Characters of the $N=4$ superconformal algebra with two central extensions.

II Massless representations.

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ABSTRACT

We complete our programme of deriving characters of the $N = 4$ algebra $\mathcal{A}_\gamma$, containing two $\widehat{SU}(2)$ Kac-Moody subalgebras, by treating the complications arising for massless representations. We find it useful to work directly with the algebra $\tilde{\mathcal{A}}_\gamma$, $\mathcal{A}_\gamma$ being the direct sum of that and four free fermions plus a $\widehat{U}(1)$ KM. Only in $\tilde{\mathcal{A}}_\gamma$ do massless representations have non-vanishing Witten index. Several consistency checks on our explicit formulas are given.

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1 Introduction

In [1] (to be referred to in the sequel as (I)), we began an investigation of characters for the $N = 4$ superconformal algebra (SCA) $\mathcal{A}_r$ introduced in [2,3], and further studied in [4,5]. We restrict ourselves to unitary highest weight representations. These are known [5] to fall into two classes: the generic massive representations and the limiting massless representations, in analogy with the singly centrally extended $N = 4$ algebra [6] studied in [7,8,9] (see also next section). In (I), explicit formulas for characters in massive representations were derived.

In this paper, we concentrate on the massless representations for which several subtleties have to be confronted. A peculiar feature of the $\mathcal{A}_r$ algebra is that the Witten index vanishes also for massless representations. This is due to the fact that $\mathcal{A}_r$ is the direct sum of the $\tilde{\mathcal{A}}_r$ algebra (introduced in [10]) of the type first studied in [11], and further analyzed in [4,5,12] and in (I), four free fermions (two complex ones) and a $U(1)$ Kac-Moody (KM) current.

The simpler algebra $\tilde{\mathcal{A}}_r$, however, is complicated by the fact that the four supersymmetry generators do not anticommute into a linear combination of the other generators: also composite operators appear (see Appendix A). We find, however, that this complication is fully tractable and, as a side result, we rederive the characters for the massive representations directly within the context of the $\tilde{\mathcal{A}}_r$ algebra, confirming the results of (I) (and refuting a fear expressed there that this might be too complicated).

The characters of (I) were found to factorize into two characters of the ordinary $N = 4$ SCA as given by [7], and thus allowing modular properties to be taken over from known results as far as massive representations go. The characters presented below in the case of massless representations are found not to enjoy a similar factorization property, thereby necessitating a new analysis of modular properties.

In section 2, we set up some notations and discuss in detail the reduction of the zero-mode multiplet structure that takes place in the massless case (Ramond sector).

In section 3, we discuss the pattern of singular vectors in the Verma module with emphasis on the new features arising for massless representations.

In section 4, we derive fermionic identities satisfied by the singular vectors in the massless case. These imply that the submodules are built with one less fermionic operator than in the massive case, the operator depending (in a fairly complicated way) on the singular vector considered. The situation is analogous to what happened in the ordinary $N = 4$ algebra, but some crucial differences occur.

In section 5, we combine all results into the final explicit character formulas in the Ramond sector, and we use the spectral flow to derive the characters in the Neveu-Schwarz sector. Also we clarify how the character in a massive representation relates to the characters in massless representations when the conformal dimension tends to its limiting massless value.

In section 6, the limit $k^- \rightarrow \infty$ (for notations, see next section) is investigated and it is verified that the massless characters of the ordinary $N = 4$ algebra emerge as expected.

Section 7 summarizes our results and presents some outlook for further work.

2 Notations. Multiplet structure

In this paper, we concentrate on the algebra $\tilde{\mathcal{A}}_r$ [10,11] obtained from $\mathcal{A}_r$ by removing the four free fermions and the $U(1)$ KM current. We shall essentially employ the notations of [4,5]. Characters for $\mathcal{A}_r$ are readily obtained from those of $\tilde{\mathcal{A}}_r$ and vice-versa (see (1)).

For notational simplicity, in this paper we shall dispense with the $\sim$ -signs wherever it is not strictly necessary.

We may describe $\tilde{\mathcal{A}}_r$ as follows: it consists of

- (i) a Virasoro-algebra with generators L_n and central charge

$$\tilde{c} = c - 3 = \frac{6k^+k^-}{k} - 3, \quad k \equiv k^+ + k^-, \quad (2.1)$$

where k^+, k^- are positive integers (levels of two $SU(2)$ KM algebras in $\mathcal{A}_r$). We define $\tilde{k}^\pm \equiv k^\pm - 1$.

- (ii) two $SU(2)$ KM algebras with generators $T_n^{\pm i}$, T_n^{-i} , $i = +, -, 3$, $n \in \mathbb{Z}$ and with levels $\tilde{k}^\pm$. These transform as dimension 1 operators under L_n .

- (iii) four supersymmetry operators $(G_a)_\pm$, where $a = +, -, +K, -K$ and $r \in \mathbb{Z}$ in the Ramond (R) sector, $r \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz (NS) sector. The G 's have conformal dimension $\frac{3}{2}$, and they transform as two doublets under $SU(2)^+$ and $SU(2)^-$, schematically represented as

$$\begin{aligned} (G_{+K}, G_-)^i &, & (G_+, -G_{-K})^i & \text{ for } SU(2)^+ \\ (G_{-K}, G_-)^i &, & (G_+, -G_{+K})^i & \text{ for } SU(2)^- \end{aligned} \quad (2.2)$$

(see (I) and above-mentioned references for more details). The G 's anticommute to a combination of the energy-momentum tensor L_n , a central charge piece, the two KM currents $T_n^{\pm i}$ and some (quadratic) composite operators including (minus) the energy-momentum tensors built from $T_n^{\pm i}$ by the Sugawara construction. These commutation relations are reproduced in Appendix A.

In the NS sector, a highest weight representation (hwr) is built on a unique highest weight state (hws) with eigenvalues (h, ℓ^+, ℓ^-) of $(L_0, T_0^{+3}, T_0^{-3})$, and annihilated by all positive mode operators as well as by $T_0^{\pm i}$.

In the R sector, a hwr is built on a multiplet of such vectors, all with the same eigenvalue h of L_0 and related by the action of zero-modes of the G 's. The multiplet may be reduced into multiplets of the $(SU(2)^+, SU(2)^-)$ zero-modes $T_0^{\pm i}$. For massive representations, the result is shown in Fig.1. There are four states which are hws of $SU(2)^+, SU(2)^-$: they are annihilated by $T_0^{\pm i}$. The states $|\Omega_+>$ and $|\Omega_->$ further satisfy

$$\begin{aligned} (G_{+K})_0|\Omega_+> &= 0 = (G_{+K})_0|\Omega_-> \\ (G_{-K})_0|\Omega_+> &= 0 = (G_{-K})_0|\Omega_->. \end{aligned} \quad (2.3)$$

For massive multiplets, we use the notation that (ℓ^+, ℓ^-) denotes the largest $SU(2)^{\pm}$ quantum numbers within the multiplet (as in (I) for $\mathcal{A}_r$):

$$\begin{aligned} T_0^{+3}|\Omega_+> &= \ell^+|\Omega_+>, & T_0^{-3}|\Omega_-> &= \ell^-|\Omega_->, \\ T_0^{-3}|\Omega_+> &= (\ell^- - 1/2)|\Omega_+>, & T_0^{+3}|\Omega_-> &= (\ell^+ - 1/2)|\Omega_->. \end{aligned} \quad (2.4)$$

Also, up to a constant normalization,

$$|\Omega_-> \propto (G_{-K})_0|\Omega_+>, \quad |\Omega_+> \propto (G_{+K})_0|\Omega_->. \quad (2.5)$$

We refer to (I) for a similar but somewhat more complicated discussion of the zero-mode multiplet for $\mathcal{A}_r$. Obviously, the Witten index, $Tr(-)^F$, vanishes for this massive case, using $\{(G_a)_\pm, (-)^F\} = 0 = [L_n, (-)^F] = [T_n^{\pm i}, (-)^F]$.

In the massive case, we have [5] the following bounds for h . For the NS sector,

$$kh > (\ell^+ - \ell^-)^2 + k^+ \ell^- + k^- \ell^+, \quad (2.6)$$

and for the R sector

$$\begin{aligned} kh - \frac{1}{4} \bar{k}^+ \bar{k}^- &> (\ell_+^+ + \ell_+^-)(\ell_+^+ + \ell_+^- + 1) = (\ell_+^+ + \ell_+^-)(\ell_+^+ + \ell_+^- + 1) \\ &= (\ell^+ + \ell^- - \frac{1}{2})(\ell^+ + \ell^- + \frac{1}{2}), \end{aligned} \quad (2.7)$$

where $\ell_\pm^{\pm}|\Omega_\pm\rangle = T_0^{\pm 3}|\Omega_\pm\rangle > T_0^{-3}|\Omega_\pm\rangle$. Thus, $\ell_+^+ = \ell^+, \ell_+^- = \ell^+ - 1/2$, $\ell_-^+ = \ell^- - 1/2, \ell_-^- = \ell^-$. These inequalities follow from considering the norm of the state $(G_{+K})_{-1/2}|hws\rangle$ in the NS case, and of $(G_{-K})_0|\Omega_+\rangle$ or $(G_{+K})_0|\Omega_-\rangle$ in the R case, where $|hws\rangle$ and $|\Omega_\pm\rangle$ have non-vanishing norm.

We first discuss the massless case in the R sector (see Fig.1). There, the states $|\Omega_+\rangle$ and $|\Omega_-\rangle$ become undistinguishable and we denote this common massless state by $|\Omega\rangle$. It satisfies

$$(G_+)_0|\Omega\rangle = 0 = (G_{+K})_0|\Omega\rangle = (G_{-K})_0|\Omega\rangle. \quad (2.8)$$

Notice that even though $(G_{+K})_0^2 = \frac{1}{4}(T^{++}T^{--})_0$ (see Appendix A) does not vanish in general, in fact it does on a singular vector; similarly with $(G_{-K})_0^2$. In the following, we define the $SU(2)^\pm$ quantum numbers of $|\Omega\rangle$ to be (ℓ^+, ℓ^-) , namely

$$T_0^{\pm 3}|\Omega\rangle = \ell^\pm|\Omega\rangle. \quad (2.9)$$

It is then easy to check that the conformal weight h of $|\Omega\rangle$ is given by

$$kh = \frac{1}{4} \bar{k}^+ \bar{k}^- + (\ell^+ + \ell^-)(\ell^+ + \ell^- + 1), \quad (2.10)$$

as a consequence of (2.8) and unitarity. The massless multiplet in the R sector also contains the state $(G_-)_0|\Omega\rangle$, which is non-zero as can be seen from the commutator $\{(G_+)_0, (G_-)_0\}$ in Appendix A, and which is a hws of the $SU(2)^\pm$ zero-mode algebras (in the massive case, the corresponding hws is $(G_-)_0|\Omega_+\rangle = -(2\ell^- + 1)T_0^{--}(G_{-K})_0|\Omega_+\rangle$). So, the states $|\Omega\rangle$ and $(G_-)_0|\Omega\rangle$ are the two $SU(2)^\pm$ zero-mode hws appearing in the massless multiplet structure, as shown in Fig.2 (for $\ell^+ > 0$ and $\ell^- > 0$).

It follows that the Witten index is (taking $|\Omega\rangle$ to be bosonic)

$$Tr(-)^F = 2(\ell^+ + \ell^- + 1). \quad (2.11)$$

In the NS sector, the conditions for masslessness are [5]

$$(G_+)^{-1/2}|\Omega\rangle = 0 \quad (2.12)$$

and

$$kh = (\ell^+ - \ell^-)^2 + k^+ \ell^- + k^- \ell^+. \quad (2.13)$$

Below, we shall deal with singular vectors of zero-norm, $|\chi\rangle = |H, L^+, L^-\rangle$ (see next section). For such states, masslessness means simultaneous fulfilment of

$$\begin{aligned} (G_{+K})_0|\chi\rangle &= (G_{-K})_0|\chi\rangle = (G_+)_0|\chi\rangle = 0 \\ kh &= \frac{1}{4} \bar{k}^+ \bar{k}^- + (L^+ + L^-)(L^+ + L^- + 1) \end{aligned} \quad (2.14)$$

in the R sector, and of

$$\begin{aligned} (G_+)^{-1/2}|\chi\rangle &= 0 \\ kh &= (L^+ - L^-)^2 + k^+ L^- + k^- L^+, \end{aligned} \quad (2.15)$$

in the NS sector. In neither sector do the two conditions follow from each other, and both have to be verified, in contrast to (2.10) which is a consequence of (2.8) and unitarity for the hws $|\Omega\rangle$.

The NS and R sectors are related by spectral flow [13] in two ways as follows.

Flow in the $SU(2)^+$ quantum number:

$$h_R = h_{NS} - \ell_{NS}^+ + \frac{\bar{k}^+}{4}; (\ell_+^+)_R = \frac{\bar{k}^+}{2} - \ell_{NS}^+; (\ell_+^-)_R = \ell_{NS}^-. \quad (2.16)$$

Flow in the $SU(2)^-$ quantum number:

$$h_R = h_{NS} - \ell_{NS}^- + \frac{\bar{k}^-}{4}; (\ell_-^+)_R = \frac{\bar{k}^-}{2} - \ell_{NS}^-; (\ell_-^-)_R = \ell_{NS}^+. \quad (2.17)$$

These follow from [13], remembering a fermionic contribution of $1/4$ in h and $1/2$ in ℓ^+ and ℓ^- . From [5], we know that when ℓ_{NS}^+ or ℓ_{NS}^- are at their maximal values $\frac{1}{2}\bar{k}^\pm$, we can *only* have massless representations. In the R sector, this is then translated into either of the following conditions:

$$\ell_+^+ = 0, \ell_+^- = \frac{\bar{k}^-}{2}, \ell_-^+ = \frac{\bar{k}^+}{2}, \ell_-^- = 0. \quad (2.18)$$

When any one of these are satisfied, we have by necessity a massless representation, but notice that this implies $\ell_+^+ = \ell_-^+ = \ell^+; \ell_+^- = \ell_-^- = \ell^-$ by the above construction.

The characters are defined as

$$Ch_{\mathcal{A}_+, I}[\bar{k}^+, \bar{k}^-, h, \ell^+, \ell^-; q, z_+, z_-] = Tr[q^{\ell_0 - c/24} z_+^{2T_0^+} z_-^{2T_0^-}], \quad (2.19)$$

where the trace is taken over the irreducible module defined by the hws for $I = NS$ and by the massive or massless multiplet for $I = R$. In the massive R case, ℓ^+, ℓ^- denote the multiplet parameters (as opposed to the state eigenvalues $\ell_\pm^+, \ell_\pm^-$). In the massless case however, ℓ^+, ℓ^- are also the quantum numbers of $|\Omega\rangle$.

Using the spectral flow, we find (cf. (I) where analogous relations were given for $\mathcal{A}_+$)

Massless representations

$$\begin{aligned} Ch_{\mathcal{A}_+, NS}[\bar{k}^+, \bar{k}^-, h, \ell^+, \ell^-; q, z_+, z_-] \\ &= q^{\bar{k}^+/4} z_+^{\bar{k}^+} Ch_{\mathcal{A}_+, R}[\bar{k}^+, \bar{k}^-, h - \ell^+ + \frac{\bar{k}^+}{4}, \frac{\bar{k}^+}{2} - \ell^+, \ell^- + \frac{1}{2}; q, q^{1/2} z_+, z_-] \\ &= q^{\bar{k}^-/4} z_-^{\bar{k}^-} Ch_{\mathcal{A}_+, R}[\bar{k}^+, \bar{k}^-, h - \ell^- + \frac{\bar{k}^-}{4}, \frac{\bar{k}^-}{2} - \ell^-, \ell^+ + \frac{1}{2}; q, z_+, q^{1/2} z_-]. \end{aligned} \quad (2.20)$$

Massless representations

$$\begin{aligned} Ch_{\mathcal{A}_+, NS}[\bar{k}^+, \bar{k}^-, h, \ell^+, \ell^-; q, z_+, z_-] \\ &= q^{\bar{k}^+/4} z_+^{\bar{k}^+} Ch_{\mathcal{A}_+, R}[\bar{k}^+, \bar{k}^-, h - \ell^+ + \frac{\bar{k}^+}{4}, \frac{\bar{k}^+}{2} - \ell^+, \ell^-; q, q^{1/2} z_+, z_-] \\ &= q^{\bar{k}^-/4} z_-^{\bar{k}^-} Ch_{\mathcal{A}_+, R}[\bar{k}^+, \bar{k}^-, h - \ell^- + \frac{\bar{k}^-}{4}, \frac{\bar{k}^-}{2} - \ell^-, \ell^+; q, z_+, q^{1/2} z_-], \end{aligned} \quad (2.21)$$

where we denote characters in massless representations by the index, o .

3 The pattern of singular vectors

For the unitary hwr we consider, the Verma module is reducible and we must appropriately subtract the contribution from submodules built on singular vectors, when forming the character of the irreducible representation. The first step consists of isolating the singular vectors. We shall work in the R sector and later convert the results for the characters to the NS sector by spectral flow (2.20)(2.21).

Singular vectors in the Verma modules are annihilated by all positive mode operators. They therefore fall in massive or massless multiplets just as described above. We denote by $|\chi, \pm\rangle$ singular vectors of the (+) type or (-) type according to whether they satisfy the same relations as $|\Omega_+\rangle$ or $|\Omega_-\rangle$ (2.3)(2.4)(2.5).

Just as in (1), but now using generators in $\tilde{\mathcal{A}}_n$, we may then easily prove the following (see also [14,7]):

If $|\chi, +; H, L^+, L^- \rangle$ is a singular vector of the (+) type with eigenvalues (H, L^+, L^-) of $(L_0, T_0^{+3}, T_0^{-3})$, then

$$\begin{aligned} \beta^- |\chi, +; H, L^+, L^- \rangle &= (T_0^{--})^{2L^-+1} |\chi, +; H, L^+, L^- \rangle \\ &= |\chi, +; H, L^+, -L^- - 1 \rangle \quad \text{and} \\ \alpha^+ |\chi, +; H, L^+, L^- \rangle &= (T_0^{++})^{\tilde{k}^+ - 2L^+ + 1} |\chi, +; H, L^+, L^- \rangle \\ &= |\chi, +; H + \tilde{k}^+ - 2L^+ + 1, \tilde{k}^+ - L^+ + 1, L^- \rangle \end{aligned} \quad (3.1)$$

are also singular vectors of the (+) type with the quantum numbers indicated. Similarly, if $|\chi, -; H, L^+, L^- \rangle$ is singular of the (-) type,

$$\begin{aligned} \beta^+ |\chi, -; H, L^+, L^- \rangle &= (T_0^{++})^{2L^+ + 1} |\chi, -; H, L^+, L^- \rangle \\ &= |\chi, -; H, -L^+ - 1, L^- \rangle \quad \text{and} \\ \alpha^- |\chi, -; H, L^+, L^- \rangle &= (T_0^{--})^{\tilde{k}^- - 2L^- + 1} |\chi, -; H, L^+, L^- \rangle \\ &= |\chi, -; H + \tilde{k}^- - 2L^- + 1, L^+, \tilde{k}^- - L^- + 1 \rangle \end{aligned} \quad (3.2)$$

are singular vectors of the (-) type with the quantum numbers indicated.

In particular, if $|\chi \rangle$ is a *massless* singular vector, as defined in (2.14), it is both of (+) and (-) type, and the above formulas provide the action of β^- and α^+ to map a massless singular vector to a massive singular vector of the (+) type, and the action of β^+ , α^- to map a massless singular vector to a massive one of the (-) type.

Using $(G_{\pm K})_0$ to map between massive singular vectors of the (+) type and the (-) types, we therefore see that we have the following maps between singular vectors. Acting on a state with quantum numbers (H, L^+, L^-) :

$$\underline{\text{Massive } (+) \rightarrow \text{Massive } (+)}$$

$$\begin{aligned} \alpha^+ &= (T_{-1}^{++})^{\tilde{k}^+ - 2L^+ + 1}, & \beta^- &= (T_0^{--})^{2L^- + 1} \\ \alpha^- &= (G_{+K})_0 (T_{-1}^{--})^{\tilde{k}^- - 2L^-} (G_{-K})_0, & \beta^+ &= (G_{+K})_0 (T_0^{++})^{2L^+} (G_{-K})_0 \end{aligned} \quad (3.3)$$

$$\underline{\text{Massless } \rightarrow \text{Massive } (+)}$$

$$\begin{aligned} \alpha^+ &= (T_{-1}^{++})^{\tilde{k}^+ - 2L^+ + 1}, & \beta^- &= (T_0^{--})^{2L^- + 1} \\ \alpha^- &= (G_{+K})_0 (T_{-1}^{--})^{\tilde{k}^- - 2L^- + 1}, & \beta^+ &= (G_{+K})_0 (T_0^{++})^{2L^+ + 1} \end{aligned} \quad (3.4)$$

$$\underline{\text{Massive } (+) \rightarrow \text{Massless}}$$

$$\begin{aligned} \alpha^+ &= (T_{-1}^{++})^{\tilde{k}^+ - 2L^+ + 1}, & \beta^- &= (T_0^{--})^{2L^- + 1} \\ \alpha^- &= (T_{-1}^{--})^{\tilde{k}^- - 2L^-} (G_{-K})_0, & \beta^+ &= (T_0^{++})^{2L^+} (G_{-K})_0. \end{aligned} \quad (3.5)$$

(To achieve greater mathematical clarity, we might decorate the operators $\alpha^\pm, \beta^\pm$ by labels specifying the nature of the states on which they act and the nature (massive or massless) of the resulting state. We refrain from such proliferation of notation in the interest of readability, hoping that no confusion will arise.)

With these definitions, it is possible in all cases (i.e. whether the original state / resulting state is massive / massless) to prove the following crucial properties when $|\chi \rangle$ is singular:

$$\begin{aligned} \alpha^+ \alpha^- |\chi \rangle &= \alpha^- \alpha^+ |\chi \rangle \\ \alpha^+ \beta^- |\chi \rangle &= \beta^- \alpha^+ |\chi \rangle \\ \beta^+ \alpha^- |\chi \rangle &= \alpha^- \beta^+ |\chi \rangle \\ \beta^+ \beta^- |\chi \rangle &= \beta^- \beta^+ |\chi \rangle, \end{aligned} \quad (3.6)$$

in complete analogy with what we found in (1).

The new features concern (i) the fact that we are dealing with $\tilde{\mathcal{A}}_n$, rather than $\mathcal{A}_n$, and therefore using the correspondingly modified (and simplified) forms (3.3) and the modified algebra (Appendix A); (ii) (3.6) are also satisfied in the new cases (3.4)(3.5).

We now define (in analogy with (1))

$$\begin{aligned} A_n^\pm &= (\beta^\pm \alpha^\pm)^n; & B_n^\pm &= (\alpha^\pm \beta^\pm)^n \\ A_n'^\pm &= (\alpha^\pm \beta^\pm)^{n-1} \alpha^\pm, & B_n'^\pm &= (\beta^\pm \alpha^\pm)^n \beta^\pm. \end{aligned} \quad (3.7)$$

$n = 1, 2, \dots$ for $A^\pm, A'^\pm$ and $n = 0, 1, \dots$ for $B^\pm, B'^\pm$.

We then obtain the following pattern of singular vectors:

$$\begin{aligned} A_n^+ A_m^- |\Omega \rangle, & A_n^+ B_m^- |\Omega \rangle, A_n^+ A_m'^- |\Omega \rangle, A_n^+ B_m'^- |\Omega \rangle \\ B_n^+ A_m^- |\Omega \rangle, & B_n^+ B_m^- |\Omega \rangle, B_n^+ A_m'^- |\Omega \rangle, B_n^+ B_m'^- |\Omega \rangle \\ A_n'^+ A_m^- |\Omega \rangle, & A_n'^+ B_m^- |\Omega \rangle, A_n'^+ A_m'^- |\Omega \rangle, A_n'^+ B_m'^- |\Omega \rangle \\ B_n'^+ A_m^- |\Omega \rangle, & B_n'^+ B_m^- |\Omega \rangle, B_n'^+ A_m'^- |\Omega \rangle, B_n'^+ B_m'^- |\Omega \rangle. \end{aligned} \quad (3.8)$$

In Appendix B, we list the quantum numbers of these states. Obviously, $A_n^+ B_m^- |\Omega \rangle = B_m^- A_n^+ |\Omega \rangle$ etc. However, the form of the quantum numbers can depend on whether $n > m$ or $n < m$. The state $|\Omega \rangle$ can be massless (as is assumed in Appendix B), or massive (of the (+) type, say). If $|\Omega \rangle$ is massless, precisely the states

$$A_n^+ A_m^- |\Omega \rangle, B_n^+ B_m^- |\Omega \rangle, A_n'^+ A_m'^- |\Omega \rangle, B_n'^+ B_m'^- |\Omega \rangle, \quad (3.9)$$

are also massless. More precisely, it follows from Appendix B that the quantum numbers of these states satisfy the masslessness criterion (2.14). Further, by construction they are annihilated by $(G_+)_0$ and $(G_{+K})_0$. Since they all have zero-norm (being singular descendants), we still have to verify explicitly that also $(G_{-K})_0$ vanish on them. As an instructive example, consider

Table 1

	A_n^+	B_n^+	A_n^+	B_n^+
A_m^+	$(G_{\pm K})_{-(m-n)}$	$(G_{\pm K})_{-(m+n)}$	$(G_{\pm})_{- m-n }$	$(G_{\pm})_{-(m+n)}$
B_m^+	$(G_{\pm K})_{-(m+n)}$	$(G_{\pm K})_{-(m-n)}$	$(G_{\pm})_{- m-n }$	$(G_{\pm})_{-(m+n)}$
A_m^-	$(G_{\pm})_{-(m+n)}$	$(G_{\pm K})_{-(m-n)}$	$(G_{\pm K})_{- m-n }$	$(G_{\pm K})_{-(m+n)}$
B_m^-	$(G_{\pm})_{-(m+n)}$	$(G_{\pm K})_{-(m-n)}$	$(G_{\pm K})_{- m-n }$	$(G_{\pm K})_{-(m+n)}$

Table 1 Operators that have to be removed from the module built on states $A_n^+ A_m^- |\Omega\rangle$ etc. when $|\Omega\rangle$ is massless (and when neither ℓ^+ nor ℓ^- vanish). The upper signs hold for $m \geq n$ and the lower signs for $m \leq n$. Thus, $(G_{-K})_{-(m-n)}$ should be removed from the module built on $B_n^+ B_m^- |\Omega\rangle$ when $m > n$, whereas $(G_{+K})_{-(m-n)}$ should be removed when $m < n$. For $m = n$, both operators are removed, but one is always trivial. For example $B_n^+ B_n^- |\Omega\rangle$ is massless and both $(G_{+K})_0$ and $(G_{-K})_0$ should be removed; $B_n^+ B_n^- |\Omega\rangle$ is not massless, but $(G_{-})_0$ should be removed, and so should $(G_{+})_0$ – as always. The indices n, m are $1, 2, \dots$ for A -operators and $0, 1, 2, \dots$ for B -operators. Thus for example $A_n^+ B_0^- |\Omega\rangle = A_n^+ |\Omega\rangle$ only involves $(+)$ -operators α^+, β^+ .

So, let us assume that Table 1 has been proven for all m, n up to some value. We consider an example of the induction step.

For definiteness, consider $A_n^+ B_m^- |\Omega\rangle$ for which we know (by the induction assumption) that there is an operator $\mathcal{O}_{-K}(m+n)$ such that (in the notation we suppress any possible dependence of $\mathcal{O}_{-K}$ on the state in question)

$$(G_{-K})_{-(m+n)} A_n^+ B_m^- |\Omega\rangle = \mathcal{O}_{-K}(m+n) A_n^+ B_m^- |\Omega\rangle, \quad (4.1)$$

where $\mathcal{O}_{-K}(m+n)$ is a complicated sum of terms, each involving products of generators with indices adding up to $-(m+n)$, but with $(G_{-K})_{-(m+n)}$ nowhere appearing. We then consider

$$\alpha^+ A_n^+ B_m^- |\Omega\rangle = A_{n+1}^+ B_m^- |\Omega\rangle, \quad (4.2)$$

and according to Table 1 we would like to show that $(G_{+})_{-(m+n+1)}$ acting on this state may be similarly replaced by an operator $\mathcal{O}_{+}(m+n+1)$ not involving $(G_{+})_{-(m+n+1)}$ itself. Let the state $A_n^+ B_m^- |\Omega\rangle$ have eigenvalues (H, L^+, L^-) of (L_0, T_0^3, T_0^3) ; it is a massive singular state of $(+)$ type. Then α^+ in (4.2) is (cf. (3.3)) $\alpha^+ = (T_{-1}^{++})^p$ with $p = k^+ - 2L^+ + 1$. Now consider for some integer $q > 0$

$$\begin{aligned} (G_{-K})_{-(m+n)} (T_{-1}^{++})^{q+1} A_{n+1}^+ B_m^- |\Omega\rangle &= (G_{-K})_{-(m+n)} (T_{-1}^{++})^{p+q+1} A_n^+ B_m^- |\Omega\rangle \\ &= (T_{-1}^{++})^{p+q+1} \mathcal{O}_{-K}(m+n) A_n^+ B_m^- |\Omega\rangle \\ &+ (p+q+1) (T_{-1}^{++})^q (G_{+})_{-(m+n+1)} \alpha^+ A_n^+ B_m^- |\Omega\rangle \end{aligned} \quad (4.3)$$

The first term on the right hand side may be rewritten as

$$\mathcal{O}_{-K}(m+n) (T_{-1}^{++})^{q+1} A_{n+1}^+ B_m^- |\Omega\rangle + [(T_{-1}^{++})^{p+q+1} \mathcal{O}_{-K}(m+n)] A_n^+ B_m^- |\Omega\rangle. \quad (4.4)$$

It is easy to see that we may always choose q large enough so that the commutator term may be written as some descendant of $A_{n+1}^+ B_m^- |\Omega\rangle$. Therefore, using (4.3) we are through, if we can find a way to remove $(T_{-1}^{++})^q$ in the last term. This is indeed possible by hitting

$$\begin{aligned} (G_{-K})_0 \beta^+ \beta^+ |\Omega\rangle &= (G_{-K})_0 (T_0^{--})^{2\ell^-} (G_{+K})_0 (T_0^{++})^{2\ell^+ + 1} |\Omega\rangle \\ &= -2\ell^- (T_0^{--})^{2\ell^- - 1} (G_{-})_0 (T_0^{--})^{2\ell^+ + 1} |\Omega\rangle \\ &+ \left[h - \frac{\tilde{k}^+ \tilde{k}^-}{4k^+} - \frac{1}{k^+} (-\ell^+ + \ell^-) (-\ell^+ + \ell^- - 1) \right] (T_0^{--})^{2\ell^+ + 1} |\Omega\rangle \\ &= 2\ell^- (2\ell^+ + 1) (T_0^{--})^{2\ell^- - 1} (T_0^{++})^{2\ell^+} (G_{-})_0^2 |\Omega\rangle \\ &+ \frac{2}{k^+} \ell^- (2\ell^+ + 1) (T_0^{--})^{2\ell^-} (T_0^{++})^{2\ell^+ + 1} |\Omega\rangle = 0, \end{aligned} \quad (3.10)$$

using (2.10) and Appendix A. The general case is dealt with similarly using induction.

If $|\Omega\rangle$ is not massless, neither is any other state in the list (3.8) (at least not for generic values of $\tilde{k}^\pm$). The states (3.9) provide plenty of examples of the situation (3.5). For example,

$$A_n^+ A_n^- |\Omega\rangle = \beta^+ A_n^+ A_n^- |\Omega\rangle, \quad (3.11)$$

showing that when acting on the massive state $A_n^+ A_n^- |\Omega\rangle$, β^+ produces a massless one. Such phenomena are responsible for the dependence on whether $n > m$ or $n < m$ in Appendix B.

We conjecture that the list (3.8) provides *all* singular vectors. Our lack of rigour here is analogous to the situation in (I) and [7].

4 Fermionic identities in massless Verma modules

The massive Verma module is generated by states obtained by acting with all combinations of negative-mode operators and the zero-mode operators T_0^{++}, T_0^{--} on the appropriate highest weight state. In the R-sector, we should act in addition with a subset of the fermionic zero-mode operators. Thus, if we build the module on (the massive state) $|\Omega_+ \rangle$ ($|\Omega_- \rangle$), we may act in addition with $(G_{-K})_0$ ($(G_{+K})_0$ for $|\Omega_- \rangle$) and $(G_{-})_0$. On the other hand, operators $T_0^{++}, T_0^{--}, (G_{+})_0$ and $(G_{-K})_0$ ($(G_{+K})_0$ for $|\Omega_- \rangle$) should not be used, since $|\Omega_+ \rangle$ ($|\Omega_- \rangle$) is annihilated by these, and therefore, any state built with $(G_{+K})_0$ may be re-expressed with other operators using the commutation relations.

In the massless module, we have *one* more restriction when neither ℓ^+ nor ℓ^- vanish. In this section we only consider this situation. The special case $\ell^+ = 0$ or $\ell^- = 0$ will be dealt with indirectly in sect.5. Anyway, in the generic case, the module built on a massless hws does not include $(G_{+K})_0$ and $(G_{-K})_0$. Notice the difference with the ordinary $N = 4$ algebra where *two additional* operators are removed [7].

Now a similar phenomenon arises for *all* the singular vectors (3.8) in the massless Verma module, and Table 1 (with table caption) summarizes, for the module built on each singular vector, the operator which is not used. The analogous situation was pointed out in [7] for the ordinary $N = 4$ algebra.

Whenever the state under consideration, say $A_n^+ B_m^- |\Omega\rangle$ is massive, we know from sect. 2 how to convert it from $(+)$ to $(-)$ type. Table 1 holds irrespective of whether the massive state is $(+)$ or $(-)$ type. Massless states are both $(+)$ and $(-)$, so again there is no ambiguity. There are obviously many subresults to prove. Here we indicate a couple of typical examples, the rest being proven very much along the same lines.

the equation with $(T_{+1}^+)^n$ and using that $A_{n+1}^{++} B_m^- |\Omega\rangle$ is a singular vector. In particular one may rather easily verify (i) that the coefficient of the term $(G_+)^{-(m+n+1)} A_{n+1}^{++} B_m^- |\Omega\rangle$ in the resulting expression never vanishes (use Appendix B), and (ii) that the operator $(G_+)^{-(m+n+1)}$ can appear nowhere else.

As our second example we consider the states

$$A_n^+ A_m^- |\Omega\rangle. \quad (4.5)$$

For $n > m$ we may write this, using (3.7) as

$$A_n^+ A_m^- |\Omega\rangle = A_{n-m}^+ A_m^+ A_m^- |\Omega\rangle = A_{n-m}^+ |\Omega\rangle, \quad (4.6)$$

where $|\Omega\rangle \equiv A_m^+ A_m^- |\Omega\rangle$ is a new *massless* state (cf. App. B and sect. 3). It is easy to verify that on such a state $(G-K)^{-(n-m)}$ is rewritable in terms of other operators (of lower mode numbers), in accordance with Table 1. The appearance of sums $(m+n)$ and differences $|m-n|$ in the table is crucial for the neat forms of our final results in sect. 5.

It might be thought that even more operators could be rewritten on the state (4.5). As an example consider $\alpha^+ |\Omega\rangle$, $\alpha^- |\Omega\rangle$ and $\alpha^+ \alpha^- |\Omega\rangle$. The first two states are massive (for $|\Omega\rangle$ massless) and $(G_+)^{-1}$ is rewritable on them. Thus it might have been thought at first that by considering $(G_+)^{-1} \alpha^+ \alpha^- |\Omega\rangle$, pushing $(G_+)^{-1}$ through α^+ , rewriting it on $\alpha^- |\Omega\rangle$ and pushing the resulting operator back through α^+ (as above), we might discover a new rewritable operator on $\alpha^+ \alpha^- |\Omega\rangle$. However, by carefully keeping track of all constant factors occurring in the commutation relations (using the composite operators in Appendix A), it turns out that unlike above, this procedure merely gives back $(G_+)^{-1} \alpha^+ \alpha^- |\Omega\rangle$. The conclusion is that any massless state (even the ones of the form $A_m^+ A_m^- |\Omega\rangle$ etc.) is only annihilated by one extra G -zero-mode operator. We may therefore always apply identities such as (4.6) to discover the one rewritable operator.

We finish by making two comments concerning the comparison with the ordinary $N = 4$ case [7]:

(i) It might be feared that the fact that we have only *one* fermionic condition on singular vectors rather than *two* as in [7] would make it impossible to recover the characters of [7] from ours in the limit $k^- \rightarrow \infty$ (cf. (I)). As we shall show in sect. 6 below, however, our results are in complete agreement with [7] in that limit.

(ii) In [7], the analogous fermionic conditions were obtained by first proving an elegant subresult ([7] Lemma 1). That result does not seem to be general, but as we have shown, it is also not necessary for the purpose at hand.

5 Character formulas

We first consider the R-sector. The character corresponding to the full reducible Verma module built on a massive state, say $|\Omega_+\rangle$, is readily written down as

$$q^{h-\varepsilon/24} z_+^{2\ell_+} z_-^{2\ell_-} F^{R,R}(q, z_+, z_-) (1 + z_+^{-1} z_-) (1 + z_+^{-1} z_-^{-1}) \cdot B^{+-}(q, z_+, z_-) (1 - z_+^{-2})^{-1} (1 - z_-^{-2})^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-1}. \quad (5.1)$$

Here (cf. (I))

$$F^I(q, z_+, z_-) \equiv \prod_{n \in \mathbb{N}_1} (1 + z_+ z_- q^n) (1 + z_+^{-1} z_-^{-1} q^n) (1 + z_+^{-1} z_- q^n) (1 + z_+ z_-^{-1} q^n) \quad (5.2)$$

is the fermionic non-zero mode contribution for $I = R$ or NS , $\mathbb{N}_{NS} = \{\text{positive half-integers}\}$, $\mathbb{N}_R = \{\text{positive integers}\}$.

$$B^{+-}(q, z_+, z_-) \equiv \prod_{n=1}^{\infty} (1 - z_+^2 q^n)^{-1} (1 - z_+^{-2} q^n)^{-1} (1 - q^n)^{-2} (1 - z_+^2 q^n)^{-1} (1 - z_-^2 q^n)^{-1} \quad (5.3)$$

is the non-zero mode contribution of the KM-currents $T_n^{\pm i}$ and finally, $\prod_{n=1}^{\infty} (1 - q^n)^{-1}$ is the contribution of the L_n 's.

When forming the character of the irreducible representation, we should *subtract* contributions such as (5.1) for each singular vector in the list (3.8) having been built by an odd number of α^+ , β^+ operators, and *add* contributions involving an even number of such operators. For a massive state this is readily completed following the techniques of (I) with the results

$$\begin{aligned} & C h_{A,R}^{A,R}(\tilde{k}^+, \tilde{k}^-, h_+, \ell^+, \ell^-; q, z_+, z_-) = q^{h-\varepsilon/24} F^{R,R}(q, z_+, z_-) B^{+-}(q, z_+, z_-) \\ & \cdot \prod_{n=1}^{\infty} (1 - q^n)^{-1} \cdot (z_+^{-1} + z_-^{-1}) (1 + z_+^{-1} z_-^{-1}) (1 - z_+^{-2})^{-1} (1 - z_-^{-2})^{-1} \\ & \cdot \sum_{n,m=-\infty}^{\infty} q^{n^2 k^+ + 2\ell^+ n + m^2 k^- + 2\ell^- m} \\ & \cdot \sum_{\epsilon_+, \epsilon_- = -1, +1} \epsilon_+ \epsilon_- z_+^{2\epsilon_+ (\ell^+ + nk^+)} z_-^{2\epsilon_- (\ell^- + mk^-)} \end{aligned} \quad (5.4)$$

$$\begin{aligned} & C h_{A,NS}^{A,NS}(\tilde{k}^+, \tilde{k}^-, h_+, \ell^+, \ell^-; q, z_+, z_-) = q^{h-\varepsilon/24} F^{NS}(q, z_+, z_-) B^{+-}(q, z_+, z_-) \\ & \cdot \prod_{n=1}^{\infty} (1 - q^n)^{-1} \cdot (1 - z_+^{-2})^{-1} (1 - z_-^{-2})^{-1} \\ & \cdot \sum_{n,m=-\infty}^{\infty} q^{n^2 k^+ + 2\ell^+ n + m^2 k^- + 2\ell^- m} q^{n+m} z_+^{-1} z_-^{-1} \\ & \cdot \sum_{\epsilon_+, \epsilon_- = -1, +1} \epsilon_+ \epsilon_- z_+^{2\epsilon_+ (\ell^+ + nk^+)} z_-^{2\epsilon_- (\ell^- + mk^-)}. \end{aligned} \quad (5.5)$$

These formulas agree with the ones obtained in a somewhat different way in (I) using the $\mathcal{A}_\gamma$ algebra.

We now turn to the massless case. Again we add and subtract contributions as in (5.1) for each of the singular vectors (3.8), using Appendix B. However, from Table 1 we read off that each term of the form (5.1) should further be divided by an appropriate factor (cf. the discussion in [7]). Thus for example for the vector $A_n^+ B_m^- |\Omega\rangle$, the operator $(G_-)^{-(m+n)}$ should not be used in building the (reducible) submodule, and hence a factor $(1 + q^{n+m} z_+^{-1} z_-^{-1})$ should be removed from (5.1).

Going over Appendix B and Table 1, we therefore build the massless character in the R-sector as a sum of 24 individual terms. These may finally be combined into the following expression (written as 4 terms):

our expression (5.6) must be valid in all cases, including the case $\ell^+ = 0$. (Notice that due to the symmetry in $+$ and $-$, (5.7) obviously has a partner where the roles of $+$ and $-$ are interchanged; a similar remark holds for (5.9)). It is important for this argument that although the multiplet structure in the massive case is also altered away from Fig. 1 when $\ell^+ = \frac{1}{2}$ (the multiplet $(\ell^+ - 1, \ell^- - \frac{1}{2})$ being absent in that case) the calculation of the character (whether being done as in the present paper or as in (1)) makes no reference to this fact.

6 The limit $\tilde{k}^- \rightarrow \infty$, $\tilde{k}^+$ fixed

As explained in detail in (1), the $\tilde{\mathcal{A}}_0$ algebra reduces, in the limit $\tilde{k}^- \rightarrow \infty$, to the ordinary $N = 4$ SCA [6] with $SU(2)^+$ subalgebra and central charge, $\hat{c} = 3 = 6\tilde{k}^+$. The non-zero modes of the $SU(2)^-$ generators must be rescaled and become oscillators of three scalar fields. They have isospin 1 under the global, unscaled T_0^{-1} 's.

The massless character formulas (5.6)(5.8) should then be closely related to the massless characters given in [7], when $\tilde{k}^- \rightarrow \infty$.

Let us examine the R sector in some detail for illustration. In the limit we consider, only the terms with $m = 0$ survive in (5.6). After some standard manipulations, the characters (5.6) can be written as

$$\begin{aligned} \lim_{\tilde{k}^- \rightarrow \infty} Ch_0^{\mathcal{A}, R}[\tilde{k}^+, \tilde{k}^-, h, \ell^+, \ell^-, q, z_+, z_-] &= q^{h-\hat{c}/24} \prod_{n=1}^{\infty} (1 - q^n)^{-1} \\ & F^R(q, z_+, z_-) B^{+-}(q, z_+, z_-) (z_-^{-1} + z_+^{-1}) (1 + z_+^{-1} z_-^{-1})^{-1} (1 - z_-^{-2})^{-1} \\ & \{ (z_-^{2\ell^+ + 1} - z_-^{-(2\ell^- + 1)}) \sum_{n=-\infty}^{n=+\infty} q^{n^2 k^+ + 2\ell^+ n} \times \\ & \left[\frac{z_+^{2(\ell^+ + nk^+)}}{(1 + q^{-n} z_+^{-1} z_-^{-1}) (1 + q^{-n} z_+^{-1} z_-^{-1})} - \frac{z_+^{-2(\ell^+ + nk^+)}}{(1 + q^{-n} z_+ z_-^{-1}) (1 + q^{-n} z_+ z_-^{-1})} \right] \\ & + (z_-^{2\ell^-} - z_-^{-2\ell^-}) \sum_{n=-\infty}^{n=+\infty} q^{n^2 k^+ + (2\ell^- - 1)n} \times \\ & \left[\frac{z_+^{2(\ell^- - 1/2 + nk^+)}}{(1 + q^{-n} z_+^{-1} z_-^{-1}) (1 + q^{-n} z_+^{-1} z_-^{-1})} - \frac{z_+^{-2(\ell^- - 1/2 + nk^+)}}{(1 + q^{-n} z_+ z_-^{-1}) (1 + q^{-n} z_+ z_-^{-1})} \right] \}. \end{aligned} \quad (6.1)$$

Comparison with the expressions given in [7] shows that the limit above is a sum of two R massless characters for representations of isospin ℓ^+ and $\ell^+ - 1/2$ at level $\tilde{k}^+$, when the algebra is $N = 4$ with $SU(2)^+$ subalgebra and global $SU(2)^-$. Indeed, in the limit $z_- \rightarrow 1$, one has (for $\ell^+ > 0$)

$$\begin{aligned} \lim_{\tilde{k}^- \rightarrow \infty} Ch_0^{\mathcal{A}, R}[\tilde{k}^+, \tilde{k}^-, h, \ell^+, \ell^-, q, z_+, z_-] &= \\ q^{-\tilde{k}^+ / 4 - 1/8} \prod_{n=1}^{\infty} (1 - q^n)^{-3} \{ (2\ell^- + 1) Ch_0^R(\tilde{k}^+, \ell^+, z_+) + 2\ell^- Ch_0^R(\tilde{k}^+, \ell^+ - \frac{1}{2}, z_+) \}, \end{aligned} \quad (6.2)$$

where the infinite product prefactor is the contribution of the oscillators (non-zero modes of the $SU(2)^-$ generators giving a central charge contribution of 3, see (1)), and the notations of [7] for the massless characters have been used on the right hand side. (In particular in [7] the characters were defined with $q^{\tilde{k}_0}$ in place of $q^{\tilde{k}_0 - \hat{c}/24}$).

It is interesting to note that in the instances where either $\ell^+ = 0$ or $\ell^+ = 0$, the limit (6.1) reduces to the first term, which is consistent with the multiplet structure (with only

$$\begin{aligned} Ch_0^{\mathcal{A}, R}[\tilde{k}^+, \tilde{k}^-, h, \ell^+, \ell^-, q, z_+, z_-] &= q^{h-\hat{c}/24} F^R(q, z_+, z_-) B^{+-}(q, z_+, z_-) \\ & \prod_{n=1}^{\infty} (1 - q^n)^{-1} \cdot (z_+^{-1} + z_-^{-1}) (1 + z_+^{-1} z_-^{-1})^{-1} (1 - z_-^{-2})^{-1} \\ & \sum_{n, m = -\infty}^{\infty} q^{n^2 k^+ + 2\ell^+ n + m^2 k^- + 2\ell^- m} \\ & \sum_{\ell_+, \ell_- = -1, +1} \epsilon_{\ell_+ \ell_-} z_+^{2\ell_+ (\ell^+ + nk^+)} z_-^{2\ell_- (\ell^- + mk^-)} (z_+^{-\ell_+} q^{-n} + z_-^{-\ell_-} q^{-m})^{-1}, \end{aligned} \quad (5.6)$$

an expression in striking resemblance with (5.4). However, the factorization property discussed in (1) is obviously lost.

It is easy to verify that for h, ℓ^+, ℓ^- satisfying the masslessness condition (2.10) and $\ell^+, \ell^- > 0$ (to begin with), we have

$$\begin{aligned} Ch_0^{\mathcal{A}, R}[\tilde{k}^+, \tilde{k}^-, h, \ell^+, \ell^-, q, z_+, z_-] &+ \\ Ch_0^{\mathcal{A}, R}[\tilde{k}^+, \tilde{k}^-, h, \ell^+ - \frac{1}{2}, \ell^-, q, z_+, z_-] &= \\ Ch_0^{\mathcal{A}, R}[\tilde{k}^+, \tilde{k}^-, h, \ell^+, \ell^- + \frac{1}{2}, q, z_+, z_-] & \end{aligned} \quad (5.7)$$

a result with an obvious interpretation according to Figs. 1 and 2.

Using the spectral flow rule, (2.21), we find

$$\begin{aligned} Ch_0^{\mathcal{A}, NS}[\tilde{k}^+, \tilde{k}^-, h, \ell^+, \ell^-, q, z_+, z_-] &= q^{h-\hat{c}/24} F^{NS}(q, z_+, z_-) B^{+-}(q, z_+, z_-) \\ & \prod_{n=1}^{\infty} (1 - q^n)^{-1} (1 - z_+^{-2})^{-1} (1 - z_-^{-2})^{-1} \\ & \sum_{n, m = -\infty}^{\infty} q^{n^2 k^+ + 2n\ell^+ + m^2 k^- + 2m\ell^-} q^{n^2 k^+} z_+^{-1} z_-^{-1} \\ & \sum_{\ell_+, \ell_- = -1, +1} \epsilon_{\ell_+ \ell_-} z_+^{\ell_+} z_-^{\ell_-} z_+^{2\ell_+ (\ell^+ + nk^+)} z_-^{2\ell_- (\ell^- + mk^-)} (1 + q^{n+m+\frac{1}{2}} z_+^{\ell_+} z_-^{\ell_-})^{-1}. \end{aligned} \quad (5.8)$$

One verifies rather easily that when the NS massless condition, (2.13) is satisfied,

$$\begin{aligned} Ch_0^{\mathcal{A}, NS}[\tilde{k}^+, \tilde{k}^-, h, \ell^+, \ell^-, q, z_+, z_-] &+ \\ Ch_0^{\mathcal{A}, NS}[\tilde{k}^+, \tilde{k}^-, h + \frac{1}{2}, \ell^+ + \frac{1}{2}, \ell^-, q, z_+, z_-] &= \\ Ch_0^{\mathcal{A}, NS}[\tilde{k}^+, \tilde{k}^-, h, \ell^+, \ell^-, q, z_+, z_-], \end{aligned} \quad (5.9)$$

as indeed also follows from (5.7) by spectral flow (2.20), (2.21).

We finally come back to the situation $\ell^+ = 0$ or $\ell^- = 0$. These cases require separate attention because then, the massless multiplet structure Fig. 2 degenerates: the multiplet $(G_-)_{0|\Omega>}$ is absent. This gives rise to a rather different structure of fermionic conditions on the system of singular vectors. However, we may circumvent redoing the entire calculation in this more complicated situation by referring to (5.7) ((5.9) in the NS case). Indeed those equations (which must continue to hold) may be applied for $\ell^+ = \frac{1}{2}$ to provide an expression for the massless character for $\ell^+ = 0$ in terms of known quantities. It follows that in fact

one multiplet in place of two as in Fig.2 for these values of the quantum numbers, cf. the discussion at the end of sec. 5).

A similar limit can be performed in the NS sector. Here one gets

$$\lim_{\tilde{k}^+ \rightarrow -\infty, z_- \rightarrow -1} Ch_o^{\mathcal{A}_+, NS}[\tilde{k}^+, \tilde{k}^-, h, \ell^+, \ell^-; q, z_+, z_-] = q^{-\tilde{k}^+/4 - 1/8} \prod_{n=1}^{\infty} (1 - q^n)^{-3} \{(2\ell^- + 1)Ch_o^{NS}(\tilde{k}^+, \ell^+, z_+) + 2\ell^- Ch_o^{NS}(\tilde{k}^+, \ell^+, z_+) + \frac{1}{2}, z_+)\} \quad (6.3)$$

which is in complete agreement with the spectral flow.

7 Conclusions and outlook

In this paper we have derived the characters for massless (and massive) representations of the $\mathcal{A}_\gamma$ algebra introduced in [10] and which is closely related to the most general, known conventional superconformal algebra $\mathcal{A}_\gamma$. The latter is a direct sum of $\tilde{\mathcal{A}}_\gamma$, four free fermions and a $U(1)$ KM algebra. Our results for the massive case (5.4) and (5.5) are in agreement with those obtained in our previous analysis [1]. The massless case requires a careful treatment of some redundant operators in the construction of the irreducible Verma module, and this is summarized in Table 1. The paper is mainly dedicated to the derivation of formulas (5.6) and (5.8), which give the massless characters in the R and NS sectors. These do not enjoy the factorizing property of the massive characters – namely that the latter are products of two massive characters of the singly extended $N=4$ – but this can be understood as the effect of the fermionic identities for the massless Verma module, derived in section 4.

In the limit where $k^- \rightarrow \infty$, our characters reproduce the known formulas for singly extended $N=4$ [7].

In order to further gain insight into the operator content of possible theories corresponding to $\mathcal{A}_\gamma$, or in order to use our results in some string compactification scheme, we ought to understand the behaviour of the characters (massive and massless) under modular transformations, as well as their reduction in terms of $N=2$ characters. In particular the case $k^- = 1$ should be interesting to analyze since it is closely related to the discrete series of $N=2$ [5].

The $\mathcal{A}_\gamma$ algebra offers a much bigger symmetry than the usual $N=4$ and it would be interesting to investigate how the modular group is represented on the characters. It is perhaps conceivable that they form a unitary representation. We leave these matters for future work.

A The anticommutator of two supersymmetry generators in the $\tilde{\mathcal{A}}_\gamma$ -algebra

Referring to [10, 5, 4, 2, 3, 11] we have (writing G_a rather than $\tilde{G}_a$ for simplicity in this paper)

$$\begin{aligned} \{(G_+)_m, (G_+)_n\} &= -\frac{2}{k}(T^{++}T^{--})_{m+n} \\ \{(G_-)_m, (G_-)_n\} &= -\frac{2}{k}(T^{+-}T^{-+})_{m+n} \\ \{(G_{+K})_m, (G_{+K})_n\} &= +\frac{2}{k}(T^{++}T^{--})_{m+n} \\ \{(G_{-K})_m, (G_{-K})_n\} &= +\frac{2}{k}(T^{+-}T^{-+})_{m+n} \\ \{(G_{+K})_m, (G_{-K})_n\} &= L_{m+n} + \frac{\tilde{k}^+ \tilde{k}^-}{k} \delta_{m+n,0} (m^2 - \frac{1}{4}) \\ &\quad + (n-m) \left[-\frac{\tilde{k}^-}{k} T^{++} + \frac{\tilde{k}^+}{k} T^{--} \right]_{m+n} \\ &\quad - \frac{1}{k} [T^{++}T^{--} + T^{+-}T^{-+}]_{m+n} \\ \{(G_+)_m, (G_-)_n\} &= L_{m+n} + \frac{\tilde{k}^+ \tilde{k}^-}{k} \delta_{m+n,0} (m^2 - \frac{1}{4}) \\ &\quad - (n-m) \left[\frac{\tilde{k}^-}{k} T^{++} + \frac{\tilde{k}^+}{k} T^{--} \right]_{m+n} \\ &\quad - \frac{1}{k} [T^{++}T^{--} + T^{+-}T^{-+}]_{m+n} \\ \{(G_+)_m, (G_{+K})_n\} &= +\frac{\tilde{k}^-}{k} (n-m) T_{m+n}^{++} - \frac{2}{k} (T^{++}T^{--})_{m+n} \\ \{(G_+)_m, (G_{-K})_n\} &= +\frac{\tilde{k}^+}{k} (n-m) T_{m+n}^{--} - \frac{2}{k} (T^{+-}T^{-+})_{m+n} \\ \{(G_-)_m, (G_{+K})_n\} &= -\frac{\tilde{k}^+}{k} (n-m) T_{m+n}^{++} - \frac{2}{k} (T^{++}T^{--})_{m+n} \\ \{(G_-)_m, (G_{-K})_n\} &= -\frac{\tilde{k}^-}{k} (n-m) T_{m+n}^{--} - \frac{2}{k} (T^{+-}T^{-+})_{m+n} \end{aligned} \quad (A.1)$$

Here

$$(AB)_N \equiv \sum_p : A_p B_{N-p} : \quad (A.2)$$

where $::$ denotes normal ordering according to modes, using for zero-modes

$$: A_0 B_0 : \equiv \frac{1}{2} \{A_0, B_0\} \quad (A.3)$$

(in our case A, B are always bosonic).

B Quantum numbers of singular vectors in the massless Verma module

We refer to (3.7) and (3.8) for definitions and find for the quantum numbers (H, L^+, L^-) of $(L_0, T_0^{+3}, T_0^{-3})$:

$$\begin{aligned}
& \frac{A_n^+ A_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- - 2n\ell^+ - m(2\ell^- + 1), \ell^+ - nk^+, \ell^- - mk^-)} \\
n \geq m : & (h + n^2 k^+ + m^2 k^- - 2n\ell^+ - m(2\ell^- + 1), \ell^+ - nk^+, \ell^- - mk^-) \\
n < m : & (h + n^2 k^+ + m^2 k^- - n(2\ell^+ + 1) - 2m\ell^-, \ell^+ + \frac{1}{2} - nk^+, \ell^- - \frac{1}{2} - mk^-) \\
& \frac{A_n^+ B_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- - 2n\ell^+ + m(2\ell^- + 1), \ell^+ - nk^+, \ell^- + mk^-)} \\
& \frac{A_n^+ A_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- - 2n\ell^+ - m(2\ell^- + 1), \ell^+ - nk^+, -\ell^- - 1 + mk^-)} \\
n \geq m : & (h + n^2 k^+ + m^2 k^- - 2n\ell^+ - m(2\ell^- + 1), \ell^+ - nk^+, -\ell^- - 1 + mk^-) \\
n < m : & (h + n^2 k^+ + m^2 k^- - n(2\ell^+ + 1) - 2m\ell^-, \ell^+ + \frac{1}{2} - nk^+, -\ell^- - \frac{1}{2} + mk^-) \\
& \frac{A_n^+ B_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- - 2n\ell^+ + m(2\ell^- + 1), \ell^+ - nk^+, -\ell^- - 1 - mk^-)} \\
& \frac{B_n^+ A_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- + n(2\ell^+ + 1) - 2m\ell^-, \ell^+ + \frac{1}{2} + nk^+, \ell^- - \frac{1}{2} - mk^-)} \\
n > m : & (h + n^2 k^+ + m^2 k^- + n(2\ell^+ + 1) + 2m\ell^-, \ell^+ + \frac{1}{2} + nk^+, \ell^- - \frac{1}{2} + mk^-) \\
n \leq m : & (h + n^2 k^+ + m^2 k^- + 2n\ell^+ + m(2\ell^- + 1), \ell^+ + nk^+, \ell^- + mk^-) \\
& \frac{B_n^+ A_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- + n(2\ell^+ + 1) - 2m\ell^-, \ell^+ + \frac{1}{2} + nk^+, -\ell^- - \frac{1}{2} + mk^-)} \\
& \frac{B_n^+ B_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- + n(2\ell^+ + 1) - 2m\ell^-, \ell^+ + \frac{1}{2} + nk^+, -\ell^- - \frac{1}{2} + mk^-)} \\
n > m : & (h + n^2 k^+ + m^2 k^- + n(2\ell^+ + 1) + 2m\ell^-, \ell^+ + \frac{1}{2} + nk^+, \ell^- - \frac{1}{2} + mk^-) \\
n \leq m : & (h + n^2 k^+ + m^2 k^- + 2n\ell^+ + m(2\ell^- + 1), \ell^+ + nk^+, -\ell^- - 1 - mk^-) \\
& \frac{A_n^+ A_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- - 2n\ell^+ - m(2\ell^- + 1), -\ell^+ + nk^+, \ell^- - mk^-)} \\
n > m : & (h + n^2 k^+ + m^2 k^- - 2n\ell^+ - m(2\ell^- + 1), -\ell^+ + nk^+, \ell^- - mk^-) \\
n \leq m : & (h + n^2 k^+ + m^2 k^- - n(2\ell^+ + 1) - 2m\ell^-, -\ell^+ + nk^+, \ell^- - \frac{1}{2} - mk^-) \\
& \frac{A_n^+ B_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- - 2n\ell^+ + m(2\ell^- + 1), -\ell^+ + nk^+, \ell^- + mk^-)} \\
& \frac{B_n^+ A_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- + n(2\ell^+ + 1) - 2m\ell^-, -\ell^+ + nk^+, -\ell^- - 1 - mk^-)} \\
n \geq m : & (h + n^2 k^+ + m^2 k^- + n(2\ell^+ + 1) + 2m\ell^-, -\ell^+ + nk^+, -\ell^- - 1 - mk^-) \\
n < m : & (h + n^2 k^+ + m^2 k^- + 2n\ell^+ + m(2\ell^- + 1), -\ell^+ + nk^+, -\ell^- - 1 - mk^-)
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
& \frac{A_n^+ A_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- - 2n\ell^+ - m(2\ell^- + 1), -\ell^+ + nk^+, -\ell^- - 1 + mk^-)} \\
n > m : & (h + n^2 k^+ + m^2 k^- - 2n\ell^+ - m(2\ell^- + 1), -\ell^+ + nk^+, -\ell^- - 1 + mk^-) \\
n \leq m : & (h + n^2 k^+ + m^2 k^- - n(2\ell^+ + 1) - 2m\ell^-, -\ell^+ + nk^+, -\ell^- - \frac{1}{2} + mk^-) \\
& \frac{A_n^+ B_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- - 2n\ell^+ + m(2\ell^- + 1), -\ell^+ + nk^+, -\ell^- - 1 - mk^-)} \\
& \frac{B_n^+ A_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- + n(2\ell^+ + 1) - 2m\ell^-, -\ell^+ + nk^+, \ell^- - \frac{1}{2} - mk^-)} \\
n \geq m : & (h + n^2 k^+ + m^2 k^- + n(2\ell^+ + 1) + 2m\ell^-, -\ell^+ + nk^+, \ell^- - \frac{1}{2} + mk^-) \\
n < m : & (h + n^2 k^+ + m^2 k^- + 2n\ell^+ + m(2\ell^- + 1), -\ell^+ + nk^+, \ell^- + mk^-) \\
& \frac{B_n^+ A_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- + n(2\ell^+ + 1) - 2m\ell^-, -\ell^+ + nk^+, -\ell^- - \frac{1}{2} + mk^-)} \\
& \frac{B_n^+ B_m^- |\Omega\rangle}{(h + n^2 k^+ + m^2 k^- + n(2\ell^+ + 1) + 2m\ell^-, -\ell^+ + nk^+, -\ell^- - \frac{1}{2} - mk^-)} \\
n \geq m : & (h + n^2 k^+ + m^2 k^- + n(2\ell^+ + 1) + 2m\ell^-, -\ell^+ + nk^+, -\ell^- - \frac{1}{2} - mk^-) \\
n < m : & (h + n^2 k^+ + m^2 k^- + 2n\ell^+ + m(2\ell^- + 1), -\ell^+ + nk^+, -\ell^- - 1 - mk^-)
\end{aligned} \tag{B.2}$$

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Figure Caption

Figure 1: Multiplet structure for a massive representation of $\hat{A}_7$ in the R-sector. The multiplet is labelled by (ℓ^+, ℓ^-) as indicated. These are *not* to be confused with the quantum numbers $(\ell_\pm^+, \ell_\pm^-)$ of $|\Omega_\pm\rangle$, cf. text. States $|\Omega_\pm\rangle$ satisfy $(2.3)(2.4)(2.5)$. The Witten index is zero. Cf. (I) (11) for the analogous picture in the A_7 algebra.

Figure 2: Multiplet structure for a massless representation in the R-sector. Here $|\Omega_+\rangle \equiv |\Omega_-\rangle \equiv |\Omega\rangle$. The picture pertains to the situation where both ℓ^+ and ℓ^- are greater than zero. When one of those vanishes, there is only a single $SU(2)^+ \otimes SU(2)^-$ multiplet. The Witten index is always $2(\ell^+ + \ell^-) + 1$.

Fig. 1

Fig. 2