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Stochastic And Topological Gauge Theories

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ABSTRACT

We show that the form of the Langevin equations for the stochastic quantization of a Yang-Mills theory in n -dimensional space is dictated by the geometrical structure. The arbitrary Lie algebra zero form, initially introduced by Zwanziger to induce a drift force along gauge orbits, is now defined as an additional gauge field component which permits stochastic time-dependent gauge transformations. The action of the functional integral representation of the Langevin equation is the same as that of an $(n+1)$ -dimensional topological quantum field theory. Covariance between the stochastic time and the physical space directions is made possible. As an application, the action which generates Donaldson invariants in four-dimensional space is shown to be identical to the supersymmetric action describing the stochastic quantization of the three-dimensional Chern-Simons theory, which generates Jones invariants.

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1. Introduction

In [1], stochastic quantization was first applied to gauge theories. Numerous works have followed [2]. In [3] the formalism of stochastic quantization was linked to that of topological quantum field theories (see also [4]). Besides, the relationship between the four-dimensional Yang-Mills topological quantum field theory introduced by Witten in [5] and the stochastically quantized three-dimensional Chern-Simons theory was pointed out in [3]. Independently of the formalism of stochastic quantization, this is a natural result, since the four-dimensional Yang-Mills topological quantum field action is a gauge fixing of the second Chern class $\int_{M_4} F_\Lambda F^\Lambda$ [6], and one has the Chern-Simons relation $\int_{M_4} F_\Lambda F^\Lambda = \int_{M_3} (A_\Lambda dA + \frac{2}{3} A_\Lambda A_\Lambda A)$.

Here, we shall refine the analysis of [3], and show more general results, presumably applicable for other gauge theories. One can define stochastic time-dependent gauge transformations, and the gauge field component along the stochastic time direction is determined from the quantity left arbitrary by the gauge freedom in the Langevin equations of gauge theories. The supersymmetric representation of the Yang-Mills Langevin equation for a gauge theory in n -dimensional space can then be identified with the generating functional of a BRST-invariant topological quantum field theory in an $(n+1)$ -dimensional space, involving the stochastic time. To obtain these properties of stochastically quantized Yang-Mills theories, we shall follow the idea that the gauge structure is strong enough to determine the formalism: we will postulate that the stochastic time evolution defined by the Langevin equations must be compatible with gauge transformations, and thus commute with the ordinary BRST symmetry.* An important ingredient is the introduction of the Faddeev-Popov ghost, on the basis of reference [8]. Before showing these results, we need to briefly recall the stochastic quantization formalism for the scalar theory.

2. Scalar theory

Consider a scalar field $\phi(x)$ and an action $S[\phi]$ defined over a (Euclidean) space of dimension n that we call physical space. x^i denote local coordinates in this space. In

* Interesting results for connecting a stochastically quantized n -dimensional gauge theory with a gauge theory in $(n+1)$ -dimensional space can also be found in [7].

stochastic quantization, the relevant object is the generating functional of stochastic time-dependent Green functions $Z[J]$, defined by functional integration:

$$Z[J] = \int [D\phi] \exp - \int dt dx (\frac{1}{2} \dot{\phi}^2(x, t) - J(x, t) \phi(x, t)), \quad (1)$$

where the field $\phi(x, t)$ is submitted to a Langevin equation:

$$E(\frac{\partial \phi(x, t)}{\partial t}, \phi) = b. \quad (2)$$

t is the inobservable stochastic time and $J(x, t)$ is the source of the t -dependent field $\phi(x, t)$. (1) means that the field b is a Gaussian noise.

Choosing $E = \frac{\partial \phi}{\partial t} - \frac{\partial S[\phi]}{\partial \phi}$, and provided the action S satisfies certain conditions, one can show that, in the limit where the stochastic time goes to infinity, $Z[J]$ relaxes to the ordinary Feynman generating functional $\int [D\phi] \exp(-S[\phi] + \int dx J(x) \phi(x))$, where the stochastic time dependence is ignored.

The supersymmetric representation of the Langevin equation is obtained by making explicit the constraint (2) within (1): assuming that the Jacobian of the transformation $\phi \rightarrow b = E(\frac{\delta \phi}{\delta t}, \phi)$ does not vanish, one inserts in (1) the identity:

$$1 = \int [D\phi] \delta(E - b) \frac{\delta E}{\delta \phi} = \int [D\phi] [D\bar{\Psi}] \delta(E - b) \exp - \int dt dx \bar{\Psi} \frac{\delta E}{\delta \phi} \Psi. \quad (3)$$

$\Psi(x, t)$ and $\bar{\Psi}(x, t)$ are a pair of anticommuting ghost fields. One obtains:

$$\begin{aligned} Z[J] &= \int [D\phi] [D\bar{\Psi}] [D\Psi] \exp - \int (dt dx (\frac{1}{2} \dot{\phi}^2 + \bar{\Psi} \frac{\delta E}{\delta \phi} \Psi - J\phi)) \\ &= \int [D\phi] [D\bar{\Psi}] [D\Psi] [D\bar{b}] \exp \int (dt dx (\frac{1}{2} \dot{\phi}^2 + bE - \bar{\Psi} \frac{\delta E}{\delta \phi} \Psi + J\phi)). \end{aligned} \quad (4)$$

It is easy to verify that $I_{GF} = \int dt dx (\frac{1}{2} \dot{\phi}^2 + bE - \bar{\Psi} \frac{\delta E}{\delta \phi} \Psi)$ is invariant under the action of the graded differential operator s_{top} , defined as follows:

$$\begin{aligned} s_{top} \phi &= \Psi \\ s_{top} \bar{\Psi} &= 0 \\ s_{top} \Psi &= b \\ s_{top} b &= 0. \end{aligned} \quad (5)$$

The supersymmetric stochastic action is invariant under the action of s_{top} because $I_{GF} = s_{top}(\bar{\Psi}(E + \frac{1}{2}b))$ and $s_{top}^2 = 0$. The relationship between stochastic quantization and topological quantum field theories was exhibited in [3], on the basis that I_{GF} is s_{top} -exact. I_{GF} can be identified as the gauge fixing of zero, or of a topological action obtained from a Lagrangian which is a pure derivative in the stochastic time. What the gauge fixing does is to concentrate the functional integral around the solutions of the Langevin equation, $E = b$.

The BRST equations (5) can be used to control the perturbative expansion of the stochastic field theory, since infinities generally occur and a regularisation must be introduced [2,7,9]. On the other hand, it is yet unknown if one can take advantage of the BRST symmetry and try other types of gauge functions than $E = \frac{\partial\phi}{\partial t} - \frac{\partial S[\phi]}{\partial\phi}$. An amusing remark is the following one. If one considers causal boundary conditions, one can use the local freedom on ϕ and formally gauge fix the topological action $I_{top} = \int_0^T dt \frac{\partial S}{\partial\phi} \frac{\partial\phi}{\partial t}$ by imposing $\phi = 0$ for t smaller than, but not equal to T . The gauge-fixed and BRST action is then $I_{GF} = \int_0^T dt \frac{\partial S}{\partial\phi} \frac{\partial\phi}{\partial t} + \int_{0 \leq t < T} dt s_{top}(\bar{\Psi}\phi) + \int_{0 \leq t < T} dt s_{top}(b\phi - \bar{\Psi}\psi)$. The fermionic integration is trivial, while b acts as a Lagrange multiplier which enforces $\phi = 0$ for $0 \leq t < T$. In this gauge I_{GF} reduces therefore to the action $S[\phi_T]$, and functional integration is confined to field configurations at a single value T of the stochastic time. One gets therefore $Z \sim \int [D\phi] \exp -S$, which is a way of understanding how the Langevin process relaxes to the ordinary path integral result.

2 . Ordinary BRST symmetry and Langevin equations for Yang-Mills theories

If one writes Langevin equations for a gauge theory, one expects a supersymmetric representation of the type (4), with a "topological" symmetry of the type (5). On the other hand, there should be a reminiscence of the ordinary gauge symmetry. To construct the full symmetry, rather than trying to improve the formulation of Parisi and Wu [1], we will construct the theory from the simplest postulate, that the ordinary Yang-Mills BRST transformations, and in particular the ordinary infinitesimal gauge transformations, must commute with the stochastic time evolution.

Our starting point is the expression of the generator s of the ordinary Yang-Mills BRST symmetry. s is a graded differential operator defined by its action on the G -valued Yang-Mills field one-form $A = A_i dx^i$, with $1 \leq i \leq n$ and n the dimension of the physical space for the theory, and on the G -valued anticommuting zero-form ghost c :

$$sA = -Dc \quad sc = -\frac{1}{2}[c, c]. \quad (6-a)$$

d is the ordinary exterior derivative $d = dx^i \partial_i$ and $D = d + [A, \cdot]$ the covariant derivative. We define the covariant BRST differential operator $S = s + [c, \cdot]$. By definition s and d anticommute. One can verify $s^2 = 0$, $S^2 = 0$ and $SD + DS = 0$. The BRST equations (6-a) can be rewritten as:

$$(d + s)(A + c) + \frac{1}{2}[A + c, A + c] = dA + [A, A]. \quad (6-b)$$

In what follows, $\partial_o = \frac{\partial}{\partial t}$ denotes the derivative with respect to the stochastic time t and all fields are assumed to depend on x^i and t . Our postulate is that s and ∂_o commute:

$$s\partial_o = \partial_o s. \quad (7)$$

One has also $s\partial_i = \partial_i s$. Applying ∂_o to both sides of (6-b), and using (7), we obtain:

$$(D + S)(\partial_o(A + c)) = D(\partial_o A). \quad (8)$$

This equation, once expanded in ghost number, gives:

$$S(\partial_o c) = 0 \quad (9-a)$$

$$S(\partial_o A) + D(\partial_o c) = 0. \quad (9-b)$$

Equations (9) can be easily solved, simply by using power counting (the dimensions and ghost numbers of A and c are as usual) and the properties $S^2 = 0$ and $SD + DS = 0$. One obtains successively:

$$\partial_o c = SA_o - \gamma = sA_o + [c, A_o] - \gamma \quad (10-a)$$

$$\partial_o A_i = D_i A_o + \frac{\delta I_{cl}[A]}{\delta A_i} + b_i, \quad (10-b)$$

where γ and b_i , which can be identified as noises for c and A_i respectively, are submitted to the BRST constraints:

$$S\gamma = 0 \quad Sb_i = D_i \gamma. \quad (11)$$

Defining $F_{oi} = \partial_o A_i - \partial_i A_o + [A_o, A_i]$, we can rewrite (10) as follows:

$$F_{oi} = \frac{\delta I_d[A]}{\delta A_i} + b_i \quad (12-a)$$

$$sA_o = D_o c + \gamma. \quad (12-b)$$

I_d can be any given functional of the A_i 's, provided it is s-invariant. In other words, I_d is a classical gauge-invariant action. (The s-invariance of $I_d[A]$ implies the s-covariance of the equation of motion, $S \frac{\delta I_d[A]}{\delta A_i} = 0$.)

A_o is an arbitrary function, left undetermined when solving (7). This freedom of the choice of A_o , and thus of part of the drift force in the Langevin equations (10), is a consequence of gauge invariance. Translating the Langevin equation (10-a) into a Fokker-Planck equation, it is possible to prove, in the limit $t \rightarrow \infty$, the independence on the choice of A_o of correlation functions of gauge-independent functionals of the A_i 's, computed from the Langevin equations (10-a). (This result is somehow explained here since A_o is introduced through a term SA_o , which is part of the trivial cohomology of S .) When A_o is chosen as a function of the A_i 's, $A_o = A_o[A_i]$, (10-a) is the modified Langevin equation introduced in [10] for inducing a drift force along the gauge orbit, without affecting the values of gauge invariant quantities. The convergence toward the usual Faddeev distribution of the Fokker-Planck equation associated to the Langevin equation (10-a) was demonstrated in [11], by using a certain functional $A_o[A_i]$.

The Langevin equation (10-b) for the ghost was introduced in [8], and generalized in [12] for other gauge theories than Yang-Mills, to obtain a stochastic definition of ghosts. (It may seem paradoxal to define stochastic evolution for the ghosts, since stochastic evolution was originally introduced in gauge theories to avoid them, however we will see that it turns out to be useful.)

When A_o is chosen as a function of the A_i 's, (10-b) can be used as a Langevin equation for the ghost, and exponentiated as a constraint in the functional representation of stochastic quantization. This was done in [13] to investigate the Ward identities of the stochastically quantized Yang-Mills theory and its renormalization, with the choice $A_o = \partial_i A^i$. (See also [14].)

If A_o is taken as a new field, independent of the A_i 's, (10-b) can no longer be used to define the evolution of the ghost along the stochastic time direction; rather, written as $sA_o = D_o c + \gamma$, (10-b) defines the BRST transforms of A_o . As a matter of fact, one sees that

A_o transforms as the component along t of a gauge field under a Yang-Mills topological gauge symmetry such as the one built in [6] (which is another way of explaining the independence of physical quantities on the choice of A_o). This is a first indication that, by covariance, we should be able to unify A_o and the A_i 's through a connection $\tilde{A} = dtA_o + dx^i A_i$. We shall in fact demonstrate this property in the next section and, moreover, see the possibility of a covariant theory in an $(n+1)$ -dimensional space, with a topological gauge symmetry $sA_\mu = D_\mu c + \Psi_\mu$ and $sc = \Phi - \frac{1}{2}[c, c]$.

2. The link between stochastically quantized and topological gauge theories

What we have in mind is to consider A_o as an additional gauge field component, and to functionally integrate on it, since physics is independent of A_o . As a first step, we construct a theory which does not involve the Faddeev-Popov ghost c . Since we take the G-valued field $A_o(x, t)$ as an independent field, we must define a stochastic time evolution for this field. We therefore complete the Langevin equation (11-a) by the following one:

$$\partial_o A_o = \mathcal{F} + b_o. \quad (13)$$

b_o is a noise, and the G-valued functional $\mathcal{F}[A]$ is arbitrarily chosen, provided the stochastic process induced by the Langevin equations (10-a) and (13) is meaningful. The correlation functions of gauge-independent functionals of the A_i 's do not depend on the choice of $\mathcal{F}$ in the limit $t \rightarrow \infty$, since they do not depend on A_o . (A natural choice is of course $\mathcal{F} = \partial_i A^i$.)

Let $J_i(x, t)$ be the sources of the A_i 's. The stochastic partition function is:

$$Z[J] = \int [D b_i][D b_o] \exp \int dt dx \text{Tr} \left(-\frac{1}{2} \dot{b}_o^2 - \frac{1}{2} b_o^2 + J_i(x, t) A_i(x, t) \right), \quad (14)$$

where the $b(x, t)$'s are submitted to the constraints (10-a) and (13).

Assuming that the Jacobian $\frac{\delta(b_o, b_i)}{\delta(A_o, A_i)}$ is not singular (which means that the dimension of the modular space of the solutions to the equation $F_{oi} - \frac{\delta I_d[A]}{\delta A_i} = 0$ has dimension zero), we have:

$$Z[J] = \int [D A_\mu][D b_\mu] \det \left(\frac{\delta(b_o, b_i)}{\delta(A_o, A_i)} \right) \delta(b_i - F_{oi} + \frac{\delta I_d}{\delta A_i}) \delta(b_o - \partial_o A_o + \mathcal{F}) \exp \left(\int dt dx \text{Tr} \left(-\frac{1}{2} b_\mu^2 + J_i A_i \right) \right) \quad (15)$$

We can introduce the G-valued ghosts $\Psi_\mu, \bar{\Psi}_i$ and $\bar{c}$ to exponentiate the determinant in (13). (In our notation, the Greek indices include the stochastic time index, for instance $A_\mu = (A_o, A_i)$ and $\Psi_\mu = (\Psi_o, \Psi_i)$, and run between 0 and n.) One has :

$$Z[J] = \int [DA_\mu][D\Psi_\mu][D\bar{\Psi}_i][D\bar{c}][D\bar{b}_\mu] \exp(I_{GF} + \int dt dx \text{Tr} J_i A_i)$$

$$I_{GF} = \int dt dx \text{Tr} \left(\frac{1}{2} b_\mu^2 + b_i (F_{oi} - \frac{\delta I_{cl}}{\delta A_i}) + b_o (\partial_o A_o - \mathcal{F}) \right. \\ \left. - \bar{\Psi}_i (D_{[0} \Psi_{i]} - \frac{\delta^2 I_{cl}}{\delta A_i \delta A_j} \Psi_j) \right. \\ \left. - \bar{c} (\partial_o \Psi_o - \frac{\delta \mathcal{F}}{\delta A_i} \Psi_i) \right). \quad (16)$$

(For going from (15) to (16), use the same manipulations as from (1) to (4).)

The BRST symmetry which leaves invariant the supersymmetric action in (14), and is comparable to that in (5), is now :

$$\begin{aligned} s_{top} A_\mu &= \Psi_\mu \\ s_{top} \bar{\Psi}_\mu &= 0 \\ s_{top} \bar{\Psi}_i &= b_i \\ s_{top} \bar{c} &= b_o \\ s_{top} b_\mu &= 0. \end{aligned} \quad (17)$$

One has $s_{top}^2 = 0$ and :

$$I_{GF} = \int dt dx s_{top} \text{Tr} \left(\bar{\Psi}_i (F_{oi} - \frac{\delta I_{cl}}{\delta A_i} + \frac{1}{2} b_i) + \bar{c} (\partial_o A_o - \mathcal{F} + \frac{1}{2} b_o) \right). \quad (18)$$

The stochastic BRST equations (17) are incomplete : the ordinary BRST invariance is not here. These equations must be improved to include the Faddeev-Popov ghost which has already occurred naturally in the determination of the Langevin equation (10) and the introduction of A_o . Moreover, in the case where the moduli space M of the solutions to the equation $F_{oi} - \frac{\delta I_{cl}}{\delta A_i} = 0$ is not empty, the generating functional (16) vanishes and the link between (14) and (16) is inconsistent.

We will thus construct a supersymmetric functional representation of the Langevin equation which involves the Faddeev-Popov ghost c . Since we have chosen A_o independent of the A_i 's, we have already indicated that (10-b) can only be considered as defining the BRST

variation of A_o . We need a new equation for defining the stochastic evolution of c . Since c is a zero-form, and since we require covariance in $(n+1)$ -dimensional space, the Langevin equation must be second order. We choose the following Langevin equation for completing (10-a) and (13) :

$$D_\mu D^\mu c = -D_\mu \Psi^\mu + \beta. \quad (19)$$

We impose that β is a Gaussian fermionic noise. We thus insert in the generating functional $Z[J]$ the identity $1 = \int [d\bar{\eta}] [d\beta] \exp(-\bar{\eta}\beta)$, with the constraint (19). The Jacobian of the transformation $(A_i, A_o) \rightarrow (F_{oi} - \frac{\delta I_{cl}}{\delta A_i}, \partial_o A_o - \mathcal{F})$ is now replaced by that of the transformation $(A_i, A_o, c) \rightarrow (F_{oi} - \frac{\delta I_{cl}}{\delta A_i}, \partial_o A_o - \mathcal{F}, D_\mu D^\mu c + \partial_\mu \Psi^\mu)$. One obtains :

$$Z[J] = \int [DA_\mu][D\Psi_\mu][D\bar{\Psi}_i][D\bar{b}_\mu][D\bar{c}][D\bar{\eta}][D\bar{\Phi}] \exp(I'_{GF} + \int dt dx \text{Tr} J_i A_i)$$

where :

$$\begin{aligned} I'_{GF} &= \int dt dx \text{Tr} \left(\frac{1}{2} b_\mu^2 + b_i (F_{oi} - \frac{\delta I_{cl}}{\delta A_i}) + b_o (\partial_o A_o - \mathcal{F}) \right. \\ &\quad \left. - \bar{\eta} (D_\mu \Psi^\mu + D_\mu D^\mu c) \right. \\ &\quad \left. - (\bar{\Psi}_i, \bar{c}, \bar{\Phi}) \begin{pmatrix} D_o \delta_{ij} - \frac{\delta^2 I_{cl}}{\delta A_i \delta A_j} & D_i & 0 \\ -\frac{\delta \mathcal{F}}{\delta A_i} & \partial_o & 0 \\ \Psi_i + D_i c & \Psi_o + D_o c & D_\mu D^\mu c \end{pmatrix} \begin{pmatrix} \Psi_i \\ \Psi_o \\ \Phi \end{pmatrix} \right) \\ &= \int dt dx \text{Tr} \left(\frac{1}{2} b_\mu^2 + b_i (F_{oi} - \frac{\delta I_{cl}}{\delta A_i}) + b_o (\partial_o A_o - \mathcal{F}) - \bar{\eta} D_\mu D^\mu c \right. \\ &\quad \left. - \bar{\Psi} (D_{[0} \Psi_{i]} - \frac{\delta^2 I_{cl}}{\delta A_i \delta A_j} \Psi_j) - \bar{c} (\partial_o \Psi_o - \frac{\delta \mathcal{F}}{\delta A_i} \Psi_i) - \bar{\eta} D_\mu \Psi^\mu \right. \\ &\quad \left. - \bar{\Phi} (D_\mu D^\mu \Phi + [\Psi^\mu, \Psi_\mu + D_\mu c] + D_\mu [\Psi^\mu, c]) \right). \end{aligned} \quad (20)$$

The G-valued fields $\Phi(x, t)$ and $\bar{\Phi}(x, t)$ are commuting fields : they are used to exponentiate the determinant of a map involving the anticommuting ghost c . All gradings are of course summarized by attributing ghost numbers to all fields : 0, 1, -1 for A_μ and b_μ , 1 for c and Ψ_μ , -1 for $\bar{c}$ and $\bar{\Psi}_\mu$, 2 for Φ and -2 for $\bar{\Phi}$, and the grading of all forms and operators is defined as the sum modulo two of the ghost numbers and form degrees.

It is easy to verify that I'_{GF} is invariant under the action of the full topological BRST Yang-Mills operator constructed in [6], (with the change of field-variables $\Psi_\mu \rightarrow \Psi_\mu + D_\mu c$, $\Phi \rightarrow \Phi - \frac{1}{2}[c, c]$), and defined as follows:

$$\begin{aligned}
s_{top} A_\mu &= \Psi_\mu \\
s_{top} c &= \Phi \\
s_{top} \Psi_\mu &= 0 \\
s_{top} \Phi &= 0 \\
s_{top} \bar{c} &= b \\
s_{top} b &= 0 \\
s_{top} \bar{\Psi}_i &= b_i \\
s_{top} b_i &= 0 \\
s_{top} \bar{\Phi} &= \bar{\eta} \\
s_{top} \bar{\eta} &= 0.
\end{aligned}
\tag{21-a}
\tag{21-b}$$

One has $s_{top}^2 = 0$, and one can write I'_{GF} as the following s_{top} -exact term:

$$I'_{GF} = \int dt dx s_{top} (\bar{\Psi}_i (F_{oi} - \frac{\delta I_{cl}}{\delta A_i} + \frac{1}{2} b_i) + \bar{c} (\partial_o A_o - \mathcal{F} + \frac{1}{2} b_o) - \bar{\Phi} (D_\mu \Psi^\mu + D_\mu D^\mu c)). \tag{22}$$

The role played by each gauge-fixing function which is in front of each antighost in (22) (in particular the gauge-fixing function for the topological ghost Ψ) has been carefully analyzed in (6). If the moduli space M is empty, I_{GF} and I'_{GF} can be used equivalently to compute gauge invariant quantities in the physical space, with the same results, since both actions only differ by BRST-exact terms. If M is not empty, only the computation with I'_{GF} makes sense, due to the further gauge fixing on Ψ , $D_\mu \Psi^\mu = 0$ when $t \rightarrow \infty$. What we have gained when changing (17) into (21) is the introduction of c and Φ , which permits the shift $\Psi \rightarrow \Psi + Dc$, and therefore the combination of the stochastic supersymmetry with the ordinary Yang-Mills invariance.

The case where the dimension of the physical space is equal to three is interesting: one gets a theory in four-dimensional space, which has a chance to be renormalizable by power counting, and, moreover, one can construct a curvature $F_{\mu\nu}$, with $F_{oi} = \partial_o A_i - \partial_i A_o + [A_o, A_i]$ and $F_{ij} = \epsilon_{ijk} \frac{\delta I_{cl}}{\delta A_k}$. Then, the Langevin equation for the A_i 's takes the form:

$$F_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = b_{\mu\nu}. \tag{23}$$

where the three components of b_i have been rearranged as the components of a self-dual two-form in four dimensions. One sees that in the limit where the stochastic time goes to

infinity, so that $b_{\mu\nu} \rightarrow 0$, the theory satisfies a kind of self-duality equation. The case of interest is that of the Chern-Simons theory, for which $F_{ij} = \epsilon_{ijk} \frac{\delta I_{cl}}{\delta A_k} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$. In this case, choosing the gauge function $\mathcal{F} = \partial_i A^i$, we get for I'_{GF} :

$$\begin{aligned}
I'_{GF} &= \int dt dx \text{Tr} s_{top} (\chi^{\mu\nu} (F_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} + \frac{1}{2} b_{\mu\nu}) + \bar{c} (\partial_\mu A^\mu + \frac{1}{2} b) - \bar{\Phi} (D_\mu \Psi^\mu + D_\mu D^\mu c)) \\
&= \int dt dx (\frac{1}{2} b_{\mu\nu}^2 + b_{\mu\nu} (F_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}) \\
&\quad + \frac{1}{2} b^2 + b (\partial_\mu A^\mu) - \bar{\eta} D_\mu D^\mu c \\
&\quad - \chi^{\mu\nu} D_{[\mu} \Psi_{\nu]} - \bar{c} \partial_\mu \Psi^\mu - \bar{\eta} D_\mu \Psi^\mu \\
&\quad + \bar{\Phi} (D_\mu D^\mu \Phi + [\Psi^\mu, \Psi_\mu + D_\mu c] + D_\mu [\Psi^\mu, c])).
\end{aligned}
\tag{24}$$

The components of $\bar{\Psi}$, have been rearranged into those of the self-dual two-form $\chi_{\mu\nu}$, with $s_{top} \chi_{\mu\nu} = b_{\mu\nu}$. One recognizes in (24) the gauge-fixing action for the second Chern class $\int M_4 \text{Tr} F \wedge F$ which generates in quantum field theory the Donaldson invariants [5,6]. We end up therefore with the interesting conclusion that the same action generates also the Jones-Witten invariants [15], through four-dimensional computations of expectation values of three-dimensional Wilson loops.

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