A theorem on $N = 2$ special Kähler product manifolds

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ABSTRACT

We consider Kähler manifolds of the special type which occurs in $N = 2$, $d = 4$ supergravity couplings of vector multiplets. A theorem is proven which says that if such a manifold is the direct product of several factors, then the manifold is one of the series $\frac{SU(n)}{U(1)} \otimes \frac{SO(n,2)}{SO(n) \otimes SO(2)}$. As an example of the use of this theorem we comment on the moduli of string theories on $(2, 2)$ vacua with enhanced gauge symmetry.

¹ This work was supported in part by the United States Department of Energy under contract DE-AT03-88ER40384, TASK E.
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CERN-TH.5543/89
UCLA/89/TEP/43
KUL-TF-89/27
September 1989
1 Introduction

Matter fields in $N = 2$, $d = 4$ supergravities can belong to hypermultiplets consisting of 4 real scalars (or a quaternion) and 4 Majorana spinors, or to vector multiplets which have a complex scalar, 4 Majorana spinors and a real vector. In the former case, the scalars parametrise a quaternionic manifold [1], in the latter case they define a Kähler space. The latter is not a generic Kähler space, but is of a special type [2]. The theorem which we will prove is applicable for such spaces. Their importance has been augmented during the last year by two developments. First, a relation has been established [3] between normal quaternionic spaces (see [4], where it is conjectured that these are all non-compact homogeneous quaternionic spaces), and such special Kähler spaces. Secondly, these manifolds occur [5,6,7] in the effective field theories around vacua of a string theory which has an internal sector with (2,2) or (1,0) world sheet supersymmetry. This concerns as well heterotic as type IIA or IIB string theories.

For Kähler manifolds, the curvature tensor is of the form $R_{ABCD}$, symmetric in the two holomorphic indices $AB$, and in the antiholomorphic indices $CD$. The manifolds which we consider have a curvature tensor of the form [8] (we raised the antiholomorphic indices using $g^{CD}$, the inverse of the metric):

$$R_{ABCD} = -2\kappa_{(A}^{(C}g_{D)B)} - Y^{-2}(z,\bar{z})Q_{ABC}(z)g^{A\bar{A}}g^{B\bar{B}}Q_{\bar{D}\bar{D}}(\bar{z})$$

(1.1)

where $Q_{ABC}$ is a symmetric tensor function of the 2 only, and $Y(z,\bar{z})$ is a positive definite function in the physical domain, where the metric was chosen to be negative definite. We further only consider this 'positivity' domain. In [9] it was proven that Bianchi identities imply that $Q$ is of the form

$$Y^{-2}Q_{ABC} = V_{A}V_{B}V_{C}X(z,\bar{z})$$

(1.2)

One can choose suitable coordinates such that

$$Q_{ABC} = \partial_{A}\partial_{B}\partial_{C}F(z).$$

(1.3)

In $N = 2$ supergravity vector multiplets the scalars are related to vectors by 2 supersymmetry transformations. In this way one defines natural coordinates for the scalars. These natural coordinates of the scalars of the vector multiplets are such that $Q$ is of the form eq.(1.3).

In the next section we consider the general case that the scalar manifold splits into independent parts. Geometrically this means that the manifold is a direct product of lower dimensional manifolds. Thus the metric $g_{AB}$ is block diagonal and in each block it depends only on the scalars belonging to that block. In other words, in the curvature tensor $R_{ABCD}$ the 4 indices belong to the same block. After proving the main theorem, we combine it in section 3 with the results of [9] to show properties of the moduli if the string model contains enhanced symmetries $U(1)^n$ beyond $E_8 \otimes E_8$. In fact it constrains the moduli neutral under these $U(1)$ factors, provided the corresponding matter fields belong to different charged sectors.

2 The theorem

Using the symmetry of the curvature sector, we will consider, as in [10], the matrix $R_{ABCD}$ as a $2(4+1)$ x $2(4+1)$ matrix. We write eq.(1.1) as

$$R_{ABCD} = -2\kappa_{A(D}^{C}g_{B)D} + W_{A(D}^{C}$$

(2.1)

where we introduced

$$W_{ABCD} = Q_{ABC}QCD\bar{Q}^{CD}Y^{-2}$$

(2.2)

and

$$\bar{Q}^{CD} = -g^{CF}g^{DG}Q_{FGH}Q_{\bar{D}\bar{G}}$$

(2.3)

We included the minus sign, such that the matrix $W$ is positive definite taking into account the negative definiteness of the metric. The above form of the curvature is in fact the only input which we need from $N = 2$ supergravity. So the result holds for the full class of theories with this type of curvature tensor.

We recall from [10] the definition of a $n \times n$ matrix $X$

$$X_{P} = \frac{1}{2}Q_{ABC}Q^{ABC}Y^{-2}$$

(2.4)

also positive definite. As remarked there, the ranks of these two matrices are equal, which implies that the rank of $W$ cannot be larger than $n$.

Lemma 1 There can be at most the following splittings

1. 1 multiplet splits from all the others
2. $n = 4$ splitting in 2 groups of 2 multiplets.

The proof is that if $p$ multiplets split from $q$ others then the curvature matrix should have at least $pq$ zero eigenvalues. But therefore $W$ should have $pq$ non-zero eigenvalues, i.e. it should be of rank $pq$. But we already know that this rank can be at most $n = p + q$. So we arrive at $pq \leq p + q$ leaving only the solutions mentioned.
Lemma 2 Case 2 (splitting in 2⊗2) is not possible

Now consider the case 2. Split the indices in the 2 pairs a = 1, 2 and α = 3, 4. To have $H_{\alpha}^{\alpha'} = 0$ and $H_{\alpha a}^{\alpha a'} = 0$ we need $W$ to look like

$$W = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.5}$$

The splitting is done as follows. The first row stands for the 3 combinations (ab), the second for the 4 combinations (ac), and the third for the 5 (abd). Only the open areas remain undetermined. But as the rank of this matrix cannot exceed 4, the open spaces should be zero also. Now this cannot be solved for $Q_{abc}$. Indeed, according to the first row we have $Q_{abc} = 0$ (as the definition of $W$ has only positive definite terms). The last row demands that $Q_{abcd} = 0$, therefore $Q = 0$, and this case is excluded.

Lemma 3 The function $F$ in eq.(1.3) has the form $x^1 x^2 x^3 \eta_{ab}$ with $a = 2, \ldots, n$.

So we are only left with case 1. Consider now again the matrix as in eq.(2.5). By the same arguments it is now

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4_{a} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.6}$$

Now one of the open areas can have rank 1. Consider first the case that the upper left corner (which is now just a number) is non-zero. Then the lower right matrix should be zero. In terms of $Q$ this implies $Q_{abc} = 0$ where now $a, b = 2, \ldots, n$. Therefore $F$ can only be linear in the $x^a$. Moreover the fact that the middle matrix is $4_{a}$ implies that all these fields occur symmetrically. Therefore, we can redefine the sum of these multiplets as one new multiplet, and that is the only one which occurs. So we are in $n = 2$, and we can give this latter multiplet label 1. In that way we can say that the upper corner of eq.(2.6) is 0. This gives us then the equations

$$Q_{111} = 0$$
$$Q_{11a} = 0$$
$$Q_{1ab} = \eta_{ab} \tag{2.7}$$

where the latter matrix is undetermined but we know

$$\eta_{ab} \theta^{b} = \delta^{b} \gamma^{a} \tag{2.8}$$

with $\psi^{a} = g^{a} \gamma^{b} \theta^{b} \omega^{c} \chi_{c}$. Therefore $\eta$ has rank $n - 1$. Then $H_{\alpha}^{\alpha} = 0$ implies $\eta_{aa} = 0$ and so also $Q_{abc} = 0$ (using non-degeneracy of the metric). Therefore $x^a$ only occurs quadratically and $x^1$ linearly.

These are the models found in [8], where it was proven that if there is a non-empty positivity domain, this fixes the manifolds such that we have the theorem:

Theorem The only $N = 2$ special Kähler manifolds which split into a direct product of manifolds are the symmetric spaces

$$SU(1, 1) \oplus SO(n - 1, 2)$$
$$U(1) \oplus SO(n - 1) \oplus SO(2). \tag{2.10}$$

Note that in the case $n = 3$ there is a further splitting in 3 blocks of 1 multiplet. In fact, in the beginning we could use the same counting argument to prove that splitting in 3 blocks is only possible with 3 multiplets ($pq + qr + pr \leq p + q + r$), and then this uses up the maximal rank, so it goes as in case 2 showing that $Q$ can have no 2 indices of the same block. However this time this allows $Q_{123} \neq 0$.

3 Application to the moduli space of string theories with enhanced gauge symmetry

In [9] it was argued that if the gauge group is enhanced beyond $F_{8} \otimes F_{8}$, the moduli space splits into different charge sectors in addition to the splitting into (1,1) and (1,2) forms. The latter splitting is related to the splitting in vector and hypermultiplets as mentioned in the introduction. We will now only consider the moduli related to one type, e.g. (1,1) forms. Each modulus is related to a 27 multiplet of matter fields. For a gauge group $H \otimes F_{8} \otimes F_{8}$ with $U = H^{1/3}$, it was proven that the $H$-preserving subspace of the moduli space is locally a direct product of smaller-dimension subspaces; each of the latter subspaces is spanned by all the neutral moduli $N^{a}$ that accompany matter fields with the same charges. In terms of the Kähler function this is expressed as

$$K(N, \overline{N}) = \sum_{a} K_{a}(N^{a}, \overline{N^{a}}). \tag{3.1}$$

On more general grounds we can say as a consequence of the previously derived theorem that $N = 2$ special Kähler manifolds only allow product spaces of the form eq.(2.10).

A first (orbifold) example in [9] had $H = U(1)^{2}$, which implies thus at least 3 different charged sectors (otherwise by a redefinition one $U(1)$...
would become trivial). In their example they had 1 modulus in each of the 3 charged sectors. By our general theorem this is not an artifact of the example, but it is the only possibility if the gauge group is $U(1)^3$. Indeed the above implies the splitting of the manifold in 3 distinct sectors. In the previous section we have proven that this can only be if there are only 3 fields. Indeed, if one considers all $Z_n$ orbifolds corresponding to $(2,2)$ vacua with enhanced gauge symmetry $U(1)^2$ or $U(1) \otimes SU(2)$, the untwisted moduli space is given by $S^1 \times S^1$ for $U(1)^2$ and $S^1 \times S^1 \otimes S^1(2)$ for the $U(1) \otimes SU(2)$ case (the latter occurs for a $Z_2$ and a $Z_2$ orbifold). These cases correspond respectively to the splitting of three and five neutral moduli into $1 \otimes 1$ and $1 \otimes 4$.

In the minimal models considered in [9] no modulus exists which is completely neutral under the enhanced gauge symmetry $H \otimes E_8 \otimes E_8$ with $H = U(1)^N$. Subgroups of $H$ exist under which all moduli are neutral; however, in this case all accompanying matter fields have the same charge, so the space of neutral moduli is not required to factorise in a direct product.

Acknowledgements

A.V.P. thanks C. Kounnas for clarifying discussions, and the U.C.L.A. at Los Angeles, for hospitality.

References


