Phase-Space Lagrangians for Null Spinning Strings

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ABSTRACT

The striking fact that normal-ordered null strings have the same critical dimension as their usual non-zero tension siblings can be understood from the observation that one must, in the tensionless case, keep all the conjugate momenta as independent dynamical variables, thus doubling the number of physical degrees of freedom. The fermionic momenta give rise to a second-class constraint which cannot be solved covariantly, but can be successfully incorporated into the first-class constraint algebra. The ghost contributions to the anomaly consist of **two** $b-c$ (and also **two** $\beta - \gamma$ systems in the supersymmetric case), of the single Virasoro sub(super)algebra for the closed null (spinning) string. In the appropriate gauge, the null (super)string is (super)chiral.

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I. Introduction

The recent interest for the very high energy behaviour of string amplitudes [1, 2, 3, 4, 5] has led to a renewed interest in the study of null strings, first introduced by Schild [6] and further studied in [7, 8, 9, 10]. In recent work [11, 12], the quantization of bosonic and null spinning strings has been thoroughly carried out both in the Lagrangian and in the BRST formalism. However, the Lagrangian formulation of the null spinning string seems to raise problems which do not arise in the bosonic case: naively, the fermionic variables fail to develop any dynamics and the constraint algebra turns out to be second-class. In order to clarify this issue we turn to the study of the phase-space approach to null (spinning) strings. Previous work has shown that phase-space Lagrangians are convenient for describing usual bosonic [13], null bosonic [14], and null space-time supersymmetric [15] string theories. The first-order approach has the advantage of yielding, in a natural way, the limiting case of vanishing string tension.

First we formulate the null bosonic case and show that, up to reparametrizations, the Schild Lagrangian is unique. In Section III we review the essential properties of spinning strings, and in Section IV we present the phase-space Lagrangian and discuss its equivalence with the usual case. In Section V, the core of this paper, we present the phase-space approach to null spinning strings, and some remarks concerning its covariant quantization are left to Section VI.

II. Phase-Space Lagrangian for Null Bosonic Strings

In this section we start from the constraint algebra to formulate the phase-space Lagrangian of null bosonic strings. We first obtain a gauge-invariant Lagrangian from which the Schild Lagrangian follows after gauge-fixing. The constraints of the null string are:

$$\mathcal{H}_\perp = P^2 = 0 \quad (2.1)$$

$$\mathcal{H}_\parallel = PX' = 0 \quad (2.2)$$

where P^μ is the conjugate momentum of X^μ . Thus the phase-space Lagrangian is given by:

$$\mathcal{L} = P\dot{X} + \lambda_\perp \mathcal{H}_\perp + \lambda_\parallel \mathcal{P}X' \quad (2.3)$$

where the momentum P^μ is supposed to be a function of X^μ . We fix the gauge by the requirement that this function be linear in the first derivatives,

$$P^\mu = a\dot{X}^\mu + bX'^\mu \quad (2.4)$$

This implies ($a \neq 0$) that $\lambda_\perp = \frac{-1}{2a}$ and $\lambda_\parallel = \frac{b}{a}$, so

$$\mathcal{L} = \frac{a}{2}\dot{X}^2 + \frac{b^2}{2a}X'^2 + b\dot{X}X' \quad (2.5)$$

The usual Schild Lagrangian corresponds to $a = 1$ and $b = 0$. Other choices for $a \neq 0$ and b amount to rotations and rescalings of the world-sheet. Thus, up to such reparametrizations, we can say that the Lagrangian of the null string is uniquely given by the Schild Lagrangian. Of particular interest is the gauge with $a = 1$ and $b = \pm 1$, for which the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\dot{X} \pm X')^2$$

is explicitly chiral. In complex notation, $\mathcal{L} = \frac{1}{2}(\partial_z X)^2$, which implies $\mathcal{H} = \frac{1}{2}\partial_z X \cdot \partial_{\bar{z}} X$. Curiously, these are the expressions for $\mathcal{H}$ and $\mathcal{L}$ of the usual string: $\mathcal{H}$ and $\mathcal{L}$ interchange roles when we reach the $\alpha' \rightarrow \infty$ limit!

In passing, note also that (2.3) and (2.4) have another qualitatively different solution, with $a = 0$. Choosing $b = 1$ and $\lambda_\perp b + \lambda_\parallel = \pm 1$, then $\mathcal{L} = X' \cdot (\dot{X} \pm X')$, which is the chiral boson Lagrangian [16].

III. Spinning String

We briefly review the well-known spinning string [17] in order to establish notation and the usual constraint algebra. The action for spinning strings is [18]

$$S = -T \int d\sigma d\tau \sqrt{-g} \left[\frac{1}{2} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{i}{2} \bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu + i \bar{\chi}_\alpha \gamma^\beta \gamma^\alpha \psi^\mu \partial_\beta X_\mu + \frac{1}{4} (\bar{\psi}_\mu \psi^\mu) (\bar{\chi}_\alpha \gamma^\beta \gamma^\alpha \chi_\beta) \right] \quad (3.1)$$

where the string tension T , which has dimensions of mass squared, will be kept explicitly throughout. This action is invariant under the local Weyl symmetry,

$$\begin{aligned} \delta c_\alpha^a &= \phi c_\alpha^a \\ \delta \psi^\mu &= -\frac{1}{2} \phi \psi^\mu \\ \delta \chi_\alpha &= \frac{1}{2} \phi \chi_\alpha \\ \delta X^\mu &= 0 \end{aligned} \quad (3.2)$$

and under the local supersymmetry transformations

$$\begin{aligned} \delta X^\mu &= i\bar{\epsilon} \psi^\mu \\ \delta \psi^\mu &= (\partial_\alpha X^\mu + i\bar{\psi}^\mu \chi_\alpha) \gamma^\alpha \epsilon \\ \delta c_\alpha^a &= -2i\bar{\epsilon} \gamma^\alpha \psi_\alpha \\ \delta \chi_\alpha &= -\nabla_\alpha \epsilon \end{aligned} \quad (3.3)$$

where $\nabla_\alpha = \partial_\alpha + \frac{1}{8}\omega_{\mu\alpha b}[\gamma^a, \gamma^b] = \partial_\alpha + \frac{1}{8}\omega_{\mu\alpha b}\epsilon^{ab}\gamma_5$ is the covariant derivative. The action (3.1) also exhibits a local fermionic symmetry, given by

$$\begin{aligned}\delta\lambda_\alpha &= i\gamma_\alpha\eta \\ \delta\epsilon_\alpha^a &= \delta\psi^\mu = \delta X^\mu = 0\end{aligned}\quad (3.4)$$

If we introduce the energy-momentum tensor using the definition

$$T_{\alpha\beta} = -\frac{1}{c}\frac{\delta S}{T}\epsilon_{\alpha\alpha}\frac{\beta}{\delta\epsilon_\alpha^a}\quad (3.5)$$

then from the Lagrangian (3.1) we get

$$\begin{aligned}T_{\alpha\beta} &= \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}g_{\alpha\beta}g^{\rho\lambda}\partial_\rho X^\mu \partial_\lambda X_\mu + \frac{i}{2}\bar{\psi}^\mu \gamma_\beta \partial_\alpha \psi_\mu - \frac{i}{2}\bar{\psi}^\mu \gamma_\alpha \partial_\beta \psi_\mu \\ &\quad + i\bar{\chi}_\alpha \gamma_\lambda \gamma_\beta \psi_\mu \partial^\lambda X^\mu + i\bar{\chi}^\lambda \gamma_\beta \gamma_\lambda \psi_\mu \partial_\alpha X^\mu - ig_{\alpha\beta}\bar{\chi}_\rho \gamma^\lambda \gamma^\rho \psi^\mu \partial_\lambda X_\mu \\ &\quad + \frac{1}{2}(\bar{\psi}_\mu \psi^\mu) \left(\bar{\chi}_\alpha \gamma_\lambda \gamma_\beta \chi^\lambda - \frac{1}{2}g_{\alpha\beta}\bar{\chi}^\rho \gamma_\rho \gamma_\lambda \chi^\lambda \right)\end{aligned}\quad (3.6)$$

The equations of motion for the propagating fields imply that $T_{\alpha\beta}$ is divergenceless and traceless. The equation of motion for the auxiliary zweibein ϵ_α^a leads more strongly to the constraint $T_{\alpha\beta} = 0$.

By virtue of the two-dimensional gamma-matrix relation $\gamma^\alpha \gamma^\beta \gamma_\alpha = 0$, the supercurrent

$$J_\alpha = \frac{\delta S}{\delta\chi^\alpha} = i\gamma^\beta \gamma_\alpha \psi^\mu \partial_\beta X_\mu + \frac{1}{2}(\bar{\psi}_\mu \psi^\mu) \gamma^\beta \gamma_\alpha \chi_\beta\quad (3.7)$$

has vanishing γ -trace: $\gamma^\alpha J_\alpha = 0$. The equation of motion for the gravitino χ^α implies the constraint relation $J_\alpha = 0$.

Scale invariance (Weyl symmetry) in two-dimensions always allows us to choose a conformally flat metric tensor. Also, the two supersymmetric and the two superconformal symmetry generators can be used locally to set all four components of χ_α equal to zero. In this gauge, the conditions $T_{\alpha\beta} = 0$ and $J_\alpha = 0$ take a simple form:

$$\partial_\alpha X^\mu \partial_\beta X_\mu + \frac{i}{2}\bar{\psi}^\mu \gamma_\beta \partial_\alpha \psi_\mu - \frac{1}{2}\eta_{\alpha\beta}(\eta^{\rho\lambda}\partial_\rho X^\mu \partial_\lambda X_\mu + i\bar{\psi}^\mu \gamma^\lambda \partial_\lambda \psi_\mu) = 0\quad (3.8)$$

$$\gamma^\alpha \gamma_\beta \psi^\mu \partial_\alpha X_\mu = 0\quad (3.9)$$

and the Lagrangian density obtained from (3.1) reduces in this gauge to

$$L = -\frac{1}{2}T(\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + i\bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu)\quad (3.10)$$

The canonical momenta are

$$\mathcal{P}_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = T\dot{X}_\mu\quad (3.11)$$

$$\begin{aligned}\pi_\mu &= \frac{\partial L}{\partial \dot{\psi}^\mu} = \frac{i}{2}T\bar{\psi}^\mu \gamma^0 \\ &= -\frac{i}{2}T\psi_\mu\end{aligned}\quad (3.12)$$

Eq. (3.12) is the usual relationship between momentum and coordinate for fermionic fields, which is to be treated carefully [19].

Using (3.11) in (3.8)-(3.9) one obtains the constraints

$$\begin{aligned}\mathcal{H}_\perp &= \mathcal{P}^2 + T^2(X'^2 + i\psi\gamma_5\psi') \approx 0 \\ \mathcal{H}_\parallel &= \mathcal{P} \cdot X' + \frac{i}{2}T\psi \cdot \psi' \approx 0 \\ \mathcal{S}_A &= \mathcal{P} \cdot \psi_A + T(\gamma_5\psi')_A \cdot X' \approx 0\end{aligned}\quad (3.13)$$

understood to be evaluated at fixed τ (from now on we drop the explicit dependence on τ). These constraints are all first-class. The bosonic ones generate reparametrizations and the fermionic ones supersymmetry transformations. In order to establish the algebra generated by these constraints, we need to know the bracket $\{\psi_A^\mu, \psi_B^\nu\}$. This is achieved in the Dirac formalism [20, 21] by covariantly eliminating the second class constraint (3.12), with result

$$\{\psi_A^\mu(\sigma, \tau), \psi_B^\nu(\sigma', \tau)\} = -\frac{i}{T}\eta^{\mu\nu}\delta_{AB}\delta(\sigma - \sigma')\quad (3.14)$$

We also need

$$\{\mathcal{P}^\mu(\sigma), X^\nu(\sigma')\} = \eta^{\mu\nu}\delta(\sigma - \sigma')\quad (3.15)$$

in order to get, after some grinding, the classical Ramond (or super-Virasoro) algebra of first-class constraints

$$\begin{aligned}\{\mathcal{H}_\perp(\sigma), \mathcal{H}_\perp(\sigma')\} &= 4T^2 \left[\mathcal{H}_\parallel(\sigma) + \mathcal{H}_\parallel(\sigma') \right] \frac{\partial}{\partial\sigma} \delta(\sigma - \sigma') \\ \{\mathcal{H}_\perp(\sigma), \mathcal{H}_\parallel(\sigma')\} &= [\mathcal{H}_\perp(\sigma) + \mathcal{H}_\perp(\sigma')] \frac{\partial}{\partial\sigma} \delta(\sigma - \sigma') \\ \{\mathcal{H}_\parallel(\sigma), \mathcal{H}_\parallel(\sigma')\} &= [\mathcal{H}_\parallel(\sigma) + \mathcal{H}_\parallel(\sigma')] \frac{\partial}{\partial\sigma} \delta(\sigma - \sigma') \\ \{\mathcal{S}_A(\sigma), \mathcal{S}_B(\sigma')\} &= -i(\delta_{AB}\mathcal{H}_\perp + 2\gamma_{AB}^5 \mathcal{H}_\parallel) \delta(\sigma - \sigma') \\ \{\mathcal{H}_\perp(\sigma), \mathcal{S}_A(\sigma')\} &= T\gamma_{AB}^5 [2\mathcal{S}_B(\sigma) + \mathcal{S}_B(\sigma')] \frac{\partial}{\partial\sigma} \delta(\sigma - \sigma') \\ \{\mathcal{H}_\parallel(\sigma), \mathcal{S}_A(\sigma')\} &= \frac{1}{2} [2\mathcal{S}_A(\sigma) + \mathcal{S}_A(\sigma')] \frac{\partial}{\partial\sigma} \delta(\sigma - \sigma')\end{aligned}\quad (3.16)$$

IV. Phase-space Lagrangian

The phase-space Lagrangian for the spinning string is

$$\begin{aligned}L &= \dot{X} \cdot \mathcal{P} + \dot{\psi} \pi + \lambda_\perp \mathcal{H}_\perp + \lambda_\parallel \mathcal{H}_\parallel + \zeta_A \mathcal{S}_A \\ &= \mathcal{P} \cdot \dot{X} + \frac{i}{2}T\psi \cdot \dot{\psi} + \lambda_\perp \left[\mathcal{P}^2 + T^2(X'^2 + i\psi\gamma_5\psi') \right] \\ &\quad + \lambda_\parallel \left(\mathcal{P} \cdot X' + \frac{i}{2}T\psi\psi' \right) + \zeta_A \left[\mathcal{P} \cdot \psi_A + T(\gamma_5\psi')_A \cdot X' \right]\end{aligned}\quad (4.1)$$

The above structure follows from the fact that the string Hamiltonian is a pure constraint, the footprint of general coordinate invariance. The equations of motion for $\mathcal{P}_\mu$, X^μ and ψ_A^μ are, respectively,

$$\dot{X}^\mu + 2\lambda_\perp \mathcal{P}^\mu + \lambda_\parallel X'^\mu + \zeta_A \psi_A^\mu = 0 \quad (4.2)$$

$$\dot{\mathcal{P}}_\mu + (2T^2 \lambda_\perp X'_\mu + \lambda_\parallel \mathcal{P}_\mu + T \zeta_A \psi_A^\mu)' = 0 \quad (4.3)$$

$$\begin{aligned} i T \dot{\psi}_A^\mu + 2i T^2 \lambda_\perp (\gamma_5 \psi'^\mu)_A - \frac{i}{2} T \lambda_\parallel \psi_A^\mu \\ + i T^2 \lambda'_\perp (\gamma_5 \psi^\mu)_A + i T \lambda_\parallel \psi_A^\mu - \zeta_A \mathcal{P}^\mu - T (\gamma_5 \zeta)_A X'^\mu = 0 \end{aligned} \quad (4.4)$$

The equations of motion for $\lambda_\perp$, $\lambda_\parallel$ and ζ_A are the constraint relations (3.13).

The bosonic Lagrange multipliers $\lambda_\perp$ and $\lambda_\parallel$ play the role of the independent components of the metric tensor $g_{\alpha\beta}$. Similarly, the fermionic ζ_A replaces the gravitino degrees of freedom. Choosing the gauge conditions

$$\begin{aligned} \lambda_\perp &= -\frac{1}{2T} \\ \lambda_\parallel &= 0 \\ \zeta_A &= 0 \end{aligned} \quad (4.5)$$

and using the customary boundary conditions, one easily finds the equations

$$\ddot{X}_\mu - X''_\mu = 0 \quad (4.6)$$

$$\dot{\psi}_A^\mu - (\gamma_5 \psi'^\mu)_A = 0 \implies \gamma^\alpha \partial_\alpha \psi_A^\mu = 0 \quad (4.7)$$

which can be derived from the Lagrangian density

$$L = \frac{T}{2} \left(\dot{X}^2 - X'^2 + \frac{i}{2} \psi^\alpha \gamma^\alpha \partial_\alpha \psi \right) \quad (4.8)$$

These are the same equations which would have been obtained if we had started from (3.10). We may thus refer to the gauge choice (4.5) as the conformal gauge: the metric is characterized by a single quantity, the conformal factor $\lambda_\perp$.

V. Null spinning strings

Setting the string tension equal to zero, the constraints (3.13) become

$$\begin{aligned} \mathcal{H}_\perp &= \mathcal{P}^2 \approx 0 \\ \mathcal{H}_\parallel &= \mathcal{P} \cdot X' \approx 0 \\ \mathcal{S}_A &= \mathcal{P} \cdot \psi_A \approx 0 \end{aligned} \quad (5.1)$$

The relationship (3.12) between fermionic coordinates and momenta is now lost. Instead, we have the constraint

$$\pi_A^\mu = 0 \quad (5.2)$$

which cannot be eliminated with the Dirac formalism as previously. Thus, the bracket $\{\psi_A, \psi_B\}$ must be considered as a Poisson (not Dirac) bracket, meaning it is strictly zero. Using this fact and the bosonic bracket (3.15), it is easy to see that the constraints (5.1) do not form a closed algebra. In [12], this difficulty was circumvented by means of a singular reparametrization of the fields in the Lagrangian so that at the end the result was a closed constraint algebra without such second class constraints and, consistently with (5.1), the fermionic variables had a trivial time evolution. In our case, closure can be enforced with the elegant trick below.

Since the constraint $\pi = 0$ cannot be eliminated via the Dirac formalism, we incorporate it to the relations (5.1) in the following most general way:

$$\begin{aligned} \mathcal{H}_\perp &= \mathcal{P}^2 + a \psi \gamma_5 \pi' + b \pi \gamma_5 \psi' \\ \mathcal{H}_\parallel &= \mathcal{P} \cdot X' + c \psi \cdot \pi' + d \pi \cdot \psi' \\ \mathcal{S}_A &= \mathcal{P} \cdot \psi_A + e \mathcal{P} \cdot \pi_A + f (\gamma_5 \pi)_A \cdot X' \end{aligned} \quad (5.3)$$

where a , b , c , d , e , and f are parameters to be determined. The set (5.1) is recovered by setting $\pi = 0$, of course.

With the bosonic brackets (3.15) and the fundamental fermionic brackets

$$\begin{aligned} \{\psi_A^\mu(\sigma), \pi_B^\nu(\sigma')\} &= -\delta_{AB} \eta^{\mu\nu} \delta(\sigma - \sigma') \\ \{\psi_A^\mu(\sigma), \psi_B^\nu(\sigma')\} &= \{\pi_A^\mu(\sigma), \pi_B^\nu(\sigma')\} = 0 \end{aligned} \quad (5.4)$$

it is a lengthy but straightforward exercise to show that the algebra closes, indeed, for the following choice of parameters:

$$\begin{aligned} a &= b = f = 0 \\ c &= d = \frac{1}{2} \end{aligned} \quad (5.5)$$

and e arbitrary. The null constraints are

$$\begin{aligned} \mathcal{H}_\perp &= \mathcal{P}^2 \\ \mathcal{H}_\parallel &= \mathcal{P} \cdot X' + \frac{1}{2} \psi \cdot \pi' + \frac{1}{2} \pi \cdot \psi' \\ \mathcal{S}_A &= \mathcal{P} \cdot \psi_A + e \mathcal{P} \cdot \pi_A \end{aligned} \quad (5.6)$$

and the classical null superalgebra is given by

$$\begin{aligned} \{\mathcal{H}_\perp(\sigma), \mathcal{H}_\perp(\sigma')\} &= 0 \\ \{\mathcal{H}_\perp(\sigma), \mathcal{H}_\parallel(\sigma')\} &= [\mathcal{H}_\perp(\sigma) + \mathcal{H}_\perp(\sigma')] \frac{\partial}{\partial \sigma} \delta(\sigma - \sigma') \\ \{\mathcal{H}_\parallel(\sigma), \mathcal{H}_\parallel(\sigma')\} &= [\mathcal{H}_\parallel(\sigma) + \mathcal{H}_\parallel(\sigma')] \frac{\partial}{\partial \sigma} \delta(\sigma - \sigma') \\ \{\mathcal{S}_A(\sigma), \mathcal{S}_B(\sigma')\} &= -2e \delta_{AB} \mathcal{H}_\perp \delta(\sigma - \sigma') \\ \{\mathcal{H}_\perp(\sigma), \mathcal{S}_A(\sigma')\} &= 0 \\ \{\mathcal{H}_\parallel(\sigma), \mathcal{S}_A(\sigma')\} &= \frac{1}{2} [2\mathcal{S}_A(\sigma) + \mathcal{S}_A(\sigma')] \frac{\partial}{\partial \sigma} \delta(\sigma - \sigma') \end{aligned} \quad (5.7)$$

The canonical choice for the free parameter $\epsilon = -\frac{1}{2}$, reproduces the null spinning string constraint algebra found previously [22, 12].

The null super Virasoro algebra (5.7) is almost what one would get from naively setting $T = 0$ in the usual super Virasoro algebra (3.16), except for the term proportional to $\mathcal{H}_{||}$ in the purely fermionic bracket. The origin of this subtlety lies in the Dirac bracket (3.14), which blows up when $T \rightarrow 0$. This was precisely the motivation for keeping the fermionic momentum π in the construction which led to the null constraints (5.6), again *not* the naive $T \rightarrow 0$ limit of the usual constraints (3.13).

Therefore, the phase-space Lagrangian density of the spinning string is given by:

$$\mathcal{L} = P \cdot \dot{X} + \dot{\psi} \cdot \pi + \lambda_{\perp} \mathcal{H}_{\perp} + \lambda_{||} \mathcal{H}_{||} + \zeta_A \mathcal{S}_A \quad (5.8)$$

Now we fix the gauge

$$\begin{aligned} P^{\mu} &= a \dot{X} + b X' \\ \pi_A &= \alpha \psi_A \end{aligned} \quad (5.9)$$

with a and α non-zero. Then

$$\begin{aligned} \mathcal{L} &= a(1 + a\lambda_{\perp})\dot{X}^2 + b(\lambda_{||} + b\lambda_{\perp})X'^2 + (b + 2ab\lambda_{\perp} + a\lambda_{||})\dot{X} \cdot X' + \alpha\dot{\psi}_A \cdot \psi'_A \\ &\quad + \alpha\lambda_{||}\psi_A \cdot \psi'_A + (1 + \alpha\epsilon)\zeta_A(a\dot{X} \cdot \psi'_A + bX' \cdot \psi_A) \end{aligned} \quad (5.10)$$

so the gauge-fixing condition (5.9) amounts to setting $\lambda_{\perp} = \frac{1}{2a}$, $\lambda_{||} = b/a$ and $\zeta_A = 0$ such that the gauge-fixed Lagrange function is:

$$\mathcal{L} = \frac{a}{2}\dot{X}^2 + \frac{b^2}{2a}X' \cdot X' + b\dot{X} \cdot X' + \alpha\dot{\psi}_A \cdot \psi'_A + \alpha\frac{b}{a}\psi_A \cdot \psi'_A \quad (5.11)$$

Therefore, choosing $a = 1$, $\alpha = 1/2$, $b = 0$, and $f = 0$, we get the Lagrange function:

$$\mathcal{L} = \frac{1}{2}(\dot{X}^2 + \psi \cdot \dot{\psi}) \quad (5.12)$$

and the constraints (with $P^{\mu} = \dot{X}^{\mu}$):

$$\begin{aligned} \mathcal{H}_{\perp} &= P^2 \\ \mathcal{H}_{||} &= P \cdot X' + \frac{1}{2}\psi \cdot \psi' \\ \mathcal{S}_A &= P \cdot \psi_A \end{aligned} \quad (5.13)$$

which is just the result obtained in [12] in the Lagrangian formalism, where this result arose after a field rescaling singular on-shell: this singularity is directly related to the singularity of the world-sheet metric. Here we have followed the constrained Hamiltonian formalism with the atypical second class constraint (5.1). In general, this constraint means that the corresponding variable is not dynamic, *i.e.*, it is a “constant of motion”.

There are two gauge choices convenient for physical applications. We choose $\zeta_A = 0$ and then either (a) $\lambda_{\perp} = -\frac{1}{2}$, $\lambda_{||} = 0$, or else (b) $\lambda_{\perp} = 0$, $\lambda_{||} = 1$. Accordingly, the Lagrangian densities are

$$\begin{aligned} \mathcal{L}^{(a)} &= \frac{1}{2}\dot{X}^2 + \dot{\psi} \cdot \psi \\ \mathcal{L}^{(b)} &= X'(\dot{X} + X') + (\dot{\psi} + \psi')\psi \end{aligned} \quad (5.14)$$

In the first case (a), the equations of motion are $\ddot{X} = 0$, $\ddot{\psi} = 0$, and there are no boundary conditions for the open string. In the second case (b), all the fields are left-movers and the open boundary conditions are the usual ones. The null gauge (a) exhibits the classical identification of the null spinning string with a set of free massless spinning particles, whereas the null chiral gauge (b) allows one to use conformal field theory methods, since all variables are left-moving (*i.e.*, holomorphic). Of course one could choose the gauge $\lambda_{||} = -1$ to get everything right-moving (anti-holomorphic). The null spinning string is obviously chiral, and one would thus expect to be able to construct all the usual “tensionful” (supersymmetric) string theories as superpositions of null (spinning) strings, as in the particle case.

VI. Quantization

Using part of the reparametrization invariance to fix the null gauge (a) above, one can expand the various dynamical variables in a Fourier series and replace the Poisson brackets by Dirac quantum brackets. The non-vanishing graded commutators are then

$$\begin{aligned} [p_m^{\mu}, x_n^{\nu}] &= -i\eta^{\mu\nu} \delta_{m+n} \\ [\psi_m^A, \pi_n^B] &= -i\eta^{\mu\nu} \epsilon^{AB} \delta_{m+n} \end{aligned} \quad (6.1)$$

The Fourier modes of the constraints are then

$$\begin{aligned} \mathcal{H}_m^{\perp} &= p_{m-k} \cdot p_k \\ \mathcal{H}_m^{\parallel} &= ikp_{m-k} \cdot x_k + i(k - \frac{m}{2})\psi_{m-k} \cdot \pi_k \\ \mathcal{S}_m^A &= p_{m-k} \cdot \psi_k^A + \epsilon p_{m-k} \cdot \pi_k^A \end{aligned} \quad (6.2)$$

and the null quantum superVirasoro algebra they satisfy has non-zero brackets

$$\begin{aligned} [\mathcal{H}_m^{\parallel}, \mathcal{H}_n^{\perp}] &= (n-m)\mathcal{H}_{m+n}^{\perp} \\ [\mathcal{H}_m^{\parallel}, \mathcal{H}_n^{\parallel}] &= (n-m)\mathcal{H}_{m+n}^{\parallel} + A_m \delta_{m+n} \\ [\mathcal{H}_m^{\perp}, \mathcal{S}_n^A] &= (n - \frac{m}{2})\mathcal{S}_{m+n}^A \\ [\mathcal{S}_m^A, \mathcal{S}_n^B] &= 2ie\epsilon^{AB}\mathcal{H}_{m+n}^{\perp} \end{aligned} \quad (6.3)$$

where A_m is the only non-vanishing anomaly in the realization of this algebra with the modes in (6.2), although there are more general anomalies consistent with the Jacobi identities [12]. Notice that there is only one Virasoro subalgebra for the closed null string.

Defining the operator ordering as is normally done, all positive-indexed operators to the right of all negative-indexed ones, and defining the vacuum to be annihilated by all positive-indexed modes (*i.e.*, by all the positive-indexed constraint modes, not by all the constraint modes), one finds [11] that the anomaly is a sum of contributions which are exactly twice the ones found in the usual string. For instance, the bosonic contribution is $\frac{D}{6}(m^3 - m)$ instead of $\frac{D}{12}(m^3 - m)$. We can now understand this, because there are twice as many independent degrees of freedom (X and P , ψ and π) as in the usual case. On the other hand, the ghost contributions are also twice what one normally has, since there are twice the number of constraints, since the distinction between left- and right-movers has now disappeared. Recall that in the usual string, $\mathcal{H}_\perp \pm 2T\mathcal{H}_\parallel$ is the single constraint in the left (right)-moving sector, whereas here both $\mathcal{H}_\perp$ and $\mathcal{H}_\parallel$ are left-moving. The same reasoning applies to the fermionic constraints and their ghosts, of course. The total anomaly is thus twice the anomaly in the usual left-moving sector of the (spinning) string and hence the critical dimension for the null (spinning) string is still 26 (10). This is in full agreement with the original analysis of [11], where naive Dirac brackets for the fermionic variables were used.

Conclusions

The construction presented in this paper puts the bosonic and fermionic variables on a more equal footing (all momenta are kept as independent dynamical variables). Let us point out that the issue of defining the phase-space Lagrangian for null strings is interesting even for the purely bosonic case. Indeed, if we naively set the tension T to zero in the usual string, then the bosonic momentum is identically zero, the constraints are trivial, and no dynamics survive. In the phase-space formalism, however, we ignore the fact that the momentum seems to be zero, and implement the Poisson brackets: a reasonable dynamical system arises. Even if we do not set the momentum to zero, we do manage to kill the second piece of $\mathcal{H}_\perp$, namely the term containing the harmonic potential X'^2 . This amounts to effecting a rescaling on X by R/T , with the dimensionful constant R of the same dimensions as T : the momentum is not proportional to T any longer, but the second term of $\mathcal{H}_\perp$ goes like T^2 , and it drops off in the limit $T \rightarrow 0$. This procedure was explicitly implemented in [1] to study the fixed angle, high- s limit of bosonic string theory. A similar trick is needed to find the reparametrization-invariant Schild action [12], where one introduces an auxiliary field with dimensions like T^{-1} . In the fermionic case, the rescaling is more

involved and, worse, it cannot be performed in a covariant way. This is ultimately due to the different dimensionalities of fermions and bosons, implying different rescalings.

Our solution to the problem of taking into account the second-class constraints of the null spinning string, namely to enlarge the phase space and to consistently generalize the constraint equations, does not depend on the particular problem we considered here. It would be interesting to see how far one could go in formalizing our approach for quite general systems. Related issues in the covariant formulation of null superstrings [15] are currently under investigation.

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