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CONFORMALLY INVARIANT CONSTRAINED FERMION MODELS

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ABSTRACT

We construct constrained fermion models within the path-integral framework. By an appropriate choice of constraints we obtain an effective theory with central charge in the minimal unitary series, its supersymmetric generalization and other well known series. We also discuss the relation between these constrained fermion models and gauged Wess-Zumino-Witten models.

SECTION I: INTRODUCTION AND RESULTS

In the last few years there has been a lot of research and progress in two dimensional Conformal Field Theories (CFTs) after Polyakov suggestion⁽¹⁾ on their relevance in providing information on the critical behaviour of statistical systems at second order phase transitions⁽²⁾.

Two dimensional CFTs are also central to string theory⁽³⁾; conformal invariance can be interpreted as the condition for finding classical solutions of string theory around which one expands in order to compute quantum corrections. In this context it turns out to give constraints on the allowed spacetime dimension and the possible internal degrees of freedom.

Concerning conformal theories with Virasoro central charge (VCC) less than unity (minimal models)^{(4),(5)} (or less than $3/2$ in the superconformal case⁽⁶⁾ or less than 2 when an extra Z_2 symmetry is added⁽⁷⁾), unitarity restricts the allowed c values to a discrete sequence and then, from the representation theory of the Virasoro algebra (VA), all critical indices can be determined⁽⁴⁾. Another discrete sequence arises for unitary representations of the VA in parafermionic models⁽⁸⁾.

The so-called Goddard-Kent-Olive⁽⁹⁾ (GKO) construction associated to homogeneous spaces G/H has proven to be extremely useful in the understanding of CFTs; finding field theoretic realizations of such "coset models" is a particularly relevant problem that has attracted much attention⁽¹⁰⁾⁻⁽¹²⁾.

Several proposals on how to formulate G/H conformal theories gauging a Mess-Zumino-Witten⁽¹⁴⁾ (MZW) action have been presented⁽¹⁰⁾. Concerning fermionic models, already in the original GHD work⁽⁶⁾ it was suggested that the coset construction could be related to constrained fermi CFTs and field theoretic realizations of c=1 models have been discussed^(11,12). In a previous letter⁽¹⁴⁾ we have presented a formulation corresponding to an explicit (1+1) constrained fermion model with VCC in the Friedan-Qiu-Shenker (FQS) series⁽¹⁴⁾.

In the present work we give a more detailed description of our proposal and also extend our treatment to other relevant models like the superconformal⁽⁶⁾ and the parafermionic⁽⁶⁾ ones and also a model in which additional Z_3 symmetry is introduced⁽⁷⁾. We also discuss the equivalence between the constrained bosonic models mentioned above (Ref.10) and the constrained fermionic ones presented here.

The paper is organized as follows: In section II we give a detailed description on how free fermion Lagrangian can be constrained⁽¹²⁾. We then obtain an explicit constrained fermionic theory with VCC corresponding to a G/H coset model with $G=U(N)$ and H certain subgroup of the original symmetry group with Kac-Moody level K_m .

In section III, by an appropriate choice of groups G and H, we obtain an explicit effective action for a constrained fermionic theory. It corresponds to free fermion fields $\psi, \bar{\chi}$, free ghosts $\eta, \varepsilon, \rho, \beta$ and WZW fields $\mathcal{H}, \mathcal{G}, h, g, \bar{h}$.

(see eq.(3.12)):

$$S_{eff} = \int d^2x \left[\bar{\Psi} : \not{\partial} \Psi + \bar{\chi} : \not{\partial} \chi + \gamma_z \partial_z \varepsilon_z + \gamma_x \partial_x \varepsilon_x + \rho_x \partial_x \beta_x + \rho_z \partial_z \beta_z + \gamma_{\mathcal{H}} \partial_z \varepsilon_{\mathcal{H}} \right] + N \left\{ S[\mathcal{H}] + S[\mathcal{G}] + S[h] + S[g] + (k+q) S[\bar{h}] \right\} \quad (1.1)$$

The corresponding VCC can be easily evaluated (see eq.(3.1)):

$$C_T = \frac{(N^2-1)k \cdot \ell (k+\ell+2N)}{(k+\ell+N)(k+N)(\ell+N)} \quad (1.2)$$

Particular values of integers N, k and ℓ , give the conformal unitary series, the superconformal unitary series, etc. (see eqs.(3.12)-(3.15) and (3.17)).

Finally in section IV we discuss the bose-fermi equivalence between fermionic models described by the effective action (1.1) and its bosonic counterpart treated in ref.[10]. We show that both bosonic and fermionic models have identical current correlation-functions and the same conformal anomalies for our election of groups and representations.

One of the relevant aspects of these results lies in the possibility of computing physical quantities for a given theory within the path integral formulation using a constrained fermionic model. In particular in the case of c=1/2, the Ising model, there exists a fermionic Quantum Field Theory, consisting in one Majorana free fermion⁽¹⁶⁾, in terms of which this model can be described at the critical point.

In the case of minimal unitary CFTs, Friedan, Qiu and Shenker⁽⁴⁾ and other authors have found, by comparison of scale dimensions h for a given value of $c(k)$

$$h_k(p, q) = \frac{[(k+3)p - (k+2)q]^2 - 1}{4(k+2)(k+3)} \quad (1.3)$$

$$(k=1, 2, \dots; p=1, 2, \dots, k+1; q=1, 2, \dots, p)$$

with the critical indices of the corresponding theories, that the CFTs with values of VCC in the FQS series correspond to some statistical models such as the Ising model, tricritical Ising model, Fotts model, etc⁽⁴⁾.

Since a quantum fermionic field description for all those models (except for the case of $c=1/2$) is lacking, we hope that effective action (1.1) could be useful in identification of primary fields and calculation of relevant physical quantities using a fermionic path-integral formulation. Work on this aspects is in progress.

SECTION II: GAUGING SYMMETRIES IN A FREE FERMION LAGRANGIAN

Given a Conformal Field Theory with a symmetry group G , one has an associated Kac-Moody algebra (KMA) with level k and a Virasoro algebra (VA) with central charge c_G . When the group G has a subgroup H with an associated k -level KMA and a VA with central charge c_H one can construct, as shown by Gelfand, Kent and Olive⁽⁵⁾ (GKO), a new VA by subtracting the Virasoro generators obtained from the Sugawara construction for G and H respectively. This algebra has a VCC $c = c_G - c_H$; when H is an invariant subgroup, it can be identified with the VA of the G/H coset model.

GKO also suggested⁽⁶⁾ that their construction could be implemented in a fermionic quantum field theory starting with a conformal theory with a symmetry group G and constraining the fermionic currents associated with a subgroup H . Following these ideas we explicitly show in this section how the coset construction can be implemented in the path integral approach starting from a free fermionic model with symmetry group G and then constraining the fermionic H -currents. The resulting fermionic models have a VCC which corresponds to the FQS series, the superconformal unitary series and others according to the choice of the constrained currents, i.e. the choice of the groups G and H , as will be shown in the next section.

Several authors^(4,6) have already considered field theoretic realizations by constraining bosonic theories; they have obtained in this way models leading to the above mentioned series.

Constrained fermionic theories have been also studied^[11,12] but the explicit effective action leading to the different series have not been yet presented (except for the particular case of the FOS series^[14]).

We start from a free fermion Lagrangian in two-dimensional Minkowski spacetime:

$$\mathcal{L}_0 = \bar{\Psi}^k i \not{\partial} \Psi^k, \quad k=1,2,\dots,N \quad (2.1)$$

where Ψ^k are Dirac fermions in the fundamental representation of a compact Lie group G . In order to obtain a coset theory, we shall constrain the fermionic currents associated with the subgroup H by imposing on the currents:

$$J_\mu^A = \bar{\Psi}^i \gamma_\mu T_{ij}^A \Psi^j \quad (2.2)$$

(T^A are the H -generators, $A=1,2,\dots,\dim(H)$) the following conditions:

$$J_\mu^A |f\rangle = 0 \quad (2.3)$$

where $|f\rangle$ represent the allowed physical states.

In the path integral formulation these conditions can be achieved introducing appropriate δ -functionals in the partition function Z :

$$Z = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{i \int d^2x \bar{\Psi}^k i \not{\partial} \Psi^k} \prod_{A \in H} \delta[J_\mu^A(x)] \quad (2.4)$$

If we now use the representation:

$$\delta[J_\mu^A] = \int \mathcal{D}A_\mu e^{i \int d^2x J_\mu^A A^{\mu A}} \quad (2.5)$$

where A_μ^A acts as a Lagrange multiplier, the generating functional can be written in the form:

$$Z = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \mathcal{D}A e^{i \int d^2x \bar{\Psi} (i \not{\partial} + A) \Psi} \quad (2.6)$$

with $A_\mu = (A_\mu^A)$.

Note that A_μ acts as a dynamical gauge field and then in order to have a well defined generating functional, the gauge must be fixed. We choose the light-cone gauge:

$$A_- = A_0 - A_1 = 0 \quad (2.7)$$

then the lagrangian takes the form:

$$\mathcal{L}[\psi_+, \psi_-, A_+] = \psi_-^\dagger (\partial_+ A_+) \psi_- + \psi_+^\dagger \partial_- \psi_+ \quad (2.8)$$

The resulting lagrangian coincides with the one presented in ref. [17] where constrained fermion models were constructed starting with Weyl fermions coupled to an A_+ field. However in the present case, expression (2.8) was obtained by fixing $A_- = 0$ and hence a Faddeev-Popov determinant has to be included,

$$\Delta_{FP} = \det \partial_- \quad (2.9)$$

Δ_{FP} can be exponentiated using ghosts η_+^c and ε^a in the adjoint representation of H with conformal weights 1 and 0 respectively.

$$L_{FP} = \eta_+^c \partial_- \varepsilon^a \quad (2.10)$$

Note that it is the need of gauge fixing that makes ghosts come into play in the present case. Since ghosts give a negative contribution to the VCC, one can expect a value of c less than unity^[17].

The complete generating functional reads:

$$Z = \int \mathcal{D}\psi_+ \mathcal{D}\psi_- \mathcal{D}A_+ \mathcal{D}\eta_+ \mathcal{D}\varepsilon \int \mathcal{L}[\psi_+, \psi_-, A_+] d^2x \int \eta_+^c \partial_- \varepsilon d^2x e^{-iCC} S[h] \quad (2.11)$$

Writing A_+ in terms of a group valued field h :

$$A_+ = i h^{-1} \partial_+ h, \quad h \in H \quad (2.12)$$

we obtain for the fermionic path integral^{[18][19]}:

$$\int \mathcal{D}\psi_+ \mathcal{D}\psi_- e^{i \int \mathcal{L}[\psi_+, \psi_-, A_+] d^2x} = \det \mathcal{D} e^{iCC} S[h] \quad (2.13)$$

where $S[h]$ is the well known WZW action^[20]:

$$S[h] = \frac{1}{4\pi} \int T_0 (\partial_- h \partial_- h^{-1})^2 d^2x + \frac{i}{12\pi} \int d^3y \varepsilon_{ijk} (h^{-1} \partial_i h \partial_j h \partial_k h) \quad (2.14)$$

and K_{WZW} is the level of the KMA satisfied by the constrained fermionic currents (2.2) which (in light-cone components) takes the form:

$$[J_+^a(x), J_-^b(y)] = 2\sqrt{2} \int_{-y}^x \omega \delta(x-y) + i \frac{c}{4} K_{\text{WZW}} \delta^{\text{AB}} \delta'(x-y) \quad (2.15)$$

The change of variables (2.12) in the generating functional leads to an additional jacobian^{[17][18][20]}:

$$\mathcal{D}A_+ = J[h] \mathcal{D}h \quad (2.16)$$

$$J[h] = \det \mathcal{D}_+ [h] = e^{-iCC} S[h] \quad (2.17)$$

representation:

$$C_H \delta^{AB} = \int \int^{A' B'} \delta^{A' B'} \quad (2.18)$$

and f^{abc} are the structure constants of H. (For details on the calculation of (2.17) see the appendix).

With all this, the generating functional takes the form:

$$Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}h \mathcal{D}\gamma_\mu \mathcal{D}\epsilon \quad : \text{Self} \quad (2.19)$$

where the effective action Self reads:

$$S_{\text{eff}} = \int d^3x \bar{\psi} (\not{\partial} - (K_m + 2C_H) S[h]) \psi + \int d^3x \gamma_\mu \partial_\mu \epsilon \quad (2.20)$$

The energy-momentum tensor associated with this action receives three independent contributions coming from free fermions, gauge fields and free ghosts:

$$\langle T_{++} \rangle = \frac{1}{2} \frac{\delta Z}{\delta g_{++}} = \langle \psi_+ \partial_+ \psi_+ + T_{++}[h] + \gamma_\mu \partial_\mu \epsilon \rangle \quad (2.21)$$

$T_{++}[h]$ is the energy-momentum tensor associated with $S[h]$ and it is given by the expression:

$$T_{++}[h] = \frac{-1}{2(K_m + C_H)} - \frac{1}{2\pi} T_2[h \partial_+ h^\mu \partial_+ h_\mu] \quad (2.22)$$

It is convenient to define a Laurent expansion of the energy-momentum tensor:

$$T_{++}(x_+) = \sum_{n=-\infty}^{\infty} x_+^{-n-2} L_n \quad (2.23)$$

In terms of these modes L_n , the VA takes the form:

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m,-n} \quad (2.24)$$

The central charge c can be obtained from the expression:

$$\partial_- \langle T(T_+(x_+), T_+(y_+)) \rangle = \delta(x_- - y_-) \langle [T_+(x_+), T_+(y_+)] \rangle \quad (2.25)$$

It can be written as:

$$C_T = C_{\text{matter}} + C_{\text{ghosts}} + C_{\text{gauge}} \quad (2.26)$$

The contribution of the free Dirac fermions is $c_{\text{matter}} = N$. The ghost contribution can be easily evaluated using (2.25), $c_{\text{ghosts}} = -2 \dim H$. Concerning the contribution of the WZW fields, it can be obtained using the Knizhnik-Zamolodchikov formula^[2]:

$$C_{\text{gauge}} = \frac{k \dim H}{k + C_H} \quad (2.27)$$

where k is the Kac-Moody level of the WZW currents. From (2.20) we have: $k = -(K_m + 2C_H)$, then:

$$C_T = \frac{- (K_m + 2C_H) \dim H}{K_m + C_H} \quad (2.28)$$

With all this, the value of c_T is then given by:

$$c_T = N - \frac{K_m \dim H}{K_m + C_H} \quad (2.29)$$

Which corresponds to the VCC of a coset model where the original symmetry group is $G = U(N)$ with a Kac-Moody level $k = 1$ and where the constrained subgroup H has level k_m .

We then conclude that the constrained theory with generating functional given by (2.6), or equivalently by (2.19)-(2.20), leads to a coset model VA. As we shall see in the next section, a careful choice of the symmetry groups G and H leads to values of c_r in the FQS, the unitary superconformal and other well known series.

SECTION III : FERMIONIC REALIZATIONS OF CONFORMAL FIELD THEORIES AND DISCRETE SERIES

We will make use of the results presented in the previous section to obtain the explicit form of the effective action for a constrained fermionic model with VCC given by the formula :

$$c_r = \frac{(N^2-1)k\ell(k+\ell+2N)}{(k+\ell+N)(k+N)(\ell+N)} \quad (3.1)$$

This value of c corresponds in the GKO construction to a theory with a symmetry group $G = SU(N)_k \times SU(N)_\ell$ and a constrained subgroup $H = SU(N)_{k+\ell}$. By an appropriate choice of integers N , k and ℓ one can obtain values of c in the FQS series, the superconformal unitary series and others.

We start with a free Dirac fermion lagrangian like (2.1) written in a suitable form

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \quad (3.2)$$

where :

$$\begin{aligned} \mathcal{L}_1 &= \bar{\psi}^{i\alpha} i \not{\partial} \psi^{i\alpha} \\ \mathcal{L}_2 &= \bar{\chi}^{\rho m} i \not{\partial} \chi^{\rho m} \end{aligned} \quad (3.3)$$

with fermions $\psi^{i\alpha}$ ($i=1, \dots, N$ and $\alpha=1, \dots, k$) and $\chi^{\rho m}$ ($\rho=1, \dots, N$

and $m=1, \dots, l$) in the fundamental representation of $U(N, k)$ and $U(N, l)$ respectively.

The Lagrangian $\mathcal{L}_1$ has a symmetry group $U(N, k)$ with currents J_μ^A given by :

$$J_\mu^A = \bar{\psi}^{i\alpha} \gamma_\mu T_{ij}^A \psi^{j\alpha} \quad (3.4)$$

We can also construct the following currents, associated with $SU(N)_k$, $SU(k)_N$ and $U(1)_1$ symmetries :

$$J_\mu^A = \bar{\psi}^{i\alpha} \gamma_\mu T_{ij}^A \psi^{j\alpha} \quad (3.5, a)$$

$$J_\mu^a = \bar{\psi}^{i\alpha} \gamma_\mu t_{\alpha\beta}^a \psi^{i\beta} \quad (3.5, b)$$

$$J_\mu = \bar{\psi}^{i\alpha} \gamma_\mu \psi^{i\alpha} \quad (3.5, c)$$

where T^A ($A=1, \dots, N^2-1$) are the $SU(N)$ generators and t^a ($a=1, \dots, k^2-1$) the $SU(k)$ ones.

Following the procedure developed in section II we are going to constrain currents (3.5, b) and (3.5, c); it is then useful to treat separately the symmetries $SU(N)_k$, $SU(k)_N$ and $U(1)_1$.

In order to obtain an effective theory with VCC given by (3.1) we begin by constraining the following currents :

$$J_\mu^a = \bar{\psi}^{i\alpha} \gamma_\mu t_{\alpha\beta}^a \psi^{i\beta}$$

$$\tilde{J}_\mu^b = \bar{\chi}^{\rho m} \gamma_\mu \tilde{t}_{mn}^b \chi^{\rho n}$$

(3.6)

where t^a ($a=1, \dots, k^2-1$) and $\tilde{t}^b$ ($b=1, \dots, l^2-1$) are the $SU(k)$ and $SU(l)$ generators respectively.

In order to eliminate the $U(1)$ factors we also need to constrain the currents :

$$J_\mu = \bar{\psi} \gamma_\mu \psi$$

$$\tilde{J}_\mu = \bar{\chi} \gamma_\mu \chi$$

(3.7)

this being achieved by introducing abelian Lagrange multipliers a and b respectively.

The value of the central charge can be evaluated using (2.25), (2.28). It is given by :

$$C = \frac{k(N^2-1)}{k+N} + \frac{l(N^2-1)}{l+N} \quad (3.8)$$

This value of VCC corresponds to a theory with a symmetry group $G = SU(N)_k \times SU(N)_l$. This group has a diagonal subgroup $H = SU(N)$; its associated KMA has a level equal to the sum of the KM levels of the factors in the direct product of G , i.e. $k+l$. (This can be easily seen from the current commutation relations).

The generators of the level- k KMA associated with $SU(N)$ are given by (3.5, a) and those with level l are given by :

$$\tilde{J}_\mu^a = \bar{\chi}^{\rho m} \gamma_\mu T_{\rho\sigma}^a \chi^{\sigma m} \quad (3.9)$$

explicit form of the corresponding effective action S_{eff} is :

$$\begin{aligned}
 S_{\text{eff}} = & \int d^2x \left[\bar{\psi}^{i\alpha} i \not{\partial} \psi^{i\alpha} + \bar{\chi}^{\rho m} i \not{\partial} \chi^{\rho m} + \eta_{\pm}^a \partial_{\pm} \varepsilon_{\pm}^a \right. \\
 & + \eta_{\Pi}^b \partial_{\pm} \varepsilon_{\Pi}^b + \rho_{\pm} \partial_{\pm} \beta_{\pm} + \rho_{\Pi} \partial_{\pm} \beta_{\Pi} + \eta_{\Pi}^i \partial_{\pm} \varepsilon_{\Pi}^i \Big] + \\
 & + N \left[S[\psi] + S[\chi] \right] + S[u] + S[g] + (k+l) S[t] \\
 & (a=1,2,\dots,k-1; b=1,2,\dots,l-1; i=1,2,\dots,n-1)
 \end{aligned} \quad (3.12)$$

With this general result it is easy to have constrained fermionic models with central charges in several well known series by a suitable choice of N, k and l (i.e. a suitable choice of the groups and km levels of the currents). These series correspond to the values of VC for conformal field theories satisfying additional conditions such as unitarity and minimality, or to conformal field theories with extra symmetries such as supersymmetry and Z_2 symmetry.

An important case corresponds to the FQS series for unitary minimal conformal models with c less than unity⁽⁴⁾:

$$c = 1 - \frac{6}{(k+2)(k+3)} \quad (3.13)$$

These models are related to statistical systems in $1+l$ dimensions such as the Ising model, Tricritical Ising model, Potts model, etc., at the critical point. Models with values of c given by (3.13) correspond to those obtained by taking $G = SU(2)_k \times SU(2)_1$ and $H = SU(2)_{k+1}$ in the effective action (3.12), leading to a value of c given by (3.1) with $N=2$ and $l=1$. That is, we have

Then the generators of the KMA associated with the diagonal $SU(N)$ subgroup are :

$$J_{\mu}^A = \bar{\psi}^{i\alpha} \gamma_{\mu} T_{ij}^A \psi^{j\alpha} + \bar{\chi}^{\rho m} \gamma_{\mu} T_{\rho\sigma}^A \chi^{\sigma m} \quad (3.10)$$

Since currents J_{μ}^A and $\tilde{J}_{\mu}^a$ commute, it is easy to see that the diagonal currents J_{μ}^A satisfy the KM relation (2.14) with level $k+l$.

The generating functional reads:

$$\begin{aligned}
 Z = & \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\bar{\chi} \mathcal{D}\chi \mathcal{D}A_{\mu} \mathcal{D}B_{\mu} \mathcal{D}a_{\mu} \mathcal{D}b_{\mu} \mathcal{D}c_{\mu} \times \\
 & \times \exp \left\{ i \int d^2x \left[\bar{\psi}^{i\alpha} \left(i \not{\partial} \psi^{i\alpha} \delta_{\mu\rho} + \delta_{ij} \not{\partial}_{\mu} \psi^{j\alpha} + \not{A}_{\mu\beta} + \not{B}_{ij} \delta_{\mu\rho} + \not{a}_{\mu\beta} + \not{b}_{ij} \delta_{\mu\rho} \right) \psi^{i\alpha} \right. \right. \\
 & \left. \left. + \bar{\chi}^{\rho m} \left(i \not{\partial} \chi^{\rho m} \delta_{\mu\nu} + \delta_{\rho\sigma} \not{\partial}_{\mu} \chi^{\sigma m} + \not{A}_{\mu\nu} + \not{B}_{\rho\sigma} \delta_{\mu\nu} + \not{a}_{\mu\nu} + \not{b}_{\rho\sigma} \delta_{\mu\nu} \right) \chi^{\rho m} \right] \right\} \quad (3.11)
 \end{aligned}$$

Following the same steps as in the previous section, (i.e. we fix the light-cone gauge for all gauge fields and change variables as in (2.12)), we obtain the partition function of a theory with the desired value of c , (3.1). The

gotten the FOS series by constraining appropriate symmetries of an originally free fermionic model.

The supersymmetric generalization of these conformal theories are called superconformal field theories¹⁰ and the unitarity condition (for $c < 3/2$) restricts the values of VCC to the series :

$$C = \frac{3}{2} - \frac{12}{(k+2)(k+4)} \quad (3.14)$$

In our approach this series can be obtained from the constrained fermionic model defined by the effective action (3.12) by taking $G = SU(2)_k \times SU(2)_2$ and $H = SU(2)_{k+2}$. Indeed from (3.1) with $N=2$ and $\ell=2$ we get (3.14).

There is another series associated with exactly solvable conformally invariant quantum field theories having also the extra symmetry generated by conserved currents of spin $3/2$.¹¹ All the models in this series possess a global Z_3 symmetry. The VCC for these models takes values in the series :

$$C = 2 - \frac{24}{(k+3)(k+4)} \quad (3.15)$$

Again, these values can be obtained from (3.1) with $N=3$ and $\ell=1$, i.e. the effective action (3.12) with $G = SU(3)_k \times SU(3)_4$ and $H = SU(3)_{k+4}$.

The simplest one of these models corresponds to $k=1$ ($C = 4/3$) and coincides with one of the minimal conformal invariant models^{12a)} ($k=3$) which is known to describe the critical point of the 3-state Potts model.^{12b)}

Another family of solvable conformal field theories, called parafermionic field theories¹⁰⁾, is described in terms of non-local, fractional spin, analytic fields called parafermions. Parafermions correspond to self-dual critical points in $2k$ symmetric statistical systems. The theory contains $2k-1$ such fields $\psi_k(z), \bar{\psi}_k(\bar{z})$ for $k=0, \dots, k-1$; ψ_0 and $\bar{\psi}_0$ correspond to identity fields. They satisfy the condition $\partial_{\bar{z}} \psi_k = 0$ ($\partial_z \bar{\psi}_k = 0$) and therefore are functions of z ($\bar{z}$) only. ψ_k has a fractional spin d_k which is also the conformal dimension of the field. The parafermionic fields are the analogous to the currents of the WZW theory. They generate some infinite-dimensional symmetry, therefore they are also referred to as parafermionic currents.

The fields ψ_k have the following operator product expansion :

$$\psi_k(z) \psi_{\ell}(\bar{z}) = C_{k,\ell} (z-\bar{z})^{d_{k,\ell}-d_k-d_{\ell}} \sum_{n=0}^{\infty} (z-\bar{z})^n \psi_k^{(n)}(z) \quad (3.16)$$

where $C_{k,\ell}$ are some constants, determined by the associativity of this expansion. The VCC for these models is given by :

$$C = 2 \cdot \frac{(N-1)}{(N+2)} \quad (3.17)$$

Within our approach, these models correspond to those obtained by taking $G = SU(N)_k \times SU(N)_4$ and $H = SU(N)_2$.

SECTION IV: BOSE-FERMI EQUIVALENCE FOR GAUGED MODELS

It is our purpose now to compare the constrained fermionic theories defined by the effective action (3.12) (leading to the series (3.13)-(3.15) and (3.17)) with certain bosonic G/H conformal theories based on gauged WZW models (also leading to these series). Indeed, in refs. [10] it has been presented a path-integral formulation of a theory with bosonic fields belonging to a group G with a level-k WZW action and a constrained subgroup H of the action global G symmetry. In this references it has been obtained a VCC, $C=C_{G,k} - C_{H,k}$ which coincides with the VCC obtained from the Gk0 construction.

In ref. [23] it has been shown that in 1+1 dimensions the effective action corresponding to massless fermions coupled to an external gauge field is identical with that produced by a gauged WZW bosonic action. Extending the analysis, we are going to study this equality in the case in which the gauge field acts as a dynamical field as in the models discussed in sections II and III.

We consider the fermionic model partition function (2.6):

$$\mathcal{Z} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A e^{i \int d^2x \bar{\psi} (i \not{\partial} + \not{A}) \psi} \quad (4.1)$$

and the bosonic one given by (4.1):

$$\mathcal{Z} = \int \mathcal{D}g \mathcal{D}A e^{i S_b[g,A]} \quad (4.2)$$

The bosonic action is the so-called gauged-WZW action:

$$S_b[g,A] = S[g] + \frac{k}{4\pi} \int d^2x \left(A_\mu \partial_\nu g g^{-1} - A_\nu \partial_\mu g g^{-1} + A_\mu g A_\nu g^{-1} - A_\nu g A_\mu g^{-1} \right) \quad (4.3)$$

with S[g] the WZW action and A_μ and A_ν the light-cone components of the gauge field A_μ , taking values in the Lie algebra of H.

In order to study the equivalence between the two models defined above, we shall analyze the corresponding generating functionals for current correlation functions and show that they are identical^[23]. We first review the analysis leading to the equivalence between the two models in the case treated in ref. [23] i.e. in which A_μ acts an external field.

We start considering the fermionic path integral:

$$\mathcal{Z}_f[A] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i S_f[\bar{\psi}, \psi; A]} \quad (4.4)$$

(We have normalized the measure $\mathcal{D}\bar{\psi} \mathcal{D}\psi$ so that the functional integral is unity in the absence of the external gauge fields).

The fermionic integration can be written in terms of a

WZW action^{(18),(19)} using a parametrization of the field A_μ like (2.12) :

$$\mathcal{Z}_F[A] = e^{-iS[h,\tilde{h}^{-1}] + i\alpha \int d^2x \, h(A,A)} \quad (4.5)$$

where $h, \tilde{h} \in H$ and α is a parameter depending on the regularization prescription used in performing the functional integral. As explained in refs. [21],[24] and [25] a local polynomial in the potential can be added to $S[h,\tilde{h}^{-1}]$ because there is no gauge invariance principle to justify in the present case its absence. This is the origin of the second term in the exponent (4.5). Later on, we shall discuss how this parameter can be chosen in connection with the equality between bosonic and fermionic generating functionals.

For the bosonic case, the counterpart of expression (4.4) is

$$\bar{\mathcal{Z}}_g[A] = \int \mathcal{D}g \, e^{iS_g[g,A]} \quad (4.6)$$

with $S_g[g,A]$ given by (4.3) and $\mathcal{D}g$ the invariant Haar measure which has been normalized, so that the functional integral is unity in absence of external gauge fields.

Using for A_μ the above-mentioned parametrization and the Polyakov-Wiegmann identity⁽¹⁸⁾:

$$S[g,h^{-1}] = S[g] + S[h^{-1}] + \frac{1}{4\pi} \int d^2x \, \text{tr} (g^{-1} \partial_\mu g h^{-1} \partial_\mu h) \quad (4.7)$$

one has:

$$S[g,A] = S[hg\tilde{h}^{-1}] - S[h,\tilde{h}^{-1}] \quad (4.8)$$

Changing variables in the functional integral

$$g' = hg\tilde{h}^{-1} \quad (4.9)$$

one finally obtains:

$$\bar{\mathcal{Z}}_g[A] = e^{-iS[h,\tilde{h}^{-1}]} \quad (4.10)$$

One can see now that the choice $\alpha=0$ in (4.5) (which corresponds to a regularization that preserves the diagonal H local gauge invariance) makes fermionic and bosonic generating functionals ((4.5) and (4.10)) identical.

Another interesting possibility corresponds to the election $\alpha=-1/4\pi$ for which the preserved symmetry is the global chiral H subgroup. This regularization choice was used by several authors⁽²³⁾ to prove the equivalence between a free fermionic model and a WZW model.

In this case the fermionic generating functional (4.5)

is equal to a bosonic one given by:

$$Z_b[A] = \int \mathcal{D}g e^{i\int \mathcal{L}(g,A)} \quad (4.11)$$

where

$$\tilde{S}(g,A) = S[g] + \frac{1}{2\pi} \int d^2x \, t_a \left[(a_g) g' A_+ - g'' \partial_g A_+ \right] \quad (4.12)$$

(see ref. [26]).

We have then demonstrated the equality between both partition functions (4.4) and (4.6) under certain regularization prescription. From this, it can be shown^[23] that the n-point current correlation functions for the fermionic and bosonic models in the background of a gauge field A coincides.

Now, we will extend this analysis to the case in which A_μ is a dynamical field, as in (2.6). Since we have introduced A_μ as a lagrange multiplier, the H currents vanish. The generating functional for fermionic current correlation functions is:

$$Z[\phi] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i\int d^2x [\bar{\psi}(\gamma_\mu \partial_\mu \psi) + \bar{\psi} \phi \psi]} \quad (4.13)$$

(we have introduced an additional external source $\phi_\mu \in 0$). Z can be written as:

$$Z[\phi] = \int \mathcal{D}A_\mu Z_f[A+\phi] \quad (4.14)$$

The B_μ projection on H can be absorbed by an appropriate change of

variables in the A_μ integration. (This explains why the H currents are zero).

The corresponding generating function for the bosonic model is:

$$\tilde{Z}[\phi] = \int \mathcal{D}A_\mu \tilde{Z}_b[A+\phi] \quad (4.15)$$

Since we have demonstrated that $Z_f[A]=Z_b[A]$ it follows that (4.14) and (4.15) are identical.

At the light of these results we can conclude that the model given by the generating functional (3.11) with a VCC given by (3.1), is equivalent, in the sense that the n-point current correlation functions coincide, to a WZW gauged model with the partition function (4.2)-(4.3) with $g \in SU(N)_k \times SU(N)_l$ and A_μ lies in the Lie algebra of $SU(N)_{k+l}$.

At this point it is interesting to give a general analysis of cases in which the symmetry group G can be written as the direct product $G=G_A \times G_B$ and the fields A_μ and B_μ in (4.13) belong to the Lie algebra of G_A and G_B respectively. In this case the parametrization (2.12) can be conveniently expressed in terms of the elements of G_A and G_B :

$$\begin{aligned} A_+ &= i h_A^{-1} \partial_+ h_A \times I_0 \\ A_- &= i \tilde{h}_A^{-1} \partial_- \tilde{h}_A \times I_0 \end{aligned} \quad (4.16)$$

$$\begin{aligned} B_+ &= I_A \times i h_B^{-1} \partial_+ h_B \\ B_- &= I_A \times i \tilde{h}_B^{-1} \partial_- \tilde{h}_B \end{aligned} \quad (4.17)$$

Integrating the fermionic variables and using the fact that

$$S[A+B] = N_A S[B] + N_B S[A] \quad (4.18)$$

$(N_{A,B} = \text{tr } I_{A,B})$. we finally obtain:

$$Z[B] = \left\{ \int \mathcal{D}A e^{-iN_A S[A]} \right\} e^{-iN_B S[B]} \quad (4.19)$$

so we were able to factorize the A_μ dependence and the partition function is a constant N_μ times the generating functional of the G_μ free fermionic currents. This can be easily seen in terms of the fermionic variables:

$$Z[B] = N_\mu \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^3x \bar{\psi} (i \not{\partial} + I_\mu \times B) \psi} \quad (4.20)$$

We can follow the same steps for the bosonic case and we obtain:

$$\bar{Z}[B] = N_\mu \int \mathcal{D}g e^{i S[g, B]} \quad (4.21)$$

Since we have seen that the generating functional for free currents are equal both for fermionic and bosonic models, it follows straightforwardly that (4.20) and (4.21) are equal.

It must be remarked that we have not yet discussed

energy-momentum tensor correlation functions. In fact there can exist two models in which current correlation functions are identical but their energy-momentum correlation functions are different⁽⁹⁾.

Comparing the expressions of energy-momentum tensors (2.21) and (2.22) for the fermionic model (2.19) with that corresponding to the bosonic model (4.21) it is easy to see that they only differ in the first term of each expressions corresponding in the fermionic case to the free fermion contribution:

$$T_{++}^F = \psi_+ \partial_- \psi_+ \quad (4.22)$$

and in the bosonic one to the free boson contribution:

$$T_{++}^B = \frac{-1}{2 + C_G} \frac{1}{2\pi} \text{tr} (\partial_+ g^{-1} \partial_+ g) \quad (4.23)$$

where C_G is the Casimir in the adjoint representation of G .

It has been shown that free fermionic and bosonic theories have the same conformal anomaly for certain groups and representations classified by Goddard, Nahm and Olive in ref. [26]. In particular, for our group and representation choices, we conclude from the results in [26] that the bosonic and fermionic contributions to (4.22) and (4.23) coincide.

With all this we conclude that the conformal anomalies both for the constrained fermionic model defined by the effective

action (3.12) and for the bosonic model defined by (4.2)-(4.3) coincide.

It must be noted that although we have demonstrated the equality of the current correlation functions and the VCC both for the fermionic and the bosonic models (4.1) and (4.2), we have not prove the complete equivalence between these two models.

As explained in ref.[27] a further necessary condition for the quantum equivalence of two theories is that equivalent primary fields should possess identical four-point functions. We shall report on this problem elsewhere.

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APPENDIX: CALCULATION OF THE FADEEV-POPOV DETERMINANT

In this appendix we calculate the Jacobian related to the change of variables (2.12):⁽¹⁷⁾⁽¹⁸⁾⁽²⁰⁾

$$A_+ = g^{-1} \partial_+ g \quad ; \quad g \in G \quad (A.1)$$

This Jacobian can be easily evaluated by considering an infinitesimal variation $\delta \varepsilon$ of g , with the corresponding change of A_+ given by:

$$\frac{\delta A_+}{\delta \varepsilon} = i D_+^{AB} \quad (A.2)$$

where D_+^{AB} is the covariant derivative in the adjoint representation:

$$D_+^{AB} = i \partial_+ + [A_+, \quad] \quad (A.3)$$

The Jacobian is then :

$$J[g] = \det D_+^{AB} \quad (A.4)$$

We can express this determinant as a path-integral over fermions

in the adjoint representation of H.

$$J[g] = \int \bar{\psi} \psi e^{i \int d^4x \bar{\psi} \gamma^\mu \partial_\mu \psi} \quad (A.5)$$

Now, $J[g]$ can be related to the determinant of the covariant derivative in the fundamental representation of H. To do this we note that after a suitable regularization (for example using the heat kernel method) the determinant can be written as [10]

$$\log \det \not{D} = -\frac{i}{4\pi} \int_0^1 dt \int d^4x \epsilon^{\mu\nu} F_{\mu\nu}^a \phi^a h(t^a t^b) \quad (A.6)$$

where $F_{\mu\nu}^a$ is the curvature associated to the field A_μ^a :

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - i [A_\mu^a, A_\nu^a] \quad (A.7)$$

with

$$\not{X}^t = i e^{-\gamma_5 \not{p} t} \not{p} e^{\gamma_5 \not{p} t} ; \quad (\not{X}^{t+1} \equiv \not{X}) \quad (A.8)$$

Now we note that the expression (A.6) is valid for all fermion representations and hence allows us to relate the results in the adjoint and the fundamental ones. In fact, both results differ only by the factor $\text{tr}(t^a t^b)$. For t^a in the fundamental representation our convention is :

$$h(t^a t^b) = \frac{1}{2} \delta^{ab} \quad (A.9)$$

Having defined in (2.18) the quadratic Casimir in the adjoint representation as $f^{abc} f^{abc} = C \delta^{ab}$ we obtain the desired result:

$$\log \det \not{D}|_{\text{adj}} = 2 C \log \det \not{D}|_{\text{fund}} \quad (A.10)$$

or

$$J[g] = e^{-2 C S[g]} \quad (A.11)$$

where $S[g]$ is the WZW action given in (2.14).

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