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Unitarized Chiral Perturbation Theory for Elastic Pion-Pion Scattering

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ABSTRACT

The low energy two-pion elastic scattering is calculated within the framework of the one-loop Chiral Perturbation Theory in combination with the Padé approximant method for the partial wave amplitudes. Elastic unitarity is satisfied to all orders. It is shown that the experimental data below 900 MeV can be described in terms of two independent parameters and that the chiral symmetry limit of massless pions gives a good description of the data above 350 MeV. With chiral symmetry breaking (finite pion mass) our calculation yields a ρ width of 155 MeV and a S-wave $I=0$ scattering length of $a_0 = 0.27m_\pi^{-1}$, in good agreement with experimental data. These results demonstrate the validity of our method as applied to the strongly interacting Higgs sector.

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Chiral Perturbation Theory (C.P.Th) [1] is now widely used to calculate matrix elements involving low energy pions. This method consists of writing down Chiral Lagrangians for the physical processes and then uses the conventional technique of the field theory to do one-loop corrections to the corresponding matrix element obtained by the lowest (tree) approximation. This procedure can then be generalized to higher loops. In this scheme, the matrix element for the physical processes is a power series expansion of the kinematical invariants. This is so because the chiral invariance of the interaction requires that the Lagrangian be constructed out of powers of the derivatives of the pion fields which make the theory non-renormalizable. In spite of this undesirable property, Weinberg [2] suggested that at each order of the perturbation series, appropriate counter terms (which are consistent with the chiral symmetry) can be introduced to renormalize the matrix element. Unitarity is therefore satisfied perturbatively by carrying out loop corrections to the tree amplitude. This program was later carried out to one-loop order by Gasser and Leutwyler [3]. Within the framework of C.P.Th., their work represents the most systematic low energy expansion of the scalar, vector pion form factors and the elastic pion-pion scattering.

One of us (T.N.T.) recently questioned the validity of the approximate perturbative unitarity used in the C.P.Th. calculation for these processes [4]. It was shown that, even at the two-pion threshold, the C.P.Th. to one-loop order for pion form factors and elastic pion-pion scattering violates the exact elastic unitarity conditions. A two-loop order computation must be carried out for the imaginary part to restore approximately the unitary condition at the two pion threshold. In order to fulfil exactly the elastic unitarity to all orders, it was shown that the inverse amplitude method or the Padé approximant method should be used. This last method is particularly convenient, because it is simply a reconstruction of an infinite geometric series out of the one-loop order amplitude and, therefore, involves no more work than the one loop C.P.Th. calculation.

Besides the violation of the unitarity, the program of C.P.Th., even if it was carried out to many-loop calculation, i.e. taking also into account the inelastic effect, without being complemented by the Padé approximant method or other unitarization schemes, would not lead to a reasonable approximation of the physical amplitudes near the ρ resonance. This is so because the resonant Breit-Wigner form cannot be approximated by an ordinary power series in momenta; the Padé approximant method, on the other hand, can easily handle this situation [5]. This explains the previous success of the calculation of the pion form factors [4].

This article deals with the application of the Padé approximant method to the elastic pion-pion scattering. We use the one-loop C.P.Th. result and unitarize the partial wave amplitude by the Padé method in order to satisfy exactly the elastic unitarity condition. The experimental test for this **Unitarized Chiral Perturbation Theory** is even more challenging, because the situation is more complex when more data are available, and also because the application of the Padé method to the scattering problem requires no special assumption on the left-hand cut discontinuity of the partial wave amplitudes. As mentioned above, a complete systematic low energy C.P.Th calculation to the one-loop level was given by Gasser and Leutwyler [3]. We simply apply the diagonal [1,1] Padé approximant method to their results.

In order to give a physical insight to our calculation and also to the physics of the C.P.Th., let us first follow reference [6] and write down a Chiral Lagrangian which contains the gross features of the low energy two-pion problem, namely the dominance of the $I=1$, $J=1$ ρ meson and a large enhancement in the $I=0$, $J=0$ $\pi\pi$ channel. This was done by using the linear σ model [7] and by adding to it a chiral invariant ρ field à la Weinberg [8]. Define the $\pi\pi$ scattering amplitude, $\pi_a(p) + \pi_b(q) \rightarrow \pi_c(p') + \pi_d(q')$, as $T_{abcd} = A(s, t, u)\delta_{ab}\delta_{cd} + A(t, s, u)\delta_{ac}\delta_{bd} + A(u, t, s)\delta_{ad}\delta_{cb}$ where s, t, u are the usual Mandelstam variables, $s = (p + q)^2$ etc.. The Weinberg ρ (non-renormalizable) Chiral Lagrangian and the linear σ model yield the tree amplitude:

$$A(s, t, u) = \frac{(s - m_\sigma^2)(m_\sigma^2 - m_\rho^2)}{f^2(m_\sigma^2 - s)} + \frac{g_{\rho\pi\pi}^2}{m_\rho^2} \left(\frac{t(s - u)}{m_\rho^2 - t} + \frac{u(s - t)}{m_\rho^2 - u} \right) \quad (1)$$

where $f = 94 \text{ MeV}$, m_σ and m_ρ , are respectively the pion, σ , and ρ (bare) masses. This amplitude has poles at $s = m_\sigma^2$ and m_ρ^2 corresponding respectively to a possible scalar meson σ and a vector meson ρ . In order to compare it with experiment, we must do one loop correction to this amplitude and then unitarize it. This procedure is however very complex. Furthermore, the existence of the scalar meson is debatable.

We choose instead a different method which is **more general** by examining the most general structure of the Chiral Lagrangian up to the quartic derivatives in the pion fields (giving rise to quadratic terms in the s, t , and u variables). In the chiral limit, the most general Lagrangian to this order of the derivatives of the pion field is:

$$\mathcal{L} = \frac{f^2}{4} \text{tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{E}{32} \text{tr}((\partial_\mu U U^\dagger, \partial_\nu U U^\dagger)^2) + \frac{G}{8} (\text{tr}(U \partial_\mu U \partial^\mu U^\dagger))^2 \quad (2)$$

with $U(x) = \exp(i\Phi_a(x)\sigma_a/f)$ where Φ_a 's are the pion fields and σ_a are the usual Pauli matrices. To the R.H.S. of eq.(2) we can add a term $\mu^2 f^2 \text{tr}(U + U^\dagger)/4$ to break the chiral symmetry and to give the pion a finite mass m .

The first term on the R.H.S. of eq.(2) is simply the usual minimal term. The second and third terms are, respectively, the Skyrme and non-Skyrme terms with $E = 1/e^2$ and $G = \gamma/e^2$ in the standard notation of Skyrmin physics. The tree-level amplitudes obtained from the quartic terms of eq.(2) can be interpreted as the low energy expansion of eq.(1) through the identification $E = 2f^2/m_\rho^2$ and $G = f^2/m_\sigma^2$. These results are simply obtained, in a more specialized language, by "integrating out" the heavy fields. At the one-loop level the quartic terms in the Lagrangian $\mathcal{L}$ provide the needed counter terms to make the non-linear σ model renormalizable at the order s^2 . In the following, we regard these terms as the Lagrangian which simulates the low energy contribution to the scattering amplitudes of the unspecified heavy fields which are "integrated out". Such heavy fields could be the above-mentioned isoscalar-scalar and/or isovector-vector bosons, or simply the quark-loop contribution to the scattering of two Nambu-Goldstone bosons etc.. The coefficients E and G are the only two free parameters in this model for the low energy pion-pion scattering problem. Resonances in the relevant channel could be generated by an appropriate choice

of the two corresponding renormalized parameters $\bar{E}_R$ and $\bar{G}_R$ which are present in the one-loop unitarized partial wave amplitudes. The result of our investigation shows an excellent agreement between the theory and the experimental data: the ρ resonance with a correct width and a large enhancement in the $I=0$ S-wave channel.

As mentioned above, Gasser and Leutwyler calculated the one-loop correction to the tree Lagrangian of eq.(2) including the chiral symmetry breaking terms. We present in the following a simplified version where the counterterms proportional to the pion mass are neglected and therefore we set the renormalized pion decay constant f_R equal to its tree value. They calculated the pion-pion scattering amplitude to one-loop order using the dimensional regularization scheme with a renormalization scale μ :

$$A(s, t, u) = \frac{s - m^2}{f^2} + \frac{\bar{E}_R(m)}{4f^4} [-2(s - 2m^2)^2 + (t - 2m^2)^2 + (u - 2m^2)^2] + \frac{\bar{G}_R(m)}{f^4} (s - 2m^2)^2 + H(s, t, u) \quad (3)$$

where

$$H(s, t, u) = \frac{1}{6f^4} [3(s^2 - m^4)\bar{J}(s) + [t(t - u) - 2m^2t + 4m^2u - 2m^4]\bar{J}(t) + [t \leftrightarrow u]\bar{J}(u) - \frac{1}{96\pi^2} [21s^2 + 5(t - u)^2 + 8m^2s - 26m^4]] \quad (4)$$

and for $s \geq 4m^2$

$$\bar{J}(s) = \frac{1}{16\pi^2} (-2\sqrt{\frac{s - 4m^2}{s}} \log \frac{\sqrt{s} + \sqrt{s - 4m^2}}{2m} + i\pi \sqrt{\frac{s - 4m^2}{s}} + 2).$$

(For other values of s the expression for $\bar{J}(s)$ can be obtained by analytic continuation; in particular $\bar{J}(0) = 0$). $\bar{E}_R(m)$ and $\bar{G}_R(m)$ are the renormalized coefficients $\bar{E}_R$ and $\bar{G}_R$ defined at the pion mass scale m . Notice that we use here the notation with Λ to distinguish them from the parameters E_R and G_R corresponding to the massless case that will be discussed below. For the other scale μ , the corresponding renormalized coefficients $\bar{E}_R(\mu)$ and $\bar{G}_R(\mu)$ are given by

$$\begin{aligned} \bar{E}_R(\mu) &= \bar{E}_R(m) - \frac{1}{6\pi^2} \log \frac{\mu}{m} \\ \bar{G}_R(\mu) &= \bar{G}_R(m) - \frac{1}{8\pi^2} \log \frac{\mu}{m}. \end{aligned} \quad (5)$$

Physical quantities calculated from eqs.(3) and (4) are, of course, scale independent.

Partial wave amplitudes in the appropriate isospin channel can be calculated from eqs.(3) and (4). It is now useful to discuss the discontinuity in the complex s plane of the J^{th} partial wave amplitude $a_{IJ}(s)$. The function $\bar{J}(s)$ gives rise to the so-called right-hand cut, with a cut on the real axis starting at $4m^2$ to ∞ . This is also called the unitarity cut. The discontinuity of this function across this cut is $i(8\pi)^{-1}[(s - 4m^2)/s]^{1/2}$ and is of the

square root singularity type (this singularity is sometimes misunderstood as the logarithm singularity). The partial wave projection of the terms involved in $\bar{J}(t)$ and $\bar{J}(u)$ gives rise to the so-called left-hand cut starting from 0 to $-\infty$ (the cut structure of the partial wave amplitude can be demonstrated from the first principles of the S-matrix theory). These two cuts are separated by a gap $0 \leq s \leq 4m^2$ where m is the pion mass. This gap shrinks to zero in the chiral limit, but an analytic continuation can be made by first giving a finite value to the pion mass and then taking the chiral limit.

For the massless pion case, the function $\bar{J}(s)$ and the partial wave projection of terms involved in $\bar{J}(t)$ and $\bar{J}(u)$ become singular as $\log(\sqrt{s}/m)$ with $m \rightarrow 0$, but this singularity can be absorbed by a redefinition of the renormalized coefficients. Instead of doing so, it is more convenient to take the chiral limit $m = 0$ in the beginning, i.e. from eq.(2), to give a more transparent result which was first given by Lehmman in [9]. Including the one-loop correction, the invariant function $A(s, t, u)$ can be written as:

$$A(s, t, u) = \frac{s}{f^2} + \frac{\bar{E}_R(\nu)}{4f^4} (-2s^2 + t^2 + u^2) + \frac{\bar{G}_R(\nu)}{f^4} s^2 + \frac{1}{(4\pi)^2} f^4 \left(-\frac{1}{12} (3t^2 + u^2 - s^2) \log\left(\frac{-t}{\nu^2}\right) - \frac{1}{12} (3u^2 + t^2 - s^2) \log\left(\frac{-u}{\nu^2}\right) - \frac{1}{2} s^2 \log\left(\frac{-s}{\nu^2}\right) \right) \quad (6)$$

where $\bar{E}_R(\nu)$ and $\bar{G}_R(\nu)$ are the renormalized coefficients that now depend on an arbitrary energy scale ν which cannot be set to zero [2, 10]. The physical one-loop amplitude eq.(6) is independent of this scale provided the evolution equations of the parameters are given by equations which are similar to eq.(5). The first three terms on the R.H.S. of eq.(6) are the same as the ones obtained from the tree amplitude of the Lagrangian eq.(2), with the change of the bare parameters E and G by the renormalized ones $\bar{E}_R(\nu)$ and $\bar{G}_R(\nu)$. The logarithmic terms in eq.(6) are typically due to the one-loop calculation. Notice that our definition of the renormalized parameters in the massless case differs by an additive constant with respect to the ones introduced in the massive case:

$$\begin{aligned} \bar{E}_R(\nu) &\leftrightarrow \bar{E}_R(\nu) + \frac{7}{72\pi^2} \\ \bar{G}_R(\nu) &\leftrightarrow \bar{G}_R(\nu) + \frac{1}{16\pi^2} \end{aligned}$$

where $\nu \neq 0$.

We are now in a position to project out the J^{th} partial wave of the isospin I channel. Let us write these partial wave amplitudes in the form $a_{IJ}(s) = a_{IJ}^{(0)}(s) + a_{IJ}^{(1)}(s)$, where $a_{IJ}^{(0)}(s)$ contains the contribution from the first term in eq.(6), that is the result from the first term in eq.(2) at tree level, and $a_{IJ}^{(1)}(s)$ are the remaining terms, for example, $a_{00}^{(0)} = s/16\pi f^2$ etc.. As was shown in ref.[4] this C.P.T.h. series violates the unitarity even at very low energy. The Padé approximant method must be used to resum this one-loop series to satisfy exactly the elastic unitarity condition. The first diagonal $[1, 1]$ Padé approximant for the J^{th} partial wave $a_{IJ}^{[1,1]}(s)$ is:

$$a_{IJ}^{[1,1]}(s) = \frac{a_{IJ}^{(0)}(s)}{1 - \frac{a_{IJ}^{(1)}(s)}{a_{IJ}^{(0)}(s)}} \quad (7)$$

and satisfies the elastic unitarity condition, i.e., $\text{Im}a_{IJ}^{(1,1)}(s) = |a_{IJ}^{(1,1)}(s)|^2$ which is in contrast with the perturbative unitarity of the C.P.Th. series, $\text{Im}a_{IJ}^{(1,1)}(s) = |a_{IJ}^{(0)}(s)|^2$. If one expanded the denominator of eq.(7) and kept only the first two terms of the series, one would arrive at the usual C.P.Th. series. As was previously shown, and will be seen later, there is a dramatic difference between the phase shift calculated from the C.P.Th. series and the fully unitarized Padé formula eq.(7).

It is straightforward to show using eqs.(6) and (7) that for $s > 4m^2$:

$$\begin{aligned} a_{00}^{(1,1)}(s) &= \frac{s}{16\pi f^2} \frac{1}{1 - \frac{s}{4f^2} \left(-\frac{4}{3} E_R(\nu) + \frac{22}{3} G_R(\nu) + \frac{1}{(4\pi)^2} \left(\frac{11}{27} - \frac{50}{9} \log \frac{s}{\nu^2} + 4i\pi \right) \right)} \\ a_{11}^{(1,1)}(s) &= \frac{s}{96\pi f^2} \frac{1}{1 - \frac{s}{f^2} \left(\frac{3}{4} E_R(\nu) - G_R(\nu) + \frac{1}{(4\pi)^2} \left(-\frac{1}{18} + \frac{i\pi}{6} \right) \right)} \\ a_{20}^{(1,1)}(s) &= \frac{-s}{32\pi f^2} \frac{1}{1 + \frac{s}{2f^2} \left(\frac{4}{3} (G_R(\nu) + \frac{E_R(\nu)}{2}) + \frac{1}{(4\pi)^2} \left(\frac{25}{54} - \frac{20}{9} \log \frac{s}{\nu^2} + i\pi \right) \right)} \end{aligned} \quad (8)$$

We begin first by examining the P-wave amplitude. The ρ resonance can be generated by requiring that the real part of the denominator of the $a_{11}^{(1,1)}$ amplitude vanish. This happens at $s = m_\rho^2$, where:

$$m_\rho^2 = \frac{3}{4} \frac{f^2}{E_R(\nu) - G_R(\nu) - \frac{1}{18(4\pi)^2}} \quad (9)$$

provided that the denominator on the R.H.S. of eq.(9) is positive. (One should not be confused with the notation of the physical ρ mass given here and that for the bare ρ mass given in eq.(1)). The ρ width can be read off from eq.(8):

$$\Gamma_\rho = \frac{m_\rho^3}{96\pi f^2} \quad (10)$$

which is the K.S.R.F. relation for the massless pion case [11]. It gives of course a too large value for the ρ width, $\Gamma_\rho = 170 \text{ MeV}$. This relation was first correctly demonstrated by Lehmann [9], who dismissed it as nothing new and gave the credit to Brown and Goble [12]. This assertion is not correct, because the Brown and Goble demonstration of the K.S.R.F. relation requires a strong assumption that the left-hand cut in the P-wave amplitude does not contribute. There is no such assumption made here or in the Lehmann paper. This point deserves further clarification.

In the expression for $a_{11}^{(1,1)}(s)$ given by eq.(8), the unitarity cut (or right hand-cut) gives rise to a term proportional to $\log(-s)$ whose discontinuity across this cut is $-2\pi i$. The partial wave projection of the $\log(-s)$ and $\log(-u)$ terms in eq.(6) gives rise also to a function $\log(s)$ which has discontinuity on the left-hand cut (see the above discussion of the partial wave projection of $\bar{J}(s)$ and $\bar{J}(t)$). For positive values of s the real part of

the right-hand cut contribution of $\log(-s)$ cancels with a similar $\log(s)$ term due to the left-hand cut. The net result is that the expression for $a_{11}^{(1,1)}(s)$ does not have a scale dependent logarithmic term. Because the demonstration of Brown and Goble relies on the assumption of the neglect of the left-hand cut, such a cancellation is not possible in the chiral limit. This reflects in the presence of the $\log(s)$ term in the Brown and Goble result, when the chiral limit is taken, and is in contradiction with the expression for $a_{11}^{(1,1)}(s)$ given here. This point was first remarked by Lehmann [9] who apparently did not notice that a strong assumption was made by Brown and Goble on the absence of the left-hand cut.

As we have two unknown parameters $E_R(\nu)$ and $G_R(\nu)$, we must need one more equation, in addition to eq.(9), to fix these parameters. This can be done by using an experimental S-wave $I=0$ phase shift at an energy sufficiently far away from the two-pion threshold to avoid the threshold effect. The following two parameters $E_R(785 \text{ MeV}) = 0.029$ and $G_R(785 \text{ MeV}) = 0.0073$ give a good description of the S and P-waves (Fig.1). It is surprising that at the scale $\nu = 785 \text{ MeV}$ the values given for E_R and G_R are approximately the same as the bare values of E and G determined from the Skyrmin physics [13]. In particular, the well-known result $E = 2f^2/m_\rho^2$ has the same value as that of $E_R(785 \text{ MeV})$.

Departure from the Chiral Symmetry.

In order to study the chiral symmetry breaking effect, we have to add to the R.H.S. of eq.(2) additional terms which are proportional to the pion mass m . Gasser and Leutwyler have studied this situation in detail. Their main conclusion is that they affect little the results given by eqs.(3) and (4). We assume here that this is the case, and use eqs.(3) and (4) with $f = 94 \text{ MeV}$. The projection of the partial wave amplitudes is done numerically and then the Padé approximant method of eq.(7) is used to calculate the pion-pion phase shifts.

The best fit to the experimental data is obtained with the following parameters: $\bar{E}_R(785 \text{ MeV}) = 0.016$ and $\bar{G}_R(785 \text{ MeV}) = -0.0009$. In Fig.2 the experimental data on the S and P wave phase shifts δ_{00} and δ_{11} are compared with the theoretical predictions for two different choices of the parameters; the upper solid curve for the $I=0$ S-wave channel and the solid curve for the $I=1$ P-wave channel correspond to taking $\bar{E}_R(785 \text{ MeV}) = 0.018$ and $\bar{G}_R(785 \text{ MeV}) = 0.0010$; the lower solid curve for the $I=0$ S-wave channel corresponds to taking $\bar{E}_R(785 \text{ MeV}) = 0.011$ and $\bar{G}_R(785 \text{ MeV}) = -0.0025$. The theoretical prediction for δ_{00} with the parameters that best fit to the experimental data (not shown) lies in between the two solid lines shown in Fig.1.a. The predictions for δ_{11} corresponding to $\bar{E}_R(785 \text{ MeV}) = 0.014$ and $\bar{G}_R(785 \text{ MeV}) = -0.0025$ and the ones corresponding to $\bar{E}_R(785 \text{ MeV}) = 0.016$ and $\bar{G}_R(785 \text{ MeV}) = -0.0009$ lie very close to each other and to the solid curve in Fig.2.b and are not shown to simplify the plot.

For the best values of the parameters mentioned above, the plot $k^2 \cot \delta_{11} / \sqrt{s}$ versus s (not shown) is a perfect straight line from the two-pion threshold to 850 MeV and yields a ρ mass of 773 MeV (defined at the position where the pion-pion P-wave phase shift passes through 90°), and a ρ width of 155 MeV to be compared with the experimental width of 153 MeV . The K.S.R.F. relation is therefore verified to a great accuracy. An earlier calculation by Jung and Willey [5] with a different physical model also yielded the same result for the ρ width.

For the best value of the parameters, the S-wave $I=0$ $\pi\pi$ scattering length is $a_0 = 0.27m^{-1}$. For the choice of $\hat{E}_\pi(785MeV) = 0.014$ and $\hat{G}_\pi(785MeV) = -0.0025$ the scattering length is $a_0 = 0.25m^{-1}$. These values have to be compared with the experimental value of $0.26 \pm 0.05m^{-1}$ obtained from the $K\pi_1$ decay [14].

For completeness, the S-wave $I=2$ phase shifts corresponding to the chiral limit are given in Fig.3, where the experimental data are not shown because they have large experimental uncertainties. In all figures 1-3 the dashed lines are the usual C.P.T. results without using the Padé method. In this last case the phases are defined by $\tan\delta_{IJ} = Im a_{IJ} / Re a_{IJ}$. It is clear that a dramatic improvement is achieved with the Padé method.

Last but not least, let us mention that the Padé [1, 1] approximant method, eq.(7), develops poles near $s = m^2/2$ for the $I=0$ S-wave amplitude, and near $s = 2m^2$ for the $I=2$ S-wave amplitude which are, in principle, not acceptable and must be removed. Due to the nearly vanishing of $a_{IJ}^{(0)}(s)$ at these poles, the residues of these poles are extremely small. The subtraction of these poles from the expression of $a_{IJ}^{(0)}(s)$ in eq.(7) will result in a tiny violation of the unitarity in the physical region and can be ignored.

We have shown in this article that the low energy two-pion problem can be well described by two independent parameters. These results strongly support our recent calculations on the strongly interacting Higgs sector [15].

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Figure Captions

Fig. 1: Comparison between the predictions for the phase shifts δ_{00} , (a), and δ_{11} , (b), from the unitarized C.P.Th. to one-loop in the chiral limit, and the experimental data: (a) $\times$ ref.[14], $\diamond$ ref.[16], $\square$ ref.[17], $\blacklozenge$ ref.[18] and $\times$ ref.[19]; (b) $\diamond$ ref.[19], $\square$ ref.[20]. Solid lines correspond to the choice of the parameters: $E_R(785\text{MeV}) = 0.029$ and $G_R(785\text{MeV}) = 0.0073$. Dashed lines are the usual C.P.Th. predictions for the phases without using the Padé approximant method and for the same value of the parameters.

Fig. 2: Same as Fig.1 but taking into account pion mass corrections from eqs.(3) and (4) ($m_\pi = 135\text{MeV}$). The data shown are the same as in Fig.1. (a) The upper solid line corresponds to $\tilde{E}_R(785\text{MeV}) = 0.018$ and $\tilde{G}_R(785\text{MeV}) = 0.0010$. The lower solid line corresponds to $\tilde{E}_R(785\text{MeV}) = 0.014$ and $\tilde{G}_R(785\text{MeV}) = -0.0025$. (b) The solid line corresponds to $\tilde{E}_R(785\text{MeV}) = 0.018$ and $\tilde{G}_R(785\text{MeV}) = 0.0010$. For the other choice of the parameters the curve (not shown) is very close to the one that is shown. The dashed lines are the results for the phases for $\tilde{E}_R(785\text{MeV}) = 0.018$ and $\tilde{G}_R(785\text{MeV}) = 0.0010$ from the C.P.Th. without using the Padé approximant method.

Fig. 3: Phase shifts predictions for the $I = 2$, $J = 0$ channel in the chiral limit. Solid line is from the unitarized C.P.Th. to one-loop corresponding to the parameters $E_R(785\text{MeV}) = 0.029$ and $G_R(785\text{MeV}) = 0.0073$. Dashed-line is the result for the phases from the C.P.Th. without using the Padé method and for the same values of the parameters.

Fig. 1

Fig. 2

Fig. 3