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Improved Bounds on Many-Body Hamiltonians II - Baryons from Mesons in the Quark Model

J.L. Basdevant

Division de Physique Théorique^{*}, Institut de Sciences Nucléaires
B.P. No 1, 91406 Orsay Cedex; and
LPTPE, T16,E1, Université P. et M. Curie, 75231 Paris Cedex 05

André Martin
CERN - Geneva

and

J.M. Richard
Institut de Sciences Nucléaires, Université Joseph Fourier
56, avenue des Martyrs, 38026 Grenoble Cedex

ABSTRACT

We derive an improved lower limit on the binding energy of a three-body system in terms of the binding energies of two-body systems with a fictitious mass. This represents a considerable improvement with respect to the previous lower limit and gives results very close to high accuracy calculations in the hyperspherical formalism for pairwise potentials r^β , $0 < \beta \leq 3$. For $\beta = 2$, the inequality becomes an equality, and this is also true for its N body generalization. This is clarified in an appendix by T.T. Wu. It is also possible to make model independent predictions on the masses of the baryons by using the known dependence of the masses of the mesons on the masses of the quarks. Possible generalizations and other applications are discussed, in particular in atomic physics.

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* Unité associée au C.N.R.S.

1. INTRODUCTION

One often feels that the three-body problem is plagued with difficulties which are not balanced by the physical output. In the present letter, we shall give a "counterexample" by setting very simple but significant and useful constraints on the three-body binding energies in terms of the two-body subsystem binding energies.

For the sake of clarity and to avoid abstract considerations, we adopt the language of hadron spectroscopy, although our results can be applied to many other fields.

Considering quark models with non-relativistic constituent quarks and pairwise interaction, the baryon spectrum is obtained by a Hamiltonian of the type

$$H_3 = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \frac{\mathbf{p}_3^2}{2m_3} + V(r_{12}) + V(r_{23}) + V(r_{31}) \quad (1.1)$$

whereas the $(Q_1\bar{Q}_2)$ meson spectrum is obtained from

$$H_2 = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V(r_{12}) \quad (1.2)$$

Within the "colour assumption" $v = 2V$, it was shown that, for equal quark masses m [1]

$$E_3(m; V) > \frac{3}{2} E_2(m; 2V) \quad (1.3)$$

where E_2 and E_3 are the ground-state energies of H_2 and H_3 , respectively. It has been made plausible [2] that inequality (1.3) might follow from QCD for baryons of spin $\frac{3}{2}$ and mesons of spin 1. For instance, one has:

$$M_\Delta = 1238 MeV > \frac{3}{2} M_\rho = \frac{3}{2} M_\omega = 1155 MeV \quad (1.4)$$

A generalization to unequal quark masses can be established [2,3], namely (still for $v = 2V$)

$$2E_3(m_1, m_2, m_3) > E_2(m_1, m_2) + E_2(m_2, m_3) + E_2(m_3, m_1) \quad (1.5)$$

Note that flavour independence is not strictly necessary so far. The above inequality holds for

$$\begin{aligned} H_2(m_1, m_2) &= \Sigma \frac{\mathbf{p}_i^2}{2m_i} + v_{ij}(r) \\ H_3(m_1, m_2, m_3) &= \Sigma \frac{\mathbf{p}_i^2}{2m_i} + \frac{1}{2} \sum_{i < j} v_{ij}(r_{ij}) \end{aligned} \quad (1.6)$$

Obviously, one can apply the inequality (1.5) for vector mesons and spin $\frac{3}{2}$ baryons, v being the potential between parallel spin states. The spin $\frac{1}{2}$ case was treated in Ref. [4], assuming a quarkonium potential which includes a central and a spin-spin piece,

$$V_{ij}(r) = u_{ij}(r) + \sigma_i \sigma_j w_{ij}(r) \quad (1.7)$$

and the above " $\frac{1}{2}$ rule" for relating baryons to mesons. Then, if one computes the (Qqq) masses of Λ type, with the qq pair in a spin 0 state, or of Σ type, where qq has spin 1, one obtains

$$\begin{aligned} m(\Lambda_Q) &> \frac{1}{2}(qq)_{s=0} + \frac{1}{4}(Qq)_{s=0} + \frac{3}{4}(Qq)_{s=1} \\ m(\Sigma_Q) &> \frac{1}{2}(qq)_{s=0} + \frac{3}{4}(Qq)_{s=0} + \frac{1}{4}(Qq)_{s=1} \end{aligned} \quad (1.8)$$

Now, the above limits on baryon masses are rather crude. For instance, in the equal mass case $m_i = 1$, using a quasi-logarithmic potential $v = r^{0.1}$, which is quite successful phenomenologically [5,6] one gets $E_3 > 1.854$ whereas the exact value is 1.880. To appreciate the crudeness of this lower limit, one may compare it to the upper limit $E_3 < 1.883$ one obtains by a simple variational calculation using a gaussian wave-function.

The reason for the discrepancy is simple. To derive (1.3) or (1.4) one writes the variational principle for each meson in terms of the baryon wave function ϕ ,

$$E_2(m_1, m_2) \int d^3r_1 d^3r_2 |\phi(r_1, r_2, r_3)|^2 < \int d^3r_1 d^3r_2 \phi^*(r_1, r_2, r_3) H_2 \phi(r_1, r_2, r_3) \quad (1.9)$$

for each r_3 , with subsequent integration on r_3 . However, the use of ϕ as a two-body trial wave-function is rather poor. Indeed, ϕ is a three-body centre-of-mass wave-function, hence

$\langle \phi | H_2 | \phi \rangle$ implicitly contains a kinetic part associated to the relative motion of the two-body system with respect to the third particle. As we show in the next section, this irrelevant kinetic energy can be explicitly removed, and this operation entails a dramatic improvement of the resulting lower bound. Technically, this amounts to replacing in the inequality observable meson masses by the binding energies of two-body quark systems where the quark masses are lighter than the actual ones. We explain in Section 3 how to obtain these two-body energies by interpolating between physical meson masses. Finally, we discuss in Section 4 some generalizations for unequal quark masses and, in Section 5, for excited states. We end by some concluding remarks.

2. Improved Limit on Ground State Baryons and N-Body Generalizations

As in the previous paper [7], we rewrite the kinetic energy of each Hamiltonian in order to separate the collective from the relative motion for each quark pair. Actually, it is as easy to write down the general case of N -body as the three-body case, so that we shall treat it right away. We call

$$E_N(M, V) \quad (2.1)$$

the binding energy of a system of N particles of mass M interacting by a two-body potential $V(r_{ij})$ for each ij pair. In the special case $N = 2$, $E_2(M, V)$ represents the binding energy of two particles of mass M , i.e., reduced mass $M/2$. E_N is the lowest energy of the Hamiltonian

$$H_N = \frac{1}{2M} \sum_i \mathbf{p}_i^2 + \sum_{i < j} V(r_{ij}) \quad (2.2)$$

for which we use the identity

$$\sum_{i=1}^N \mathbf{p}_i^2 = \frac{1}{N} \left[\sum_i (\mathbf{p}_i - \mathbf{p}_j)^2 + \left(\sum_i \mathbf{p}_i \right)^2 \right] \quad (2.3)$$

We can disregard $\sum \mathbf{p}_i$, whose expectation value vanishes for a system at rest and notice that

the conjugate momentum to the relative distance $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ is

$$\pi_{ij} = \frac{\mathbf{p}_i - \mathbf{p}_j}{2} \quad (2.4)$$

Hence

$$H_N = \sum_{i < j} \left[\frac{4\pi_{ij}^2}{2MN} + V(r_{ij}) \right] \quad (2.5)$$

Then, taking as in the previous section the expectation of H_N in the ground state ϕ_N of the whole system, we get

$$E_N(M, V) \geq \frac{N(N-1)}{2} E_2\left(\frac{NM}{2}, V\right) \quad (2.6)$$

The generalization to different two-body potentials is obvious

$$E_N(M, V_{ij}) \geq \sum_{i < j} E_2\left(\frac{NM}{2}, V_{ij}\right) \quad (2.7)$$

In the special case of the baryon to meson comparison, we get

$$E_3(M, V) \geq 3E_2\left(\frac{3M}{2}, V\right) = \frac{3}{2}E_2\left(\frac{3M}{4}, 2V\right) \quad (2.8)$$

which constitutes clearly an improvement over inequality (1.3) because the masses of the constituents are decreased by a factor $\frac{3}{4}$ and therefore the energy is algebraically increased. To see how big is the effect, we can take the special case of power-law potentials $\epsilon(\beta)r^\beta$, for which the lower limit $\frac{3}{2}E_2$ (positive for $\beta > 0$, negative for $\beta < 0$) is multiplied by $(\frac{4}{3})^{\frac{\beta}{\beta+2}}$. The numerical comparison is shown in Table 1, where we list the naive lower limit (1.3), the improved lower limit (2.7), the exact binding energy E_3 obtained by an elaborate hyperspherical expansion [6] and a simple upper bound derived from the variational ansatz $\psi = \alpha x p - \alpha(r_{12}^2 + r_{23}^2 + r_{31}^2)$.

It is worth noticing that the new lower limit (2.7) becomes exact in the case of the harmonic oscillator. As pointed out by T.T. Wu [8], this result is true for general N and the harmonic oscillator is the only potential for which inequality (2.7) becomes an equality. This is shown in the Appendix.

Naturally the equality also holds to lowest order in perturbation for potentials

$$V = ar^2 + b + \lambda v \quad (2.9)$$

As a consequence, the variational approximations for $E_3(M, V)$ and $E_2(\frac{3M}{4}, 2V)$ with gaussian trial wave functions satisfies the equality. For instance, for $v = 2V = \epsilon(\beta)\rho^\beta$, one obtains

$$\tilde{E}_3(M, V) = \frac{3}{2}\tilde{E}_2(\frac{3M}{4}, 2V) = \frac{3\alpha_0}{m}(1 + \frac{2}{\beta}) \quad (2.10)$$

with

$$\alpha_0 = \left[\frac{m|\beta|\Gamma(\frac{3}{2} + \frac{\beta}{2})}{4\Gamma(\frac{3}{2})} \right]^{\frac{2}{\beta+2}} \quad (2.11)$$

Within this gaussian approximation, the r.m.s. separation of the pair ij , $d_{ij} = \sqrt{\langle r_{ij}^2 \rangle}$ experiences the same value in the baryon and in the pseudo-meson with quark mass $\frac{3M}{4}$. This is of course exact for $V = \mu r^2$ and can only be approximate for a different potential. We compare in Table 2 the r.m.s. separation in the pseudo-meson and in the baryon. For $\beta \neq 2$, the quarks are more tightly connected in the baryon. The reason is the following. If one starts from the gaussian approximation (2.9) which corresponds to rewriting the potential as in (2.8), and treats the anharmonicity λv in perturbation, the pseudo-mesons, in comparison to baryons, receive larger contributions from higher states of the harmonic oscillator. This results into larger shifts, hence into the inequality (2.7), and also into larger radii.

3. Connection to Physical Mesons

Here we keep the “ $\frac{1}{2}$ rule”, connecting two-body potentials in baryons and mesons

$$V_{QQ} = \frac{1}{2}V_{QQ} \quad (3.1)$$

but we try to connect baryon masses to physical meson masses without using a specific potential. For simplicity we shall ignore spin effects. However, a pure spin-spin force could be included in our considerations, since it leaves the orbital momentum ℓ as a good quantum number and does not introduce ℓ dependence beside the centrifugal barrier in the radial equations.

The problem is that in the new inequality (2.7), we use the unphysical quark mass $\frac{3M}{4}$. However, we notice that, by the Feynman-Hellmann theorem

$$\frac{dE(m, 2V)}{dm} = -\frac{T(m)}{m} \quad (3.2)$$

where T is the expectation value of the kinetic energy in the ground state, and hence

$$E_2(\frac{3M}{4}, 2V) = E_2(M, 2V) + \int_{\frac{3M}{4}}^M T(m, 2V) \frac{dm}{m} \quad (3.3)$$

There is an inequality on $T(m)$ by Berthmann and one of us [9]

$$T(m) \geq \frac{3}{4}[E_2(\ell = 1, m) - E_2(\ell = 0, m)] \quad (3.4)$$

where the energies appearing in the right-hand side are those of ground state for angular momentum $\ell = 1$ and $\ell = 0$, respectively, in particular $E_2(m) \equiv E_2(\ell = 0, m)$.

Furthermore, if we impose the mild restriction that the potential satisfies

$$\frac{dV}{dr} > 0 \quad \text{and} \quad \Delta V \geq 0 \quad (3.5)$$

which is an expression of asymptotic freedom (the colour charge seen increases with distance)

and is valid for all existing models, phenomenological or QCD motivated, one can prove [9]:

$$\frac{d}{dm} \frac{T(m)}{m} < 0 \quad (3.6)$$

Putting (2.7), (3.3), (3.4) and (3.6) together results into

$$E_3(M, 1^+) > \frac{3}{2} [E_2(M, 2, 1^+, \ell = 0) + \frac{3}{16} (E_2(M, 2, 1^+, \ell = 1) - E_2(M, 2, 1^+, \ell = 0))] \quad (3.7)$$

This inequality applies to binding energies as well as to hadron masses since one can add to both sides three times the quark constituent mass M .

For applications the problem is that we cannot ignore spin completely. If the spin-spin potential is regularized, Eqs. (3.2) - (3.4) can be generalized for a spin-triplet state. To eliminate more or less the tensor and spin-orbit effects, one should average over the three spin-triplet P -states. Utilisation of (3.6) is more questionable. However, the range of integration in (3.3) is only a quarter of the mass, so that the uncertainty on $\frac{T(m)}{m}$ is not very large.

We can try to apply this to the $\rho - \Delta$ comparison. If we follow the prescriptions of the Particle Data Group [10], the $J^G = 1^+ \text{ triplet } P\text{-states } a_J$ are:

$$a_0(980 MeV), a_1(1270 MeV), a_2(1320 MeV) \quad (3.8)$$

This gives an average mass of 1265 MeV and leads, with $m_\rho = 770 MeV$, to

$$M_\Delta > 1294 MeV \quad (3.9)$$

We overshoot a little bit since $M_\Delta = 1238 MeV$ [10], but this was expected because of the crude treatment of spin effects, the neglect of the coupling to decay channels, etc...

If we consider the centre of gravities of the hyperfine multiplets and incorporate the $b(1233)(1^{+-})$ and the $\rho - \pi$ c.o.g. (assuming that the hyperfine splittings are in the ratio 1 : -3 we obtain

$$\frac{M_\Delta + M_N}{2} (= +1087) \geq 1096 \quad (3.10)$$

in very close agreement.

For the $\phi - \Omega$ comparison, we need the ss P -state average energy and here, we have only two candidates [10], $f_1(1420 MeV)$ which is $J^{PC} = 1^{++}$ and $f'_2(1525)$, $J^{PC} = 2^{++}$. Let us assume that they are not too far from the centre of gravity. Then we get, with $M_\phi = 1020 MeV$

$$M_{\Omega^-} > 1659 MeV \quad (3.11)$$

in agreement with the experimental $M_{\Omega^-} = 1672 MeV$ [10].

We can also make predictions for the masses of the ccc and bbb baryons. With $M_{J/\psi} = 3095 MeV$ and $M_{\chi}(triplet) = 3520 MeV$, we obtain

$$M_{ccc} > 4762 MeV \quad (3.12)$$

and, with $M_\tau = 9460 MeV$ and $M_{\chi}(triplet) = 9900 MeV$,

$$M_{bbb} > 14314 MeV \quad (3.13)$$

The observability of the ccc baryon is considered as possible by Bjorken [11], and would be very interesting [12].

4. The Case of Unequal Constituent Masses

The extension of inequality (2.7) to baryons bearing different flavours is straightforward, but its practical consequences deserve some discussion.

One easily rewrites the kinetic energy as

$$\frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \frac{\mathbf{p}_3^2}{2m_3} + = \frac{1}{2} \frac{(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)^2}{m_1 + m_2 + m_3} + \frac{1}{2} \sum_{i < j} \frac{(m_j \mathbf{p}_i - m_i \mathbf{p}_j)^2}{(m_i + m_2 + m_3)m_i m_j} \quad (4.1)$$

Thus if one introduces

$$\mathbf{p}_{ij} = \frac{m_j \mathbf{p}_i - m_i \mathbf{p}_j}{m_i + m_j} \quad (4.2)$$

$$\mu_{ij} = \frac{2m_i m_j}{m_i + m_j}$$

$$\tilde{\mu}_{ij} = \mu_{ij} \frac{m_1 + m_2 + m_3}{m_i + m_j}$$

one can rewrite the reduced Hamiltonian of the baryon as

$$h_3 = \frac{1}{2} \sum_{i < j} \left[\frac{\mathbf{p}_{ij}^2}{\tilde{\mu}_{ij}} + v(r_{ij}) \right] \quad (4.3)$$

leading to the lower limit

$$E_3(m_1, m_2, m_3; \frac{v}{2}) \geq \frac{1}{2} \sum_{i < j} E_2(\tilde{\mu}_{ij}, v) \quad (4.4)$$

which usually corresponds to a very good approximation. There are, however, some new features:

1. the limit is not saturated anymore in the harmonic oscillator case $\sum \mathbf{r}_{ij}^2$ with equal strengths, but it is for strengths proportional to the product of the masses, $\sum m_i m_j \mathbf{r}_{ij}^2$, as shown in the Appendix;
2. the new limit (4.4) is not always better than the naive limit (1.5).

Let us discuss this latter point in some detail, by considering two different masses, say $m_1 = m_2 = M$ and $m_3 = M$. We have to compare the two inequalities

$$E_3(m, m, M; \frac{v}{2}) > \frac{1}{2} E_2(m, v) + E_2(\frac{2mM}{m+M}, v) \quad (4.5)$$

$$E_3(m, m, M; \frac{v}{2}) > \frac{1}{2} E_2(\frac{m}{2} + \frac{M}{4}, v) + E_2(\frac{mM(M+2m)}{(M+m)^2}, v) \quad (4.6)$$

For $M < 2m$, the constituent masses in the new inequality (4.6) are always lighter than in (4.5)

, resulting into a better limit. For very large M , the above inequalities give respectively

$$E_3 > \frac{1}{2} E_2(m) + E_2(2m) \quad (4.7)$$

$$E_3 > \frac{1}{2} E_2(\infty) + E_2(m)$$

and the former is larger than the latter, because the ground-state energy E_2 in a concave function of the inverse reduced mass [9,13]. For a power-law potential, r^β , the limits (4.5) and (4.6) are proportional to

$$A = \frac{1}{2} + \left(\frac{2x}{1+x} \right)^\gamma \quad (4.8)$$

$$B = \frac{1}{2} \left(\frac{1}{2} + \frac{x}{4} \right)^\gamma + \left[\frac{x(2+x)}{(1+x)^2} \right]^\gamma$$

respectively, where $\gamma = -\frac{\beta}{\beta+2}$ and $x = \frac{M}{m}$. For $x \geq 8.4$ in the Coulomb case $\beta = -1$, for $x \geq 12$ in the case of a smooth $r^{0.1}$ confinement, or for $x \geq 17.7$ in the harmonic oscillator case, one has $A > B$, so that the old limit is better than the new one.

To illustrate the inequality (4.4), we choose in Table 3 several power-law potentials and the sets of constituent masses $(m_1, m_2, m_3) = (1, 1, 0.2)$ and $(1, 1, 5)$ which are representative of the mass ratios involved in double-charm or single-charm baryons.

5. Generalization to Excited States

Extending the inequalities to excited states, or to the sum of the first energies does not appear as an easy task. The "minimax" principle [14] is not of immediate use here: for fixed $\mathbf{r}_3$, the wave function of a ground-state baryon, $\phi_0(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ and one excited state, $\phi_1(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ are not orthogonal with respect to integration over r_1 and r_2 . For very special cases (harmonic oscillator, for instance) and in particular values of $\mathbf{r}_3$, ϕ_0 and ϕ_1 are not even linearly independent.

If one considers, however, our inequality (2.4) as relating the first symmetric level of the baryon spectrum and the first even level of the pseudomeson with quark mass $\frac{3m}{4}$, then a similar

inequality holds between the first baryon with antisymmetric (A) spatial wave function and the first odd level of the pseudoscalar, i.e.,

$$E_A(m, V, A) \geq \frac{3}{2} E_2(\frac{3}{4}m, 2V, \ell=1) \quad (5.1)$$

In the quark model this (A) state exists only for u and d quarks. A spin $\frac{1}{2}$ and an isospin- $\frac{1}{2}$ wave-function, with mixed permutation symmetry can be arranged in an antisymmetric spin-isospin wave-function, which, in turn, can be combined to the colour wave-function and this (A) spatial wave-function in order to fulfil the Pauli principle. In the harmonic oscillator model, this state with spatial wave-function

$$\psi_A = (\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1) \exp -\alpha(r_1^2 + r_2^2 + r_3^2) \quad (5.2)$$

is usually referred to by specialists as the $[20, 1^+]$, $N=2$ state [15]. With anharmonic confinement, this (A) state occurs at the top of the multiplet of positive parity excitations [15].

6. Conclusions

We started our investigations by a simple improvement of the crude lower limit on baryon masses proposed in Refs. [1-3]. We discovered that the new lower limit provides in fact a very good approximation of the three-body binding energy in terms of simple two-body binding energies.

So far, the applications were restricted to self-gravitating bosons [7] and to baryon spectroscopy. It is very tempting to use our inequalities in other fields. We first consider the $({}^4He)$ trimer, using the so-called $H-D-2$ potential [16] which reads

$$\begin{aligned} r \leq r_2 \quad V &= \epsilon(\exp 2\epsilon(1 - \frac{r}{r_1}) - 2 \exp \epsilon(1 - \frac{r}{r_1})) \\ r > r_2 \quad V &= \frac{A}{r^6} + \frac{B}{r^8} \end{aligned} \quad (6.1)$$

V and its derivative being continuous at $r = r_2$. The parameters, in the usual units of the field,

are

$$\begin{aligned} \epsilon &= 1.484 \times 10^{-15} \text{ erg} = 10.748 \text{ K} \\ r_1 &= 3.0238 \bar{A} \quad r_2 = 3.68280 \bar{A} \\ c &= 6.12777 \quad \frac{\hbar^2}{M} = 12.02 K \bar{A}^2 \end{aligned} \quad (6.2)$$

The three-body problem, when solved via the ATMS method [17], provides a binding $E_3 = -0.105 \text{ K}$ [16]. Our inequality (2.7) gives a lower limit $E_3 > -0.87 \text{ K}$. Such a discrepancy is not too surprising. Due to the very hard core of radius $3\bar{A}$ in the potential (6.1) both two-body and three-body wave-functions differ appreciably from the gaussian approximation for which the inequality (2.7) would become an equality.

We anticipate that a similar effect should affect the bounds derived for the 3H or 3He nuclei in term of two-body binding energies. As already stressed, flavour-independence is not required for our inequality. So, schematically,

$$E_3({}^3H) > \frac{1}{2} E_2(\frac{3}{4}m_N, 2V_{pp}) + E_2(\frac{3}{4}m_N, 2V_{np}) \quad (6.3)$$

This is currently under study, using a variety of realistic nucleon-nucleon potentials [18].

In atomic physics, we are dealing with even less symmetric three-body systems, with a mixture of Coulomb attraction and repulsion: while opposite charges lead to cluster, charges of the same sign are anticorrelated, and our quasi-two-body approximation is very crude. Our inequality (4.4) indicates that there is room for binding, for $e^+e^-e^-$ or pe^+e^- , which are indeed bounded, but also for pe^+e^- which is not.

Actually, for the positronium ion, it gives very poor results:

$$E(e^+e^-e^-) > -\frac{3}{4}m\alpha^2 \quad (6.4)$$

while it is known that the binding energy is only a fraction of electron-Volt (0.6 eV) below the

positronium binding energy [19], which is

$$E(e^+e^-) = -\frac{1}{4}m\alpha^2 = -6.8eV \quad (6.5)$$

A slightly better bound is obtained by splitting the Hamiltonian in two pieces only and disregarding the repulsion between the electrons:

$$E(e^+e^-e^-) > -\frac{2}{3}m\alpha^2 \quad (6.6)$$

In short, we have no predictive power for these delicate systems. However, the product of quasi-two-body wave-functions corresponding to our lower bounds may provide an improved starting point for algorithms where the three-body wave-function is constructed iteratively [17].

We remark that the inequality (2.6) can be extended to larger sub-clusters. The reduced N body Hamiltonian H_N (2.5) can be rewritten as

$$H_N(M, V) = \frac{1}{N-2} \sum_{i=1}^N H_{N-1}^{(i)}(M \frac{N}{N-1}, V) \quad (6.7)$$

where the $(N-1)$ body Hamiltonian $H_{N-1}^{(i)}$ does not contain the i th constituent. Then

$$E_N(M, V) \geq \frac{N}{N-2} E_{N-1}(M \frac{N}{N-1}, V) \quad (6.8)$$

relating the N body to the $(N-1)$ body binding energy. One can iterate this equation and, in particular, recover the inequality (2.6) relating the N body binding energy to two-body energies. The relation between the N -body energy E_N and the k -body subenergy is:

$$E_N(M, V) \geq \frac{N(N-1)}{k(k-1)} E_k(M \frac{N}{k}, V) \quad (6.9)$$

If we apply this to the $(e^+e^+e^-e^-)$ system, considering that all **three-body** subsystems are bound (with a binding energy of $\sim 7.4eV$), and taking into account the scaling emerging from

the Coulomb interaction (the hyperfine interaction energy is negligible $\sim 10^{-3}eV$), one obtains

$$E(e^+e^+e^-e^-) \geq \frac{8}{3} E(e^+e^-e^-) \quad (6.10)$$

that is to say

$$E(e^+e^+e^-e^-) \geq (-13.6 - 6.2)eV \quad (6.11)$$

where we have split off twice the positronium binding energy. Since the bound (6.11) is noticeably lower than twice the positronium binding energy, and since our up-to-date calculations have shown that the lower bound is within 20% of the exact result, the numerical value (6.11) is very strongly suggestive that the $(e^+e^+e^-e^-)$ ground state is **stable** against a decay into two positronium systems.

Finally, let us mention that in the case of **unequal masses** it is easy to show that the N -body centre-of-mass kinetic energy can be expressed in terms of the k -body centre-of-mass kinetic energies each of which being affected with a coefficient m_k/M , m_k being the total mass of the k -body subsystem and $M = \sum m_i$ the total N -body mass. Therefore, our results can be extended with no difficulty to the unequal mass case. In the previous paper [7], three-body clusters have been used to improve the lower bound on the energy of an assembly of self-gravitating bosons, but other applications are possible.

APPENDIX A

We present here the proof by T.T. Wu that the only harmonic oscillator potentials saturate inequality (2.7). It is followed by a more pedestrian proof, based on more conventional Jacobi co-ordinates and Brody-Moshinsky coefficients.

The ground state energy $E_N(M, V)$ of the N -body harmonic oscillator potential

$$V = k \sum_{i < j} r_{ij}^2 \quad (\text{A1})$$

satisfies

$$E_N(M, V) = \frac{N(N-1)}{2} E_2\left(\frac{NM}{2}, V\right) \quad (\text{A2})$$

If one considers, indeed, the N -body translation invariant gaussian wave-function

$$\psi_N = \exp -\alpha_N \sum_{i < j} r_{ij}^2 \quad (\text{A3})$$

one finds

$$\frac{1}{2M} \sum_i \Delta_i \psi = -\frac{6\alpha_N N(N-1)}{2M} \psi + \frac{2\alpha_N^2 N}{M} \left(\sum_{i > j} r_{ij}^2 \right) \psi \quad (\text{A4})$$

so $\alpha_N = \sqrt{\frac{Mk}{2N}}$ and $E_N(M, V) = 3N(N-1) \sqrt{\frac{k}{2NM}}$, q.e.d.

We notice that (A3) can be rewritten as

$$\psi_N = \exp(\alpha_N \frac{N}{2} r_{12}^2) \exp -\alpha_N \left[\frac{N-2}{2} (r_1 + r_2) \bullet \sum_{i \geq 3} r_i + \sum_{3 \leq i < j} r_{ij}^2 \right] \quad (\text{A5})$$

i.e.,

$$\psi_N = \psi_2(r_1 - r_2) \phi(r_1 + r_2, r_3, \dots, r_N) \quad (\text{A6})$$

To show the uniqueness, we can restrict without loss of generality to **one** space dimension. If inequality (2.7) is to be saturated, it means that if we look at a given two-body Hamiltonian

in the sum (2.5), say for instance the pair (1, 2), it should act only on the relative co-ordinate $x_{12} = x_1 - x_2$. This means that it is possible to decompose the wave-function (in the equal mass case) as

$$\Psi(x_1, x_2, \dots, x_N) = \psi(x_1 - x_2) \phi(x_1 + x_2, x_3, \dots, x_N) \quad (\text{A7})$$

Since ψ is nodeless, being the ground-state, we can define

$$f(x_1, x_2, \dots, x_N) = \log \psi(x_1, x_2, \dots, x_N) \quad (\text{A8})$$

Hence

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) f = 2 \frac{\partial}{\partial x_1} \log \psi(x_1 - x_2) \quad (\text{A9})$$

and this implies

$$\frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) f = 0, \quad i = 3, \dots, N \quad (\text{A10})$$

and

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) f = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) f = 0 \quad (\text{A11})$$

Let

$$f_i = \frac{\partial f}{\partial x_i} \quad (\text{A12})$$

we have, from (A10)

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_i}{\partial x_3} = \dots = \frac{\partial f_1}{\partial x_N} \quad (\text{A13})$$

if we introduce the variable

$$S = x_1 + x_2 + x_3 + \dots + x_N \quad (\text{A14})$$

We deduce from (A13) that f_i is only a function of x_1 and S :

$$f_i(x_1, x_2, \dots, x_N) = \phi_i(x_1, S) \quad (\text{A15})$$

Since $\frac{\partial L}{\partial x_2} = \frac{\partial L}{\partial x_1}$, we have

$$\frac{d}{dS}\phi_1(x_1, S) = \frac{d}{dS}\phi_2(x_2, S) \quad (\text{A16})$$

Integrating over S (S can be taken as an independent variable for $N \geq 3$!) we get

$$\phi(x_1, S) = K_1(x_1) = \phi_2(x_2, S) = K_2(x_2) \quad (\text{A17})$$

which implies that $\phi_1(x_1, S) = K_1(x_1)$ is only a function of S since the right-hand side of (A17) is independent of x_1 . Therefore,

$$f(x_1, x_2, \dots, x_N) = K_1(x_1) + \psi_1(S) \quad (\text{A18})$$

Integrating over x_1 we get

$$f(x_1, x_2, \dots, x_N) = h(x_2, x_3, \dots, x_N) + g_1(x_1) + X(S) \quad (\text{A19})$$

However, one can replace x_1 by any other x_i . Then the conclusion is that f has the form

$$f = g_1(x_1) + g_2(x_2) + \dots + g_N(x_N) + X(S) \quad (\text{A20})$$

If we use now (A11), we get $g_1'' = g_2'' = \dots = g_N'' = C$.

Bose symmetry imposes

$$f = \lambda \Sigma x_i^2 + X(\Sigma x_i),$$

but the requirement of translation invariance of the wave function gives

$$f = -\lambda \left[\sum_{i=1}^N x_i^2 - \frac{1}{N} (\Sigma x_i)^2 \right] \quad (\text{A21})$$

This agrees with the desired structure (A7). It means that the only possible two-body interaction saturating (2.7) is our r^2 potential.

In the unequal mass case the requirement that the two-body Hamiltonian acts only on the relative co-ordinate of the two-body system is

$$\Psi = \psi(x_1 - x_2) \phi(m_1 x_1 + m_2 x_2, x_3, \dots, x_N) \quad (\text{A22})$$

and the equivalent of (A9) becomes

$$\left(\frac{1}{m_2} \frac{\partial}{\partial x_1} - \frac{1}{m_2} \frac{\partial}{\partial x_2} \right) f = \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \frac{\partial}{\partial x_1} \log \psi(x_1 - x_2) \quad (\text{A23})$$

while (A11) becomes

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \left(\frac{1}{m_1} \frac{\partial}{\partial x_1} - \frac{1}{m_1} \frac{\partial}{\partial x_2} \right) f = 0 \quad (\text{A24})$$

There is essentially no change in the argument and we get

$$f = -\lambda [\Sigma m_i \Sigma m_j x_i^2 - (\Sigma m_i x_i)^2] \quad (\text{A25})$$

When one calculates from f the corresponding potential, one finds that the two-body interaction is of the form

$$\text{const} \times m_i m_j (r_{ij})^2, \quad (\text{A26})$$

which leads to saturation of the inequality (generalized to unequal couplings) but explains also why it cannot be saturated in the case of equal coupling for all pairs.

An alternative approach is the following. Let us introduce the $(N-1)$ relative Jacobi variables

$$y_1 = r_1 - r_2, \quad y_2 = \frac{2r_3 - r_1 - r_2}{\sqrt{3}}, \quad (\text{A27})$$

$$y_3 = \frac{3r_4 - r_1 - r_2 - r_3}{\sqrt{12}},$$

and their conjugate momenta $q_1 = \frac{p_1 - p_2}{2}$, etc. Disregarding the centre-of-mass motion, the

harmonic oscillator Hamiltonian can be rewritten as

$$H_N = \sum_{i=1}^{N-1} \frac{\mathbf{q}_i^2}{m} + \frac{Nk}{2} \sum_{i=1}^{N-1} y_i^2 \quad (\text{A28})$$

so that one recovers the ground-state energy $E_N = 3(N-1)\sqrt{\frac{Nk}{2m}}$ and the equality (A2).

Let us suppose now that a two-body potential $V(1)$ saturates the inequality (2.7). As in Eq. (2.8), we separate explicitly the optimal harmonic approximation to V . Thus the potential

$$V = ar^2 + b + \lambda v \quad (\text{A29})$$

should saturate (2.7) to lowest order in λ .

A typical second-order contribution to the energy shift of the two-body subsystem

$$h_{12} = \frac{\mathbf{p}_{12}^2}{\left(\frac{mN}{2}\right)} + a\mathbf{r}_{12}^2 + b + v(r_{12}) \quad (\text{A30})$$

is

$$\delta E_2 = \langle \phi_0 | v | \phi_1 \rangle = - \sum_{n=2}^{\infty} \frac{|\langle 0 | v | n \rangle|^2}{4n\sqrt{\frac{2a}{mN}}} \quad (\text{A31})$$

where $|\phi_0\rangle = |0\rangle$ in the zeroth order wave-function, $|\phi_1\rangle$ is its first-order correction, expanded in terms of excited oscillator states $|n\rangle$, n being the number of nodes. When one starts from variational equilibrium, there is no $n=1$ contribution [19].

The corresponding shift for

$$H_N = \sum_{i=1}^{N-1} \frac{\mathbf{q}_i^2}{m} + \frac{aN}{2} \sum_{i=1}^{N-1} y_i^2 + \frac{N(N-1)}{2} b + \lambda \sum_{i<j} v(r_{ij}) \quad (\text{A32})$$

is

$$\delta E_N = \sum_{i<j} \sum_{k<l} \langle \phi_0 | v(r_{ij}) | \psi_1^{kl} \rangle \quad (\text{A33})$$

or

$$\delta E_N = \frac{N(N-1)}{2} \sum_{k<l} \langle \psi_0 | v(r_{12}) | \psi_1^{kl} \rangle \quad (\text{A34})$$

Here, the zeroth-order wave-function is a product of $(N-1)$ gaussians

$$|\psi_0\rangle = |0\rangle \otimes |0\rangle \otimes \dots |0\rangle = \psi_0(y_1, \psi_0(y_2) \dots \quad (\text{A35})$$

Its first-order correction results from the independent contribution of each $v(r_{kl})$ term

$$|\psi_1\rangle = \sum_{k<l} |\psi_1^{kl}\rangle = \sum_{k<l} \frac{\langle n | v | 0 \rangle}{4n\sqrt{\frac{aN}{2m}}} |n^{kl}\rangle \quad (\text{A36})$$

The (12) pair induces admixture of states of the type

$$|n^{12}\rangle = |n\rangle \otimes |0\rangle \otimes \dots |0\rangle \otimes \dots |0\rangle \quad (\text{A37})$$

States $|n^{kl}\rangle$ are defined similarly by choosing r_{kl} as the first Jacobi variable and are induced in the wave-function by the $v(r_{kl})$ piece of the perturbation. $|n^{kl}\rangle$ can be extended in the original basis adapted to our choice (A27) of relative variables, say

$$|n^{kl}\rangle = a_{kl} |n\rangle \otimes |0\rangle \otimes \dots |0\rangle \otimes b_{kl} |n \dots 1\rangle \otimes |1\rangle \otimes \dots + \dots \quad (\text{A38})$$

or

$$|n^{kl}\rangle = a_{kl} |n^{12}\rangle + \dots \quad (\text{A39})$$

since obviously, only the first term will contribute to (A34). Here, it is not necessary to know the full machinery of the Brody-Moshinsky coefficients [20]. Obviously, $a_{ij} = 0$ if the (ij) pair

does not share any indice with (1, 2). The calculation of, say, d_{13} , is easy. One writes

$$r_{13} = r_1 - r_3 = \frac{1}{2}(r_1 - r_2) - \frac{\sqrt{3}(2r_3 - r_1 - r_2)}{2\sqrt{3}} \quad (\text{A40})$$

So, thanks to the simple polynomial structure of the harmonic oscillator wave-functions, $|n^3\rangle$ has a component $\frac{1}{2^n}$ on $|n^2\rangle$, i.e., $a_{13} = \frac{1}{2^n}$. We thus obtain

$$\delta E_N = -\frac{N(N-1)}{2} \sum_{n=2}^{\infty} \frac{|\langle 0 | v | n \rangle|^2}{4n\sqrt{\frac{2n}{mN}}} \frac{2}{N} [1 + (2N-4)\frac{1}{2^n}] \quad (\text{A41})$$

where the contribution of $v(r_{12})$ to (A34) is explicitly separated from that of the $(2N-4)$ neighbouring ones.

As expected,

$$|\delta E_N| \leq \frac{N(N-1)}{2} |\delta E_2| \quad (\text{A42})$$

and one gets equality only if $\langle 0 | v | n \rangle = 0 \forall n$, i.e., if V is harmonic, q.e.d.

β	$\frac{3}{2}E_2(1)$	$\frac{3}{2}E_2(\frac{3}{4})$	E_3	$\tilde{E}_3$
-1	-0.37500	-0.28125	-0.26675	-0.23873
-0.5	-0.65759	-0.59746	-0.59173	-0.57964
0.1	1.85359	1.87916	1.88019	1.88278
0.5	2.75009	2.91296	2.91654	2.92590
1	3.50716	3.86013	3.86309	3.87114
2	4.5	5.19615	5.19615	5.19615
3	5.17584	6.15098	6.15591	6.17147

TABLE 1

Ground state energy E_3 of $H_3 = \sum \frac{E_i}{2} + \frac{1}{2}\epsilon(\beta) \sum r_{ij}^\beta$ compared to the naive limit $\frac{3}{2}E_2(1, r^\beta)$, the improved lower limit $\frac{3}{2}E_2(\frac{3}{4}, r^\beta)$ and a simple variational approximation $\tilde{E}_3$ obtained with a gaussian wave function.

β	d_2	d_3
-1	4.6188	4.5077
-0.5	5.0905	5.0008
0.1	7.2883	7.2261
0.5	2.8182	2.8051
1	1.8794	1.8761
2	1.3161	1.3161
3	1.1069	1.1059

TABLE 2

Numerical comparison of the r.m.s. interquark distance d_3 , in the baryon governed by $h_3 = \sum \frac{E_i}{2} + \frac{1}{2} \sum_{i=j} \epsilon(\beta) r_{ij}^\beta$ and d_2 , in the subsystem governed by $h_2 = \frac{2}{3} p_{ij}^2 + \epsilon(\beta) r^\beta$.

β	$m = 1, M = 0.2$			$m = 1, M = 5$		
	$\frac{1}{2}\Sigma E_2(\mu_{ij})$	$\frac{1}{2}\Sigma E_1(\bar{\mu}_{ij})$	"exact"	$\frac{1}{2}\Sigma E_2(\mu_{ij})$	$\frac{1}{2}\Sigma E_1(\bar{\mu}_{ij})$	"exact"
-1	-0.2083	-0.1451	-0.1398	-0.5117	-0.4618	-0.3848
0.1	1.9200	1.9432	1.9452	1.8239	1.8390	1.8486
1	4.5412	4.8982	4.9392	3.1411	3.3303	3.4379
2	6.6962	7.4498	7.5730	3.8238	4.1764	4.3729
3	8.3958	9.4978	9.7389	4.2712	4.7496	5.0166

TABLE 3

Ground-state energy of (Qqq) compared to the lower limits of Eqs. (1.4) and (4.4) for some power law potentials $2(\beta)r^{\beta}$. The constituent masses are $m(q) = 1$ and $M(Q) = 0.2$ or $M(Q) = 5$. The "exact" value of E_3 corresponds to an hyperspherical expansion up to a grand orbital momentum $L = 8$.

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