in local coordinates :

$$\begin{aligned} & A_0 \\ & g_{ij} F^{ij} + D_i L \\ & g^{ij} D_i \Psi_j \\ & g^{ij} \partial_i A_j \end{aligned} \quad (7)$$

We have defined $F_i = \epsilon_{ijk} F^{jk}$. Flat potentials over M_3 mean $F_i = 0$.

Note that what we have done so far could be done for any compact M_4 with boundary M_3 in a tubular neighbourhood of M_3 .

In section 3 we replace the gauge functions (7) by other gauge functions, non-covariant on M_3 . By doing so we will obtain an effective bidimensional quantum field theory.

To implement the gauge functions (7) the last term in (6) becomes :

$$\begin{aligned} \int_{M_4} d^4 x \{ \dots \} = \int_{M_4} d^4 x s Tr \{ & \bar{\Psi}_0 A_0 \\ & + \bar{\Psi}_j g^{ij} (F_i + D_i L) + \frac{1}{2} \bar{\Psi}_j g^{ij} \beta_i \\ & + \bar{\Phi} g^{ij} D_i \Psi_j + \bar{c} g^{ij} \partial_i A_j + \frac{1}{2} \bar{c} b \} \end{aligned} \quad (8)$$

In this action g_{ij} is a metric inert under s transformations : $s g_{ij} = 0$. Using the definition of s , we can rewrite (8) as follows :

$$\begin{aligned} \int_{M_4} d^4 x Tr \{ & \beta_0 A_0 - \bar{\Psi}_0 (\Psi_0 + D_0 c) \\ & + \beta_j g^{ij} (F_i + D_i L) + \frac{1}{2} g^{ij} \beta_i \beta_j \\ & - \bar{c}^{ijk} \bar{\Psi}_i D_j \Psi_k - \bar{\Psi}_i g^{ij} [F_j, c] \\ & - g^{ij} \bar{\Psi}_j D_i \eta - g^{ij} \bar{\Psi}_j [\Psi_i + D_i c, L] \\ & - \bar{\eta} g^{ij} D_i \Psi_j + \bar{\Phi} g^{ij} D_i D_j \Phi - \bar{\Phi} g^{ij} [\Psi_i + D_i c, \Psi_j] \\ & - \bar{c} g^{ij} \partial_j (\Psi_i + D_i c) + \frac{1}{2} b^2 + b g^{ij} \partial_i A_j \} \end{aligned} \quad (9)$$

The equation of motion with respect to β_0 enforces the condition $A_0 = 0$. The algebraic dependence on the fields β_i and b permits us to replace these fields by their field equations.

The fields Ψ_0 and $\bar{\Psi}_0$ can be also be eliminated by a trivial Gaussian integration. Using the Bianchi identity $D_i F_j g^{ij} = 0$ we can thus rewrite the action (9) as follows :

$$\begin{aligned} \int_{M_4} d^4x \text{Tr} \{ & \beta_0 A_0 - \frac{1}{2} F_{ij} F^{ij} - \frac{1}{2} D_i L D^i L \\ & - \epsilon^{ijk} \Psi_i D_j \Psi'_k - \bar{\Psi}_i [\Psi'_i, L] \\ & - \bar{\Psi}^i D_i \eta \\ & - (\bar{\eta} + \bar{c}) D_i \Psi'^i + \bar{c} [A_i, \Psi'^i] \\ & + \bar{\Phi} D_i D^i \Phi - \bar{\Phi} [\Psi'_i, \Psi^i] \\ & + \bar{\eta} D_i D^i c \\ & - \frac{1}{2} (\partial_i A^i)^2 \} \end{aligned} \quad (10)$$

(We have defined $\Psi' = \Psi + Dc$.) This action does not involve derivatives with respect to the time direction. Therefore, in the context of quantum field theory, the dependence upon the variable x^0 can be forgotten and, up to an overall coefficient that we shall denote as α , we can replace (10) by the same action but with all fields only functions of the spatial coordinates x^i and M_3 replacing M_4 . (There are many formal arguments for this property. They are simply based on the meaning of the functional integral as a sum over all paths with a factorized measure. We shall not display these arguments here.) Finally, by redefining Ψ_i as $\Psi_i + D_i c$ our gauge-fixed action $\int_{M_4} \text{Tr}(F_\Lambda F + s\{\dots\})$ becomes :

$$\begin{aligned} I_{GF} = \int_{M_3} d^3x \text{Tr} (& A_\Lambda dA + \frac{2}{3} A_\Lambda A_\Lambda A) \\ & + \alpha \int_{M_3} \text{Tr} \{ -\frac{1}{2} F_{ij} F^{ij} - \frac{1}{2} D_i L D^i L - \frac{1}{2} (\partial_i A^i)^2 \\ & - \epsilon^{ijk} \bar{\Psi}_i D_j \Psi_k - \bar{\Psi}_i [\Psi_i - D_i c, L] - \bar{\Psi}^i D_i \eta \\ & - (\bar{\eta} + \bar{c}) D_i \Psi'^i + \bar{c} [A_i, \Psi'^i] \\ & + \bar{\Phi} D_i D^i \Phi - \bar{\Phi} [\Psi_i - D_i c, \Psi^i] \\ & + \bar{\eta} D_i D^i c \} \end{aligned} \quad (11)$$

The functional integral is now done with respect to the functional measure $\int [dA_i][d\bar{c}][d\bar{\Psi}_i][d\Psi_i][d\bar{\Phi}][d\Phi][d\eta][d\bar{\eta}]$ defined for fields on M_3 . All other fields have been eliminated by their algebraic equations of motion. The action (11) makes sense perturbatively because

its quadratic field approximation is given by an elliptic operator (the virtue of our gauge-fixing is to yield a second-order action) ; the propagators exist. If one restricts the action to its ghost-independent part, it is minimized by flat potentials over M_3 as desired. Hence, in the weak coupling limit, the functional integration is concentrated around the moduli space M of flat potentials defined modulo gauge transformations. Under special assumptions on M_3 , M will be a discrete set.

The metric dependence which is necessary to obtain a second-order action and define quantum field theory is due to the part of the action proportional to α . Certain expectations values should, however, not depend on the choice of the metric g_{ij} since it has been introduced through a BRST exact gauge fixing terms in M_4 . The Ward identities resulting from the remaining BRST invariance in (11) should be enough to prove metric independence for the Wilson loops over paths in M_3 by a generalization of the method used in [3] for proving metric independence of Donaldson invariants [3].

3. Reduction to a 2-dimensional quantum field theory

In [2] Witten interpreted the quantum field theory for a pure Chern-Simons action in terms of a two-dimensional conformal field theory (using the classical constraints before quantizing). We have not been able to prove that the three-dimensional supersymmetric action (11) is equivalent to a two-dimensional quantum field theory. Rather, we have been led to start again from the topological action in four dimensions with a more radical change of the gauge functions into ones which are more directly appropriate to two-dimensional Yang-Mills-Higgs systems. The gauge symmetry of the topological action in four dimensions $\int_{M_4} \text{Tr} F_\Lambda F$ is in fact large enough to accommodate such a gauge-fixing.

We assume that $M_3 \equiv M_2 \times S^1$ and we label v the one form whose restriction to S^1 is $d\theta$. We can now choose the temporal gauge alover again. That is, we choose A_0 and $v \cdot A$ as gauge functions. Then we need further gauge functions to gauge-fix $\int_{M_4} \text{Tr} F_\Lambda F$ down to a topological Yang Mills-Higgs 2-dimensional action, possibly spontaneously broken. We shall use the middle ghost of ghost L and the antighost of ghost $\bar{\Phi}$ as components of a Higgs fields doublet. We accept the possibility of terms with ghost number zero modulo 2 in the action. This is a generalization consistent with the rules of functional integration and permits the use of $(1 - [L, \bar{\Phi}])^2$ as a Higgs potential. We arrange the fields L and $\bar{\Phi}$ into the object $\phi^\alpha = (L, \bar{\Phi})$ and the antighosts $\bar{\Psi}_1$ and $\bar{\Psi}_2$ as off-diagonal elements of a 2×2 matrix $\bar{\Psi}_{\alpha\beta}$

with vanishing diagonal elements (the Greek indices $\alpha, \beta, \gamma, \dots$ take the values 1 or 2, one has for instance $\frac{1}{2}\epsilon^{\alpha\beta}[\phi_\alpha, \phi_\beta] = [L, \tilde{\Phi}]$).

Since the Bogomolny equations are of the type $\frac{1}{2}\epsilon^{\alpha\beta}F_{\alpha\beta} - (1 - [L, \tilde{\Phi}])^2 = 0$ and $D_\alpha\Phi^\beta + \epsilon^\beta_\alpha D_\gamma\Phi^\gamma = 0$, we consider the following gauge functions :

$$\begin{aligned} A_0 & \\ A_3 & \\ \frac{1}{2}\epsilon^{\alpha\beta}F_{\alpha\beta} - (1 - \frac{1}{2}\epsilon^{\alpha\beta}[\phi_\alpha, \phi_\beta])^2 & \quad (12) \\ D_\alpha\phi^\beta + \epsilon^\beta_\alpha D_\gamma\phi^\gamma & \\ g^{\alpha\beta}\partial_\alpha A_\beta + \Phi & \end{aligned}$$

This choice amounts to writing $s\{..$ in (6) as follows :

$$\begin{aligned} \int_{M_4} s\{...\} = \int_{M_4} d^4x s \text{Tr} \{ & \bar{\Psi}_0 A_0 + \bar{c} A_3 \\ & + \bar{\Psi}_\beta (D_\alpha \phi^\beta + \epsilon^\beta_\alpha D_\gamma \phi^\gamma) + \frac{1}{2} \rho \bar{\Psi}_\alpha g^{\alpha\beta} \beta_\beta \\ & + \bar{\Psi}_3 (\frac{1}{2} \epsilon^{\alpha\beta} F_{\alpha\beta} - (1 - \frac{1}{2} \epsilon^{\alpha\beta} [\phi_\alpha, \phi_\beta])^2) + \frac{1}{2} \rho' \bar{\Psi}_3 \beta_3 \\ & + c(g^{\alpha\beta} \partial_\alpha A_\beta + \frac{1}{2} \Phi) \} \end{aligned} \quad (13)$$

ρ and ρ' are parameters that we can fix at will. By expanding this s -exact action and eliminating the fields with algebraic equations of motion one finds the following Lagrangian :

$$\begin{aligned} s\{...\} = \text{Tr} \{ & -\frac{1}{2\rho} (\frac{1}{2} \epsilon^{\alpha\beta} F_{\alpha\beta} - (1 - \frac{1}{2} \epsilon^{\alpha\beta} [\phi_\alpha, \phi_\beta])^2)^2 \\ & - \frac{1}{2\rho'} (D_\alpha \phi^\beta + \epsilon^\beta_\alpha D_\gamma \phi^\gamma)^2 - \frac{1}{2} (g^{\alpha\beta} \partial_\alpha A_\beta + \frac{1}{4} [c, c])^2 \\ & \dots \} \end{aligned} \quad (14)$$

The dots on the right-hand side of (14) stand for fermionic ghost terms, and one can verify that all propagators are defined in (14). Φ has been eliminated by its equation of motion $\Phi = g^{\alpha\beta} \partial_\alpha A_\beta + \frac{1}{4} [c, c]$. One sees the possibility of spontaneous symmetry breaking due to the quartic potential involving the fields L and Φ , and that the functional integral will be concentrated around the solutions to the Bogomolny equations. For the full gauge-fixed action $I_{GF} = I_{top} + s\{..$ the only source of derivatives with respect to the coordinate 3 is

the topological action which has been gauge-fixed into the free action $\int_{M_3} d^3x \text{Tr} \epsilon^{\alpha\beta} A_\alpha \partial_3 A_\beta$ due to the gauge condition $A_3 = 0$. Otherwise I_{GF} involves only derivatives with respect to the coordinates 1 and 2 stemming from (14). It follows that the corresponding topological quantum field theory can be formally recognized as a supersymmetric Yang-Mills-Higgs system defined over a manifold of dimension 2, the term $\int_{M_3} d^3x \text{Tr} A_\alpha \partial_3 A_\beta$ acting as a mass term.

Finally, if we suppress the Higgs potential by simply changing the gauge function $\frac{1}{2}\epsilon^{\alpha\beta}F_{\alpha\beta} - (1 - [L, \tilde{\Phi}])$ into $\frac{1}{2}\epsilon^{\alpha\beta}F_{\alpha\beta}$ and choosing the gauge parameter ρ equal to zero, the A integration reduces to the domain for which F_{12} vanishes. Following the formal known argument, one can solve this constraint by $A_\alpha = g^{-1} \text{partial}_\alpha g$. Substituting this expression in the Chern-Simons action one gets a Wess-Zumino-Witten action, which indicates the possible relation with a conformal field theory.

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