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A TOPOLOGICAL EXPANSION FOR HIGH ENERGY HADRONIC COLLISIONS

II. - S CHANNEL DISCONTINUITIES AND MULTIPARTICLE CONTENT *)

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A B S T R A C T

In a companion paper, we have defined a topological expansion for high energy hadronic processes, and in particular discussed its asymptotic relation with Reggeon field theory from a t channel point of view. Here we turn to the study of s channel unitarity, which is automatically incorporated in our scheme. We first give a simple topological classification of s channel discontinuities at high energies, for which some interference terms are argued to be negligible. We then find that the unitarity equation is decomposed in a sum of positive definite terms related to various classes of events (generalized gap expansion). In the enhanced graph approximation we derive a cut Reggeon field theory having three Pomeron fields and four couplings, which is automatically consistent with the previous positivity requirements. Unitarity to all orders in $1/N$ [where $U(N)$ is the internal symmetry group] constrains all couplings to be proportional, consistently (except for a possible ambiguity) with the cutting rules of Abramovskii, Gribov and Kancheli.

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1. INTRODUCTION

Recent results in the Reggeon field theory ^{1),2)} (RFT) and absorptive models ³⁾ have suggested that the Gribov calculus with $\alpha_P(0) = 1$ and $g_P(0) \neq 0$ is possibly consistent with both t and s channel unitarity. In a previous paper ⁴⁾ (hereafter referred to as I) we have motivated and analyzed a crossing symmetric topological expansion for strong interactions ⁵⁾ and we have shown that its asymptotic behaviour is consistent with t channel discontinuity equations for Pomerons ⁶⁾.

Besides giving this asymptotic relation to RFT, the topological expansion allows a precise definition of the Reggeons, the bare Pomeron, and Pomeron couplings ⁵⁾. The s channel (planar) unitarity equation for Reggeons, analyzed to lowest order in $1/N$ [where $U(N)$ is the internal symmetry group] fixes the expansion parameter to be $g^2 N = O(1)$ ⁷⁾ and the bare triple Pomeron coupling to be non-vanishing and of order $g_P(0) \sim g^2 \sim 1/N$ ⁸⁾.

The field theoretical (or dual) models underlying the topological expansion fulfil perturbatively both t and s channel unitarity. It is not easy, however, to give a simple topological expansion of both sides of the unitarity equation, as it can be done instead in unitarity sum rules ⁸⁾. In fact, complicated output topologies can be constructed from the sum over states in TT^+ , even if simple ones are put in T . A careful study is therefore needed in order to disentangle some non-leading contributions (subsections 2a and 3a).

In this way we first obtain a topological classification of s channel discontinuities (subsection 2a) and we prove the existence of a generalized fireball expansion which is automatically consistent with positivity of total and partial cross-sections (subsection 2b). We show, in particular, that the single fireball includes the polyperipheral events first found in a ladder model by Abramovskii, Gribov and Kancheli ⁹⁾.

We then analyze the asymptotic behaviour of the discontinuities in the enhanced regions of phase space (subsection 3a) and we show that it is given by a cut Reggeon field theory (CRFT) involving three fields (among which the cut Pomeron) and four trilinear couplings. We give, in particular, a direct meaning in the theory to the triple-Pomeron coupling measured in diffractive excitation of large masses. The positivity of the fireball expansion, proved before on topological grounds, is here made explicit by proving it on the basis of CRFT (at least in a particular case) (subsection 3b). Finally, we show that in this topological approach (where a single expansion parameter, $1/N$, exists) unitarity constrains all couplings in the CRFT to be proportional to a single parameter g_P , thus recovering previous results ^{9),10)} for

Feynman diagram discontinuities. While in a different approach ¹¹⁾ the optical theorem was imposed only at infinite energy, here it turns out to be needed at all perturbative orders in g_P (subsection 3c). A proof of this result and other details on CRFT are found in the Appendices.

The outcome of this analysis -- summarized and discussed in Section 4 -- is that, under some dynamical conditions, the topological expansion is asymptotically consistent with both t channel discontinuity equations for Reggeons and s channel unitarity. The unresolved questions, also listed and discussed in Section 4, concern mainly the renormalized Pomeron intercept and the relative importance of non-enhanced versus enhanced contributions.

2. s CHANNEL UNITARITY AND INTERMEDIATE STATES

Let us consider in the topological expansion the four-point amplitude A_P of Eq. (3.8) of I for the Pomeron sector (i.e., with two boundaries). The s channel unitarity equation can then be written as

$$2 \operatorname{Im} A_P = \sum_n \sum_{(\pm)} \int d\phi_n A_n^{(+)} A_n^{(-)} \quad (2.1)$$

where $A_n^{(\pm)}$ are n particle production amplitudes with the prescription corresponding to the topological surfaces $S_n^{\pm}$. The indices $(\pm)$ characterize $S_n^{\pm}$, and represent the number of handles h_+ , h_- , the number of boundaries, and the possible assignments of external and intermediate particles to the various boundaries. As explained in I, $A_n^{(\pm)}$ is the sum of all the orientable "quark" graphs which can be embedded in $S_n^{\pm}$.

Since in the topological approach, h and b are the relevant parameters, it would be desirable to have a relation among $h_{\pm}$, $b_{\pm}$ in the right-hand side and the number of handles h in the left-hand side. However, there are interference terms in Eq. (2.1) for which this relation is not simple. Consider, in fact, the simplest case $b_{\pm} = 1$ and $h_{\pm} = 0$. The contribution to (2.1) in which the n intermediate particles are attached to $S_n^{(+)}$ and $S_n^{(-)}$ in the same order, gives $h=0$ in the left-hand side (Fig. 1a). However, if two intermediate particles in $S_n^{(+)}$ are exchanged, the contribution to the left-hand side will have, in general, $h=1$ (Fig. 1b). Since other permutations of the n intermediate particles give rise to higher values of h in the left-hand side, it is difficult to guess a relation between $h_{\pm}$, $b_{\pm}$ and h from this point of view. It is important to note, however, that in this case ($h_+ = h_- = 0$) all exchange graphs

correspond asymptotically to multi-Regge production amplitudes, so that each of them behaves at most as the $h=0$ surface. According to I they are therefore non-leading contributions to $h>0$.

2a Topological classification of discontinuities

A simpler s channel approach is to start fixing h in the left-hand side of Eq. (2.1) and then use the Cutkosky rules for taking the imaginary part. This corresponds to cutting with a plane the surface S_h into two disjoint surfaces $S_{h+b_+}^{(+)}$ and $S_{h+b_-}^{(-)}$.

The simplest example is the bare Pomeron amplitude ($h=0$), corresponding to a cylinder with $b=2$ (I, Section 3). In this case $S^{(+)}$ and $S^{(-)}$ are planar with $h_+=h_-=0$, $b_+=b_-=1$ (Fig. 2). The planarity of $S^{(\pm)}$ has been used¹²⁾ to argue that $\alpha_P^{(0)}(0) \approx 1$ and that particles are produced in $S^{(+)}$ in two sets, uniformly distributed in rapidity, and weakly correlated. The relation among h_+ , b_+ and h is simple in this case, because we do not have intermediate particle exchanges (Fig. 1b) which are considered at higher topological orders.

In general, an arbitrary contribution to the s channel discontinuity will consist of cutting n particle propagators, i.e., n quark-antiquark pairs. The cut $q, \bar{q}$ lines will belong to f_c distinct faces or to the two boundaries. Two cases are possible.

- a) $n = f_c + 2$, when each face and boundary is cut in only two regions above (+) or below (-) the cut. In this case, one must have $b_+ = b_- = b$.
- b) $n > f_c + 2$, when a face or a boundary is cut in more than two regions. Both $b_+ = b_-$ and $b_+ \neq b_-$ can occur in this case.

We shall argue, however, that the latter contributions are asymptotically non-leading (or non-enhanced).

The Euler characteristics $E = f + v - p$ ^{*)} of the complete surface are related to those $E_-(E_+)$ of the surface below (above) the cutting plane by

$$E = E_+ + E_- + f_c - n \leq E_+ + E_- - 2, \quad (2.2)$$

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*) f , v and p denote the number of faces, vertices and propagators in a Feynman graph triangulation of the surface (cf. I, Section 2). For the purpose of this discussion the boundaries are considered to be open, i.e., they are not counted as faces but appear in the expression for the Euler characteristics.

where the $\langle \rangle$ sign holds for case (a) [(b)]. By replacing in Eq. (2.2) the expression ¹³⁾ $E_i = 2 - 2h_i - b_i$ we get

$$h = h_+ + h_- + b - 1 \quad (b_+ = b_- = b) , \quad (2.3a)$$

$$h > h_+ + h_- + \frac{1}{2}(b_+ + b_-) - 1 , \quad (2.3b)$$

for the cases (a) and (b), respectively. We shall base on the relation (2.3a) our topological classification of the s channel discontinuities.

The case (b) with $b_+ = b_- = b$ has, according to (2.3b), $h > h_+ + h_- + b - 1$. This was considered previously (Fig. 1b) in the example with $b = 1$ and $h_+ = h_- = 0$. In general, h gives the order in $1/N^2$ of the graph, but the energy dependence (factors of $\ln s$) is characteristic of a surface with $\bar{h} = h_+ + h_- + b - 1 < h$. Moreover, a class of these terms arises from exchange contributions to unitarity, for which some momentum transfers are large. For both reasons, they are therefore non-enhanced contributions to A_h .

An example of class (b) with $b_+ \neq b_-$ is shown in Fig. 3 with $h = 1$, $b_+ = 1$, $b_- = 2$, $h_+ = h_- = 0$. Asymptotically, this also is a non-enhanced correction to the bare Pomeron. Let us, in fact, label the intermediate particle rapidities on the boundary B_- as in Fig. 3. Particles in $(0, y)$ $[(y', Y)]$ are absorbed in B_1^+ (B_2^+) and no particles are produced in (y, y') . In this last region we therefore have an interference between a planar surface in $B^{(-)}$ (a Reggeon) and a cylindric one in $S^{(+)}$ (a Pomeron), which is non-leading or non-enhanced according to whether $y' - y$ is large or not. Similarly, higher order interferences with $b_+ \neq b_-$ are shown to have "unsaturated" Reggeon exchanges which make them non-leading when all rapidity differences are large. Therefore, all the enhanced graphs belong to class (a).

If Eq. (2.3a) is assumed, it allows us to rewrite Eq. (2.1) with simple topological combinations, in the form

$$2 \operatorname{Im} A_h \simeq \sum_n \sum_{p, b, h_+, h_-} \int d\phi_n A_{n, b h_+}^{(+)} A_{n, b h_-}^{(-)} \delta_{h_+ + h_-, h - b + 1},$$

$$2 \operatorname{Im} \sum_h A_h \simeq \sum_n \sum_{p, b} \int d\phi_n \left(\sum_{h_+} A_{n, b h_+}^{(+)} \right) \left(\sum_{h_-} A_{n, b h_-}^{(-)} \right), \quad (2.4)$$

where p labels the possible assignments of the external particles to the various boundaries.

The topological expansion of unitarity in (2.4) has the important property of consisting, for $t=0$ and fixed b and p of a sum of positive terms. In fact for fixed b , the surfaces S_h^- and S_h^+ are topologically identical, while the corresponding functions are, at $t=0$, complex conjugate one to the other. The same characteristic could have been obtained starting directly from Eq. (2.1), but the elimination of non-leading contributions would have been less transparent.

2b Gap expansion of final states

Since the terms of fixed b give a positive definite contribution to unitarity, we shall define an expansion in classes of boundaries. However, since the final state particles can be permuted among different boundaries, we have to classify all topologically distinct ways of arranging the b boundaries on the cutting plane. For instance, if $h=1$, we have, according to Eqs. (2.3) and (2.4), the following cases (Fig. 4): $b=1$ (with either $h_+=1, h_-=0$ or vice versa); $b=2$ ($h_+=h_-=0$) and one boundary enclosing the other; $b=2$ ($h_+=h_-=0$) and disjoint boundaries.

In general, for a given b we have several configurations $C_i^{(b)}$ in which the boundaries are classified according to their connectedness structure and to their inclusion relations. All contributions to Eq. (2.4) are obtained by adding, for a given $C_i^{(b)}$, any number of handles above the cut (h_+) or below (h_-).

The topological configurations of the boundaries on the cutting plane can be put in correspondence with the type of production events. Consider first the case $b=1$. Particles produced on this boundary fill up the available rapidity phase space $(0,Y)$. This contribution to σ_{tot} is called fireball ¹⁴⁾, and roughly corresponds, in a multiperipheral model (MPM) to production amplitudes with no diffractive excitation (purely inelastic events). It includes, however, all possible absorptive corrections due to the sum over h_+ and h_- , and in a MPM has been studied in detail ³⁾ (cf. Appendix B), showing in this case uniform rapidity density and existence of critical behaviour. For example, the term $h=h_+=h_-=0$ is just the bare Pomeron (Fig. 2); the one with $h=1$ ($h_+=1, h_-=0$ or vice versa) is a contribution to the torus (Fig. 4a).

A natural generalization of the fireball concept to $b>1$ is given in the topological approach by the sum of all connected boundaries. The simplest non-trivial example is still the torus ($h=1$) with $h_+=h_-=0$

and $b=2$ (Fig. 4b). A more complicated one is shown in Figs. 5a and 5b where three connected boundaries first appear. In these cases the produced particles still populate the whole phase space $(0,Y)$, but not uniformly. The density of particles is enhanced where the internal boundary(ies) overlaps the external one, as first realized by Abramovskii, Gribov and Kancheli⁹⁾ (polyperipheral events).

The generalized gap expansion can be defined as the sum over disconnected clusters of boundaries. A general term of this sum can give rise to events in which particles are produced with rapidity gaps among the clusters of boundaries (diffractive events). The simplest examples are shown in Fig. 4c for the torus, and Figs. 5c and 5d for $b=3$.

It is important to remark that the term with $b=2$ (of which Fig. 4c is the simplest example) includes the elastic contribution to s channel unitarity. Therefore, in Eq. (2.4) all partial waves are bounded. This is not automatically satisfied if only enhanced graphs are kept (cf. Section 4).

3. THE REGGEON CALCULUS FOR S CHANNEL DISCONTINUITIES

In paper I we have discussed directly the asymptotic behaviour of the amplitude $A^{(h)}$ on the left-hand side of Eq. (2.4). Here we want to discuss the asymptotic behaviour of the right-hand side, and (as in I) we analyze in particular the enhanced contributions. We therefore need to define the phase space in rapidity and transverse momentum. A useful set of co-ordinates are the following. 1) The handle coordinates $k_i^\pm, y_i^\pm, y_i'^\pm$ defined as in I for any of the uncut handles h_+ and h_- . This was done in I by considering the Feynman (or dual) graphs embedded in the surface, by singling out the momenta $k_i^\pm$ carried by the handles, and by defining

$$|M_{a_i}^\pm|^2 = |(p_a - k_i^\pm)|^2 \approx s_0 e^{y_i^\pm}, |M_{b_i}^\pm|^2 = |(p_b + k_i^\pm)|^2 \approx s_0 e^{Y - y_i'^\pm}, \quad (3.1)$$

where p_a and p_b are the two incoming momenta. 2) The total transverse momenta k_i and extreme rapidities y_i, y_i' of b boundaries defined in terms of the same quantities for the intermediate particles (these include the available rapidity Y and momentum transfer Q , which are fixed). The enhanced region of phase space is the one for which all rapidity differences are large and transverse momenta are small. The analysis of the asymptotic behaviour is then based on the results (cf. paper I) for the bare quantities, that is:

- i) at high energy the transverse momenta are strongly damped;
- ii) the cylindric surface (cut along two edges) has a leading factorized ^{*)} Regge behaviour with intercept close to one (the Pomeron);
- iii) unitarity sum rules lead to a "coupling theorem", i.e., the triple Pomeron coupling is non-vanishing, even at zero momentum transfers.

The actual derivation of the asymptotic behaviour of Eq. (2.4) is involved, essentially because the cutting boundaries can be deformed in various shapes. The idea is as in I to connect the number h of handles to the number of loops in a properly chosen Gribov graph. Here, however, one has three types: the uncut handles above (h_+) or below (h_-) the cutting plane, and the cut ones defined as $h_c = b - 1$. Their sum gives, according to Eq. (2.3a), the total loop number h .

The correspondence of h_+ and h_- with two uncut Pomerons P_+ and P_- seems natural and straightforward. The introduction of a cut Pomeron P_c (a cylinder cut along two edges, Fig. 2) is made complicated by the fact that a "cylindrical" portion of a surface can be cut along more than two edges (Fig. 6b). The resulting $+$ and $-$ portions can be connected for sufficiently complicated topologies (Fig. 6a). Fortunately, these contributions will now be shown to be non-enhanced.

Consider, for example, the cut of Fig. 6a for $h=2$ which divides the central cylindric part (with rapidity between y and y') in four different regions. While in the case of Fig. 2 (where only two edges are cut), we have intermediate states of the bare Pomeron with normal multiplicity, in the case of Fig. 6a the particle density of the Pomeron part in $y-y'$ is doubled. This could be called a Pomeron with double multiplicity, and may lead to interesting effects at present energies. It leads, however, to non-enhanced contributions.

Let us, in fact, denote by $i=1,2,3,4$ the four sets of n_i particles produced in the region of rapidity $(y-y')$ at the two boundaries in Fig. 6a, and by $T_{n_i n_j}^{(\pm)}$ the planar production amplitude occurring in Fig. 6b. Then the contribution of the "double multiplicity Pomeron" to the unitarity integral is given by

$$\text{Im } P^{(2)} \propto \sum_{n_i} \int \prod_{i=1}^4 d\phi_i T_{n_1 n_2}^{(+)} T_{n_2 n_3}^{(-)} T_{n_3 n_4}^{(+)} T_{n_4 n_1}^{(-)} . \quad (3.2)$$

^{*)} According to the usual simplified treatment of the Pomeron helicity ¹⁾, we make here, and in the following, the assumption that cutting the Pomeron does not spoil its factorization properties.

The partial sum

$$M_{n_i n_j} = \sum_{n_k} \int d\phi_k T_{n_i n_k}^{(+)} T_{n_k n_j}^{(-)} \quad (3.3)$$

turns out to be a positive definite operator (with respect to the scalar product defined by $\sum_n \int d\phi$). Equation (3.2) has therefore the property

$$\text{Im } P^{(2)} \propto \text{Tr}(M^2) \leq (\text{Tr } M)^2, \quad (3.4)$$

where the last quantity is proportional to the discontinuity of the two-Pomeron cut with normal multiplicity. In fact

$$\text{Im } P \otimes P \propto \sum_{n_1 n_2} \int |T_{n_1 n_2}^{(+)}|^2 d\phi_1 d\phi_2 \cdot \sum_{n_3 n_4} \int |T_{n_3 n_4}^{(+)}|^2 d\phi_3 d\phi_4 = (\text{Tr } M)^2. \quad (3.5)$$

The inequality in (3.4) proves then that a discontinuity, cutting a handle in more than two regions, leads to non-enhanced contributions.

3a Cut Reggeon field theory

The asymptotic behaviour of a discontinuity graph containing h_+ , h_- uncut handles and $h_c = b - 1$ cut ones, can now proceed as in I (Section 4). However, in this case, even after the elimination of the multiply cut Pomeron we have three fields: two uncut Pomerons P_+ , P_- and a cut Pomeron P_c . Moreover, these three Pomerons can interact in various possible ways (Fig. 7):

- i) the vertices for $P_{\pm} \rightarrow P_{\pm} + P_{\pm}$ (and vice versa, Fig. 7a) are complex conjugate because of the $\pm i\epsilon$ prescription, and pure imaginary because of the usual argument ⁵⁾. We denote them by $\pm ig_P$;
- ii) the vertex for $P_c \rightarrow P_+ + P_-$ (and vice versa, Fig. 7b) is the one measured in diffractive excitation of large masses. Since, for example, the graph of Fig. 4c contributes positively to the total cross-section, this is a real coupling which we denote by g_D ;
- iii) the vertex $P_c \rightarrow P_c + P_{\pm}$ (and vice versa, Fig. 7c) is the one coming from absorptive corrections to the pure inelastic production. Since these are known to give a negative contribution to σ_T (e.g., Fig. 4a), this is an imaginary coupling, $\pm ig_A$, corresponding to the $\pm i\epsilon$ prescription;

iv) the vertex for $P_c \rightarrow P_c + P_c$ (and vice versa, Fig. 7d) occurs in double multiplicity events (polyperipheral graphs, e.g., Fig. 4b) and contributes positively to the single fireball. It is therefore a real coupling g_I .

Other couplings (as, for example, $P_+ + P_c \rightarrow P_-$) which cannot be excluded on topological grounds, are forbidden by phase space considerations ¹⁾.

The occurrence of the four vertices listed above modifies in some instances the counting of various graphs. More precisely:

- a) the counting involving only P_+ (or P_-) interacting with itself is the same as in I. The same counting holds also for vertices involving only P_c ;
- b) the counting of a graph involving two g_D or two g_A , when taking the discontinuity of a loop containing only two identical lines, is changed by a factor of 2 (i.e., the factor $\frac{1}{2}$ of the loop disappears). This is because (e.g., for g_D) either a cut with two vertices of Fig. 8a occurs, or a cut with two vertices of Fig. 8b, giving the same result. Analogously for g_A .

From (a) and (b) we deduce that while the interactions due to vertices $\pm ig_P$, g_I (Figs. 7a and 7d) carry the factor $\frac{1}{2}$ typical of two identical particles, the interactions for g_A , g_D (Figs. 7b and 7c) do not.

The last step for deducing the actual Lagrangian comes from the remark that a cut Pomeron line P_c is given, according to the Cutkosky rules, by $(1/i)\text{Disc } P = 2 \text{ Im } P$. If we normalize the P_c propagator to give $\text{Im } P$, this factor of 2 is to be absorbed in the vertices by a wave function re-normalization. This implies

$$g_A \rightarrow g_A, \quad g_D \rightarrow \frac{1}{\sqrt{2}} g_D, \quad g_I \rightarrow \sqrt{2} g_I. \quad (3.6)$$

By denoting with $\psi_{\pm}$, ψ_c the various fields corresponding to $P_{\pm}$, P_c , we finally get

$$\begin{aligned} \mathcal{L}_{CP} = & \mathcal{L}_+(\psi_+) + \mathcal{L}_-(\psi_-) + \mathcal{L}_0(\psi_c) + \\ & + ig_A \psi_c^\dagger \psi_c (\psi_+ - \psi_- + \text{h.c.}) + \frac{1}{\sqrt{2}} g_D (\psi_c^\dagger \psi_+ \psi_- + \text{h.c.}) + \\ & + \frac{1}{\sqrt{2}} \psi_c^\dagger \psi_c (\psi_c + \psi_c^\dagger), \end{aligned} \quad (3.7)$$

where $\mathcal{L}_{\pm}(\psi)$ denotes the usual RFT Lagrangian with triple Pomeron couplings $\pm ig_P$, and $\mathcal{L}_0(\psi)$ its free part.

An important remark is that Eq. (3.7) does not allow transitions of a single P_+ (or P_-) into P_c 's and P_- 's (or P_+ 's). More generally, a Green's function $\langle 0 | \psi_+ \dots \psi_+ \psi_{i_1}^+ \dots \psi_{i_n}^+ | 0 \rangle$ is non-vanishing only for $i_1 = \dots = i_n = +$, and analogously for P_- . This proves:

- a) that the P_+ (P_-) Green's functions are the same (except for a complex conjugation) as in the usual RFT, as is expected;
- b) that the incoming (outgoing) particles are always connected by a $+$ ($-$) surface. No disconnected portions (e.g., containing only P_+) are possible. In this way, we avoid the problems of previous attempts ^{11), 15)} to derive a field theory with only one uncut Pomeron.

3b Gap expansion and positivity

The advantage of this approach, based on Eq. (3.7), is that we can deal directly with definite classes of final states. Even if we shall argue that in the topological expansion all the g_i 's are related to g_P , it is possible to select a set of events by formally varying the g_i 's.

In order to see this, let us count the powers of g_i 's coming from a configuration $C_i^{(b)}$ of the cutting boundaries (subsection 2b). Consider first the case $h_+ = h_- = 0$. Denoting by $\{b_1 \dots b_{n+1}\}$ the $(n+1)$ clusters of b_i boundaries, it is easy to see that only powers of g_I and g_D will occur, as follows:

$$(g_D^2)^n (g_I^2)^{\sum_{i=1}^{n+1} (b_i - 1)} = (g_D^2)^n (g_I^2)^{h_c - n}, \quad (h_c \equiv b - 1). \quad (3.8)$$

If we now take $h_+, h_- \neq 0$, the maximum power of g_A involved is $(g_A^2)^{h_+ + h_-}$, obtained when all the uncut handles are attached to a cut boundary. A power of $\pm ig_A (g_A^2)$ in a given Gribov graph can, however, be replaced by a $\pm ig_P$ (or by a $g_I \cdot g_D$) in a similar graph by the topological transformations illustrated in Fig. 9. We see therefore that the general term of the gap expansion, having $(n+1)$ clusters of boundaries, contributes a factor

$$(-g_A^2)^{h_+ + h_-} \left(\frac{g_P}{g_A}\right)^{j_+ + j_-} \left(\frac{g_I g_D}{g_A^2}\right)^k (g_I^2)^{h_c - n} (g_D^2)^n \quad (3.9)$$

where $j_{\pm} \leq h_{\pm}$, $k < \text{Min}(h_+, h_-)$.

Starting from Eq. (3.9) one can study, in principle, the various classes of events. For instance, the generalized fireball (a cluster of connected discontinuity boundaries) corresponds to setting $n=0$, and the generalized gap expansion to a power expansion in g_D^2 , keeping g_I and $g_I g_D / g_A^2$ fixed. In the literature, only the case $g_I=0$ has been studied ³⁾. The critical indices for this case are derived in Appendix B.

It should be possible to prove, starting directly from the Lagrangian in Eq. (3.7), that each term of the gap expansion is positive definite at $t=0$, provided only enhanced graphs are kept. Note that this has already been shown in subsection 2b for the topological expansion, Eq. (2.4), which includes also some non-enhanced graphs and, in particular, the elastic contribution to unitarity. Here we limit ourselves to give a proof for the single fireball term with $g_I=0$.

This contribution ($b=1, g_I=0$) is given in Fig. 10, in terms of the (possibly disconnected) Green's functions $G_{n_{\pm}}^{(\pm)}$ for $n_{\pm}$ uncut Pomerons and of $G_{n_+ n_-}^{(0)}(v_0, V)$ defined as

$$G_{n_+, n_-}^{(0)}(v_0, V) = \prod_{i=0}^{n_+} \left(i g_A \frac{\delta}{\delta \varphi_+(v_i)} \right) \prod_{j=0}^{n_-} \left(-i g_A \frac{\delta}{\delta \varphi_-(v_j)} \right) G_{v_0 V}^{(0)}(\varphi_+ + \varphi_-) \quad (3.10)$$

$\varphi_+ = \varphi_- = 0$

Here $v_0 = (0, 0)$, $V = (B, Y)$, $v_i = (b_i, y_i)$ denote configuration space variables (impact parameter and rapidities), and $G_{v_0 V}^{(0)}(\varphi)$ is the bare P_c propagator function in the external field φ , defined as

$$G^{(0)}(\varphi) = [1 - D_c^{(0)} \varphi]^{-1} D_c^{(0)}, \quad (3.11)$$

where $D_c^{(0)}$ is the bare P_c propagator, and an operator product in v space is understood.

The central point is now to prove that $G^{(0)}(\varphi)$ has, in the enhanced graph approximation, the form

$$G_{v_0 V}^{(0)}(\varphi_+ + \varphi_-) \simeq \sum_n \int dv_1 \dots dv_n M_{v_0 V}^{(n)}(v_1, \varphi_+) M_{v_0 V}^{(n)}(v_n, \varphi_-)^* \quad (3.12)$$

for a proper choice of $M^{(n)}$. If this is in fact the case, the operations on φ_+ and φ_- in Eq. (3.10) and Fig. 10 are done, independently, on $M^{(n)} [M^{(n)*}]$, thus preserving the positive structure of Eq. (3.12).

In order to prove Eq. (3.12), we expand Eq. (3.11) both in terms of the φ 's and of the $(1+\varphi)$'s as follows

$$\begin{aligned} G_{v_0 v}^{(0)}(\varphi) &= \sum_n \int d\nu_1 \dots d\nu_n \prod_{i=0}^n D_c^{(0)}(\nu_i, \nu_{i+1}) \prod_{i=1}^n \varphi(\nu_i) = \\ &= \sum_n \int d\nu_1 \dots d\nu_n \prod_{i=0}^n K(\nu_i, \nu_{i+1}) \prod_{i=1}^n [1 + \varphi(\nu_i)] , \end{aligned} \quad (3.13)$$

where, for a pure pole form of $D_c^{(0)}$, we have

$$K(\nu) = (1 + D_c^{(0)})^{-1} D_c^{(0)} = \lambda e^{-\lambda y} D_c^{(0)}(\nu) > 0, \quad (3.14)$$

and λ is related to the wave function renormalization constant. We can therefore define

$$\begin{aligned} |M_{v_0 v}^{n,0}(\nu_i)|^2 &= \prod_{i=0}^n K(\nu_i, \nu_{i+1}) , \\ M_{v_0 v}^{(n)}(\nu_i, \varphi) &= M_{v_0 v}^{n,0}(\nu_i) \prod_{i=1}^n [1 + \varphi(\nu_i)] , \end{aligned} \quad (3.15)$$

and we get, by Eq. (3.13),

$$\begin{aligned} G_{v_0 v}^{(0)}(\varphi_+ + \varphi_-) &= \sum_n \int d\nu_1 \dots d\nu_n M_{v_0 v}^{(n)}(\nu_i, \varphi_+) M_{v_0 v}^{(n)}(\nu_i, \varphi_-)^* - \\ &- \sum_n \int d\nu_1 \dots d\nu_n |M_{v_0 v}^{n,0}(\nu_i)|^2 \prod_{i=1}^n \varphi_+(\nu_i) \varphi_-(\nu_i) . \end{aligned} \quad (3.16)$$

The last expression in Eq. (3.16) involves the product of two fields φ_+ , φ_- at the same point, and therefore generates all kinds of non-enhanced graphs with some four-Pomeron interactions. This allows the use of Eq. (3.12) in Eq. (3.10), and proves positivity for the single fireball (with $g_I = 0$) in the enhanced graph approximation.

3c Relation between the two field theories

The coupling constants g_i occurring in the Lagrangian of the CRFT, Eq. (3.7), do not need to be, in general, the same as g_P (the one occurring in the usual RFT), because they describe different physical processes ^{3),11)}, and the only constraint posed by the optical theorem is that the P_0 and P_+ , P_- propagators have to coincide at infinite energy.

However, the topological approach that we follow here has a single expansion parameter, $1/N$. Therefore, unitarity is to be satisfied to each order in $1/N$. Since the g_i 's are all of order $1/N$, and because of the enhanced graph approximation, unitarity implies that the equality

$$D_c(\omega, k; g_i) = D(\omega, k; g_P) \quad , \quad (\omega \equiv J-1) \quad (3.17)$$

is to be satisfied to any given perturbative order.

Using the Dyson equations for P_0 , this implies for the second order correction to D_0 ,

$$2g_A^2 - g_I^2 - \frac{1}{2}g_D^2 = \frac{1}{2}g_P^2 \quad , \quad (3.18)$$

and, for the vertex functions renormalization constants ^{*)}, to the same order,

$$\begin{aligned} 2g_A g_P + 2g_A^2 - 2g_I^2 - \frac{1}{2}g_D^2 \left(\frac{g_P}{g_A} + 1 \right) + g_I g_D &= 2g_P^2, \quad (g_A) \\ 4g_A^2 - 2g_I^2 - g_D^2 \frac{g_A^2}{g_I g_D} &= g_P^2, \quad (g_I) \\ 2g_A g_P &= 2g_P^2, \quad (g_D) \end{aligned} \quad (3.19)$$

Since the last equation gives $g_A = g_P$, (3.18) and (3.19) imply

$$2g_I^2 + g_D^2 = 3g_P^2 \quad , \quad g_I g_D = g_A^2 = g_P^2 \quad , \quad (3.20)$$

^{*)} There is, in principle, also an α'_0 renormalization constant, but the condition for this to be trivial is still Eq. (3.18).

which admits two solutions

$$A) \quad g_I = g_D = g_A = g_P, \quad (3.21)$$

$$B) \quad g_A = g_P, \quad g_I = \frac{1}{\sqrt{2}} g_P, \quad g_D = \sqrt{2} g_P.$$

Solution (A) is, according to Eq. (3.18), the one consistent with the AGK⁹⁾ prescription for the ratio of A, I, and D discontinuities of the two-Pomeron cut (-4:2:1), and is found to be true for the simplest Feynman diagrams. Solution (B) emphasizes diffractive excitation more than inelastic rescattering and therefore changes the multiplicity distributions¹⁵⁾ in a way reminiscent of purely absorptive models.

An important question is whether the solutions (3.21) which satisfy (3.18) and (3.19), and therefore (3.17) to 2nd and 4th order, also guarantee Eq. (3.17) to all orders. This non-trivial result can be proved by a partial diagonalization of the Lagrangian (3.7), performed by means of a linear similarity transformation. More precisely, by defining the new fields ϕ_α ($\alpha = +, -, 0$) by

$$\begin{aligned} \psi_i &= \sum_\alpha (D^{-1})_{i\alpha} \phi_\alpha = S \phi_i S^{-1}, \\ \psi_i^+ &= \sum_\alpha \phi_\alpha^+ D_{\alpha i} = S \phi_i^+ S^{-1}, \end{aligned} \quad (3.22)$$

where

$$S = \exp\left(\int dv \phi^+ \ln D \phi\right), \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -i\sqrt{2} r_I \end{bmatrix}, \quad (3.23)$$

$$r_I(A) = 1, \quad r_I(B) = 1/\sqrt{2},$$

one can prove (Appendix A) that, provided (3.21) is satisfied, the Lagrangian becomes

$$\begin{aligned} \mathcal{L}' &= S^{-1} \mathcal{L} S = \sum_\alpha \mathcal{L}_0(\phi_\alpha) + i \frac{g_P}{2} \sum_\alpha \phi_\alpha^+ \phi_\alpha (\phi_+ + \phi_-) + \mathcal{L}'' , \\ \mathcal{L}'' &= i g_P \left[\left(1 - \frac{1}{2r_I^2}\right) \phi_0^+ (\phi_+^+ + \phi_-^+) - \frac{1}{2r_I^2} \phi_+^+ \phi_-^+ \right] (\phi_0 - \phi_+ - \phi_-) + i g_P \phi_0^+ (\phi_+^+ \phi_+ + \phi_-^+ \phi_-). \end{aligned} \quad (3.24)$$

Owing to the asymmetric form of $\mathcal{L}''$ with respect to $\phi_\alpha = \phi_\alpha^+$ (because of the absence of some vertices) and also to the diagonal form of the remaining interaction, one can show (Appendix A) that

$$\langle 0 | \phi_\alpha \dots \phi_\alpha \phi_{\beta_1}^+ \dots \phi_{\beta_n}^+ | 0 \rangle = \delta_{\alpha\beta_1} \dots \delta_{\alpha\beta_n} G^{(m,n)}, \quad (3.25)$$

where $G^{(m,n)}$ are the Green's functions of the usual RFT, and α is any value $[+, -, 0]$ not summed over.

From Eq. (3.25), applied to the propagator and vertex functions, we obtain, in particular,

$$\begin{aligned} \langle 0 | \psi_i \psi_j^+ | 0 \rangle &= \sum_{\alpha, \beta} (D^{-1})_{i\alpha} D_{j\beta} \langle 0 | \phi_\alpha \phi_\beta^+ | 0 \rangle = \\ &= \delta_{ij} G^{(1,1)} \equiv \delta_{ij} D, \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} (-)^{n_+ + n_-} \langle 0 | \psi_i \psi_j^+ \psi_k^+ | 0 \rangle^* &= \langle 0 | \psi_i^+ \psi_j \psi_k | 0 \rangle = \\ &= \sum_{\alpha, \beta, \gamma} (D^{-1})_{i\alpha} D_{\beta j} D_{\gamma k} \langle 0 | \phi_\alpha \phi_\beta^+ \phi_\gamma^+ | 0 \rangle = i g_{ijk} G^{(1,2)}, \end{aligned} \quad (3.27)$$

where $\frac{i}{2} g_{ijk} (\psi_i \psi_j^+ \psi_k^+ + \psi_i^+ \psi_j \psi_k)$ denotes the original interaction Lagrangian.

Equation (3.26) implies in particular Eq. (3.17), and Eq. (3.27) means that also the vertex functions are identically proportional to the one of usual RFT. The ambiguity of Eq. (3.21) only affects the $\mathcal{L}''$ term in Eq. (3.24) and therefore seems to persist, as far as the unitarity constraint of Eq. (3.17) is concerned ^{*)}. It may be interesting to see whether some models realize it, or whether instead it is incompatible with other requirements such as analyticity and transverse momentum damping.

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^{*)} We have also explicitly checked on the Dyson equations that the ambiguity persists to the crossed diagram level of the Bethe-Salpeter kernel.

4. CONCLUSIONS

In this section we first summarize the results obtained both in paper I and in the present paper. We then discuss some of the most interesting open questions.

We have started from the idea ⁵⁾ that in strong interaction physics the relevant perturbative parameter is not the coupling constant, but the topology of the graph. In some theories this is equivalent to expanding in the parameter $1/N^2$, where $U(N)$ is the internal symmetry group. The lowest order terms in this expansion have been discussed elsewhere ^{7),8)} and lead naturally, at high energy, to a definition of a bare Pomeron pole related through unitarity to multiparticle production events with short-range correlations. This pole has intercept close to one ¹²⁾ and a non-vanishing triple Pomeron coupling ⁸⁾.

In our two papers, we have analyzed the structure of the theory as a systematic perturbative expansion in $1/N^2$. The exciting outcome is the possibility of relating such an expansion to the Reggeon calculus first guessed by Gribov ¹⁾ on the basis of simple Feynman diagrams. The disadvantages of the original approach are mainly the ambiguous definition of the bare Pomeron and, consequently, the difficulty in putting in correspondence classes of Feynman graphs and Gribov diagrams. This problem is completely solved in our scheme, at least in principle.

The correspondence that we find (I, Section 4) is not one to one. A simple topological term with h handles corresponds to all possible Reggeon graphs of order $2h$ in the triple Pomeron coupling g_P . Actually, the topological term contains also all the so-called non-enhanced contributions. These could be represented in the Reggeon calculus by other coupling constants with 4, 5, etc., Pomerons, or in general by a set of couplings $g^{(n,m)}$ between n Pomerons and m external particles. For the order of the topological term g_P^{2h} to match all combinations of these couplings, one has to have $g^{(n,m)} = O(g_P^{n+m-2})$. Since $g_P = O(1/N)$, the resulting interaction Lagrangian for the Pomeron field ψ has the form $N^2 V(\psi/N)$.

The main novel feature of our approach comes from the fact that the topological expansion is automatically unitary in all channels, in particular, the s channel. The resulting Reggeon calculus should therefore satisfy unitarity.

In order to see how this works, we have examined in Section 2, the s channel discontinuities of the topological terms and we have classified them. After elimination of some "interference terms" (II, subsection 2a),

the topological expansion of unitarity of Eq. (2.4) can be written. This approximate unitarity consists of a sum of positive terms, including the elastic one, which can be put in correspondence with production events (generalized gap expansion, subsection 2b). By neglecting the "multiply cut" Pomeron and using the enhancement approximation, we were able to write a cut Reggeon field theory (CRFT, subsections 3a and b), i.e., a Lagrangian which generates the cut Reggeon diagrams [each corresponding to a contribution to the right-hand side of the unitarity equation (2.4)]. This Lagrangian involves three Pomeron fields and four coupling constants. Imposing unitarity to all orders in $1/N$ does restrict, however, the four couplings to be proportional to each other with ratios given by the AGK rules⁹⁾ (with a possible ambiguity, subsection 3c). The Cutkosky rule holds then to all orders in $1/N$, as best seen by performing a similarity transformation on the CRFT Lagrangian (subsection 3c and Appendix A).

We now turn to the discussion of the main questions which, in our opinion, are left open in our approach.

4a Fulfilment of s channel unitarity

Since we start in our approach with a theory which is unitary in all channels, the real question is: can we lose s channel unitarity when we replace topological graphs at asymptotic energies with Gribov diagrams?

The first observation to make is that if we include all possible Gribov diagrams, enhanced and non-enhanced, s channel unitarity should be satisfied at high energy, since only non-leading Regge poles and cuts have been neglected. Enhanced and non-enhanced couplings should, of course, be related in our approach. The classification of the inelastic contributions to the unitarity bound of Fig. 11a would, however, be difficult, unless we argue (subsection 2a) that some interference terms are negligible at high energy. If these terms are really less important than other non-enhanced contributions included in (2.4), from this equation one should be able to prove unitarity in the form of Fig. 11b, where all types of Pomeron vertices are included.

The final drastic simplification occurs if we assume that enhanced graphs dominate over non-enhanced ones at asymptotic (certainly not at present) energies. In such a case, the asymptotic theory is a RFT and a CRFT with only trilinear couplings. For the enhancement assumption to be self-consistent, one has to require that (i) the renormalized intercept be $\alpha_P(0) \leq 1$, and (ii) if $\alpha_P(0) = 1$, the partial waves be asymptotically vanishing, i.e., $\ell > 0$ ^{*)} if $\psi \sim y^{-\ell} f(b^2/y^v)$. Only in such a case will

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*) An argument for this to be true in RFT has been given by Cardy and Sugar¹⁷⁾.

the elastic or eikonal type contributions be negligible for the renormalized Pomeron.

Under these dynamical conditions on RFT we can conclude that s channel unitarity is maintained in our approach: at high energies left- and right-hand sides of the unitarity equation can be estimated by a RFT and a CRFT, respectively.

4b The renormalized Pomeron intercept

Many recent developments in RFT are based on the assumption that $\alpha_p(0) = 1$ exactly which, by use of renormalization group methods ^{2), 16)}, implies $\Lambda_0 = \Lambda_{crit}^0 = \text{const}(g_p^2/\alpha') \ln(\Lambda/g_p^2)$ where $1 + \Delta_0$ is the bare Pomeron intercept and Λ a cut-off parameter.

A peculiar aspect of our approach is that $\Lambda_0 = O(g^2 N)$ and is possibly small only for dynamical reasons ¹²⁾, whereas $g_p^2 = O(1/N^2)$. It is therefore somewhat artificial to expect a precise relation like $\Lambda_0 = \Lambda_{crit}^0(g_p^2)$ to exist between the two. (Note that non-enhanced graphs cannot help, being also of order $1/N^2$.) Actually, it has been observed ¹⁸⁾ that if Δ_0 has to explain phenomenologically the rise of total cross-sections, g_p^2 turns out to be experimentally too small for the critical point condition to be satisfied.

It is therefore interesting to ask the question of whether the renormalized intercept can be $\Delta \leq 0$ both for $\Lambda_0 \leq \Lambda_{crit}^0$ and for $N \rightarrow \infty$, i.e., $\Delta_0 > 0$ fixed and $g_p^2 \rightarrow 0$ ($\Lambda_0 \gg \Lambda_{crit}^0$). For the first case we know that if $\Delta_0 = \Lambda_{crit}^0$, the proper self-energy part $\Sigma(g_p^2; \omega = \Delta; \Lambda, \Lambda) = \bar{\Sigma}(g_p^2, \Delta)$ reaches Δ_0 for $\Delta = 0$ and is expected to be smooth around $\Delta \leq 0$ when $\Delta_0 < \Lambda_{crit}^0$ if renormalization group arguments hold ¹⁶⁾ [Fig. 12, curve (a)]. On the other hand, if $\Delta \leq 0$ is assumed also in the latter case ($N \rightarrow \infty$), $\bar{\Sigma}$ has to have a singular behaviour close to $\Delta = 0$ [Fig. 12, curve (c)], so that $\bar{\Sigma} \simeq \Delta_0 \gg \Lambda_{crit}^0$ even for $g_p^2 \rightarrow 0$. This is possible, provided in this case the renormalized vertex function is still hard enough so that the perturbative infra-red divergence $[\bar{\Sigma} \sim g_p^2/\alpha' \ln(1/|\Delta|)]$ is not completely cancelled.

If the latter behaviour is conjectured, a possible matching of curves (a) and (c) is given in Fig. 12 by curve (b) which is also consistent with some recent arguments ¹⁹⁾ on the behaviour of $\bar{\Sigma}$ close to the critical point ($\Delta_0 \gtrsim \Lambda_{crit}^0$). This is, in any case, an unresolved question open to future investigations. [It is possible, for instance, that conditions (i) and (ii) of point (a) are violated by RFT, and eikonalization is the way s channel unitarity is achieved.]

4c Relevance of the theory at finite energies

The topological expansion appears to be very well suited for use at finite as well as asymptotic energies. This is because the expansion itself contains all non-enhanced contributions as well as non-leading Reggeon exchanges. If, however, the bare Pomeron intercept $1+\Delta_0$ turns out to be larger than one ($\Delta_0 > 0$), then the $1/N$ expansion is expected to converge only at finite energies. An estimate of the scale at which it breaks down due to enhancement in the cut contribution, is however hard to make because it depends on the way s channel unitarity is achieved in the theory. For instance, if curve (c) of Fig. 12 applies (singular perturbation theory), the ratio of the enhanced two Pomeron cut to the pole contribution behaves like $(g_P^2/16\pi B)(\ln^2 s)$ for small Δ_0 and α' , where B is some slope parameter. If instead curve (a) applies (renormalization group estimate), it is rather $(g_P^2/16\pi \alpha')(\ln s)$ which matters^{2),20)}. Semi-enhanced terms (triple Regge contributions) could also be important. All that we can say is that in our case N^2 seems to set the scale of $\ln s$ at which non-perturbative effects become important. One would then be led to think that up to present FNAL and ISR energies, a decent theory could just include the $h=0$ and $h=1$ terms of the expansion.

A precise estimate of the values of σ_{tot} and σ_{el}/σ_{tot} coming out of the expansion is important, and work is in progress in this direction. For meson-meson scattering, one has roughly $\sigma_{tot} \approx (16\pi^2 \alpha'/N^2)$, and there are indications for $\Delta_0 > 0$. On the other hand, baryon-baryon scattering has not been looked at from our point of view. It is clear that more work is needed before attempting a comparison of the theory with experimental data.

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APPENDIX A - PARTIAL DIAGONALIZATION OF CUT REGGEON FIELD THEORY

Here we want to derive the similarity transformation of the text relating the CRFT Lagrangian (3.7) to the simpler Lagrangian (3.24) for which the propagator and vertex relations are trivial. With the minor redefinition $\psi_- \rightarrow -\psi_-$ the original Lagrangian can be written as

$$\mathcal{L} = \psi_i^\dagger K \psi_i + \frac{i}{2} g_{ijk} (\psi_i^\dagger \psi_j \psi_k + \psi_i \psi_j^\dagger \psi_k^\dagger) \quad (\text{A.1})$$

where K is a proper operator, g_{ijk} is symmetrical in the last two indices, and takes the non-zero values

$$g_{+++} = g_{---} = g_P, \quad g_{0\pm 0} = g_A, \quad g_{0+-} = \frac{i}{\sqrt{2}} g_D, \quad g_{000} = -i\sqrt{2} g_I. \quad (\text{A.2})$$

We first seek a linear transformation [Eq. (3.22)]

$$\phi_\alpha = D_{\alpha i} \psi_i, \quad \phi_\alpha^\dagger = \psi_i^\dagger (D^{-1})_{i\alpha} \quad (\text{A.3})$$

such that the quadratic form $\psi_i^\dagger \psi_i$ is left invariant, and the bilinear form $g_{ijk} \psi_i^\dagger \psi_j \psi_k$ is exactly diagonalized to the form $i(g_P/2) \phi_\alpha^\dagger \phi_\alpha^2$. We therefore require

$$g_P (D^{-1})_{i\alpha} D_{\alpha j} D_{\alpha k} = g_{ijk} \quad (\text{A.4})$$

or, in more detail, by (A.2)

$$\begin{aligned} \text{(a)} \quad & (D^{-1})_{i\alpha} D_{\alpha\pm}^2 = \delta_{i,\pm}, \\ \text{(b)} \quad & (D^{-1})_{i\alpha} D_{\alpha+} D_{\alpha-} = \frac{i}{\sqrt{2}} r_D \delta_{i,0}, \\ \text{(c)} \quad & (D^{-1})_{i\alpha} D_{\alpha 0}^2 = -i\sqrt{2} r_I \delta_{i,0}, \\ \text{(d)} \quad & (D^{-1})_{i\alpha} D_{\alpha\pm} D_{\alpha 0} = r_A \delta_{i,0}, \end{aligned} \quad (\text{A.5})$$

where $r_i = g_i/g_p$, so that one has

$$\begin{aligned}
 (a) \quad D_{\alpha\pm}^2 &= D_{\alpha\pm} \quad , \\
 (b) \quad D_{\alpha+} D_{\alpha-} &= \frac{i}{\sqrt{2}} r_D D_{\alpha 0} \quad , \\
 (c) \quad D_{\alpha 0}^2 &= -i \sqrt{2} r_I D_{\alpha 0} \quad , \\
 (d) \quad D_{\alpha\pm} D_{\alpha 0} &= r_A D_{\alpha 0} \quad .
 \end{aligned} \tag{A.6}$$

Since, by (a) in Eq. (A.6), we have $D_{\alpha\pm} = 1$ or 0 , (d) implies $r_A = 1$. Analogously (b) and (c) imply $r_I r_D = 1$, so that we get the last two conditions (3.20) for (A.6) to be satisfied, while r_I is still a free parameter. A solution to (A.5) is then

$$D(r_I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -i\sqrt{2} r_I \end{bmatrix}, \quad D^{-1}(r_I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-i}{\sqrt{2} r_I} & \frac{-i}{\sqrt{2} r_I} & \frac{i}{\sqrt{2} r_I} \end{bmatrix} \tag{A.7}$$

We then calculate the transform of the form $g_{ijk} \psi_i \psi_j^+ \psi_k^+$, to get

$$\frac{i}{2} \bar{g}_{\alpha\beta\gamma} \phi_\alpha \phi_\beta^+ \phi_\gamma^+, \quad \bar{g}_{\alpha\beta\gamma} = (DD^T)^{-1}_{\sigma\alpha} (DD^T)_{\sigma\beta} (DD^T)_{\sigma\gamma} \tag{A.8}$$

Let us remark that a complete diagonalization of (A.8) is impossible, because it would require $DD^T = 1$ which is incompatible with (A.7). We can, however, impose the condition (with β not summed over)

$$\bar{g}_{\alpha\beta\beta} = g_p \delta_{\alpha\beta} \tag{A.9}$$

which in particular gives $\phi_\alpha \phi_\alpha^{+2}$ the same coupling as $\phi_\alpha^2 \phi_\alpha^+$. Equation (A.9) implies

$$[(DD^T)_{\alpha\beta}]^2 = (DD^T)_{\alpha\beta}, \quad \text{with } DD^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2(1-r_I^2) \end{bmatrix} \tag{A.10}$$

which gives the two solutions .

$$(A) \quad v_I = 1$$

$$(B) \quad v_I = 1/\sqrt{2} \quad . \quad (A.11)$$

After some simple algebra we can now calculate $\bar{g}_{\alpha\beta\gamma}$ from (A.8) and get the transformed Lagrangian of the text, i.e.,

$$\mathcal{L}'_{int} = \phi_\alpha^\dagger \phi_\alpha (\phi_+ + \phi_-^\dagger) + \mathcal{L}'' \quad , \quad (A.12)$$

where

$$(A) \quad \mathcal{L}'' = \frac{ig_P}{2} \left[\phi_0^\dagger \phi_+^\dagger (\phi_0 + \phi_+ - \phi_-) + \phi_0^\dagger \phi_-^\dagger (\phi_0 - \phi_+ + \phi_-) - \phi_+^\dagger \phi_-^\dagger (\phi_0 - \phi_+ - \phi_-) \right]$$

$$(B) \quad \mathcal{L}'' = ig_P \left[\phi_0^\dagger (\phi_+^\dagger \phi_+ + \phi_-^\dagger \phi_-) - \phi_+^\dagger \phi_-^\dagger (\phi_0 - \phi_+ - \phi_-) \right] \quad (A.13)$$

for solutions (A) and (B), respectively. Besides the diagonal vertices, Eq. (A.13) allows the transitions

$$\phi_0, \phi_+, \phi_- \rightarrow (\phi_0, \phi_+), (\phi_0, \phi_-), (\phi_+, \phi_-), \quad (A.14)$$

but not vice versa. It is therefore clear that the Green's function (α not summed)

$$\langle 0 | \phi_\alpha \cdots \phi_\alpha \phi_{\beta_1}^\dagger \cdots \phi_{\beta_n}^\dagger | 0 \rangle \quad (A.15)$$

for which all the m fields on the left have the same α , can only be non-vanishing if $\beta_1 = \beta_2 = \dots = \beta_n = \alpha$ and, in such a case it equals the usual one of RFT.

Besides the propagator and vertex equations (3.26), (3.27) of the text, Eq. (A.15) implies a more general relation among Green's functions of the CRFT, as follows (with α not summed over)

$$\begin{aligned} D_{\alpha i_1} \cdots D_{\alpha i_n} \langle 0 | \psi_{i_1} \cdots \psi_{i_n} \psi_{j_1}^+ \cdots \psi_{j_m}^+ | 0 \rangle = \\ = D_{\alpha j_1} \cdots D_{\alpha j_m} G^{(n, m)} \end{aligned} \quad (A.16)$$

with $D_{\alpha i}$ given by (A.7) with the proviso (A.11). In particular, we get

$$\langle 0 | \psi_i \psi_{j_1}^+ \cdots \psi_{j_n}^+ | 0 \rangle = \sum_{\alpha} (D^{-1})_{i\alpha} D_{\alpha j_1} \cdots D_{\alpha j_n} G^{(1, n)},$$

and therefore, after simple algebra,

$$G^{(i=0; n_+, n_-, n_c)} = (-i\sqrt{2} r_I)^{n_c-1} G^{(1, n)} \quad (A.17)$$

which, for the choice $r_I = 1$ is consistent with the AGK rules for cutting n_c Pomerons out of n . Equation (A.16) generalizes this relation to the $n \rightarrow m$ Pomeron case.

APPENDIX B - SINGLE FIREBALL FIELD THEORY

The various couplings g_P , g_A , g_D and g_I in the CRFT Lagrangian Eq. (3.7) are all related in the topological expansion due to the existence of the single parameter $1/N$ (Section 3.c). We have noticed, however, that by varying formally their values one can select classes of diagrams contributing to the different terms in the gap expansion of Eq. (2.4). In particular, by setting $g_D = g_I = 0$ one obtains all the graphs which correspond to the single fireball (SF) contribution ^{*}). The resulting Lagrangian is [cf. Eq. (3.7)]

$$\mathcal{L}_{SF} = \mathcal{L}_+(\psi_+) + \mathcal{L}_-(\psi_-) + \mathcal{L}_0(\psi_{SF}) + i g_A \psi_{SF}^\dagger \psi_{SF} (\psi_+ - \psi_- + h.c.) \quad (B.1)$$

where ψ_{SF} is the field which corresponds to the cut Pomeron when only absorptive corrections are considered. In this Appendix, we discuss the scaling form of the Green's function of (B.1) and compute the values of the critical indices in the first order ϵ expansion. Moreover, we show that the renormalized SF intercept α_{SF} is smaller than one (if $\alpha_P = 1$).

Let us consider the various Green's functions of (B.1). First note that the Dyson equations for the ψ_+ or ψ_- fields decouple from ψ_{SF} and are the same as in RFT, which, if the renormalized Pomeron intercept $\alpha_P(0)$ is at one ($\Delta = \alpha_P - 1 = 0$), have the usual ²⁾ scaling behaviour (paper I, Section 4),

$$\begin{aligned} \Gamma^{(2)}(\omega, k) \equiv D(\omega, k) &\simeq \omega^{D-2\ell} g_2(k^2/\omega^\nu), \\ \Gamma^{(3)}(\omega_i, k_i) &\simeq i \omega_i^{D-3\ell} g_3(k_i^2/\omega_i^\nu, \omega_j/\omega_i), \end{aligned} \quad (B.2)$$

with $D = 1 + (d/2)\nu$ and d the number of transverse dimensions. In the usual notation ²⁾ $\eta = D - 2\ell - 1$, $\gamma = D - 3\ell$, which satisfy the bootstrap relation ¹⁾:

$$2\gamma - 3\eta + \frac{d}{2}\nu = 2 \quad (B.3)$$

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^{*}) This case has been studied in detail in Ref. 3) for a purely absorptive model, which, however, did not admit a Lagrangian formulation. The results are similar.

On the contrary the ψ_{SF} field satisfies a complicated set of coupled Dyson equations for which the corresponding ω discontinuity equations (Fig. 13) are typical of an RFT with two coupled fields ²¹⁾. The ω discontinuity equations that we use (cf. paper I, Section 4) make scaling properties more transparent and the perturbative computation of the critical indices more direct since only renormalized quantities appear.

Since the ψ_{SF} field in (B.1) has no self-interaction, the renormalized intercept α_{SF} ($\Delta_{SF} = \alpha_{SF} - 1$) can be freely shifted by defining

$$\Omega = \omega - \Delta_{SF} \quad (B.4)$$

Let us now seek a solution to the discontinuity equations for the propagator and vertex function having the scaling behaviour

$$\begin{aligned} \Gamma_{SF}^{(2)}(\Omega, k) &\simeq \Omega^{1+\eta'} g_2'(k^2/\Omega^\nu, k^2/\Omega^{\nu'}) , \\ \Gamma_{SF}^{(3)}(\Omega_i, k_i) &\simeq i \Omega_i^{\gamma'} g_3'(k_i^2/\Omega_i^\nu, k_i^2/\Omega_i^{\nu'}, \Omega_i/\Omega_i) . \end{aligned} \quad (B.5)$$

It is clear, first of all, that only the region with k^2/Ω^{ν_M} fixed is important, with $\nu_M = \max(\nu, \nu')$. Therefore, the scale invariance of the loop integrations in Fig. 13 requires, in place of Eq. (B.3), the bootstrap condition

$$-2\eta' - \eta + 2\gamma' + \frac{d}{2} \nu_M = 2 . \quad (B.6)$$

[Solutions to this equations can also be parametrized in terms of two scale dimensions ℓ and ℓ' as $\eta' + 1 = D' - 2\ell'$, $\gamma' = D' + \frac{1}{2}(D - D_M) - 2\ell - \ell'$, where $D_i = (d/2)\nu_i + 1$ and $\nu_i = \nu, \nu', \nu_M$].

Secondly, in the scale invariant form of (B.2) and (B.5), a pure pole solution can occur only for $\epsilon = 4 - d = 0$, where also the coupling constants are $g_P = g_{SF} = 0$, so that a perturbative calculation of (B.2) and (B.3) must be done in the ϵ expansion ²²⁾. This enables us to derive the critical indices to first order in ϵ from the ω discontinuity equations (Fig. 13) (cf. paper I, Section 4).

For the propagator equation (Fig. 13), we have

$$\text{Disc. } \Gamma_{SF}^{(2)}(\Omega, \underline{k}) = -2 \int \frac{d\Omega' d^{(d)}\underline{k}'}{2\pi i (2\pi)^d} \left[\Gamma^{(2)}(\Omega, \Omega', \underline{k}, \underline{k}') \right]^2 \cdot \text{Disc} \left[\Gamma_{SF}^{(2)}(\Omega', \underline{k}') \right]^{-1} \text{Disc} \left[\Gamma^{(2)}(\Omega - \Omega', \underline{k} - \underline{k}') \right]^{-1} + O(g_A^4) \quad (\text{B.7})$$

As in I, the first contribution can be estimated by putting in the left-hand side the "free" form of $\Gamma^{(2)}$ and $\Gamma^{(3)}$ [cf. paper I, Eq. (4.14)]. Some care should, however, be taken in this case according to the sign of $v - v'$. In fact, the momentum dependence in $\Gamma_{SF}^{(2)}$ ($\Gamma^{(2)}$) is negligible if $v' < v$ ($v' > v$). By introducing then a parameter r

$$r = \begin{cases} 0 & (v' > v) \\ \infty & (v' < v) \\ r_0 & (v' = v) \end{cases}, \quad (\text{B.8})$$

we have [cf. paper I, Eq. (4.14)]

$$2\pi \left[\eta' \Omega + (\eta' + 1 - v') \alpha_{SF}' k^2 \right] \simeq 2g_{SF}^2 \frac{1}{16\pi\alpha_{SF}'} \left(\Omega + \frac{r}{1+r} \alpha_{SF}' k^2 \right) \quad (\text{B.9})$$

which gives

$$\begin{aligned} \eta' &= \frac{1}{32\pi^2\alpha_{SF}'} 2g_{SF}^2 + O(\epsilon^2), \\ v' &= 1 + \frac{1}{1+r} \eta' + O(\epsilon^2). \end{aligned} \quad (\text{B.10})$$

A similar calculation for the vertex equation of Fig. 13 [cf. paper I, Eq. (4.16)] gives

$$\begin{aligned} \gamma' &= \frac{1}{32\pi^2\alpha_{SF}'} \left(2g_{SF}^2 + \frac{1+3r}{1+r} g_{SF} g_F \right) + O(\epsilon^2) \\ &\simeq \eta' + \frac{1+3r}{1+r} \sqrt{\eta\eta'}. \end{aligned} \quad (\text{B.11})$$

Equations (B.10) and (B.11) together with (B.6) give then the set of solutions

$$\begin{aligned}
 a) \quad r=0 \quad (v' > v) : & \quad \begin{cases} \eta' = (4 \pm \frac{1}{2}\sqrt{15}) \eta > \eta, \\ \nu' = 1 + \eta', \quad \gamma' = \frac{7}{2} \eta, \end{cases} \\
 b) \quad r=\infty \quad (v' < v) : & \quad \eta' = \eta, \quad \nu' = 1, \quad \gamma' = 4\eta, \\
 c) \quad r=r_0 \quad (v' = v) : & \quad \begin{cases} r_0 = \frac{2}{3} + \frac{1}{3}\sqrt{21} \simeq 2.2, \\ \eta' = \frac{1}{2}(1+r_0) \eta > \eta, \\ \gamma' = [3 + \frac{1}{2}(1+r_0)] \eta. \end{cases} \quad (B.12)
 \end{aligned}$$

The stability nature of the solution must be investigated with the renormalization group methods ²⁰⁾ from which one can verify that solutions (a), (b) are stable (there is a neighbourhood of attraction for the effective charges $\bar{r}$ and $\bar{g}_1$) while (c) is unstable (there is only a line of attraction).

We conclude this section by showing that, to first order in ϵ , the renormalized SF intercept α_{SF} is smaller than 1 when the renormalized intercept is $\alpha_P=1$.

In fact, if $\Delta=0$, the bare intercept $\alpha_P^0 = 1 + \Delta_0$ and the SF one $\alpha_{SF} = 1 + \Delta_{SF}$ are related by the renormalization group to the wave function renormalization constants by the relations ¹⁶⁾

$$\begin{aligned}
 \Delta_0 &= \int_0^{\wedge} d\omega' [1 - Z^{-1}(\omega')] , \\
 \Delta_{SF} &= \Delta_0 - \int_0^{\wedge} d\omega' [1 - Z_{SF}^{-1}(\omega')] = \\
 &= \int_0^{\wedge} d\omega [Z_{SF}^{-1}(\omega) - Z^{-1}(\omega)] , \quad (B.13)
 \end{aligned}$$

where Z and Z_{SF} are the renormalization constants of the $\psi_{\pm}$ and ψ_{SF} fields, respectively, and are given in terms of the effective charges as follows.

For Z , to the first order, in ϵ , we have

$$-\omega \frac{\partial}{\partial \omega} \ln Z(\omega) = u^2(\omega) + O(\epsilon^2) , \quad (B.14)$$

where $u(\omega)$ is the effective charge (ω_0 is the normalization point)

$$u(\omega_0) = \frac{1}{8\pi} g_p(\omega_0) \frac{\omega_0^{-\epsilon/4}}{\alpha'(\omega_0)^{d/4}} ,$$

which moves according to the renormalization group equation ²⁾ but remains positive. This enables Sugar and White ¹⁶⁾ to conclude that $Z > 1$ so that, from (B.13), $\Delta_0 > 0$.

Analogously, for Z_{SF} we have

$$-\omega \frac{\partial}{\partial \omega} \ln Z_{SF}(\omega) = 4 u_A^2(\omega) + O(\epsilon^2) \quad (B.15)$$

with

$$u_A(\omega_0) = \frac{1}{8\pi} g_A(\omega_0) \omega_0^{-\epsilon/4} \left(\frac{2}{\alpha'(\omega_0) + \alpha'_{SF}(\omega_0)} \right)^{d/4} .$$

The effective charge $u_A(\omega)$ moves as $\omega \rightarrow 0$ according to the renormalization group equations. Its motion can be followed ¹¹⁾ from Fig. 14, where, by defining $\bar{r}(\omega) = \alpha'(\omega)/\alpha'_{SF}(\omega)$, we have indicated in the various regions of the $u_A, \bar{r}$ plane the directions of the flow vector

$$\left(-\omega \frac{\partial u_A}{\partial \omega} , -\omega \frac{\partial \bar{r}}{\partial \omega} \right)$$

as computed to first order in ϵ . We see then that if we start from a normalization point ω_0 with the condition [see Eq. (3.19)]

$$u_A(\omega_0) = u_p(\omega_0) , \quad \bar{r}(\omega_0) = 1 , \quad (B.16)$$

the inequality $u^2(\omega) < 4u_A^2(\omega)$ will be kept for any $\omega \leq \omega_0$, so that

$$Z_{SF}(\omega) > Z(\omega) , \quad \Delta_{SF} = \Delta_{SF} - 1 < 0 . \quad (B.17)$$

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FIGURE CAPTIONS

Fig. 1 : Example of a contribution to $\text{Im } A$ in Eq. (2.1) from $S_n^{(\pm)}$ with $b_{\pm} = 1, h_{\pm} = 0$.

- a) Particles are attached in an ordered way. The resulting form of the surface of A has $h=0$: a cylinder.
- b) Two intermediate particles are exchanged. The resulting form of the surface of A has $h=1$: a torus.

Fig. 2 : s channel discontinuity of the bare Pomeron. Full lines denote boundaries; dashed lines indicate the shape of the surface, and planar surfaces are shaded. The intermediate particles connecting the two boundaries of $S^{(+)}$ and $S^{(-)}$ are not shown.

Fig. 3 : Example of a discontinuity cut for the torus ($h=1$) which divides a face in more than two parts. Intermediate particles connecting the boundary $B^{(-)}$ with $B_1^{(+)}$ are not shown. The rapidity of the produced particles is also indicated.

Fig. 4 : All the possible cuts of the torus ($h=1$) in:

- a) $b=1$ boundaries (absorptive cut with $h_+ = 1, h_- = 0$);
- b) $b=2$ boundaries (polyperipheral cut $h_+ = h_- = 0$); and
- c) $b=2$ boundaries (diffractive cut $h_+ = h_- = 0$).

Fig. 5 : Possible clusterings of $b=3$ boundaries.

Fig. 6 : A particular discontinuity of the $h=2$ surface which divides the central cylindric part [with rapidity (y, y')] in four regions. In the rapidity interval (y, y') , four sets of intermediate particles are cut by the discontinuity plane leading to a Pomeron with double multiplicity.

Fig. 7 : Various vertices for the three distinct Pomerons P_+, P_- and P_c . Full lines represent the discontinuity plane crossing the vertices.

Fig. 8 : The vertex $P_c \rightarrow P_+ + P_-$ and its two topological distinct discontinuity cuts.

Fig. 9 : Topological transformations showing the replacement of $g_A(g_A^2)$ with $g_P(g_I g_D)$. Full lines represent the P_c propagator, and the wavy lines the $P_{\pm}$ propagators.

Fig. 10 : Contribution to the renormalized propagator for the cut Pomeron D_c from the class of diagrams with $g_I = g_D = 0$. $G_{n_{\pm}}^{(\pm)}$ represent the Green's functions for the $P_{\pm}$ Pomerons (wavy lines). The full lines represent the (bare) cut Pomeron propagator $D_c^{(0)}$. V and v_0 are the configuration space variables $v = (\underline{b}, y)$.

Fig. 11 : a) General form of unitarity including all non-enhanced contributions.

b) Approximate form when non-enhanced interference terms are neglected. Full (wavy) lines denote cut (uncut) Pomerons.

A term $n_c = 0$ (elastic contribution) is also present.

Fig. 12 : Possible behaviour of the proper self-energy part giving $\Delta \leq 0$ away from the critical point. The value of Δ is given by the intersection of the drawn curve $\omega = \bar{\Sigma}(\Delta)$ with the straight line $\omega = \Delta_0 - \Delta$.

Fig. 13 : Discontinuity equations for inverse propagator and vertex function in the single fireball field theory. The big dots denote renormalized quantities.

Fig. 14 : Lines of zero of $\beta_r = -\omega(\partial/\partial\omega)r(\omega)$ (solid) and of $\beta_A = -\omega(\partial/\partial\omega)u_A(\omega)$ (dashed). The arrows denote the direction of the vector (β_A, β_r) which also gives the flow of (u_A, r) as $\omega \rightarrow 0$.

a

b

FIG. 1

FIG. 2

FIG. 3

FIG. 4

FIG. 5

FIG. 6

FIG. 7

FIG. 8

FIG. 9

FIG. 10

FIG. 11

FIG. 12

$$\begin{aligned}
 \text{Disc } (\text{---}\bullet\text{---})^{-1} &= \text{---}\bullet\text{---} + \text{---}\bullet\text{---} + \dots \\
 \Omega_1, k_1 \text{---}\bullet\text{---} \Omega_2, k_2 &= \text{---}\bullet\text{---} + \text{---}\bullet\text{---} + \dots \\
 &+ \text{---}\bullet\text{---} + \text{---}\bullet\text{---} + \dots
 \end{aligned}$$

FIG. 13

FIG. 14