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CONFORMAL FIELD THEORY FOR FOUR-DIMENSIONAL STRINGS*

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ABSTRACT

The purpose of these lectures is to summarize what can be learned about four-dimensional strings from a general point of view, using only conformal field theory, and no particular construction method. It is especially emphasized how the bosonic string map and currents with simple fusion rules streamline the discussion.

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1. INTRODUCTION

In the last few years many ways of constructing four-dimensional string theories have been presented. In the beginning most work was focused on understanding special properties of each construction, but after a while general features started to emerge. One may quarrel about the elegance, simplicity, or usefulness of particular constructions, but it is now well understood that they are all just “representations” of the same set of two-dimensional symmetries, known as (super)-conformal field theory. The purpose of these lectures is to give an introduction to the construction of four-dimensional strings from the point of view of (super)-conformal field theory, rather than to explain any particular construction in detail.

There are clear advantages to seeing the greater picture instead of studying explicit constructions. This can be illustrated with many papers dealing with explicit constructions in which “theorems” are proved which in the worst cases are manifestly false, and in other cases are either trivial consequences of conformal field theory that could just as easily have been proved in general, or are only true for special constructions and therefore not very interesting. There are many features of four-dimensional strings that allow a completely general discussion, such as the presence of gravitons and dilatons, the possible spins of massless particles, the possibilities for extended supersymmetries, the absence of tachyons and the vanishing of the one-loop partition function for superstrings, the fact that massless vector bosons are gauge bosons, the possibility or impossibility of having Higgs bosons in the adjoint representation, properties of so-called (2,2) models such as the presence of moduli, anomaly cancellation, etc. Many of these properties have been “proved” for special constructions, sometimes with great effort, but even with a very limited understanding of conformal field theory most of them are obvious.

One fact that only becomes clear by studying explicit constructions is the extent of the problem. Indeed, the number of explicit realizations of the consistency conditions of string theory is huge, even if one does not count continuous parameters. One usually thinks of all these theories as different vacua of one and the same theory (or perhaps three theories, the universal type-II, heterotic and bosonic string theories). This point of view is strongly supported by many known connections between different theories, especially via compactification to lower dimensions. Unfortunately there does not yet exist a formulation of some universal string theory that admits this interpretation. The idea that all four-dimensional strings are vacua of the same theory is philosophically appealing since one can still maintain that the theory (though not its vacuum) is unique, but it does not help much with the practical problem of finding the vacuum relevant to our world, assuming it exists.

There exist differences of opinion regarding the relevance of the presently available constructions of four-dimensional strings. Very few people doubt that the supersymmetric strings are consistent to all orders in perturbation theory, although this has not been proved rigorously. However, when it comes to non-perturbative effects one is left to speculation. Undoubtedly non-perturbative effects must exist, first of all because they exist in theories like QCD that are supposed to be embedded in string theory, and secondly because some perturbative string physics (unbroken supersymmetry and a massless dilaton) is in manifest disagreement with reality. Is it reasonable to assume that non-perturbative effects will

only solve our problems without affecting our successes? Of course no-one can answer this question, and if one wants to be pessimistic one could even argue that the complete set of good features of heterotic strings might be wiped out by non-perturbative effects. People who take this attitude should not work on string theory. On the other hand, optimists who want to continue working on string theory have no choice but to assume that the promising features of heterotic strings are not some sort of *fata morgana* leading an entire generation of physicists astray. By studying conformal field theory we are studying classical solutions that are stable to all orders in perturbation theory. Such an exercise is usually worthwhile, and much better than one can do in many other situations in physics.

Apart from its use in applications to string theory, there are other motivations for studying conformal field theory. It is a remarkable fact that in an entirely different field of physics, statistical mechanics, exactly the same mathematical structures appeared as in string theory. This is probably an indication that these structures are interesting and important in their own right. Finally, perhaps the most important and least often mentioned reason for working on conformal field theory is that it is simply a lot of fun.

These notes consist of three parts, essentially corresponding to the three lectures (although slightly more material is included). The three parts correspond to the three main consistency conditions for four-dimensional strings: conformal invariance, modular invariance and super-conformal invariance. The discussion in the first two chapters is limited to bosonic strings, for the simple reason that all relevant features of conformal and modular invariance already appear there. Fermionic strings are then discussed in the third chapter.

There are already many reviews on conformal field theory, and I will avoid duplicating them as much as possible. Some basic knowledge of string theory is assumed, and some familiarity with at least one method for constructing string theories will help greatly in appreciating the general results. The basics of string theory are explained for example in [1]. A standard reference to the essential concepts of conformal field theory is [2]. Many issues discussed here are discussed in more detail and illustrated for the special case of lattice constructions in [3]. Reviews emphasizing in part other aspects of conformal invariance and the construction of string theories are for example [4,5], and a nice recent review of conformal field theory with more emphasis on statistical mechanics is [6]. Conformal field theory is used in string theory for two different (though not unrelated) purposes, namely the description and classification of “string ground states”, and as a tool in string perturbation theory. The latter application will not be discussed here, but has been the subject of many review papers; see for example [7,8].

2. CONFORMAL INVARIANCE

2.1 Conformal invariance for bosonic strings

We begin by explaining the basic idea of four-dimensional string construction for closed bosonic strings. The propagation of such a string is governed by a function $X^\mu(\sigma^0, \sigma^1)$ that specifies the space-time position of every point of the string (denoted by σ^1) for any value of the proper-time, σ^0 . Here μ labels the space-time coordinates, $\mu = 1, \dots, d$, where d is the space-time dimension. For closed strings this function satisfies the boundary condition $X^\mu(\sigma^0, 0) = X^\mu(\sigma^0, \pi)$. When it propagates through space-time the string sweeps out a two-dimensional surface parametrized by σ^0 and σ^1 . This surface is called the *world-sheet*, in analogy to world lines for point particles. World-sheets without boundaries correspond to vacuum graphs, without in and out-going states. They are distinguished topologically by their number of handles; this number is called the *genus*. The world-sheet of a propagating non-interacting closed string has the shape of a cylinder.

One can also regard the functions $X^\mu(\sigma_0, \sigma_1)$ as a set of d scalar fields, which are functions of a two-dimensional coordinate σ^α , $\alpha = 0, 1$. To specify the propagation of the string we have to specify an action, which from the two-dimensional point of view defines a two-dimensional scalar field theory with d scalar fields. An important restriction on this action is that it should be invariant if one chooses a different set of coordinates on the world-sheet. This is called *reparametrization invariance*, and it is of course the same as two-dimensional general coordinate invariance. The simplest action having this property is

$$S \propto \int d^2\sigma \sqrt{(-g)} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu, \quad (2.1)$$

where $g_{\alpha\beta}$ is the world sheet metric and g its determinant. This action is not only general coordinate invariant, but has an additional invariance under Weyl-transformations $g_{\alpha\beta} \rightarrow \Lambda(\sigma^\alpha) g_{\alpha\beta}$.

Using general coordinate transformations, one can bring the world sheet metric to the form $e^{\phi(\sigma_0, \sigma_1)} \delta_{\alpha\beta}$. It is instructive to write this in terms of complex coordinates $z = \sigma_0 + i\sigma_1$, in which the metric may be written as

$$g_{zz} = g_{\bar{z}\bar{z}} = 0; \quad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2} e^\phi.$$

This form is preserved by *conformal transformations*

$$z \rightarrow f(z), \quad \bar{z} \rightarrow f(\bar{z}),$$

where f is an analytic function. Such a transformation does change the function ϕ , but ϕ can always be brought back to its original form by a Weyl rescaling. Hence the theory has a symmetry, which is called *conformal invariance*. This symmetry is of great importance for making string theory manageable, and we will therefore insist on maintaining it in what follows.

The point is now that (2.1) is by no means the only action that has the required symmetries. To understand the possible generalizations, we will first rewrite the conformal transformations in a more algebraic form. One usually formulates conformal field theory on the complex plane by making a conformal transformation $z = e^w$. Hence the point $z = 0$ corresponds to $\sigma_0 = -\infty$, and loops around the cylinder become circles around the origin. Conformal transformations on the complex plane are generated on the fields in the theory by the energy-momentum tensor $T(z)$ (we will discuss this in more detail in a moment). For example, the energy momentum tensor for free bosons can be derived from the action by standard techniques, and one finds $T(z) = -\frac{1}{2} \partial X(z) \partial X(z)$. Actually this is only half of the energy momentum tensor, since there is also an anti-holomorphic term, $T(\bar{z})$. The full energy-momentum tensor is thus $T(z, \bar{z}) = T(z) + T(\bar{z})$, but for most practical purposes one can concentrate on the first term alone (for simplicity we are ignoring here a technical complication having to do with the zero modes). The energy-momentum tensor can be expanded in modes: $T(z) = \sum_n z^{-n-2} L_n$. These modes satisfy the *Virasoro algebra*

$$[L_n, L_m] = (n-m) + \frac{c}{12} n(n^2 - 1) \delta_{n+m}. \quad (2.2)$$

There is a separate Virasoro algebra for left- and right-movers. In the following, quantities with a bar refer to left-movers, so that for example the left-moving Virasoro generators are denoted as $\bar{L}_n$.

The parameter c appearing in (2.2) is called the central charge. If one computes the commutator classically using Poisson brackets, one finds that c vanishes. In the quantized theory the central charge of d free bosons does not vanish, but is equal to d . This modification of the classical commutation relations by quantum corrections is an anomaly, called the conformal anomaly. If it is not cancelled, it ruins conformal invariance, which makes string quantization considerably more difficult, if not impossible. Fortunately the d bosons are not the only field that contribute to the central charge. If one carefully quantizes the theory one has to introduce Faddeev-Popov ghosts for two-dimensional coordinate invariance, a local symmetry. The corresponding *reparametrization ghosts* b and c have their own Virasoro generators L_n^{ghost} , which must be added to the Virasoro generators of the free bosons. It is the total Virasoro generator that embodies conformal invariance, and hence the sum of the central charges of the d bosons and the Faddeev-Popov ghosts must vanish. Since the ghosts contribute -26 , this implies that we must have 26 bosons, and hence 26 space-time dimensions.

However, nothing forces us to start with a two-dimensional theory consisting of just d free bosons. There are many other choices one can make for the two-dimensional action. The presence of the bosons X^μ was dictated by their rôle as space-time coordinates. But if we are interested in four-dimensional strings, we need only four of them, not 26. We can simply replace the remaining 22 bosons by something entirely different, which gives the same contribution to the conformal anomaly. In other words, we look for different representations of the same algebra, with central charge 22. We will refer to the bosons X^μ plus the ghost b and c as the *space-time* sector of the theory, and to all the remaining field as the *internal*

sector. The latter is some conformal field theory $C_{26-d;26-d}$, where the subscripts denote the conformal anomaly of the left- and right-moving internal Virasoro algebras.

The internal conformal field theory has to satisfy some further conditions. One of them is unitarity. What we mean by this is that the internal CFT's must be built out of unitary highest weight representations of the Virasoro algebra. These are representations in a Hilbert space such that the Virasoro generators L_n satisfy $L_n^\dagger = L_{-n}$, and with an L_0 spectrum that is bounded from below. One consequence of unitarity is that c_L and c_R cannot be negative, which limits the space-time dimensions d from above by 26. The only non-unitary CFT's that we allow to appear in bosonic strings are the b, c ghost system, and the time-like component of X^μ . Their non-unitarity is precisely needed to get a unitary space-time theory. Any additional violation of unitarity of the world-sheet theory would probably ruin the unitarity of the space-time theory.

Furthermore the internal CFT should be compact, i.e. the spectrum of L_0 should be discrete. If this were not the case the mass spectrum observed in d dimensions would be continuous, which to a d -dimensional observer would indicate the presence of extra, uncompactified (though not necessarily flat) dimensions. There is nothing fundamentally wrong with this, but it is not what we would call a d -dimensional string theory.

2.2 States and primary fields

Consider the internal CFT $C_{26-d;26-d}$. The left-moving and right-moving Virasoro generators satisfy (2.2) with $c = 26 - d$. The Hilbert space is most conveniently described in terms of the eigenvalues of the operator L_0 . Obviously $L_n, n < 0$ increases the L_0 eigenvalue when acting on a state. Since, by assumption, the L_0 spectrum is bounded from below, one can distinguish two kinds of states: the *highest weight states* $|\bar{h}, h\rangle$, which satisfy

$$\begin{aligned} L_0 |\bar{h}, h\rangle &= h |\bar{h}, h\rangle \\ \bar{L}_0 |\bar{h}, h\rangle &= \bar{h} |\bar{h}, h\rangle \\ L_n |\bar{h}, h\rangle &= \bar{L}_n |\bar{h}, h\rangle = 0, \quad n > 0, \end{aligned}$$

and all the other states that are not annihilated by all the positive L_n 's. Hence all the latter states can be obtained from the highest weight states by the action of the operators $L_{-n}, n > 0$. These states are called the *descendants*.

The complete set of states obtained by acting on a given highest weight state with the operators $L_{-n}, n > 0$, form the *Verma-module* V_h of that highest weight state, i.e.

$$V_h = \{L_{-n_1} L_{-n_2} \dots\} |\bar{h}, h\rangle \quad n_i > 0.$$

(Some of these states may have zero or negative norm; we will discuss this later.) A Verma module is a representation of the Virasoro algebra, and is completely determined by the values of c and h , in the same way as Lie algebra representations are completely

determined by the highest weights (hence the name *highest weight states*). Since the right- and left-moving Virasoro algebras act independently, the complete set of states generated from $|\bar{h}, h\rangle$ is given by the direct product $\bar{V}_h \otimes V_h$ of the Verma-modules for left- and right-movers.

Among the highest weight states one always has a state $|0, 0\rangle$ with $h = \bar{h} = 0$. This state is annihilated not just by all $L_n, n > 0$, but also by L_0 and L_{-1} (and similarly for left-movers). This state is called the $SL(2, \mathbb{C})$ -invariant vacuum because these three operators generate a sub-algebra of the Virasoro algebra that is isomorphic to $SL(2, \mathbb{C})$.

The highest weight states $|\bar{h}, h\rangle$ are in one-to-one correspondence with *primary fields* or *conformal fields* $\phi_{h,h}(\bar{z}, z)$, which, by definition, must transform in the following way under conformal transformations $z \rightarrow f(z)$

$$\phi_{h,h}(\bar{z}, z) \rightarrow \phi'_{h,h}(\bar{z}, z) = \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \left(\frac{\partial f}{\partial z}\right)^h \phi_{h,h}(\bar{f}, f), \quad (2.3)$$

where $\bar{f}$ denotes $f(\bar{z})$. The parameters h and $\bar{h}$ are called the conformal weights of the field ϕ (the bar on h serves only to distinguish left- and right-movers, and does not mean complex conjugation; in fact h and $\bar{h}$ are real). In general conformal fields are not simply a product of a left- and a right-moving field, although in some special cases they do factorize in that way.

The highest weight states are obtained by acting with the primary fields on the vacuum at $z = 0$

$$|\bar{h}, h\rangle = \lim_{z \rightarrow 0} \phi_{h,h}(\bar{z}, z) |0, 0\rangle. \quad (2.4)$$

It is easy to verify that the requirements that $L_n, n > 0$ should annihilate a highest weight state and that $\phi_{h,h}$ is a primary field are equivalent. The descendants correspond to combinations of derivatives of the primary fields. Their operator product with the energy momentum contains poles of higher order in $(z - w)$.

An important piece of knowledge are the operator products of the primary fields, which have the general form

$$\begin{aligned} \phi_a(\bar{z}, z) \phi_b(\bar{w}, w) &= \sum_c (\bar{z} - \bar{w})^{\bar{h}_c - \bar{h}_a - \bar{h}_b} (z - w)^{h_c - h_a - h_b} \\ &\times C_{abc} \phi_c(\bar{w}, w). \end{aligned} \quad (2.5)$$

In such expressions z and w are always *radially ordered* (i.e. $|z| > |w|$) and all operators are normal ordered with respect to the $SL(2, \mathbb{C})$ invariant vacuum. The fields ϕ_c on the right-hand side can be primary fields or descendants, but knowledge of the coefficients C_{abc} for the primary fields is enough to determine all others.

As has been mentioned in section (2.1), the energy momentum tensor generates infinitesimal conformal transformations $z \rightarrow z + \epsilon(z)$. The precise meaning of this statement is the following operator product of the energy momentum tensor with a primary field

$$T(z)\phi_{h,k}(\bar{w}, w) = \frac{h}{(z-w)^2}\phi_{h,k}(\bar{w}, w) + \frac{1}{z-w}\partial_w\phi_{h,k}(\bar{w}, w) + \text{finite terms}.$$

Here "finite terms" are terms that form a power series in $(z-w)$, starting with a constant. One is usually only interested in the singular terms in the limit $z \rightarrow w$, since they contribute to contour integrals. There is an analogous expression for the left-moving component of the energy momentum tensor. With this operator product, one gets

$$\oint \frac{dz}{2\pi i} \epsilon(z) T(z) \phi_{h,k}(\bar{w}, w) = \delta_\epsilon \phi_{h,k}(\bar{w}, w),$$

where the right-hand side denotes the infinitesimal form of (2.3).

It is instructive to verify some of the above for a single free boson $X(z)$ (we consider here a *chiral* boson, which depends holomorphically on z). We have already seen that the energy momentum tensor is $T(z) = -\frac{1}{2}\partial X(z)\partial X(z)$. One can perform explicit computations of operator products by expanding the bosonic field into modes:

$$X(z) = q + ip \ln z + i \sum_{n \neq 0} \frac{1}{n} \alpha_n z^{-n}. \quad (2.6)$$

The modes satisfy the commutation relations

$$[q, p] = i \quad [\alpha_n, \alpha_m] = m\delta_{m+n,0}.$$

One can now compute the operator products explicitly by means of Wick's theorem, plus the propagator of a free boson

$$X(z)X(w) = -\ln(z-w) + :X(z)X(w):$$

which can be derived easily from (2.6). The normal-ordered term on the right-hand side is finite for $z \rightarrow w$, and vanishes of course if one takes the vacuum expectation value of both sides. It is left as an exercise to check that $\partial_z X(z)$ is a conformal field of weight 1, e^{ikX} a conformal field of weight $\frac{1}{2}k^2$, and that $\partial_z^2 X(z)$ and $T(z)$ are not conformal fields. An elementary explanation of the computation of operator products for free bosons may be found in lecture notes of Hermann Nicolai in the proceedings of this meeting three years ago [9].

2.3 The bosonic string spectrum

Quantization of a compactified bosonic string works in essentially the same way as for uncompactified strings, with which the reader is assumed to be familiar (see e.g. [1] for more details). The ground state is $c(0)|0\rangle$, where $c(z)$ is a reparametrization ghost field and $|0\rangle$ the $SL(2, \mathbb{C})$ invariant vacuum, annihilated by L_0, L_1 and L_{-1} . The states in the theory are created by vertex operators of the form $V(z)e^{ikX(z)}$, where V consists of fields of the internal sector, combined with polynomials of operators of the form $\partial^\mu X^\mu$ from the space-time sector. The physical states $c(0)V(0)e^{ikX(0)}|0\rangle$ of the bosonic string must be annihilated by the BRST operator Q , which has the form [7]

$$Q = \oint \frac{dz}{2\pi i} [cT + bc\partial c + \frac{3}{2}\partial^2 c]. \quad (2.7)$$

Here T is the energy momentum tensor T^{int} of the internal sector plus the energy momentum tensor T^{st} of all d fields X^μ (but not the ghosts). All manipulations involving the BRST charge depend only on the operator products of the energy-momentum tensor and the ghosts, which we required before to remain unchanged if part of the space-time sector is replaced by an internal one.

Since $Q|0\rangle = 0$, Q will annihilate all physical states if it commutes with the operators $cV e^{ikX}$, which leads to the condition that $V e^{ikX}$ must be a conformal field of weight one (in the left-moving as well as the right-moving sector).

The BRST conditions on the physical states can be solved explicitly by choosing light-cone gauge. This leads to an expression of ∂X^- in terms of the transverse components X^i ,

$$\partial X^- = \frac{1}{2p^+} \sum_{i=1}^{d-2} \partial X^i \partial X^i + \frac{1}{p^+} T^{\text{int}}, \quad (2.8)$$

This implies that the oscillator excitations of ∂X^- are a function of the transverse oscillators, and hence they do not create new physical states. For the zero-modes one gets the condition $L_0^{\text{st}} + L_0^{\text{int}} - 1|\text{phys}\rangle = 0$, which reduces in light-cone gauge to

$$\begin{aligned} L_0^{\text{int}} + \mathcal{N} - 1|\psi\rangle &= \frac{1}{2}m_R^2 \\ \bar{L}_0^{\text{int}} + \bar{\mathcal{N}} - 1|\psi\rangle &= \frac{1}{2}m_L^2 \\ m^2 &= m_L^2 = m_R^2, \end{aligned} \quad (2.9)$$

where

$$\mathcal{N} = \sum_{n=1}^{\infty} \alpha_n^\dagger \alpha_n^i,$$

which is nothing but the transverse part of L_0^{st} , expressed in terms of oscillators. The only way in which all this differs from the corresponding expressions for uncompactified strings

is that the sum on i runs only to $d-2$; the remaining terms ($i = d-1, \dots, 24$) have been replaced by the internal energy momentum tensor T^{int} and its modes L_n^{int} .

The lowest state in the spectrum is the vacuum itself, which is a tachyon with $m^2 = -2$. We will see in the next chapter that this state is inevitable in a modular invariant theory, and we have chosen its multiplicity equal to one. The space-time excitations of the vacuum yield a state $\alpha_{-1}^i \bar{\alpha}_{-1}^j$ of zero mass. This tensor product consists of a symmetric traceless tensor (the graviton), an anti-symmetric tensor (usually denoted B_{ij}) and a trace (the dilaton). Their multiplicity is equal to that of the vacuum, so that they are present in any bosonic string theory.

The rest of the spectrum depends on the choice one makes for the internal conformal field theory. Without trying to be specific, one can say the following. Apart from particles mentioned above, the only other states of zero mass or less that can occur are scalars $|h, h\rangle$, with $h \leq 1$ (these scalars are tachyons if $h < 1$) and vector bosons $\alpha_{-1}^i |1, 0\rangle$ or $\bar{\alpha}_{-1}^i |0, 1\rangle$. We will see in section (2.6) that the massless vector bosons are always gauge bosons of some compact Lie algebra, which in general is some semi-simple algebra plus some $U(1)$'s.

2.4 Virasoro Representation Theory

The representation theory of Virasoro algebras can in principle be worked out using techniques that are very similar to those used for determining the unitary representations of $SU(2)$. The discussion is more complicated, but the complications are only technical ones, not fundamental ones.

The derivation of the unitary representations of $SU(2)$ is done by defining highest weights $|m\rangle$ with $J_+ |m\rangle = 0$, and acting on them with J_- . One finds that $(J_-)^n |m\rangle$ is a state of negative norm for $n > 2m+1$, and has zero norm for $n = 2m+1$. Hence one gets a unitary representation only for m half-integer, by setting the zero-norm state $(J_-)^{2m+1} |m\rangle$ equal to zero.

The analogous procedure for Virasoro representations works as follows. The states in a Verma module do not necessarily all have positive norm. Those values of c and h for which negative norm states appear are of no interest for the internal sector of lower-dimensional strings, since the corresponding representations are not unitary. A necessary condition for the absence of negative norm states is $c \geq 0, h \geq 0$. A necessary and sufficient condition is $c \geq 1, h \geq 0$. In the region $0 < c < 1$ one obtains unitary representations only for special values of c and h [2], [10-12], corresponding to the so-called *minimal conformal field theories*

$$\begin{aligned} c &= 1 - \frac{6}{m(m+1)}, \quad m = 3, 4, 5, \dots \\ h &= \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}, \\ p &= 1, 2, \dots, m-1; \quad q = 1, 2, \dots, p. \end{aligned} \quad (2.10)$$

Finally for $c = 0$ the only unitary highest weight representation is the trivial one, $L_n |0\rangle = 0$ for all n [13].

A Verma module may furthermore contain *null vectors*, that is vectors $|\xi\rangle$ with zero norm. Such vectors satisfy $L_n |\xi\rangle = 0, n > 0$, and hence they are highest weight vectors of a submodule of V_h . The converse is also true: if V_h is not irreducible, it contains null vectors which are highest weight vectors of a submodule. One can always consistently set all null-vectors equal to zero to obtain an irreducible representation of the Virasoro algebra that is free of null-vectors.

Null-vectors of the Virasoro algebra appear only in the following cases

$$- c, h \text{ as in (2.10)}$$

$$- c = 1, h = \frac{1}{4}k^2, k \in \mathbb{Z}$$

Furthermore the vector $L_{-1} |\bar{h}, 0\rangle$ is always a null-vector, as one can easily show directly from the algebra. Therefore L_{-1} should always annihilate any state with $h = 0$. This shows in particular that the state $|0, 0\rangle$ is $SL(2, \mathbb{C})$ invariant, as claimed in section (2.2).

Note that for $c \geq 1$ we do not get any constraints on the allowed spectrum of h -values. Indeed, already for $c = 1$ all h values can occur. An explicit representation in terms of a single free boson is given by e^{ikX} , where $h = \frac{1}{2}k^2$, and k can take any real value.

2.5 Extended Virasoro Algebras

For $c < 1$ the Virasoro algebra alone is enough to determine all correlation functions on the sphere and presumably also on higher genus surfaces. This is due to the fact that one has a finite number of primary fields, but, more importantly, to the existence of null-vectors which imply that the correlation functions satisfy certain differential equations. If all the correlation functions (at arbitrary genus) can be computed exactly (at least in principle) one speaks of an exactly solvable conformal field theory.

Unfortunately the case of interest in string theory is $c \geq 1$ with $c \in \mathbb{Z}$, for which the minimal conformal field theories are not immediately useful. For $c \geq 1$ exact solvability is always related to some extra structure on top of the Virasoro algebra. This extra structure is usually due to some set of *chiral operators*, which are primary fields with either $\bar{h} = 0$ or $h = 0$. A chiral operator $\phi_{0,h}$ must commute with $\bar{L}_{-1}$, and hence it follows that $\partial_z \phi_{0,h} = 0$, i.e. the operator depends only on z . Therefore we can concentrate on the right-moving sector and we will henceforth denote the operator as $\phi_h(z)$. We are not requiring that $h \in \mathbb{Z}$ or that the ground state $|0, h\rangle$ is part of the spectrum. The set of operators $\phi_h(z)$ generates an operator product algebra. The structure of this algebra depends on the value of h . The most well-known cases are $h = \frac{1}{2}$ (free fermions), $h = 1$, (Kac-Moody algebras), $h = \frac{3}{2}$ (super-Virasoro algebras) $h = 2$ (which includes the Virasoro algebra) and $h = 3$ (the W -algebra [14]). Algebras with generators that have neither integer nor half-integer spin ("para-fermions") have also been considered [15]. One of the consequences of modular invariance, to be discussed in the next section, is that all operators in the theory (i.e. those primary fields $\phi(z, \bar{z})$ with the property that the state $\phi(0, 0) |0, 0\rangle$ appears in the spectrum) must satisfy $h - \bar{h} \in \mathbb{Z}$. It follows that fractional spin chiral operators cannot be part of the theory themselves. Nevertheless they may still play a rôle in determining the spectrum and the correlation functions.

The operator ϕ_h can be expanded in modes,

$$\phi_h(z) = \sum_r z^{-r-h} \phi_r; \quad \phi_r = \oint \frac{dz}{2\pi i} z^{r+h-1} \phi_h(z). \quad (2.11)$$

The allowed values of r depend on the state on which ϕ acts, and can be determined by computing the operator product of ϕ with the primary field σ creating the state. If

$$\phi(z)\sigma(0) = z^\alpha \sigma'(0) + \dots, \quad (2.12)$$

then $r \in \mathbf{Z} - \alpha - h$, in order to make the mode-integral well-defined. Note that descendants of σ yield the same condition on the modes. If α is not an integer the operator product is called *non-local*.

If the operator product is as in (2.12), the field ϕ picks up a phase $e^{2\pi i \alpha}$ when it moves along a contour around the position of σ . This behavior of ϕ is called the *monodromy* of ϕ with respect to the *twist field* σ ([4][16,17]). In general, not every phase is allowed. Obviously the z -dependence of the higher order terms on the right-hand side of (2.12) should differ by integral powers from the leading term; otherwise it makes no sense even to discuss any possible rôle of ϕ in the theory. For the same reason, the right-hand side and left-hand side of the operator product of the fields ϕ_h with each other must have the same monodromy. In other words, only twists that are an automorphism of the algebra are allowed.

The modes of ϕ have the following commutation relation with the Virasoro generators

$$[L_n, \phi_r] = \{n(h-1) - r\} \phi_{r+n}.$$

Using this relation and the unitarity of the Virasoro representation one can show that ϕ is Hermitian, i.e. $\phi_r^\dagger = \phi_{-r}$ (more precisely, when there is more than one field with a given value of h one can always choose Hermitian linear combinations.)

The operators ϕ_r may form a new algebra that extends the Virasoro algebra. Such an *extended Virasoro algebra* may organize the representations in a more efficient way. Typically, one defines primary fields with respect to the extended algebra rather than just the Virasoro algebra. Such fields must satisfy the conditions for Virasoro primary fields, but in addition they must have a certain operator product with the operators $\phi_k(z)$, in such a way that ϕ_r , $r > 0$ annihilates the corresponding ground states. In operator language this means that the most singular term in the operator product of $\phi_k(z)$ with a primary field must behave like $(z-w)^{-\eta}$, $\eta \leq h$. Descendants are defined by acting on these ground states with the operators L_{-n} , $n > 0$ and ϕ_{-r} , $r > 0$. In this way, some of the primary fields of the Virasoro algebra become descendants of the extended Virasoro algebra: the representations become larger, but there are fewer of them. Furthermore one typically gets additional null-vectors, and hence more constraints on the correlation functions.

If the number of representations of the algebra is finite one speaks of a *rational* conformal field theory (RCFT).

In the remainder of this chapter we give a brief description of some frequently used classes of conformal field theories.

2.6 Kac-Moody algebras

The most familiar example is the extension of the Virasoro algebra by $h = 1$ chiral operators. Consider a set of such operators with local mutual operator products, and denote them as J^a , $a = 1, \dots, D$. The operator product of two such operators must have the form

$$J^a(z)J^b(w) = \frac{\frac{1}{2}k\delta^{ab}}{(z-w)^2} + \frac{1}{(z-w)}if_{abc}J^c + \text{finite terms}. \quad (2.13)$$

The form of the first term is a matter of normalization. It has been shown in [14] that in a unitary conformal field theory the coefficients f_{abc} are (completely anti-symmetric) structure constants of a Lie-algebra. They are normalized here so that $f_{abc}f_{abc} = C(A)\delta_{ab}$, where $C(A)$ is the quadratic Casimir eigenvalue of the adjoint representation (normalized to N for $SU(N)$, $N+1$ for $Sp(2N)$, $N-2$ for $SO(N)$ and 4, 9, 12, 18, 30 for respectively G_2, F_4, E_6, E_7, E_8). This fixes the normalization of J_a , and leaves an as yet undetermined central charge k .

The modes of $J^a(z)$, defined as in (2.11), satisfy

$$[J_n^a, J_m^b] = if_{abc}J_{m+n}^c + \frac{1}{2}km\delta^{ab}\delta_{m,-n},$$

so that the zero-modes generate a Lie-algebra $\mathcal{G}$. The complete algebra is called an *affine* algebra or *Kac-Moody* algebra (for reviews of this subject see [18], [19]). One usually reserves this terminology for the non-abelian part of the algebra $\mathcal{G}$; there may of course also be $U(1)$ factors, for which f_{abc} is absent, so that k can be normalized to 1.

We have seen in section (2.2) that massless vector bosons in bosonic string spectra are always due to the presence of states $|0,1\rangle$ or $|1,0\rangle$. These highest weight states correspond to chiral operators with conformal spin 1, which always generate a Kac-Moody algebra (plus some $U(1)$ factors). This implies that the vector bosons interact like gauge bosons of the corresponding Lie group. Conversely, all Kac-Moody generators in the internal sector of a bosonic string manifest themselves via vector bosons.

Primary fields of the Kac-Moody extended Virasoro algebra are defined as

$$T(z)\phi(w) = \frac{h}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w}$$

$$J^a(z)\phi(w) = \frac{1}{z-w}t^a \phi(w),$$

where t^a is a representation matrix of an irreducible representation R of the Lie algebra $\mathcal{G}$. The primary fields create ground states from the vacuum which are annihilated by all

the positive modes of $J^a(z)$ as well as $T(z)$. One can build representations of the algebra by acting with the negative modes of $J^a(z)$ and $T(z)$. Using null-vectors one can find a relation between the value of h and the representation R

$$h(R) = \frac{C(R)}{C(A) + k}, \quad (2.14)$$

where $C(R)$ is the quadratic Casimir eigenvalue of the representation R in the normalization given above.

Unitary representations are obtained only for special choices of the representation R , and are completely determined by this representation. The Lie-algebra representation in its turn is completely determined by a highest weight vector $\vec{\lambda}$. The condition for unitarity of the highest weight representation of the Kac-Moody algebra is

$$\vec{\lambda} \cdot \vec{\psi} \leq k, \quad (2.15)$$

where $\vec{\psi}$ is the highest root (i.e. the highest weight of the adjoint representation, which is always a long root). We normalize it so that $\vec{\psi} \cdot \vec{\psi} = 2$. If one chooses this normalization, unitarity restricts the values of k to positive integers (in the literature one very often finds the definition $m = \frac{2k}{\vec{\psi} \cdot \vec{\psi}}$, where m is called the *level* of the representation. With our normalization, $m = k$).

In contrast to pure Virasoro algebras, the number of unitary highest weight representations is thus always finite for Kac-Moody algebras. This is especially helpful if the entire internal sector is spanned by products of KM algebras. To verify this one can compute the central charge of the Sugawara energy momentum tensor of the Kac-Moody algebra, which is an explicit realization of $T(z)$ in terms of the currents:

$$T(z) = -\frac{1}{2[C(A) + k]} \sum_a J^a(z) J^a(z), \quad (2.16)$$

where the product is defined by point-splitting and subtraction of the singular terms. It is straightforward to compute the central charge:

$$c(\mathcal{G}, k) = \frac{k \dim \mathcal{G}}{C(A) + k}. \quad (2.17)$$

In the previous expressions we assumed that $\mathcal{G}$ is simple. If it consists of more than one factor one simply has to sum the central charge over all factors. If the result equals $26 - d$ the entire internal sector can be written in terms of KM representations.

We know now what the possible gauge representations are for massless (and tachyonic) scalars in the bosonic string. In principle, one may encounter any representation R with $h(R) \leq 1$, where R must be a primary field (for tachyonic scalars the upper bound on $h(R)$ is of course lower, and the tachyonic ground state must be a singlet). For $k \rightarrow \infty$ (which is possible only for small groups, since otherwise c becomes too large) all representations are allowed in principle. Of course both c and h may receive contributions from additional factors of the internal sector, which with some cleverness may be tailored to complete their values to the desired answer. A trivial but somewhat important consequence is that massless scalars in the adjoint representation can occur only for Kac-Moody algebras of level 2 or larger.

There are also twisted realizations of Kac-Moody algebras. As explained in section (2.4), the allowed monodromies of the currents with respect to twist fields must be automorphisms of the algebra. In this case they must thus be automorphisms of the Lie algebra. The resulting algebras have currents $J_{r,r} \notin \mathbb{Z}$ and are known as twisted Kac-Moody algebras. For simple Lie algebras there are two kinds of automorphism: inner automorphisms (which are generated by elements of the Weyl group) and outer automorphisms (which correspond to symmetries of the Dynkin diagram, and which can be either the interchange of conjugate representations, or triality rotations in $SO(8)$).

2.7 Free Bosons

The foregoing discussion simplifies if the currents generate a product of $U(1)$ factors. Then each such $U(1)$ generator can be represented by a free boson X , so that $J(z) = \partial_z X$. Each $U(1)$ contributes 1 to the central charge. There is now an infinite number of primary fields that yield unitary representations: $\phi_p(z) = e^{ipX(z)}$, with $h = \frac{1}{2}p^2$, and charge p (where "charge" refers to the eigenvalue of J_0 on ϕ_p). Even though the number of primary fields is infinite, it is still advantageous to describe the spectrum in terms of $T(z)$ and $J(z)$ rather than $T(z)$ alone. One gets in that way the usual description of the bosonic string in terms of the modes α_n of the $U(1)$ currents $J(z)$. These modes act independently without creating null-vectors.

One may also consider twisted bosons. The possible twist fields are restricted by the requirement that they should respect the operator product $\partial X^a(z) \partial X^b(w) = \delta^{ab}(z-w)^{-2}$. For N bosons this allows all $O(N)$ rotations of the bosons into each other. For one boson there is obviously just one non-trivial possibility for the monodromy with respect to the twist field, namely $X \rightarrow -X$.

2.8 Free fermions

One may think of free fermions as an extension of the Virasoro algebra by operators with $h = \frac{1}{2}$. On general grounds one can conclude that the operator product of a set of fermions ψ^a can only have the form

$$\psi^a(z) \psi^b(z) = \frac{\delta^{ab}}{z-w}, \quad (2.18)$$

where of course we have chosen an orthonormal basis. This implies that the modes b_r^a satisfy $\{b_r^a, b_s^b\} = \delta^{ab}\delta_{r,-s}$. The monodromy of the fermions is restricted by the condition that (2.18) must be respected. This implies that the set of fermions may transform by an orthogonal transformation, $\psi^a \rightarrow \sum_b R_{ab}\psi^b(z)$, around some twist field σ . For a given σ the matrix R can be diagonalized by means of a unitary transformation. The fact that R is orthogonal implies that the eigenvalues come in complex conjugate pairs, plus in general some eigenvalues ± 1 .

If one has just one fermion, the only non-trivial automorphism is multiplication by -1 , i.e. fermions can be periodic or anti-periodic around $z = 0$. Anti-periodic fermions are said to belong to the Ramond sector, periodic ones to the Neveu-Schwarz sector (note that because of the conformal factor, all boundary conditions of half-integer spin fields are reversed when going from the cylinder to the complex plane). Primary fields for the Virasoro algebra, combined with one free fermion ψ can have at most a $(z-w)^{-\frac{1}{2}}$ singularity with ψ . There are just two possibilities for the value of h , namely $h = 0$ (for the vacuum) and $h = \frac{1}{16}$ (ψ is a Virasoro primary field, but it is not primary with respect to itself; extended symmetry generators never are.) The $h = \frac{1}{16}$ primary field has a square root branch cut in its operator product with ψ . Acting on the vacuum it creates thus a ground state for the Ramond sector.

2.9 Tensor products

A simple way to obtain solvable conformal field theories for $c > 1$ is to tensor several copies of elements of the discrete series (or other solvable CFT's). The Virasoro generators of the tensor product are simply the sum of the Virasoro generators of the components, and hence the central charge is the sum of the central charges of the components. One obtains automatically a class of extended Virasoro algebras, since the “orthogonal” linear combinations of the Virasoro-generators are chiral operators in the theory. For example, if one considers the tensor product of two commuting Virasoro algebras with generators L_n^1 and L_n^2 and central charge c_1 and c_2 , it is easy to see that the state $(c_2 L_{-m}^1 - c_1 L_{-m}^2)|0\rangle$ is annihilated by all the positive modes of $L_n = L_n^1 + L_n^2$. Hence all these states correspond to primary fields of the Virasoro algebra, but they become descendants if one adds the $h = 2$ operator $c_2 T^1(z) - c_1 T^2(z)$ (transforming like a conformal field with respect to $T(z)$) to the operator algebra. Obviously the only primary fields with respect to this extended algebra are the products of the primary fields of the original algebras, and they are finite in number if c_1 and c_2 are less than one. On the other hand, the unextended Virasoro algebra of the tensor product clearly has an infinite number of primary fields.

2.10 The coset construction

The coset construction is in a sense the opposite of the foregoing, since one attempts to use the difference rather than the sum of two Virasoro generators to define a new conformal field theory. It is not so clear how to make sense of such an operation in general, but one can define it for the special case of Kac-Moody algebras G and H , with H embedded in G . It has been shown in [12] that the operator $L_G - L_H$ defines a unitary CFT with central charge $c_G - c_H$. The primary fields are in principle labelled by two indices Λ and λ , belonging to primary fields of G and H respectively. Thus the labelling is exactly as for the tensor product of G and H . However, there is a complication. If the Kac-Moody decomposition of Λ does not contain λ at all, the corresponding primary field does not exist. We return to this in the next section.

3. MODULAR INVARIANCE

3.1 Consistency conditions for string loops

The conditions we have imposed so far (conformal invariance, unitarity, ...) are sufficient to guarantee consistency of the theory in local coordinate patches on the world-sheet. Already at tree level there are additional constraints to be satisfied, because for example the four-point functions of closed strings must be invariant under duality transformations (exchanges of the s , t and u channels). Consistency at higher loops imposes still more constraints, which have to do with the non-trivial topology of the world sheet. We summarize all these consistency conditions by the name “modular invariance”.

On all surfaces with at least one handle one can make coordinate transformations that cannot be connected continuously to the identity. These global transformations must be “divided out” of the path integral, so that one only integrates over inequivalent surfaces. If the partition and correlation functions were not invariant under modular transformations, one could not choose in an unambiguous way a region of integration that covers all inequivalent surfaces precisely once. Then the theory has a global anomaly, and does not make sense. One should not just have modular invariance for each genus separately, but the amplitudes at different genera should also have the correct relation with each other. In the limit where a surface falls apart into lower genus surfaces, the partition and correlation functions should factorize into the corresponding lower genus expressions. Finally, the contribution of the states in the spectrum to the one-loop partition function should be positive integers for space-time bosons and negative integers for space-time fermions, i.e. one should have a correct spin statistics relation.

These modular invariance conditions are certainly not all independent. For example, two-loop modular invariance is probably enough to guarantee multi-loop modular invariance; modular invariance of the partition function plus factorization probably guarantees modular invariance of the correlation functions; the spin statistics relation follows in special cases (see e.g. [20]) and probably in general, from multi-loop modular invariance. It was shown in [21] that a sufficient set of consistency conditions is duality of the four point function plus modular invariance of the one point functions at one loop.

The first non-trivial modular invariance condition appears for one-loop diagrams without external lines. To obtain the amplitude one computes a Polyakov path integral, where the path is a closed loop swept out by a closed string, *i.e.* a torus. In Euclidean space this path integral has the form

$$\int \mathcal{D}g \mathcal{D}\Phi e^{-S_E(\Phi)} .$$

Here Φ denotes all two-dimensional fields, and $\mathcal{D}g$ denotes the integral over all two-dimensional metrics, *i.e.* the different shapes of the torus. For an uncompactified bosonic string Φ is just the collection of fields X^μ , but in general it could be something far more complicated. In fact there are many conformal field theories where no formulation in terms of a Euclidean action S_E is known, and where the following manipulations have only a formal meaning.

To compute the path-integral one has to perform a Faddeev-Popov gauge fixing and introduce an integral over the ghost field b and c . The effect of this is to reduce the integral over the metrics to an integral over only those metrics that are not gauge-equivalent. On a torus, the latter are parametrized by a complex parameter τ , which must lie in the complex upper half plane (more details regarding the definition of τ can be found in nearly every review on string theory).

This leaves us with the Φ integral. To compute it, one can use a standard formula for Euclidean path integrals, relating the path integral to a trace over the exponential of the Hamiltonian. For a system with just one degree of freedom q the result is

$$\int \mathcal{D}q e^{-S_E(q)} = \text{Tr } e^{-\beta H} ,$$

for a Euclidean time-interval β . The only complication for string theory is that there is an infinite number of degrees of freedom, labelled by the parameter σ that parametrizes the point on the string. This is not an essential complication however, and one can show that for a torus of a shape parametrized by τ the path integral is

$$\int_{\tau} \mathcal{D}\Phi e^{-S_E} = \text{Tr } e^{2\pi i \tau L_0} e^{-2\pi i \tau \bar{L}_0} . \quad (3.1)$$

Here $L_0 + \bar{L}_0$ is just the two-dimensional Hamiltonian (the difference $L_0 - \bar{L}_0$ generates shifts in the σ_1 direction, and enters because the torus can close into itself with a displacement in the σ_1 direction.) Even if one does not have a Euclidean action, one can still use (3.1) to define the partition function.

In applications to lower-dimensional strings L_0 consists of a space-time term and a term from the internal sector. These two terms lead to a factorized expression for the partition function, so that the one-loop vacuum amplitude for a lower-dimensional bosonic string

theory has the form

$$\int \frac{d^2 \tau}{(\text{Im} \tau)^2} \mathcal{P}^{\text{st}}(\tau) \mathcal{P}^{\text{int}}(\tau) . \quad (3.2)$$

The fact that the internal and space-time contributions factorize will be proved in section (3.3).

The condition of modular invariance arises when one attempts to do the τ -integral. If one integrates over the entire upper half plane one is over-counting the set of inequivalent tori. The reason is that although we have factored the local coordinate transformations out of the path integral, there are also global ones (coordinate transformations that are not continuously connected to the identity) that have not yet been taken into account.

The upper half plane is covered by an infinite number of regions, each of which represents all truly inequivalent tori precisely once. These regions are mapped into each other by modular transformations, which for the torus have the simple form

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} , \quad a, b, c, d \in \mathbf{Z} , \quad ad - bc = 1 .$$

This entire group of transformations is generated by two basic transformations,

$$\begin{aligned} T : \quad \tau &\rightarrow \tau + 1 \\ S : \quad \tau &\rightarrow -\frac{1}{\tau} . \end{aligned}$$

We would like to restrict the τ integration to just one of the regions that covers all metrics. Clearly this makes sense only if one gets the same answer for every region. Hence the integrand in (3.2) must be invariant under modular transformations. It is sufficient to check invariance under just the two transformations S and T , since they generate everything. Note that the measure, $d^2 \tau (\text{Im} \tau)^{-2}$, is modular invariant by itself.

3.2 Characters

For each irreducible representation V of an (extended) Virasoro algebra one can define a character χ_V

$$\chi_V(q) = \text{Tr } q^{L_0 - \frac{c}{24}} , \quad (3.3)$$

where the trace is over all non-null states in the representation. From the expansion of $\chi_V(q)$ in powers of q one can read off the number of states with a certain L_0 eigenvalue.

In general only a subset of all the possible highest weight states appears in a given string theory. One can choose a set of non-negative integers M_{kl} which give the multiplicity of the state $|h_k, h_l\rangle$ in the spectrum of the internal CFT. These integers are constrained by the requirement of modular invariance.

From (3.1) and (3.3) one sees that the one-loop partition function for the internal sector can be expressed in terms of the characters and their multiplicities (after including a zero-point subtraction $\frac{\epsilon}{24}$ in (3.1))

$$\mathcal{P}^{\text{int}}(\tau) = \sum_{k,l} M_{kl} \mathcal{X}_k(\bar{q}) \mathcal{X}_l(q) , \quad (3.4)$$

where $q = e^{2\pi i \tau}$. We are assuming here that the left- and right-moving algebras are the same, so that for the left- and right-moving sectors one has the same set of characters. In general that does not have to be true.

To check modular invariance, we need the transformation of the characters under T and S . The transformation of the characters under T is obvious

$$\mathcal{X}_l(\tau + 1) = e^{2\pi i(h_l - \frac{c}{24})} \mathcal{X}_l(\tau) . \quad (3.5)$$

This implies that if $M_{kl} \neq 0$, then $h_l - \bar{h}_k \in \mathbb{Z}$. The characters have a more complicated transformation under S :

$$\mathcal{X}_l(-\frac{1}{\tau}) = \sum_k S_{lk} \mathcal{X}_k(\tau) , \quad (3.6)$$

where S is a unitary and symmetric matrix [22]. Furthermore the matrices S and T must form representations of the modular group. Hence

$$S^2 = C; \quad (ST)^3 = C .$$

Here C is the charge conjugation matrix of the two-dimensional field theory. A field ϕ_a is either self-conjugate ($C_{aa} = 1$, $C_{ab} = 0$ for all $b \neq a$) or conjugate to some other field ϕ_b ($C_{ab} = C_{ba} = 1$, all other matrix elements involving a and b vanish). [Note that applying $\tau \rightarrow -\frac{1}{\tau}$ twice yields τ , so that one might have guessed that $S^2 = 1$. However the transformation of τ does not provide a faithful representation of the modular group: it hides the fact that the direction of proper time on the world-sheet is inverted if S is applied twice. Hence in general $S^4 = 1$, but $S^2 = C$].

Invariance under the modular transformation S leads to the matrix condition $SM S^\dagger = M$, which is obviously satisfied if M is the identity (but in general this is by no means the only solution). The modular invariant $M_{ij} = \delta_{ij}$ is called the diagonal invariant. It exists only if the left- and right-moving algebras and characters are the same. If they are not the same it is usually quite difficult to find *any* modular invariant partition function.

An interesting consequence of invariance under S and positivity of the matrix M is that the multiplicity of the vacuum (M_{00}) is non-zero:

$$M_{00} = \sum_{k,l} S_{0k} M_{kl} S_{l0} . \quad (3.7)$$

All terms in this sum have the same sign, because of the following inequality [22], valid for

unitary CFT's

$$\frac{S_{0k}}{S_{00}} = \lim_{q \rightarrow 0} \frac{S_{kl} \mathcal{X}_l(q)}{S_{0l} \mathcal{X}_l(q)} = \lim_{q \rightarrow 1} \frac{\mathcal{X}_k(q)}{\mathcal{X}_0(q)} > 0$$

(with implicit sums on l).

3.3 The space-time sector.

The partition function for the space-time sector is completely determined by physical considerations. A flat space interpretation of the theory implies that every state can appear with any transverse momentum $k^i \in \mathbb{R}^{d-2}$. Hence the vertex operators $V(\vec{k}) = e^{i\vec{k} \cdot X(z, \bar{z})}$ must be part of the set of mutually local operators in the theory for any value of k^i . The existence of these operators implies that also the operators $\partial_z X$ and $\partial_{\bar{z}} X$ must be present, since they appear in operator products. Hence one is inevitably led to the usual description of the space-time sector in terms of oscillators α_n^i and $\bar{\alpha}_n^i$. One can describe the primary fields of the space-time sector in terms of the $(\partial_z X^i, \partial_{\bar{z}} X^j)$ charges (p_L^i, p_R^j) . The vertex operators $V(\vec{k})$ have $\vec{p}_L = \vec{p}_R = \vec{k}$. Locality of any other primary fields with respect to $V(\vec{k})$ for arbitrary $\vec{k}$ implies that $\vec{p}_L = \vec{p}_R$, so that nothing new can be added.

This demonstrates that the conventional way to construct the closed, oriented bosonic string and to combine it with the internal sector is indeed the only possible way. The Hilbert space of physical states is simply a tensor product of the internal CFT and a space-time CFT built out of $d-2$ left- and right-moving transverse oscillators. The contribution $\mathcal{P}^{\text{st}}$ to the partition function (3.2) can now be computed as follows. It is simply the diagonal invariant of the $U(1)^{d-2}$ algebra, written in terms of characters

$$\mathcal{X}_p(q) = q^{\frac{1}{2} p^2} \eta(q)^{-d-1} . \quad (3.8)$$

Since there is now an infinite number of characters, the diagonal invariant is an integral instead of a sum

$$\begin{aligned} \mathcal{P}^{\text{st}} &= \int \frac{d^{d-2} k}{(2\pi)^{d-2}} \prod_i \mathcal{X}_k(0|q) \mathcal{X}_k(0|\bar{q}) \\ &= (2\pi\sqrt{2} \text{Im}\tau)^{2-d} \eta(\tau)^{2-d} \eta(\bar{\tau})^{2-d} , \end{aligned}$$

which is indeed modular invariant.

Since the partition function of the space-time sector is separately modular invariant, the internal one must be separately modular invariant as well. We will now discuss various ways of constructing partition functions for the internal sector.

3.4 Modular invariants for free theories

It is fairly well understood how to build modular invariant partition functions for free bosons and fermions. The most general partition function for N free bosons with the property that all N operators ∂X appear in the spectrum as left- and right-moving chiral fields is given by an even Lorentzian self-dual lattice [23]. This is extensively explained elsewhere [3], and I will not repeat that discussion here.

The class of conformal field theories described by free bosons can be generalized further, from lattices to *orbifolds*. (Historically the name was associated with singular manifolds used in heterotic string compactification, but it has acquired a broader meaning.) Self-dual lattices are the most general CFT's in which all $2N$ $U(1)$ bosons are present in the spectrum as chiral fields. Orbifolds are characterized by the presence of twist fields, with respect to which some of the currents ∂X have non-trivial monodromy. This prevents these currents from being part of the spectrum, but not from governing some properties of the theory.

The simplest method to obtain modular invariant theories of this type is to choose a self-dual lattice which has some automorphism (which can be a discrete rotation (*twist*) or a translation (*shift*), or a combination of such operations). A new theory is then made by “modding out” by the automorphism, that is to remove all states from the spectrum that are not invariant under it. Whenever such an operation is performed in string theory, modular invariance requires that new states are added to replace the ones that were removed. These states (the *twisted sectors*) are built on ground states created by twist operators. The addition of twisted sectors is needed to guarantee invariance under the S -transformation. Invariance under T leads to an additional constraint on the allowed twist, known as *level matching* [24].

The problem of finding all positive modular invariant combinations of free fermion partition functions has been solved to a large extent in [25,26]. In order to construct modular invariants one has to define, in addition to the normal characters for the Neveu-Schwarz and Ramond ground states, also characters with an additional factor $(-)^F$ in the trace, where F is the two-dimensional fermion number. States created with an odd number of fermionic oscillators contribute to such characters with a $-$ sign. Altogether we thus get four characters, which correspond to the four spin-structures (the periodicities along the two independent non-contractible loops on the torus).

A single fermion has $c = \frac{1}{2}$. To get larger values of c one can tensor many such systems with each other, for left-movers as well as right-movers. For simplicity one may still allow only periodic and anti-periodic boundary conditions, although for more than one fermion this is not the most general solution. For $2N$ left- and right-moving fermions ($c = N$) one obtains 4^{2N} possible combinations of boundary conditions. The partition function is a linear combination of the corresponding 4^{2N} characters, and the coefficients must be chosen so as to get a modular invariant sum. This is known as *spin structure summation*. One can generalize this to boundary conditions defined by allowing arbitrary $SO(2N)$ rotations in the monodromy, for left- and right-movers independently. For a given ground state such a matrix can always be diagonalized (on a complex basis), so that one gets pairs of fermions with conjugate monodromy phases, plus, in general, some periodic and anti-periodic ones.

It is known how to construct modular invariant partition functions in the special case that all monodromy matrices of all ground states are simultaneously diagonalizable [25] [27].

In addition one may consider tensor products of free fermions and free bosons, and construct orbifolds by means of automorphisms that act on both. It is clear that the resulting set of modular invariant partition functions is huge, and that there are still many dark corners that have not yet been explored. The exploration has only been completed for $c = 1$ (relevant for 25-dimensional strings) [28-30].

3.5 Fusion rules

We now want to consider more general conformal field theories. To discuss their partition functions, we first have to develop some tools. A very helpful piece of information is provided by the *fusion rules* of the conformal field theory.

In the operator product of any two primary fields in a conformal field theory one finds on the right-hand side some combination of the primary fields of the theory. The fusion rules govern which fields can appear. One usually writes this as a formal expression

$$\phi_a \cdot \phi_b = \sum_c N_{abc} \phi_c. \quad (3.9)$$

Here N_{abc} is a positive integer, not to be confused with the operator product coefficient C_{abc} of (2.5). If $N_{abc} = 0$ the field ϕ_c cannot appear, i.e. $C_{abc} = 0$. If $N_{abc} = 1$ the field ϕ_c may appear, and hence C_{abc} may be non-vanishing. If N_{abc} is an integer larger than 1 this indicates that there is more than one possible coupling of ϕ_a and ϕ_b to ϕ_c . Roughly speaking, one may compare (3.9) with the Clebsch-Gordan series for tensor products of group representations.

A remarkable and very useful observation due to E. Verlinde [31] is the fact that the fusion coefficients can be expressed in terms of the matrix elements of the modular transformation matrix S :

$$N_{abc} = \sum_n \frac{S_{an} S_{bn} S_{cn}^*}{S_{0n}}$$

This relation was conjectured by Verlinde on the basis of intuitive arguments, and proved by Moore and Seiberg [32].

3.6 Modular invariant partition functions

One of the many unsolved problems in conformal field theory is that of finding all modular invariant partition functions for a given theory. In other words, given a set of characters with modular transformation matrices S and T , what are all the matrices M with positive integral coefficients so that $\chi M \chi$ is modular invariant?

Before summarizing what is known about this question, it should be pointed out that such a matrix M does not yet guarantee that one has a sensible conformal field theory. The matrix M only implies the existence of the one-loop vacuum diagram without external lines, whereas one should prove that all correlation functions can be defined, at arbitrary loop order (nevertheless, in most cases that have been studied the existence of M is sufficient). From now on we will focus on finding solutions for M , but one should keep in mind that there are other questions to be addressed.

One can distinguish two types of modular invariants. The first, which we will call *integer spin invariants*, have in the simplest cases the following form

$$\sum_j \left| \sum_{i=0}^{N_j-1} \chi_i^j \right|^2, \quad (3.10)$$

i.e. it is a sum of squares, each square containing N terms, where N is some integer. The characters χ_i^j are some subset of the total set of characters of the original conformal field theory, which must contain the identity (denote here as χ_0^0).

The matrices M belonging to the second class have the form $M_{ij} = \Pi_{ij}$, where Π is some permutation of order 2 of the characters. Note that in the latter case all characters appear, whereas in the first case some are “projected out”.

Modular invariants of the first type can be regarded as diagonal invariants of a larger chiral algebra. One can regard the primary fields belonging to the characters in the identity block ($\chi_i^0, i = 1, \dots, N-1$), as extra chiral currents, that are added to the algebra one already had. All the other squares in the sum are characters of this extended algebra, written as a sum of characters of a sub-algebra.

Modular invariants of the second type are always related to a discrete automorphism of the fusion rules, i.e. $N_{abc} = N_{\Pi(a)\Pi(b)c}$. It has been shown by Moore and Seiberg [21] that any modular invariant can be regarded as a diagonal invariant of some extended algebra, “twisted” by some automorphism of the fusion rules.

In practice one often encounters invariants that look more complicated than these two basic types. The first complication that can occur for integer spin modular invariants is that instead of (3.10) one gets something of the form

$$\sum_j \frac{N_0}{N_j} \left| \sum_{i=0}^{N_j-1} \chi_i^j \right|^2, \quad (3.11)$$

where N_j is a divisor of N_0 , the number of characters in the identity block. We will refer to character combinations for which $N_j < N_0$ as *fixed points* (for reasons that will be explained later). The coefficients $\sqrt{N_0/N_j}$ ensure that the matrix M is idempotent, up to normalization. If the coefficients are anything else, one can make new modular invariants by constructing polynomials in M . One can then always construct a polynomial so that one

gets a positive modular invariant without a vacuum state, which is in contradiction with (3.7).

Even if one extends the algebra, the partition function (3.11) is not manifestly a diagonal invariant. If $\sqrt{N_0/N_j}$ happens to be an integer, one can try to absorb this factor into the characters it multiplies. This is sometimes indeed the correct interpretation. However, if $\sqrt{N_0/N_j}$ is not an integer this clearly cannot work. For example the Kac-Moody algebra $SU(2)$, level 4 has a modular invariant $|\chi_0 + \chi_4|^2 + 2|\chi_2|^2$, where the subscripts denote twice the spin of the $SU(2)$ representation of the primary field. The correct answer in this case turns out to be that the last term represents two distinct primary fields χ_2^a and χ_2^b with the same character so that the modular invariant partition function should be interpreted as $|\chi_0 + \chi_4|^2 + |\chi_2^a|^2 + |\chi_2^b|^2$. We call this procedure “resolving the fixed point”. This is the correct answer in many cases, but note that this solution is in principle also available if $\sqrt{N_0/N_j}$ is integer.

To resolve such ambiguities it is sometimes helpful to construct the matrix S for the characters of the extended algebra. This can be done without any problems if all coefficients $\sqrt{N_0/N_j}$ are equal to one, but as soon as some characters correspond to more than one field the problem becomes far more difficult. The problem is to find the matrix elements of S among the fixed points. Having found a solution, one may check the fusion rules, to see if it is a sensible solution.

A second reason why modular invariant partition functions may sometimes look more complicated than the ones discussed so far is that one can have several modular invariants on top of each other. For example, one can use the characters of an extended algebra obtained from an integer spin modular invariant to build a new, “secondary” modular invariant. One may also multiply different matrices M , or take linear combinations of them. In terms of the original characters the result may look quite different from the simple cases discussed above. Not all such manipulations lead to a sensible conformal field theory, even though the matrices M are manifestly modular invariant.

3.7 Simple currents

The fusion rules contain more or less the same information as S , but since the fusion coefficients are integers, they are more pleasant to deal with. It was shown in [33] how one can derive many modular invariant partition functions directly from the fusion rules. The main ingredients in this construction are the *simple currents*, defined as follows.

Consider a conformal field theory C , which we will assume to be unitary and rational. Suppose there exists a primary field J which has unique fusion rules with all other fields in the theory, i.e.

$$J \cdot \Phi = \Phi'. \quad (3.12)$$

with just one term appearing on the right-hand side for any primary field Φ in the theory. Then we will refer to such a primary field J as a *simple current*.

There are also other, equivalent characterizations of simple currents. In general, any primary field Φ has a conjugate Φ^c , which is the (unique) field whose operator product with

Φ contains the identity (Φ^c may be equal to Φ). If the operator product $\Phi \cdot \Phi_c$ contains only the identity, then Φ is a simple current [34]. Yet another characterization involves the matrix S . In general, one can show that for any primary field Φ_a $S_{0a} \geq S_{00}$ [22]. If this inequality is saturated, then Φ_a is a simple current.

Consider now the fusion rule of J with itself. The right-hand side is unique and is either the identity or a new primary field, which we will denote as J_2 (it is easy to show that $J_2 \neq J$, and that J_2 is itself a simple current). One can now act with J on J_2 , and show by similar arguments that the result is either the identity, or a new simple current J_3 . By repeating this process one defines an orbit of simple currents, which inevitably (since the total number of primary fields is finite) returns to the identity after a finite number of steps. Let the order N of J be the smallest positive integer so that $J_N = 1$. Obviously the fusion rules for the set of fields $J_n, n = 0, \dots, N-1$ (where $J_0 = 1$, and $J_1 = J$) generate a group isomorphic to $\mathbf{Z}_N$.

The orbit generated by J may not exhaust the set of simple currents in the theory. Obviously, the complete set of simple currents in a conformal field theory forms an abelian group, with the fusion rule as group operation, and charge conjugation as the inverse. This group is isomorphic to $\mathbf{Z}_{N_1} \times \dots \times \mathbf{Z}_{N_k}$, and we will call it the center of the conformal field theory.

Consider now any simple current J of order N . Acting on the other primary fields, J and its powers J_n produce orbits which close back into themselves after N steps. The fields Φ_n produced by acting with $J_n, n = 1, \dots, N$ on a primary field $\Phi \equiv \Phi_0$ need not all be different, but clearly the total number of different fields that appears must be a divisor of N . Hence the current J combines all fields in the theory into orbits, whose length is N or a divisor (which may be 1) of N . These orbits can be thought of as representations of an extended algebra of which the current J is the generator.

Uniqueness of the fusion rules (3.12) implies that every field Φ has definite monodromy with the current J (the converse is of course not true). If $J(z)\Phi(w) \propto (z-w)^{-\alpha}\Phi'(w)$, then the monodromy of Φ with respect to J is $\alpha \bmod 1$. Because $J_N = 1$, Φ must have trivial monodromy with respect to N currents J , so that $\alpha = \frac{t}{N}, t \in \mathbf{Z}$.

Applying this to the monodromy of the current with respect to itself, we can derive an expression for the h -values of the currents, modulo integers. We get the following expression for $h_n \equiv h(J_n)$

$$h_n = \frac{rn(N-n)}{2N} \pmod{1}, \quad (3.13)$$

where for odd N the parameter r is defined modulo N , whereas for even N it is defined modulo $2N$. The precise value of r depends on the conformal field theory under consideration.

The L_0 spectrum on an orbit generated by acting on some field Φ is completely determined by its monodromy $\frac{t}{N}$ with J and its L_0 eigenvalue $h(\Phi)$. For currents satisfying the

charge conjugation condition given above, the result is

$$h(\Phi_n) = h(\Phi) + \frac{rn(N-n)}{2N} - \frac{tn}{N} \pmod{1}. \quad (3.14)$$

We will call $\frac{t}{N} \pmod{1}$ the charge $Q(\Phi)$ of the field Φ . It is easy to show this charge is conserved in correlation functions. Using (3.14) one can compute the charges of all other fields on the orbit. The result is

$$Q(\Phi_n) = \frac{t}{N} + \frac{rn}{N}. \quad (3.15)$$

One can define such a charge for any current. The charge Q_n belonging to the current J_n is simply equal to $nQ \pmod{1}$.

3.8 Modular invariants from simple currents

In the previous section we have shown how the matrix elements of T on the same orbit are related to each other. One can also derive such a relation for the matrix S [33][34]. To simplify the notation we denote a primary field Φ_a as “ a ”, and the field $J^n\Phi_a$ as J^na . For S we then find the following expression.

$$S_{J^na, J^nb} = e^{2\pi i \beta Q(a)} e^{2\pi i \alpha Q(b)} e^{2\pi i \frac{rn}{N}} S_{ab}. \quad (3.16)$$

This formula can now be used to show that the following matrix defines a modular invariant [33]

$$M_{J^na, J^nb} = \sum_{l=1}^N \delta_{ab} \delta_{\beta, \alpha+l} \delta^l [Q(a) + \frac{2\alpha+l}{2N}r], \quad (3.17)$$

for r even. Note that for odd N one can always choose r even, since r is defined modulo N . For even N and odd r however there is no modular invariant partition function. (This can be understood in terms of a level matching condition for orbifolds). It is not difficult to show that this matrix M defines a partition function that is also invariant under T .

There is also a more intuitive orbifold-like construction of most of these invariants. Consider a simple current J and the corresponding charge Q . This charge is conserved, and defines a symmetry that can be used as if it were a target space symmetry in orbifold constructions (which in general it is not). One starts (for example) with the diagonal invariant. The untwisted sector consists of all fields with $Q = 0$, paired with themselves. Now one adds the twisted sectors, by acting on the states in the untwisted sector $N-1$ times with $J_L J_R^c$ (here N is the order of J and J^c its conjugate). This obviously leads to an off-diagonal left-right pairing.

If the current J has integral spin, one gets a modular invariant partition function of the form (3.10). If there are orbits of length N_a smaller than N , the corresponding square appears with a factor N/N_a . In the latter case the current J (or some power of J) must have a fixed point: $J \times \phi = \phi$.

If the current has fractional spin, one gets an automorphism invariant, or a combination of an integral spin invariant and an automorphism invariant (these combinations appear whenever some power of J has integer spin).

In the remainder of this section we summarize what is known about the modular invariant partition functions of the most commonly used conformal field theories.

3.9 Modular invariants of Kac-Moody algebras

Although a fairly simple expression exists for the matrix S that represents $\tau \rightarrow -\frac{1}{\tau}$ on the Kac-Moody characters [35], the problem of finding all modular invariant partition functions is still far from solved.

In principle one would like to know all modular invariant partition functions for all possible left-right combinations of Kac-Moody algebras. Most work on this subject has however been concentrated on invariants for identical and simple left- and right-moving algebras, which is unfortunately by no means the most general case.

At present, the following modular invariant partition functions for simple Kac-Moody algebras are known.

- The diagonal invariant.
- The charge conjugation invariant. This is an automorphism invariant, which exists because charge conjugation defines an automorphism of the fusion rules. Indeed, the matrix $M = C$ defines a manifestly modular invariant partition function.
- Modular invariants generated by simple currents [33] [36] [37]. These can be integer spin or automorphism invariants, or combinations thereof, depending on the spin of the simple current. (These invariants correspond probably to Wess-Zumino-Witten models on group manifolds divided by the center. [19] [38].)
- Conformal sub-algebras. If a Kac-Moody algebra H is embedded in another Kac-Moody algebra G with the same central charge, H is called a conformal sub-algebra of G . The diagonal invariant (or any other invariant) of G looks non-diagonal when written in terms of characters of H . The extended algebra is simply the extension of H to G by spin-1 currents [39]. All such invariants can be constructed systematically, since the conformal sub-algebras have been classified [40].
- Exceptional automorphism invariants. There are three known examples of automorphism of the fusion rules that are not related to simple currents. They occur for $SU(2)$, level 16, $SU(3)$ level 9 [21] and $SU(5)$, level 5 [41]. In all these cases, the invariant is built on top of an integral spin invariant obtained with simple currents.
- “Very exceptional” integer spin invariants. There is one strange example of an integer spin invariant that is not explicable by simple currents, nor by conformal embeddings. This example occurs for F_4 , level 6 [33] [42].

In some cases the first three categories overlap, so that the same modular invariant has more than one interpretation. In addition to the above, one may also consider linear combinations of products of different matrices. For example, one can always multiply any solution M with C to get a new invariant that is formally different.

Only for the $SU(2)$ KM-algebra have all positive invariants been classified for all values of k . The solutions admit a so-called *A-D-E classification* in terms of the Dynkin diagrams of the simply laced algebras (i.e. A_n, D_n and E_6, E_7, E_8) [43], but this is only to be understood as a labelling of the solutions; the simply-laced algebras themselves do not play any known rôle. In terms of the list shown above, these invariants can be described as follows. The A invariants are diagonal; the D -invariants can be obtained by simple currents; the E_6 and E_8 invariants correspond respectively to the conformal embeddings $SU(2) \subset SO(10)$ and $SU(2) \subset G_2$ and the E_7 invariant is the exceptional automorphism invariant at level 16 that was mentioned above. It is an extremely interesting question how this generalizes to other Kac-Moody algebras.

3.10 Modular invariants for tensor products

An obvious way to obtain modular invariant partition functions for tensor products is to tensor modular invariants of the products. However, this is certainly not the most general solution. Using simple currents one can construct many invariants that are not of this type. The product of two simple currents J_1 and J_2 of different factors in the tensor product is of course a simple current. In general, the modular invariant partition function it generates is *not* the tensor product of the ones generated by J_1 and J_2 . Thus the space of modular invariant partition functions can become very large for tensor products. One has not only $M(J_1)$ and $M(J_2)$, but also $M(J_1 J_2)$ plus all possible combinations of products of these three matrices.

Since it is not true that the simple current invariants factorize, there is no reason to expect this to happen for other invariants either, and indeed they do not. For example, the tensor product of the G_2 and F_4 level-1 Kac-Moody algebras has a modular invariant that corresponds to E_8 level-1, even though the two algebras separately have only diagonal invariants. This invariant can be derived from the conformal embedding $E_8 \supset G_2 \times F_4$ (one can get many other examples in this way), but there might very well exist exceptional invariants that can not even be obtained in that way. If true, this is an unpleasant complication since it means that to classify all modular invariant partition functions of tensor products, it is not sufficient to do so for the factors separately (the simple current invariants do not give rise to such classification problems, since for them we *do* have a systematic construction).

3.11 Modular Invariants for coset theories

There is a close analogy between the problem of constructing modular invariant partition functions for tensor products and for coset theories. The characters of the coset theory are usually defined in terms of the *branching functions* of the Kac-Moody embedding $G \supset H$. We write the characters of G as χ_Λ and those of H as χ_λ , where Λ and λ denote primary fields of these algebras. The branching functions b satisfy

$$\chi_\Lambda(\tau) = \sum_{\lambda} b_{\lambda}^{\Lambda}(\tau) \chi_{\lambda}(\tau). \quad (3.18)$$

These functions can in principle be computed by decomposing the Kac-Moody representation Λ level-by-level into representations of the Lie algebra of H , and recombining these representations into primary fields of H and their descendants. From (3.18) we can read off the modular transformation properties of the branching functions in terms of those of the characters of G and H .

$$\begin{aligned} b_{\lambda}^{\Lambda}(\tau + 1) &= e^{2\pi i(h_{\Lambda} - h_{\lambda})} b_{\lambda}^{\Lambda}(\tau) \\ b_{\lambda}^{\Lambda}\left(-\frac{1}{\tau}\right) &= \sum_{\lambda', \lambda''} S_{\Lambda\lambda'}^G (S_{\lambda\lambda'}^H)^* b_{\lambda'}^{\Lambda'}(\tau), \end{aligned} \quad (3.19)$$

in a self-explanatory notation.

Thus we see that the branching functions transform as if they were characters of a tensor product of G with some CFT H^c (which we call the *complement* of H) whose modular transformation matrices are $(S^H)^*$ and $(T^H)^*$. Hence if we understand the problem of finding modular invariant partition functions for tensor products, we understand it for coset theories as well.

There is a technical complication however. It happens very often that some branching functions vanish for the simple reason that the H -representation λ never appears in the decomposition of the G representation Λ or its descendants. This in its turn leads to a second complication: if some branching functions vanish, some others will be identical to each other (this phenomenon is called *field identification*) [44]. None of this affects the modular invariant partition functions directly: they will still be modular invariant. However, since in particular the identity branching function b_0^0 is identical to one or more others, one ends up with a partition function in which the multiplicity of the identity is larger than 1. If one is lucky one can divide the entire partition function by an integer to normalize M_{00} to one, but there are many cases where this division would introduce unacceptable fractional coefficients.

A solution to the latter problem is proposed in [41]. The conclusion is that first of all one can get all the modular invariant partition functions by treating the coset theory formally as a tensor product $G \times H^c$. There is a subclass of modular invariant partition functions in which the vanishing branching functions do not appear at all. This is achieved by building

an integer spin modular invariant that precisely “projects out” the vanishing branching functions, while combining the identical ones into multiplets. This subclass contains all distinct matrices M_{ij} that yield modular invariant partition functions. So there are no real subtleties with the matrices M , but there is a problem with the characters. It turns out that it is *not* true that they are simply equal to the branching functions: in general, some of them have to be modified in a definite and rather interesting way.

4. SUPERCONFORMAL INVARIANCE

4.1 Conformal Invariance for fermionic strings

In this chapter we will consider string theories which have space-time fermions in their spectrum (we will refer to them as *fermionic strings* in the following; some authors use this terminology for strings constructed using world-sheet fermions). Fermionic strings must, in addition to reparametrization invariance, have another two-dimensional gauge invariance, local supersymmetry. In principle one can have local (p, q) supersymmetry [45], where p is the number of supersymmetries of the left-moving modes and q the number of supersymmetries of the right-movers. In order to get four-dimensional strings one needs $p \leq 1, q \leq 1$. The existence of a supersymmetry in one or both chiral sectors of the world-sheet theory implies the existence of extra world-sheet fields. The scalars X^μ need fermionic partners ψ^μ , the Neveu-Schwarz-Ramond (NSR) fermions. Furthermore the existence of an additional local symmetry implies the existence of additional ghosts, usually denoted β and γ . There is one such pair of ghosts for every supersymmetry.

In the same way as for bosonic strings, we will refer to the two-dimensional fields described so far as the *space-time* sector, as it determines the Lorentz representations of the space-time states. Thus there are two possible space-time sectors for each world-sheet chirality, namely an $N = 0$ system (with fields X^μ, b, c), and an $N = 1$ system with fields $X^\mu, b, c, \psi^\mu, \beta, \gamma$. In addition there may again be an internal sector, which is some conformal field theory whose conformal anomaly cancels that of the space-time sector (in units so that each scalar X contributes 1 to the conformal anomaly, each Majorana-Weyl fermion ψ contributes $\frac{1}{2}$, b and c together -26 and β and γ together 11). Denoting the internal sector as $C_{d,c;n}$, where c_L and c_R are the conformal anomalies in the left- and right-moving sector respectively, we can have the following three classes of d -dimensional string theories

$$\begin{array}{lll} \text{Bosonic:} & (N=0)_L \ (N=0)_R & + \ C_{26-d;26-d} \\ \text{Heterotic:} & (N=0)_L \ (N=1)_R & + \ C_{26-d;15-\frac{3d}{2}} \\ \text{Type II:} & (N=1)_L \ (N=1)_R & + \ C_{15-\frac{3d}{2};15-\frac{3d}{2}} \end{array}$$

This also specifies our convention for left- and right-movers in the heterotic string. The internal sectors are again unitary CFT's, with positive central charges. Hence $d \leq 26$ for the bosonic string, and $d \leq 10$ for heterotic and type-II strings.

4.2 The bosonic string representation

In constructions of fermionic strings (i.e. heterotic and type-II) strings, one encounters two new problems: the ψ^μ, β, γ system, and the presence of an $N = 1$ world-sheet supersymmetry. The first of these problems is quite easy to deal with. It implies the following

general structure of the partition function for heterotic strings

$$\mathcal{P}(\bar{\tau}, \tau) = \mathcal{P}^{\text{st}} \sum_{\alpha} \mathcal{P}_{\alpha}^{\text{NSR}}(\tau) \mathcal{P}_{\alpha}^{\text{ghost}}(\tau) \mathcal{P}_{\alpha}^{\text{int}}(\bar{\tau}, \tau) .$$

Here $\mathcal{P}^{\text{st}}$ is identical to the corresponding function for the bosonic string, $\mathcal{P}^{\text{NSR}}$ represents the contribution of $d-2$ NSR-fermions ψ^μ , $\mathcal{P}^{\text{ghost}}$ contains the contribution of the remaining two fermions plus the β, γ ghosts, and $\mathcal{P}^{\text{int}}$ represents the contribution of the internal sector. The summation is over all spin-structures, and the general form of the expression is valid for any genus (with the period matrix denoted as τ). Our main interest is in the contribution of the internal sector. In general, this contribution may be different for different NSR spin structures. Under modular transformations these factors must transform into each other in such a way that the complete expression is invariant.

The main technical problem is the factor $\mathcal{P}^{\text{ghost}}$. Fortunately we do not need to know an explicit expression for it, but only its transformation properties. It is reasonable to assume that those are the same for any fermionic string since the ghost and fermion determinants are universal.

It is in fact convenient to replace $\mathcal{P}_{\alpha}^{\text{ghost}}$ by something simpler that transforms in the same way. The most useful replacement is [9] [46,47]

$$\mathcal{P}_{\alpha}^{\text{ghost}} \rightarrow \mathcal{P}_{\alpha}^{D_4} \mathcal{P}^{E_8} , \quad (4.1)$$

where the first factor is the character of a level-1 D_4 Kac-Moody algebra for spin structure α , and the second factor is the character of E_8 , (level 1). Note that the conformal anomaly of the β, γ ghosts plus two fermions is 12, the same as that of a level-1 $D_4 \times E_8$ KM-algebra. Hence this replacement preserves both conformal and modular invariance. It maps the non-unitary ghost system to a unitary conformal field theory. The resulting partition function satisfies all the consistency conditions to serve as the partition function of a bosonic string. The effect of (4.1) is to extend the $SO(d-2)$ KM-algebra generated by the transverse NSR-fermions to $SO(d+6)$, while keeping the same correlation with the spin-structure labels of the internal sector. Conversely any bosonic string theory with level-1 $SO(d+6) \times E_8$ KM-algebra (plus a realization of $N = 1$ world-sheet supersymmetry, to be discussed below) corresponds to a heterotic string theory. To summarize, for any heterotic string there exists a bosonic string which faithfully represents its properties. We will call this the *bosonic string representation* of a heterotic string, and it will often be convenient to work in this representation.

There is a similar map for type-II strings. The generic form of a type-II partition function is

$$\begin{aligned} \mathcal{P}(\bar{\tau}, \tau) &= \mathcal{P}^{\text{st}}(\bar{\tau}, \tau) \sum_{\alpha\beta} \mathcal{P}_{\alpha}^{\text{NSR}}(\tau) \mathcal{P}_{\alpha}^{\text{ghost}}(\tau) \\ &\times \mathcal{P}_{\beta}^{\text{NSR}}(\bar{\tau}) \mathcal{P}_{\beta}^{\text{ghost}}(\bar{\tau}) \mathcal{P}_{\alpha\beta}^{\text{int}}(\bar{\tau}, \tau) . \end{aligned}$$

Here one has the option of replacing one ghost system to obtain a heterotic string partition

function, or both to obtain a bosonic string partition function.

Hence the classification problem of heterotic and type-II strings is contained in that of bosonic strings, in the following way

$$\text{Bosonic Strings} \supset \text{Heterotic Strings} \supset \text{Type-II Strings} \quad (4.2)$$

We have discussed the conditions for modular invariance of bosonic strings in the previous section. Once the problem is understood there, one never has to worry about it again for fermionic strings.

4.3 Super-Virasoro algebras

The right-moving sector of heterotic strings, and both sectors of type-II strings should not just form representations of the Virasoro algebra, but of an $N = 1$ super-Virasoro algebra. This is an extension of the Virasoro algebra by a current T_F , which is a conformal field of weight $\frac{3}{2}$. It must have the following operator product with itself

$$T_F(z)T_F(w) = \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2T(w)}{(z-w)} \quad (4.3)$$

The form of T_F in the space-time sector is determined by Lorentz-invariance, so that the general form of T_F is

$$T_F(z) = i\psi^\mu \partial X_\mu + T_F^{\text{ghost}} + T_F^{\text{int}} \quad (4.4)$$

For the ghost part see [7]. Our main interest is in the internal part, which must satisfy (4.3) with $c = 15 - \frac{3d}{2}$.

Since T_F is a half-integer spin operator with $(\bar{h}, h) = (0, \frac{3}{2})$, it cannot appear in the spectrum itself. Hence one may expect that it has non-trivial monodromy with respect to some field in the theory. The possible twists are restricted by two conditions: they must form an automorphism of (4.3), and T_F must have definite monodromy with respect to all primary fields in the theory. The only possible automorphism is $T_F \rightarrow -T_F$, so that the states on which T_F can act fall into two classes, just as in theories constructed with free fermions: those on which T_F acts without generating branch cuts form the Neveu-Schwarz sector, and those on which T_F has a square root branch cut form the Ramond sector.

The super-Virasoro generator T_F is integer-moded on R-states and half-integer moded on NS states. The modes G_r of T_F have the following anti-commutator

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s} \quad (4.5)$$

The Ramond sector has a zero-mode operator G_0 which satisfies $G_0^2 = L_0 - \frac{c}{24}$. For unitary representations this implies that in the Ramond sector $h \geq \frac{c}{24}$. This lower bound is saturated

by all states that are annihilated by G_0 . These states are called the Ramond ground states. There may not be such a state; if there is not, supersymmetry is spontaneously broken. This does not mean that the resulting string theory is inconsistent, but it does imply that there are no massless fermions in the spectrum. Note that (4.5) also implies that the $SL(2, \mathbb{C})$ -invariant vacuum $|0, 0\rangle$ is annihilated by $G_{-1/2}$.

Primary fields with respect to the super-Virasoro algebra must be primary with respect to the Virasoro algebra, and have at most a $(z-w)^{-\frac{3}{2}}$ singularity in their operator product with T_F . In the NS-sector one gets then

$$\begin{aligned} T_F(z)\phi(w) &= \frac{1}{(z-w)}\phi'(w) \\ T_F(z)\phi'(w) &= \frac{2h}{(z-w)^2}\phi(w) + \frac{\partial_w\phi(w)}{z-w} \end{aligned} \quad (4.6)$$

Here h is the conformal weight of ϕ (this equation does not hold for $h = 0$, i.e. $\phi = 1$). The additional field ϕ' appearing here is itself a Virasoro primary field of weight $h + \frac{1}{2}$. The effect of the super-extension of the Virasoro algebra is thus to combine Virasoro primary fields into supermultiplets, whose members have weights h and $h + \frac{1}{2}$.

In the Ramond sector the super-Virasoro primary fields have a $(z-w)^{-3/2}$ singularity in their operator product with T_F , with the exception of the ground states, which have only a $(z-w)^{-1/2}$ singularity. This is easily seen to imply that all excited states come in pairs that are mapped into each other by G_0 . Only the ground state escapes this pairing. This pairing plays an important rôle in establishing Lorentz invariance in the chiral sector of the theory. Lorentz invariance forces massive fermions to appear in pairs of opposite chirality. The existence of a chiral partner for every massive fermion is guaranteed by world-sheet supersymmetry, which implies a pairing of all excited states as explained a moment ago.

4.4 World-sheet supersymmetry and modular invariance

So far we have argued that a d -dimensional heterotic string is obtained by constructing a d -dimensional bosonic string whose internal sector is

$$C_{26-d;26-d} = C_{26-d,15-\frac{3d}{2}} \times (SO(d+6) \times E_8)_R \quad (4.7)$$

where the right-moving part of the internal sector must be built out of representations of an $N = 1$ super-Virasoro algebra with $c = 15 - \frac{3d}{2}$. The supercurrent (4.4) is the sum of an internal and a space-time term, and must have definite monodromy with all fields in the theory.

One cannot simply tensor two $N = 1$ models to get a new $N = 1$ model. The tensor product would have fields that are products of Neveu-Schwarz and Majorana fields, on which the sum of the two supercurrents does not have definite monodromy. Clearly one would like to keep only those fields that are products of NS fields or products of R fields.

At first this requirement may seem incompatible with modular invariance, but the problem can be solved as follows. Think of the $N = 1$ model not in terms of super-*Virasoro* representations but in terms of *Virasoro* representations. This means that all the super-*Virasoro* doublets are split into two fields. In this theory the supercurrent is a primary field, and furthermore it is a simple current, since it takes any field to its supersymmetric partner.

Therefore we can build an integer spin modular invariant in the tensor product theory by using the product of the two supercurrents as a simple current. This current has spin 3, and projects out all products of primary fields that are not purely Neveu-Schwarz or Ramond. Hence we have now achieved our objective of building a modular invariant partition function with a complete $N = 1$ supersymmetry in the right-moving sector.

We are implicitly assuming here that there is already at least one modular invariant partition function known. If the left- and right-moving algebras are the same one can start with the diagonal invariant. However, there is no reason why the left- and right-moving algebras should be the same; indeed, this goes against the spirit of heterotic strings. However, if by any method one has managed to construct a modular invariant partition function, then one can always make it compatible with world-sheet supersymmetry in the way explained above.

4.5 Physical states

Just as for the bosonic string, the physical states of the fermionic string are subject to the BRST condition. The BRST charge of fermionic strings contains a few extra terms. First of all there is a term just like (2.7), but with an energy momentum tensor T that also includes the NSR fields ψ^μ and the β, γ ghosts. There are two additional terms: one term of the form $\oint \frac{dz}{2\pi i} \gamma(z) T_F(z)$, plus a pure ghost term that plays no rôle in the following.

The discussion of the consequences of the extra term involving the supercurrent is simplest if one bosonizes the $\beta\gamma$ ghost system as follows [7]

$$\gamma(z) = e^{i\phi(z)} \eta(z); \quad \beta(z) = e^{-i\phi(z)} \partial \xi(z).$$

Here ϕ is a boson with opposite metric, $\phi(z)\phi(w) = \log(z-w)$, and η and ξ form an auxiliary system whose properties do not matter in the following.

The canonical form of the vertex operators is now $e^{-i\phi} c V e^{ikX}$. The commutator with the first term in the BRST charge, of the form (2.7), vanishes as before if $e^{-i\phi} V e^{ikX}$ is a conformal field of weight 1 in the *Virasoro* algebra of all fields except b and c . Since the factor $e^{-i\phi}$ contributes $\frac{1}{2}$ to the weight, $V e^{ikX}$ must thus have weight $\frac{1}{2}$. The commutator with the second term in the BRST charge vanishes if and only if $V e^{ikX}$ is a super-*Virasoro* primary field. Hence the physical states created by $V e^{ikX}$ are annihilated by all positive modes of L_n and G_r . This imposes two constraints on the Hilbert space of states that can both be solved explicitly by going to light-cone gauge. This leads to an expression of ∂X^- and ψ^- in terms of the light-cone fields. There is no essential difference between

“compactified” and uncompactified strings here, except that the expressions for ψ^- and ∂X^- include fields from the internal sector.

One could in principle have started with vertex operators of the form $e^{iq\phi} V e^{ikX}$, and apply the BRST conditions to them. This would also select a set of physical states, but the constraints are in general realized in a far more complicated way. The choice $q = -1$ is the unique one for which the vertex operators are super-*Virasoro* primary fields. This choice properly which allows us to solve the constraints explicitly in light-cone gauge. This choice $q = -1$ is sometimes called the canonical ghost charge (other choices are connected to this one via the “picture changing” operation). In the Ramond sector the analogous canonical choice is $q = -\frac{1}{2}$.

4.6 Physical states in the bosonic string representation

In the bosonic string representation introduced above the procedure of choosing a ghost charge in order to determine the light-cone states can be performed extremely easily. Consider the $SO(d+6)$ KM-algebra that is created from the $SO(d-2)$ light-cone algebra by the map (4.1). One simply decomposes $SO(d+6)$ to $SO(d-2) \times SO(8)$, and keeps only those states that are 8_v or 8_s ground states of the $SO(8)$ Kac-Moody algebra (and that are also ground states of the additional E_8 factor in (4.1)). The states associated with 8_v transform as tensors of $SO(d-2)$, while those associated with 8_s transform as $SO(d-2)$ spinors (see [3] for a more detailed explanation).

One can arrive at the same answer by computing the light-cone partition function. The functions P_α^{ghost} can be calculated explicitly at genus 1. One finds

$$P_3^{\text{ghost}} = -P_4^{\text{ghost}} = -P_2^{\text{ghost}} = \pm P_1^{\text{ghost}},$$

where we have labelled the spin structures by θ -function indices. The relative signs between these functions are consistent with the rule derived above from the canonical ghost charge assignments: they imply that the states associated with 8_v of $SO(8)$ are indeed the bosons that contribute to the one-loop partition function. Furthermore they ensure that space-time spinors contribute with an extra $-$ sign relative to space-time tensors. Thus the correct spin-statistics relation is automatically obtained if one has the correct spin-statistics relation in the bosonic string.

The mass-formula for the fermionic string states is also most easily described in terms of their bosonic string representation: it is simply the bosonic string mass-formula (2.9). Here one should include the contribution $+\frac{1}{2}$ due to the 8_v and 8_s ground states.

Clearly there are only five $SO(d+6)$ representations that can give rise to massless states, namely the primary fields $(0), (v), (s)$ and (c) and the descendant $J_{-1}^a |0\rangle$ which transforms in the adjoint representation. All other descendants have an L_0 eigenvalue larger than 1, too high for a massless state. The singlet does not give physical states, since it does not contain an octet representation of $SO(8)$; the other four give respectively scalars, spinors, conjugate spinors and vectors. Of course these spins may still be modified by left-moving space-time excitations.

4.7 Classification of space-time supersymmetries

The heterotic and type-II string theories discussed so far are consistent at tree level, and up to one loop in perturbation theory if they are free of tachyons. It is a common belief that in order to have consistency to all orders in perturbation theory one should have local space-time supersymmetry. This is equivalent to having at least one gravitino in the spectrum.

In both heterotic and type-II strings gravitinos have to get their spinor index from the right-moving sector, and their vector index from the left-moving one (of course in type-II strings it may also be the other way around). The vector index is obtained by acting with $\tilde{\alpha}_{-1}^\mu$ in heterotic strings, and by acting with the left-moving $SO(d+6)$ KM-generator in type-II strings. In both cases a gravitino is thus necessarily generated from a state with $h = 1, \bar{h} = 0$, transforming in the spinor representation of the right-moving $SO(d+6)$ factor.

The primary field corresponding to this state is a $(0,1)$ operator, (i.e. an operator with $h = 1, \bar{h} = 0$) which is therefore a Kac-Moody generator. Thus $SO(d+6)$ must be an index-1 embedding in a level-1 KM-algebra such that some of the roots transform as $SO(d+6)$ spinors. It is elementary to find all the possibilities, and they are listed in table I (from [48]).

Type	Space-Time Dimension									
	10	9	8	7	6	5	4	3	2	
E_8	1	1	1	2	2	4	4	8	(8, 8)	
E_7	—	—	—	—	1	2	2	4	(4, 4)	
E_6	—	—	—	—	—	—	1	2	(2, 2)	
F_4	—	—	—	—	—	—	—	1	(1, 1)	
$Spin(M)$	—	—	—	—	—	—	—	—	$(M-8, 0)$ or $(0, M-8)$	

Table I: Complete classification of extended supersymmetry in heterotic string theories. Column 1 gives the Lie algebra into which $SO(d+6)$ is embedded. The other columns give the number of space-time supersymmetries in each dimension.

The generalization to type-II strings is completely straightforward. One simply gets all possible combinations of left- and right-moving supersymmetries allowed by table I. For example in four dimensions one can have type-II strings with $N = 0, 1, 2, 3, 4, 5, 6$ and $N = 8$ supersymmetry (there are two possible realizations of $N = 2$ and $N = 4$). It is not *a priori* obvious that all these possibilities can indeed be realized, but examples have recently been constructed [49,50].

Looking at table I one may have the impression that on every line all entries except the first one are just torus compactifications. This is indeed true as far as the supermultiplet structure is concerned. However, this does not mean that for example an $N = 4$ string theory in four dimensions (i.e. a theory of E_8 type) is necessarily a torus compactification (in a “stringy” sense) of a ten-dimensional theory. A counterexample to this tempting conjecture can be constructed as follows: take the ten-dimensional $E_8 \times E_8$ ten-dimensional heterotic string, compactify it to four dimensions on the $(D_6)_L \times (D_6)_R$ Lorentzian self-dual lattice, and make a twist that interchanges the two E_8 factors, combined with a shift by a vector weight of $(D_6)_R$ (this is analogous to construction of the E_8 level 2 string in [51]). The resulting theory satisfies all consistency conditions, has $N = 4$ space-time supersymmetry, but has a gauge group of rank 14 (namely $E_8(\text{level } 2) \times B_6$), whereas any torus compactification has a gauge group of rank 22.

4.8 Space-time supersymmetry and modular invariance

The foregoing discussion classifies all possible space-time supersymmetries, but it does not yet tell us how to build such theories. We face a similar problem as previously with world-sheet supersymmetry, namely how one can get a modular invariant partition function containing the $(0,1)$ field S that yields the gravitino. All this has to be achieved without ruining world-sheet supersymmetry. Clearly we are now looking for a modular invariant partition function of the conformal field theory (4.7) where the right-moving algebra is extended by two currents: the product of the space-time and internal supercurrents, and the $(0,1)$ field S .

The latter is built out of the $SO(d+6)$ spin field S^{st} (with $h = \frac{d+6}{16}$) plus some field S^{int} from the internal sector, which must have $h = \frac{10-d}{16}$, and belong to the Ramond sector. This value of h is precisely the smallest possible one in the Ramond sector, so that S^{int} is a Ramond ground state (this also shows that there can never be fermionic tachyons as a consequence of world-sheet supersymmetry).

We can easily put S in the extended algebra if it is a simple current. The space-time part of S is a simple current for even dimensions, which is fine since our interest is mainly in four dimensions. The remaining questions is thus whether the internal part of S , a Ramond ground state of an $N = 1$ super-Virasoro algebra, is a simple current. We will now show that there is such a Ramond ground state if and only if the $N = 1$ world-sheet supersymmetry in the internal sector can be extended to an $N = 2$ algebra (for simplicity we will only consider four-dimensional theories with $N = 1$ space-time supersymmetry here).

An $N = 2$ super-Virasoro algebra is an extension of the Virasoro algebra by two spin $\frac{1}{2}$ currents and a spin-1 current, which is the generator of a $U(1)$. It is well known that in such algebras the NS and R sectors are connected by spectral flow [52]. In particular the vacuum is connected to one of the Ramond ground states (with $h = \frac{c}{24}$). The structure of the fusion rules is preserved under spectral flow. In particular it follows that the Ramond ground state connected to the identity has simple fusion rules, because of course the identity has that property. This completes the first half of the proof.

* With thanks to L. Dixon for a bitnet-discussion from which this example resulted.

Suppose now that the $N = 1$ Ramond ground state has simple fusion rules. Then it can be used to build a string theory with $N = 1$ supersymmetry in four dimensions. In the description of table I this corresponds to an embedding $SO(10) \subset E_6$, which implies the existence of an additional $U(1)$ current ∂H in the theory, extending the Cartan sub-algebra of $SO(10)$ to that of E_6 . It is easy to see that the operator product of this current with the supercurrent T_F generates a new supercurrent T_F' . These two supercurrents generate an $N = 2$ world-sheet super-Virasoro algebra (note that only one of these two supersymmetries is local; T_F' generates a global supersymmetry). The current ∂H is the $U(1)$ generator in that algebra.

The second part of the argument shows that we get in any case an extension of the $N = 1$ algebra to $N = 2$ [53], even if we somehow manage to extend $SO(10)$ to E_6 with a current that is not simple. Most $N = 2$ algebras have many Ramond ground states, and only a few of them are simple currents. The simple currents can in any case be used for S^{int} , but nothing we have said rules out the possibility of using other Ramond states in “exceptional” invariants (although that possibility is unlikely and difficult to investigate).

Thus we conclude that if we are interested in building four-dimensional string theories with space-time supersymmetry, we can assume without loss of generality that the internal sector is a representation of a right-moving $N = 2$ super-Virasoro algebra. One must know at least one modular invariant partition function of this theory. Any modular invariant partition function can always be modified by simple currents so that it has both space-time and world-sheet supersymmetry.

There are many ways of building $N = 2$ CFT's that can be used for the internal sector. One can make them out of free fermions [54], free bosons [55-57], out of special coset theories [58] (this includes the minimal $N = 2$ theories with $c < 3$), and one can tensor such theories [59] with each other.

4.9 The one-loop superstring partition function

One would expect a string theory with gravitinos in its spectrum to be supersymmetric at all levels. For all supersymmetries in table I that are associated with exceptional algebras this is indeed easy to prove. One simply notes that physical states are obtained by projecting on the (8_v) and (8_s) representations of an $SO(8)$ sub-algebra, and that $SO(8)$ triality becomes an inner automorphism when $SO(8)$ is embedded in an exceptional algebra. For more details see [3][48]. This implies vanishing of the one-loop cosmological constant for all supersymmetric string theories (with $d \geq 3$) that have been discussed so far in the literature. At higher loops less rigorous statements can be made at present, but one can derive identities, similar in structure to Riemann-identities, that are satisfied by the internal sector partition functions of all superstrings based on exceptional algebras [48]. At genus one, such identities have also been obtained in [60,61]. For a discussion of the vanishing of higher genus partition functions see [62].

4.10 Four-dimensional heterotic string spectra

In table II we have listed all massless states and tachyons that one can in principle get in four-dimensional heterotic strings.

Right-movers		Left-movers		Mass	Particles
Space-time	Internal	Space-time	Internal		
$SO(10) \times E_8$	α_{-1}^μ	$\tilde{\alpha}_{-1}^\mu$	$\tilde{h}$		
$(45, 1)$	—	yes	0	0	graviton, $B_{\mu\nu}$, dilaton
$(45, 1)$	—	—	1	0	gauge boson (KM)
$(10, 1)$	—	—	$h + \frac{1}{2}$	$h - \frac{1}{2}$	tachyonic scalar
$(10, 1)$	—	—	$\frac{1}{2}$	1	massless scalar
$(10, 1)$	—	yes	0	0	gauge boson (super-KM)
$(16, 1)$	—	—	1	0	massless fermion (chirality +)
$(16, 1)$	—	yes	0	0	gravitino
$(\overline{16}, 1)$	—	—	1	0	massless fermion (chirality -)
$(\overline{16}, 1)$	—	yes	0	0	gravitino

Table II. All massless states and tachyons that can appear in heterotic string spectra.

This table should be fairly self-explanatory, and can easily be derived using the results of section (4.6). The mass is given in units so that the bosonic string tachyon has mass -1 . Tachyons can only occur in non-supersymmetric heterotic strings (and typically, though not always, they do). In superstrings they cannot occur: space-time supersymmetry requires tachyonic scalars to have fermionic partners, but world-sheet supersymmetry forbids fermionic tachyons (see section (4.8)).

As discussed in section (2.6), any $(0,1)$ operator in the theory is a Kac-Moody generator. In the bosonic string representation of any heterotic string one already has a level-1 $SO(10) \times E_8$ Kac-Moody algebra. Any additional $(0,1)$ operator transforms according to some representation of $SO(10)$ (and must be an E_8 -singlet). There are only four $SO(10)$ representations that can appear, namely (1) , (10) , $(\overline{16})$ and (16) since the others have $h > 1$. If all extra $(0,1)$ operators are $SO(10)$ singlets they generate a Kac-Moody algebra that appears as a direct product with $SO(10) \times E_8$. If there are extra $(0,1)$ operators that are not $SO(10)$ -singlets this implies that $SO(10)$ appears as a sub-algebra of a bigger algebra. By simple group-theoretical arguments one finds that the only possibilities are the embedding of $SO(10)$ in E_6 , E_7 or E_8 (leading to $N = 1, 2$ and 4 supergravity as discussed in section (4.7)), plus the embedding $SO(10) \subset SO(10 + M)$. In all these cases the spectrum is organized in terms of representations of the extended algebra, and one can obtain the contents of each such “supermultiplet” by simply decomposing representations and using table II.

The representations that are relevant for massless states are for $E_6 \supset SO(10)$

$$\begin{aligned}(27) &= (16) + (10) + (1) \\ (\bar{27}) &= (\bar{16}) + (10) + (1) \\ (78) &= (45) + (16) + (\bar{16}) + (1)\end{aligned}$$

for $E_7 \supset SO(10)$

$$\begin{aligned}(56) &= (16) + (\bar{16}) + 2(10) + 4(1) \\ (133) &= (45) + 2(16) + 2(\bar{16}) + 2(10) + 4(1)\end{aligned}$$

for $E_8 \supset SO(10)$

$$(248) = (45) + 4(16) + 4(\bar{16}) + 6(10) + 15(1)$$

for $SO(10 + M) \supset SO(10)$ (M even)

$$\begin{aligned}(10 + M) &= (10) + M(1) \\ (\text{spinor}) &= 2^{\frac{M}{2}} - 1 [(16) + (\bar{16})] \\ (\text{spinor}') &= 2^{\frac{M}{2}} - 1 [(16) + (\bar{16})] \\ (\text{adjoint}) &= (45) + M(10) + \frac{M(M-1)}{2}(1)\end{aligned}$$

and finally for $SO(10 + M) \supset SO(10)$ (M odd)

$$\begin{aligned}(10 + M) &= (10) + M(1) \\ (\text{spinor}) &= 2^{\frac{M-1}{2}} [(16) + (\bar{16})] \\ (\text{adjoint}) &= (45) + M(10) + \frac{M(M-1)}{2}(1)\end{aligned}$$

Note that the singlets never give rise to massless particles. From these decompositions one can immediately read off the content of each $N = 1$, $N = 2$ or $N = 4$ supermultiplet. It is clear that the extensions of $SO(10)$ to E_7 , E_8 and $SO(10 + M)$ can never give a chiral spectrum.

4.11 Gauge groups

There can be two kinds of gauge bosons in heterotic string theories. They are distinguished by the fact that they get their vector index from either the right- or the left-moving sector. The gauge group of the first kind is embedded in a Kac-Moody algebra, exactly as for vector bosons in the bosonic string. This Kac-Moody algebra is associated with the left-moving sector.

Since the central charge of the left-moving sector is 22, the Kac-Moody algebra must have $c < 22$ (it may be tensored with other CFT's without spin 1 currents to complete the central charge to 22). This means in particular that the rank of the gauge group must be smaller than 22. (Since the standard model has rank 4, heterotic strings are thus ruled out experimentally if one finds more than 18 extra "Z"-bosons. This is a completely general bound, but unfortunately not a very tight one.) The dimension of the gauge group can then be at most 946 (the dimension of $SO(44)$). The possible Kac-Moody levels are also bounded because the Kac-Moody central charge (2.17) should not exceed 22, but for small groups (such as $SU(2)$ and $SU(3)$) this still allows all possible levels. Nearly all model-building attempts in string theory have focused on the simplest possibility, namely that the standard model is a level-1 Kac-Moody algebra. However, the Kac-Moody level is an unknown parameter of the standard model, and there is *a priori* no reason why the level could not be larger than 1.

The second kind of gauge bosons are associated with $(0, \frac{1}{2})$ operators in the internal sector, that appear in combination with the vector conjugacy class of $SO(10)$. The combination yields a $(0,1)$ operator, which implies an extension of $SO(10)$ to $SO(10 + M)$, as we have just seen. The $(0, \frac{1}{2})$ operators are super-Virasoro primary fields, whose superpartners are a set of Kac-Moody generators of the internal sector. The zero-mode part of this Kac-Moody algebra is the gauge algebra of the M vector bosons. The dimension of this algebra can at most be 18. Thus we see that space-time vector bosons "gauge" a Kac-Moody algebra, although it works here in a slightly more complicated way than for the bosonic string (this system of Kac-Moody generators with the supercurrent is called a super-Kac-Moody algebra). All this is mainly of academic interest, since the extension of $SO(10)$ to $SO(10 + M)$ always implies that there are no chiral fermions in the spectrum. Gauge bosons of this kind occur also in $N = 2$ and $N = 4$ strings.

4.12 Representations

The restrictions on the possible representations of massless fermions and scalars in heterotic strings are exactly the same as for the bosonic string (we restrict ourselves here to gauge symmetries associated with the left-moving sector). The allowed representations correspond to the set of primary fields with $h \leq 1$ (see (2.14)). For sufficiently large level (always respecting the bound $c < 1$ of course) all representations can in principle occur. If on the other hand one chooses a level-1 realization of the standard model, then one can only have massless singlets and doublets of $SU(2)$ and singlets and triplets of $SU(3)$. Since this is precisely all that has been observed in experiment it is understandable why level 1 is the

favorite choice. However, even though higher level Kac-Moody algebras have primary fields in different representations, they need not appear in the massless sector of the theory.

Some authors prefer to obtain the standard model via an intermediate “grand unification” gauge group (usually E_6 , $SO(10)$ or $SU(5)$), which is broken below the Planck scale. In field theory approaches to grand unification one has the luxury of being able to choose the set of Higgs bosons one needs. This is not possible in string theory however. For any given theory that satisfies all consistency conditions the set of massless scalars is given, and cannot simply be modified. There has in particular been some interest in the question whether one can have scalars in the adjoint representation of the gauge group. The answer is extremely simple: it is not possible for level-1 Kac-Moody algebras, since the adjoint representation does not appear as a primary field; it is possible in principle in higher level Kac-Moody algebras since the adjoint representation is a primary field with $h < 1$. (In a recent paper it was shown that examples with massless scalars in the adjoint representation can indeed be constructed [63].)

4.13 The standard approach to the standard model

Up to now we have only imposed conditions on heterotic strings that are absolutely necessary for consistency and rudimentary phenomenology. In many papers a more restricted class of heterotic string theories is considered, which is more easily manageable in most construction methods, and which furthermore automatically has a reasonable spectrum. One should remember that this is only a special case, however.

These theories are usually called “(2,2) theories”, for the following reason. A generic heterotic string has no left-moving supersymmetries and one local right-moving one. One can denote this situation as $(0,1)$ world-sheet supersymmetry. It has become customary to denote also the presence of additional global supersymmetries in this way. For example, the four-dimensional heterotic strings with $N = 1$ space-time supersymmetry have an additional (global) world-sheet supersymmetry, and are therefore called $(0,2)$ theories. (This notation is somewhat inadequate, since $N = 2$ world-sheet supersymmetry is a necessary, but not sufficient condition for space-time supersymmetry. It is implicitly assumed that one has chosen a modular invariant partition function that contains a gravitino).

One can also formally define global supersymmetries associated with the left-moving sector. The best way to do that is to think of heterotic strings as indicated in (4.2). A very small subset of the class of heterotic strings can be regarded as the image under the bosonic string map of a type-II string. Type-II strings are only consistent if they have at least a local $(1,1)$ supersymmetry. Since the internal sector of the type-II string is not affected by the map, this property is preserved. This defines a small class of heterotic strings with $(1,1)$ world-sheet supersymmetry, which always have a gauge group containing at least $SO(d+6) \times E_8$. The existence of a world-sheet supersymmetry, inherited from the type-II string, implies that the Ramond and Neveu-Schwarz states of $SO(d+6)$ are paired respectively with the Ramond and Neveu-Schwarz states of the internal sector of the type-II string.

If the heterotic string has $N = 1$ space-time supersymmetry in four dimensions, the world-sheet supersymmetry is promoted to $(1,2)$. If furthermore the $SO(10)$ gauge group is a sub-algebra of $(E_6)_L$, the heterotic string is said to be a $(2,2)$ theory. In that case the type-II theory of which it is the image has $N = 2$ supersymmetry, with one gravitino coming from the left- and one from the right-moving sector. When the heterotic string is mapped to a bosonic string, such a bosonic string has an E_6 gauge-group in the left- as well as the right-moving sector.

There are two reasons for considering such theories. The first is that one automatically gets an E_6 (level 1) gauge group, which, although not ideal, is not a bad starting point for phenomenology. The second is that this class of conformal field theories can be constructed in a left-right symmetric way, starting from a bosonic string with an internal sector

$$(E_6 \times E_8)_R \times (E_6 \times E_8)_L \times C_{9,9} . \quad (4.8)$$

The internal CFT has both a left-moving and a right-moving $N = 2$ supersymmetry. Hence one can choose identical left- and right-moving representations of the $N = 2$ algebra and start with a diagonal modular invariant. This is of great help, since genuine “heterotic” modular invariants for left- and right-moving theories that are different are hard to find. It goes without saying that nature may not limit its choice of modular invariant partition functions to the easily accessible ones. Indeed, we know already that it does not, since (4.8) would imply a gauge group $E_6(\times E_8)$, which is not the observed standard model gauge group. Thus we need to break E_6 at the Planck scale or below, which in any case ruins the apparent left-right symmetry in (4.8). Note that the appearance of E_6 as a gauge group in these diagonal invariants is directly linked to the appearance of E_6 in the right-moving sector, which in its turn is present as a consequence of space-time supersymmetry and the bosonic string map. This is exactly the reason why E_6 appeared in Calabi-Yau compactification [64] after “embedding the spin connection into the gauge group”. It remains a miraculous fact that one gets so close to a phenomenologically reasonable spectrum in such a simple way.

The presence of left-moving world-sheet supersymmetries leads to some interesting consequences. Because of the $N = 1$ world-sheet supersymmetry on the left, one has an operator $\psi^a(\varepsilon)T_F(\varepsilon)$ in the theory, where ψ^a is a fermion in the vector representation of $SO(d+6)$, and T_F is the supercurrent in the left-moving internal sector of the type-II string (this operator is essentially the product of the internal and space-time supercurrent, but with the factor ∂X removed from the latter).

This operator has $h = 2$ and is therefore not present in the massless spectrum itself. However, its existence does have a less direct consequence, since it can be shown to map some massless states to other massless states. In particular a massless (27) or (27) representation is always mapped to a massless E_6 singlet.

This can be seen as follows. Decomposing E_6 to $SO(10)$ we find in every (27) or (27) a vector of $SO(10)$. Since we are considering massless states, this $SO(10)$ vector representation must appear in combination with a $h = \frac{1}{2}$ field $\Phi(\varepsilon)$ in the internal sector, which must be a primary field of the left-moving super-Virasoro algebra (because of world-sheet

supersymmetry Φ must be a NS-field, but a NS-field with $h \leq \frac{1}{2}$ cannot be a descendant). Then acting with $\psi^\alpha(z)T_F(\bar{z})$ on $(SO(10)$ vector, Φ) one gets $(SO(10)$ singlet, Φ'), where Φ' denotes the super-Virasoro descendant of Φ . The corresponding state is massless, since the mass is increased by $\frac{1}{2}$ in the internal sector, but decreased by the same amount in $SO(10)$.

In an $N = 1$ heterotic string with $(E_6)_L$ gauge group, a left-moving (27) or $(\bar{27})$ with $m_L = 0$ can only correspond to a physical state if it appears in combination with a chiral $N = 1$ multiplet (i.e. a (27) or $(\bar{27})$ of $(E_6)_R$ in the bosonic string representation) of the right-moving sector with $m_R = 0$. These massless chiral multiplets are the "families" and "anti-families", consisting of fermions and scalars in the (27) and $(\bar{27})$ of $(E_6)_L$. Because of the existence of the aforementioned spin-2 operator in the left-moving sector, each such family or anti-family is accompanied by a massless chiral multiplet, which is a singlet of $(E_6)_L$.

The scalars in these supermultiplets are called the *moduli*, a name that originates from Calabi-Yau compactification. They correspond to continuous parameters of the space of conformal field theories. This means that if one gives a vacuum expectation value to any of these scalars one obtains another consistent $(2, 2)$ string theory. Indeed, as one might expect, the potential of the moduli is flat to all orders in string perturbation theory. For reviews and references, see e.g. [4], [65].

Calabi-Yau compactifications provided the first examples of four-dimensional $(2, 2)$ theories, and here the moduli have a simple physical interpretation: they correspond to zero-modes of the metric, and parametrize the moduli-space of deformations of the manifold. It has been conjectured that in fact all $(2, 2)$ heterotic strings correspond to points in the moduli space of some Calabi-Yau manifold [66].

4.14 Anomalies

As a consequence of modular invariance, the fermion polygon anomalies of all chiral fermions in all theories discussed here factorize exactly in the same way as Green and Schwarz [67] discovered in ten dimensions:

$$A(F, R) = (Tr F^2 - Tr R^2)[X_{d-2}(F, R)], \quad (4.9)$$

where $A(F, R)$ denotes the usual character polynomial which parametrizes the anomaly and X_{d-2} denotes a polynomial of order $\frac{d-2}{2}$ in the gauge and curvature two-forms F and R . The traces are in the vector representation, or normalized to the vector representation.

This result was proved in [68] for theories constructed out of free fermions or bosons. It was pointed out in [69] that it holds for all gauge symmetries corresponding to Kac-Moody algebras, because of the modular transformation properties of the Weyl-Kac character formula. Since the extension to additional $U(1)$'s is trivial, the factorization of the anomaly has now been established for all string theories discussed in the present literature.

The remaining anomaly is cancelled by the cooperation of two terms in the effective action, both involving the rank 2 anti-symmetric tensor $B_{\mu\nu}$. One, the Chern-Simons term,

appears at string tree level and can be calculated quite straightforwardly for any string theory in any dimension. This term is always the same as in ten dimensions. The other term appears at one-loop order in string perturbation theory, and depends non-trivially on the details of the theory. It has been calculated in [70], but only for one kind of heterotic string construction, the covariant lattice construction of [46]. Needless to say, the result confirms the expectation of Green and Schwarz, and the anomalies cancel. It should be easy to redo this calculation with abstract Kac-Moody generators instead of the Frenkel-Kac operators used in [70], in order to establish also this part of the calculation for the entire class of heterotic strings (anomalies in type-II theories can be discussed in an analogous way).

An interesting phenomenon related to anomaly cancellation occurs in four dimensions. Here the factor X_2 in (4.9) can only be $Tr F$, and hence it can only be non-vanishing if some $U(1)$ generator in the theory is not traceless. If this term is present, the corresponding anomaly-cancelling term triggers a Higgs mechanism, so that the $U(1)$ gauge-boson becomes massive by absorbing the $B_{\mu\nu}$ field (which is a scalar in four dimensions). In a supersymmetric theory, this is a first indication of supersymmetry breaking, and indeed it can be shown that in such theories supersymmetry is broken by a Fayet-Iliopoulos D -term [71]. At two loops this leads to a non-vanishing cosmological constant [72], and at still higher loops presumably to infinities. Hence four-dimensional theories with such non-traceless $U(1)$'s are not good string theories. This is in fact the only known example where a string theory that is supersymmetric at tree level develops problems in perturbation theory.

Fortunately there are many four-dimensional strings that do not have this problem. Obviously it does not occur for theories with only simple factors in their gauge groups. Less obvious is the fact that it cannot occur in any string theory that has a level-1 E_8 factor in its spectrum (which includes all $(2, 2)$ theories). The reason is that modular invariance dictates that the traces in (4.9) must be over the entire gauge group, including the E_8 factor. But there is no non-trivial E_8 representation that can have a $U(1)$ charge and still have $h = 1$.

4.15 Type-II spectra

Almost all aspects of type-II spectra follow in a straightforward way from the discussion of heterotic strings given above. To get the massless states (and tachyons, if any are present) one simply combines the adjoint, vector and spinor representations of $SO(d+6)$ in all possible ways. The left-right combination of the adjoint representations yields the usual graviton, dilaton and $B_{\mu\nu}$, which are again inevitable. Combinations of a left-moving $SO(d+6)$ adjoint with a right-moving $SO(d+6)$ vector plus $(\bar{h}, h) = (0, \frac{1}{2})$ superfield yield gauge bosons (and analogously with left and right interchanged). The possible gauge groups are thus exactly the same as the ones that can be obtained from the right-moving sector of a heterotic string. Such gauge bosons do not have to be present, but if they are present in one chiral sector they prohibit chiral fermions from that sector. Consequently, the only chance of constructing chiral type-II strings in four dimensions (where a gauge group must be present in a chiral theory) is to have a gauge group in only one sector.

There is in fact another source of vector bosons in four dimensions, namely the tensor product of a left- and right-moving massless spinor. However, it can easily be shown that

these vector bosons do not couple to any other field (including themselves) in the low-energy limit. In particular they are thus gauge bosons of a set of $U(1)$'s.

The restrictions on the spectrum that follow from superconformal invariance are far more severe than for heterotic strings. In fact it has been proved that no superconformal type-II vacuum has the spectrum required for the standard model [73]. It is thus not correct to say, as is often done, that string theory has no verifiable predictions: an entire class has already been ruled out by its predictions.

5. FINAL REMARKS

In these lectures I have tried to concentrate on the forest and not on the trees. One can get a long way without ever considering any particular model or class of models, but at some point one has to start constructing concrete examples. There are some important properties that one can say only very little about from a general point of view. The most important among them is the number of massless fermion generations. The only thing one can presently say about this without explicit constructions is that there is nothing that forbids their existence. It requires special constructions to show that there actually are four-dimensional string theories with 3 generations and a "reasonable" gauge group (see e.g. [74-77] and [50]). One might hope that our understanding of conformal field theory develops to a point where it becomes possible to classify all such theories, but despite an extensive literature on the "classification" of conformal field theories, we are still very far from that.

The number of presently known theories with 3 generations is relatively small. This has certainly no fundamental significance, and may simply mean that we have only explored a rather special corner of the space of parameters. Nevertheless, if the number does not go up too much, it may just be possible to confront all of these theories with more stringent phenomenological requirements. One can only hope that at least some of them are in clear contradiction with experiment. Given the large number of unexplored possibilities, if some model does not disagree with experiment then this says probably more about our inability to extract accurate predictions from string theory than about the correctness of that model. (Unless, of course, we are extremely lucky.)

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