CERN-TH.5513/89

SEMI-RELATIVISTIC STABILITY AND CRITICAL MASS OF A SYSTEM
OF SPINLESS BOSONS IN GRAVITATIONAL INTERACTION

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ABSTRACT

The N particle Hamiltonian

$$\sum_{i=1}^N (\vec{p}_i^2 + m^2)^{\frac{1}{2}} - \sum_{i < j \leq N} Gm^2/r_{ij}$$

implies a critical mass M_{cr} , a bosonic analogue of the Chandrasekhar limit, beyond which there must be relativistic collapse. We prove that $M_{cr} > 4[3/3 Gm]^{-1} \sim 0.77(Gm)^{-1}$, consistent with the bound $M_{cr} < 1.51(Gm)^{-1}$ obtained by one of us (A.M.). Comparison with the Kaup-Ruffini-Bonazzola statistical curved space Klein-Gordon equation result $M_{cr} \approx 0.633(Gm)^{-1}$ raises interesting questions.

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1. - INTRODUCTION

The non-relativistic stability of ordinary matter, namely that $H_N > \text{constant} \times N$ for $N \rightarrow \infty$, where N is the total number of particles (electrons and nuclei) with Coulomb interactions only, was proved by Dyson and Lenard [1], and the bound dramatically improved by Lieb and Thirring [2]. With gravitational interactions included there is instability, with $H_N \sim -AN^3$ for bosons and $H_N \sim -AN^{5/3}$ for fermions [3]. The corresponding relativistic problem was studied by Lieb and Thirring [4] replacing $p^2/(2m)$ for each particle of mass m in the non-relativistic Hamiltonian by the non-local operator $(p^2 + m^2)^{1/2} - m$. In this formalism, Lieb and Yau proved that many-particle stability requires a bound on the fine structure constant [5]. For gravitational interactions, one finds the phenomenon of relativistic collapse, i.e., even for finite N , if N is large enough, the relativistic Hamiltonian is unbounded below, a phenomenon directly linked to the Chandrasekhar mass limit discovered in the statistical treatment [6].

Recently, Thirring [7] proved that non-relativistic instability implies relativistic collapse for large enough N if inverse square law forces (Coulomb and gravitational) only are included. One of us (A.M. [8]) proved, for the same interactions, a more quantitative result: if the non-relativistic ground state energy is less than $(-mN/2)$, then there must be relativistic collapse. One immediate consequence is that for bosons with only gravitational interaction [9]

$$M_{cr} < 1.51 (Gm)^{-1/2}, \quad (1)$$

where M_{cr} is the critical mass beyond which there is relativistic collapse and G is the gravitational constant. In the present work, we start from the N -particle Hamiltonian

$$H_N = \sum_{i=1}^N (p_i^2 + m^2)^{1/2} - \sum_{i < j \leq N} Gm^2/r_{ij}, \quad (2)$$

considered also in Refs. [4-9], and derive lower bounds on it. We thus deduce sufficient conditions for relativistic stability (i.e., finiteness of the lower bound). It follows that

$$M_{cr} \geq 4 [3\sqrt{3} Gm]^{-1/2} \approx 0.77 (Gm)^{-1/2}. \quad (3)$$

This bound is consistent with (1), but is apparently inconsistent with the approximate value of M_{cr} obtained by Kaup and by Ruffini and Bonazzola [10]:

$$M_{cr} \approx 0.633 (Gm)^{-1/2}. \quad (4)$$

It must be remembered, however, that the result (4) is obtained in the very different framework of Klein-Gordon field theory in curved space, and in a simple approximate ansatz for the ground state. Ruffini and Bonazzola have emphasized that they have treated the particles in a statistical way, with no allowance for the details of particle-particle coupling. On the other hand, we have used the non-local Hamiltonian (2) and neglected general relativity effects. It will be extremely interesting to understand the real source of the discrepancy between (3) and (4).

2. - UPPER BOUNDS ON H_N AND THE CRITICAL MASS

We recall briefly [8] that replacing each $(p_i^2 + m^2)^{1/2}$ in H_N by an upper bound on it, we have the operator upper bound

$$H_N < N \left[\frac{m^2 + M^2}{2M} \right] + \left[\sum_i \frac{p_i^2}{2M} - \sum_{i < j} \frac{Gm^2}{r_{ij}} \right], \quad (5)$$

valid for arbitrary real M . Due to scaling, the ground-state energy of the operator in the square brackets is $M(E_{\text{non-rel}}/m)$, where $E_{\text{non-rel}}$ is the ground-state energy for $M = m$. If $E_{\text{non-rel}} + mN/2 < 0$, it follows by letting $M \rightarrow \infty$ that the ground-state mass $M_N = \infty$, i.e., there must be relativistic collapse [8]. If $E_{\text{non-rel}} + mN/2 > 0$, then optimization over M gives the bound

$$M_N < mN \sqrt{1 + 2E_{\text{non-rel}}/(mN)}. \quad (6)$$

Using the variational bound [9]

$$E_{\text{non-rel}} < -0.0542 N(N-1)^2 Gm^5, \quad (7)$$

we conclude that there must be relativistic collapse if

$$N-1 > 3.037 / Gm^2 \quad (8)$$

If $N-1 < 3.037/(Gm^2)$, M_N has the upper bound (6) with $E_{\text{non-rel}}$ replaced by the right-hand side of (7). Maximizing the bound over N , we get the upper bound on the critical mass

$$M_{cr} < 1.51/(Gm) \quad (9)$$

3. - LOWER BOUNDS ON H_N AND CRITICAL MASS

The condition (8) above is a sufficient condition for relativistic collapse. To obtain sufficient conditions for relativistic stability, we need lower bounds on H_N . Using the operator inequality [11,12]

$$1/r_{ij} \leq \frac{\pi}{2} \left| \frac{\vec{p}_i - \vec{p}_j}{2} \right|, \quad (10)$$

and $|\vec{p}_i - \vec{p}_j| < |\vec{p}_i| + |\vec{p}_j|$, we obtain

$$H_N \geq \sum_i \left[(p_i^2 + m^2)^{1/2} - Gm^2(N-1) \left| \vec{p}_i \right| \frac{\pi}{2} \right]. \quad (11)$$

This lower bound is clearly non-negative if

$$N-1 < (Gm^2)^{-1} 4/\pi. \quad (12)$$

Thus (12) gives a sufficient condition for relativistic stability. If this condition holds, we may minimize the square bracket in (11) over each $|\vec{p}_i|$ and obtain

$$H_N \geq Nm \left[1 - \{ Gm^2(N-1)\pi/4 \}^{1/2} \right]. \quad (13)$$

Replacing $(N-1)$ by N in the curly brackets, we obtain a weaker bound; this bound has a maximum value $[6m\pi/2]^{-1}$ at $N = (Gm^2)^{-1} 2/2\pi$, which is inside the range (12) of relativistic stability. Hence the critical mass M_{cr} has a lower bound

$$M_{cr} \geq [Gm\pi/2]^{-1}. \quad (14)$$

We now show that in a part of the stability region (12) the bound (13) can be improved, resulting in an improved lower bound on M_{cr} . Using an adaptation of the non-relativistic technique [3], we have

$$H_N = \frac{1}{N-1} \sum_{i \neq j} \left[(p_i^2 + m^2)^{1/2} - \frac{1}{2} Gm^2(N-1)/r_{ij} \right]. \quad (15)$$

Though the operators in square brackets for different pairs i, j do not necessarily commute, it is clear that a sum of lower bounds on them provides a lower bound on H_N . Each square bracket is of the form

$$H = (p^2 + m^2)^{1/2} - \lambda/r \quad (16)$$

with

$$\lambda = Gm^2(N-1)/2 \quad (17)$$

Herbst [12] showed that for $\lambda < 2/\pi$, H has the lower bound

$$H \geq m \left[1 - (\pi\lambda/2)^2 \right]^{1/2} \quad (18)$$

We shall prove that for $\lambda < 1/2$, H has the improved lower bound [13]

$$H \geq m \left[\frac{1}{2} + \left\{ \frac{1}{4} - \lambda^2 \right\}^{1/2} \right] \quad (19)$$

It is elementary to see that (19) improves (18). In addition, it can be shown [13] that for $\lambda \rightarrow 0$ the right-hand side of (19) can differ from the exact ground state energy of H only by terms of $O(\lambda^5)$. Let E and $\phi(r)$ denote respectively the ground-state energy and eigenfunction of H ,

$$H \phi(r) = E \phi(r) \quad (20)$$

Then, denoting $\vec{p} = \vec{E} \vec{r}$ and $\vec{p}_\rho = -i\vec{\nabla}_\rho = \vec{p}/E$, we obtain

$$-\frac{m^2}{E^2} = \frac{\int d^3p \phi^*(p) [p_\rho^2 - (1+\lambda/p)^2] \phi(p)}{\int d^3p |\phi(p)|^2} \quad (21)$$

The Rayleigh-Ritz variational principle [14] then implies that for $0 < \lambda < 1/2$

$$-m^2/E^2 \geq -m^2/\mu^2 \quad (22)$$

where $(-m^2/\mu^2)$ denotes the lowest eigenvalue of the operator $[p_\rho^2 - (1+\lambda/p)^2]$. In fact, μ is just the lowest positive eigenvalue of the Klein-Gordon Schrödinger equation

$$(\vec{p}^2 + m^2)\psi = (\mu + \frac{\vec{p}^2}{2})\psi, \quad (23)$$

and is known to be [15] equal to the right-hand side of Eq. (19). Since (18) gives $E > 0$, (22) implies $E > \mu$, thus proving the bound (19).

Substituting the bound (19) on the right-hand side of (15), we obtain

$$H_N \geq m N \left[\frac{1}{2} + \left\{ \frac{1}{4} - \left(\frac{Gm^2(N-1)}{2} \right)^{1/2} \right\}^{1/2} \right] \quad (24)$$

provided that $(N-1) < (Gm^2)^{-1}$. In this region, the bound (24) improves the bound (13) which is valid over the larger region (12). If, inside the curly brackets in (24), we replace $N-1$ by N , we obtain a weaker bound, which has a maximum value $(3/3) Gm^2/4)^{-1}$ at

$$N = (Gm^2)^{-1/2} \approx \sqrt{2}/3, \quad (25)$$

which is inside the region of validity of (24). We conclude that

$$(1) \quad M_{cr} \geq (Gm)^{-1} 4/(3\sqrt{3}), \quad (26)$$

(ii) relativistic stability must hold in the region (12) where H_N has the lower bound (13);

(iii) in the sub-region $(N-1) < (Gm^2)^{-1}$, H_N has the improved lower bound (24).

4. - IMPROVED LOWER BOUNDS ON H_N

In the region (12) of relativistic stability, the bound (13) will now be improved using techniques quite different from that used in getting (24). Their advantage will be their larger region of validity. They will not, however, give any improvement of the bound (26) for the critical mass.

To get the lower bound (11) on H_N , we used the inequality $|\vec{p}_i - \vec{p}_j| < |\vec{p}_1| + |\vec{p}_j|$. Clearly this inequality cannot be an equality for all i, j . For example, in the momentum representation all pairs of vectors $\vec{p}_i, \vec{p}_j$ cannot be antiparallel. If we only use the bound (10) on $1/r_{ij}$, we obtain

$$H_N \geq \sum_{i=1}^N (\vec{p}_i^2 + m^2)^{1/2} - \sum_{i < j \leq N} |\vec{p}_i - \vec{p}_j| Gm^2 \pi / 4. \quad (27)$$

The right-hand side can be rewritten as a sum of three-body Hamiltonians, to give

$$H_N \geq \frac{2}{(N-1)(N-2)} \sum_{i < j < k \leq N} H_{ijk} \quad (28)$$

where

$$H_{ijk} = K_i + K_j + K_k - (p_{ij} + p_{jk} + p_{ik}) Gm^2 (N-1) \frac{\pi}{8}, \quad (29)$$

with

$$K_i = (\vec{p}_i^2 + m^2)^{1/2}, \quad p_{ij} = |\vec{p}_i - \vec{p}_j|. \quad (30)$$

The sum of lower bounds on the H_{ijk} will provide a lower bound on H_N . The H_{ijk} are of the form of the right-hand side of (27) for $N = 3$, with $G + G(N-1)/2$. The essential task is then to obtain a lower bound on H_3 using (27).

For given $\vec{p}_i$'s, it is easy to see that the kinetic energy is reduced if the $\vec{p}_i$'s ($i = 1, 2, 3$) are replaced by their projection on the 1, 2, 3 plane. Let us keep $\vec{p}_2, \vec{p}_3$ and p_{12}, p_{13} fixed. Then $\vec{p}_1$ moves on an ellipse. The kinetic energy will be stationary only if $\vec{p}_1$ lies on the bisector of $\vec{p}_2 - \vec{p}_3$ and $\vec{p}_2 - \vec{p}_3$. This should hold for all $\vec{p}_i$'s. Then elementary geometry tells us that $p_{12} = p_1 \cos \alpha_1 + p_2 \cos \alpha_2$, where $2\alpha_1, 2\alpha_2, 2\alpha_3$ are the angles of the triangle 1, 2, 3 (see the Figure), so that $\alpha_1 + \alpha_2 + \alpha_3 = \pi/2$. We also have $p_1 \sin \alpha_1 = p_2 \sin \alpha_2 = p_3 \sin \alpha_3$, but we shall not use this latter constraint in the final minimizations.

Hence

$$H_3 \geq \sum_{i=1}^3 \left(\sqrt{p_i^2 + m^2} - Gm^2 \frac{\pi}{2} \cos \alpha_i \right) \\ > m \sum_{i=1}^3 \left[1 - \left(Gm^2 \frac{\pi}{2} \cos \alpha_i \right)^2 \right]^{1/2} \quad (31)$$

We use $\sum \alpha_i = \pi/2$ to eliminate α_2 , and we minimize (31) over α_1 , considering α_3 as given, to obtain

$$H_3 \geq N \min_{\alpha_3} \left[(1-x^2 \cos \alpha_3)^{\frac{1}{2}} + (1-x^2)^{\frac{1}{2}} + (1-x^2 \sin^2 \alpha_3)^{\frac{1}{2}}, (1-x^2 \cos^2 \alpha_3)^{\frac{1}{2}} + 2 \left\{ 1 - \frac{x^2}{2} (1 + \sin^2 \alpha_3) \right\}^{\frac{1}{2}} \right]$$

where $x \equiv Gm^2/2$, and $\min[a,b]$ means the smaller of a and b . A straightforward calculation gives

$$H_3/m \geq \min \left[1 + 2\sqrt{1-x^2}, 3\sqrt{1-\frac{3}{4}x^2} \right],$$

of which the first term corresponds to $\alpha_1 = \alpha_3 = 0$ and the second to $\alpha_1 = \alpha_3 = \pi/6$.

Hence

$$H_3 \geq 3m \sqrt{1 - \frac{3}{4}(Gm^2\pi/2)^2} \quad \text{if } (Gm^2\pi/2)^2 \leq \frac{96}{121} \quad (32)$$

and

$$H_3 \geq m \left[1 + 2\sqrt{1 - (Gm^2\pi/2)^2} \right] \quad \text{if } \frac{96}{121} < (Gm^2\pi/2)^2 \leq 1 \quad (33)$$

It may be seen that in the minimizing situations we obtained, the constraints involving the p_i 's which we ignored are nevertheless obeyed. Notice that in (33) we have a symmetry-breaking situation in which the α 's are not all equal.

The bounds on H_{ijk} are obtained from those on H_3 by letting $G \rightarrow G(N-1)/2$. Thus we obtain

$$H_N \geq Nm \left[1 - \frac{3}{4}(Gm^2(N-1)\pi/4)^2 \right]^{\frac{1}{2}}, \quad \text{if } (Gm^2(N-1)\pi/4)^2 < \frac{96}{121} \quad (34)$$

and

$$H_N \geq \frac{Nm}{3} \left[1 + 2 \left\{ 1 - (Gm^2(N-1)\pi/4)^2 \right\}^{\frac{1}{2}} \right] \quad \text{if } \frac{96}{121} < (Gm^2(N-1)\pi/4)^2 \leq 1. \quad (35)$$

These bounds give $M_{cr} > (Gm)^{-1}(4/(\pi/3))$ which is slightly poorer than (26). How-

ever, the bounds (34) and (35) cover the whole region (12) of relativistic stability.

It is likely that by looking at four-body subsystems one can improve (26). Though the four-body system is much more complicated, it can still probably be handled, because the p_{ij} 's are still independent.

At the present stage there is already a conflict between the semi-relativistic treatment and the general relativistic treatment. There are various differences between the two treatments. Our approach has the advantage of making absolutely no assumption about the structure of the wave function, but a weak point is that, at least in the boson case, matter is highly concentrated in a small region, and hence curvature of the space cannot be ignored. This might be the origin of the discrepancy.

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FIGURE CAPTION

The configuration of momenta $\vec{p}_1, \vec{p}_2, \vec{p}_3$ which maximize $|\vec{p}_1 - \vec{p}_2| + |\vec{p}_2 - \vec{p}_3| + |\vec{p}_3 - \vec{p}_1|$ for given $|\vec{p}_1|, |\vec{p}_2|, |\vec{p}_3|$.

FIG. 1