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SUPERCONFORMAL MINIMAL MODELS ON THE TORUS

T. Jayaraman

International Centre for Theoretical Physics
34014 Trieste, Italy

M.A. Namazie

Rutherford Appleton Laboratory
Chilton, Didcot, Oxon. OX11 0QX, United Kingdom

K.S. Narain

CERN - Geneva
and
International Centre for Theoretical Physics
34014 Trieste, Italy

C. Nuñez

Inst. de Astro. y Fis. del Espacio
1428 Buenos Aires, Argentina

and

M.H. Sarmadi

CERN - Geneva

ABSTRACT

Correlation functions for the discrete unitary series of $N = 1$ and $N = 2$ superconformal models on the torus are constructed using a Coulomb gas description. The monodromy and modular properties of the resulting conformal blocks are discussed.

1 Introduction

Recently, explicit expressions for correlation functions of $c < 1$ conformal models have been presented^{1,2)}. The construction uses a generalization of the Coulomb gas formalism of Dotsenko and Fateev³⁾ to the torus (and in principle a surface of arbitrary genus). These results ought perhaps to be viewed as complementing, in the sense of providing explicit realizations of, the more general 'constructive' approaches of several authors⁴⁻¹²⁾ which have already led to significant progress in our understanding of rational conformal field theory.

In the present work and its sequel we consider superconformal models, in particular the discrete unitary series for both the $N = 1$ and $N = 2$ superconformal algebras. We shall present several examples of two-point correlation functions in both the Neveu-Schwarz and Ramond sectors on the torus. The $N = 2$ case is particularly interesting in the light of the connection between such classes of conformal field theories and string propagation in Calabi-Yau manifolds^{13,14)}. The explicit form of higher genus superconformal correlation functions is likely to be a necessary ingredient for the study of string perturbation theory in these models.

More generally, these minimal model correlation functions and their modular and monodromy properties are important for a further understanding of the structure underlying two-dimensional conformal field theory *per se*.

In the next section we commence with a brief summary of the Coulomb gas description of the $N = 1$ superconformal discrete unitary series and its extension to the torus, after which we construct the conformal blocks for the primary field of type $\phi_{1,s}$. To illustrate the method we concentrate on the specific case of the Ramond field $\phi_{1,2}$ which involves a single screening contour. We go on to discuss the consistency checks that the conformal blocks should satisfy and examine some of its properties.

The analogous $N = 2$ problem is taken up in section 3. Conformal blocks corresponding to two-point functions in both the $N = S$ and R sectors are constructed, each involving at most a single screening operator. To facilitate the construction the $N = 2$ characters are rewritten in such a way as to make the 'solitonic' contributions of the bosons in the Coulomb gas picture manifest. Again various monodromy and modular properties of the resulting blocks are established, while an appendix deals specifically with the modular transformation of the lattice part of the conformal blocks.

In the sequel to this paper we turn to somewhat more involved examples of the general procedure including a two-contour problem in the $N = S$ sector of the $N = 1$ model and the twisted sector of $N = 2$ theories. In addition, we shall detail the construction of modular and monodromy invariant combinations of the conformal blocks by incorporating left and right sectors.

2 Two-Point functions for the $N = 1$ discrete unitary series on the torus.

In the present section we consider two-point functions in $N = 1$ superconformal minimal models formulated on the torus, in the Neveu-Schwarz and Ramond sectors.

In the Coulomb gas approach^{3,15,16,17)} to the unitary models in the discrete series, the central charge $\hat{c}(= 2c/3)$ is given by:

$$\hat{c} = 1 - 16\alpha_0^2 = 1 - \frac{8}{p(p+2)}, \quad p = 3, 4, \dots \quad (2.1)$$

(A fixed charge $-2\alpha_0$ resides at the point at ∞ on the extended complex plane.) The allowed conformal weights of the primary fields are correspondingly:

$$\Delta_{r,s} = \Delta_{p-r,p+1-s} = \frac{[r(p+2) - sp]^2 - 4}{8p(p+2)} + \frac{1}{32}[1 - (-)^{r-s}];$$

$$1 \leq r < p-1, \quad 1 \leq s < p+2 \quad (2.2)$$

($r-s$ is even in the $N = S$ sector, while it is odd in the R sector). Moreover, in the $N = S$ sector, the (super) primary fields consist of a supermultiplet with respect to the $N = 1$ superconformal algebra whose components $\phi_{r,s}$ and $\phi_{r,s}^S$ carry conformal weights $\Delta_{r,s}$ and $\Delta_{r,s} + 1/2$ respectively. The 'superdescendent' fields $\phi_{r,s}^S$ are however primary with respect to the Virasoro sub-algebra.

In order to compute correlation functions, the fields are represented by vertex operators of the form:

$$\begin{aligned} N_{\Delta_{r,s}} &= : e^{i\alpha_{r,s}\phi(z)} : \\ N_{\Delta_{r,s}}^S &= N_{\Delta_{r,s}+1/2} = : \psi(z) e^{i\alpha_{r,s}\phi(z)} : \\ R_{\Delta_{r,s}} &= : \sigma(z) e^{i\alpha_{r,s}\phi(z)} : \end{aligned} \quad (2.3)$$

where ϕ and ψ are a free boson and Majorana fermion. The field $\psi(z)(\Delta_\psi = 1/2)$ and spin field $\sigma(z)(\Delta_\sigma = 1/16)$ together constitute an Ising $c = 1/2$ conformal

system (more precisely, its holomorphic part) and have the usual fusion rules. The parameter $\alpha_{r,s}$ in eqns.(2.3) is related to the conformal dimension of the primary fields by:

$$\Delta_{r,s} = \alpha_{r,s}(\alpha_{r,s} - 2\alpha_0) \quad (2.4)$$

and is given explicitly as:

$$\alpha_{r,s} = \frac{1}{2}[(1-r)\alpha_+ + (1-s)\alpha_-] \quad (2.5)$$

where we have introduced the ‘charges’

$$\alpha_{\pm} = \alpha_0 \pm \sqrt{\alpha_0^2 + 1/2}, \quad \alpha_+ = \sqrt{\frac{p+2}{2p}} \quad \text{and} \quad \alpha_- = -\sqrt{\frac{p}{2(p+2)}} \quad (2.6)$$

In general, constructing non-vanishing correlation functions requires the introduction of non-local screening operators with zero conformal weight. Following ref.15), we take these to be:

$$Q_{\pm} = \oint_C dz V_{\alpha_{\pm}} = \oint_C dz \psi(z) e^{i\alpha_{\pm} \phi(z)} \quad (2.7)$$

in both the $N-S$ and R sectors (with ψ satisfying the appropriate boundary conditions in each case). A choice of integration contour has yet to be specified. We shall also need the $N=1$ superconformal characters for the theories given by eqns.(2.1) and (2.2); these are given in ref.18) as:

$$\chi_{\lambda}^{NS} = \left[\frac{\theta_3(0)}{\eta(\tau)} \right]^{\frac{1}{2}} [K_{\lambda}(\tau) - K_{\bar{\lambda}}(\tau)] \quad (2.8a)$$

$$\tilde{\chi}_{\lambda}^{NS} = \left[\frac{\theta_4(0)}{\eta(\tau)} \right]^{\frac{1}{2}} [K_{\lambda}(\tau+1) - K_{\bar{\lambda}}(\tau+1)] \quad (2.8b)$$

$$\chi_{\lambda}^R = \left[\frac{\theta_2(0)}{\eta(\tau)} \right]^{\frac{1}{2}} [K_{\lambda}(\tau) - K_{\bar{\lambda}}(\tau)] \quad (2.8c)$$

where $\theta_{\nu} = \theta[\frac{\nu}{2}]$ are the usual Jacobi θ -functions, ν labelling the spin structure with $(a,b) = (1/2, 1/2), (1/2, 0), (0, 0)$ and $(0, 1/2)$ corresponding to $\nu = 1, 2, 3$ and 4 respectively. The one-dimensional lattice difference contribution is:

$$K_{\lambda}(\tau) = \frac{1}{\eta(\tau)} \sum_{n=-\infty}^{\infty} q^{\frac{(nN+\lambda)^2}{4N}} \quad (2.9)$$

where $\lambda = r(p+2) - sp, \bar{\lambda} = -r(p+2) - sp, N = 2p(p+2)$, $q = e^{2\pi i \tau}$ and (r,s) labels the primary block $[\phi_{r,s}]$ in which the trace is evaluated. Of course,

the remaining character $\tilde{\chi}_{\lambda}^R(\sim \theta_1(0)^{1/2})$ vanishes since each Ramond state, being doubly degenerate, contributes to it twice but with opposite sign. However, two-point functions are in general non-zero in the $\nu = 1$ sector. To take account of this here, for purposes of exposition, we formally define

$$\tilde{\chi}_{\lambda}^R = \lim_{\delta \rightarrow 0} \left[\frac{\theta_1(\delta)}{\eta(\tau)} \right]^{\frac{1}{2}} [\tilde{K}_{\lambda}(\tau) - \tilde{K}_{\bar{\lambda}}(\tau)] \quad (2.8d)$$

where $\tilde{K}_{\lambda}(\tau) \sim 1/\eta(\tau) \sum_{n=-\infty}^{\infty} (-)^{np} q^{(nN+\lambda)^2/4N}$. While this may appear to be somewhat *ad hoc*, it should be emphasized that it can be justified by using the fact that under a-cycle monodromy, the conformal block in the $\nu = 2$ sector transforms into the $\nu = 1$ conformal block and vice versa (see below).

This brief introduction aimed mainly at defining notation and conventions. In the following we shall present torus two-point functions involving fields $\phi_{1,a}$ and consider various consistency checks that they should satisfy. In particular we shall concentrate on the fields $\phi_{1,2}$, $\phi_{1,3}$ and its superdescendent $\phi_{1,3}^S$. For the case $p = 3, c = 7/10$ viz. the Tricritical Ising Model, $\phi_{1,2}$ is the leading magnetic spin operator whereas $\phi_{1,3}$ is the energy operator; $\phi_{1,3}^S$ describes the vacancy density.

2.1 The two-point function $\langle \phi_{1,2} \phi_{1,2} \rangle$

For the sake of simplicity we shall first consider the correlation function $\langle \phi_{1,2} \phi_{1,2} \rangle$ since this is a one-contour problem. Correlation functions for the Ising spin fields have been constructed in ref.19) both on the plane and torus. Here we use some of the results of these authors to construct the correlators for Ramond primary fields on the torus. The lattice contribution to this correlation function is an immediate generalization of the results of ref.1) on torus correlation functions for minimal conformal models.

The conformal blocks corresponding to the correlator $\langle \phi_{1,2}(z_1) \phi_{1,2}(z_2) \rangle$ are given by the expression:

$$G_i(r,s) = \oint_{C_i} dz < \sigma(z_1) \sigma(z_2) \psi(z) > < e^{-i\frac{\alpha_-}{2}\phi(z_1)} e^{-i\frac{\alpha_-}{2}\phi(z_2)} e^{i\alpha_+ \phi(z)} > \quad (2.10)$$

where the basis of closed contours C_i will be specified below.

Consider first the fermionic part of eqn.(2.10). The ‘holomorphic square root’ of the correlation function of $2n$ Ising spin fields on the torus given in ref.18)

has the form:

$$< \sigma(z_1) \sigma(z_2) \dots \sigma(z_{2n}) > \sim \left[\sum_{\substack{\epsilon_i = \pm 1 \\ \sum \epsilon_i = 0}} \theta_\nu \left(\frac{\sum \epsilon_i z_i}{2} \right) \prod_{i < j} \theta_1(z_i - z_j)^{\frac{\epsilon_i \epsilon_j}{2}} \right]^{\frac{1}{2}} \quad (2.11)$$

The relative signs of the various terms on the R.H.S. can in principle be fixed by a consideration of its monodromy properties, but this is not essential for our present purpose. Specializing to the case $n = 2$, i.e. a correlation function of four σ 's, one may take the limit $z_1 \rightarrow z_2$. From the Ising model fusion rules one expects two intermediate states: the identity channel and ψ (see fig 1). Setting $z_1 = z + \frac{\epsilon}{2}$, $z_2 = z - \frac{\epsilon}{2}$ and Taylor expanding, one may isolate the residue of the $\epsilon^{+\frac{1}{2}}$ singularity in the expansion. This clearly corresponds to the three-point function $< \psi(z) \sigma(z_3) \sigma(z_4) >$.

In this way one obtains the expression:

$$\begin{aligned} A_\nu &\equiv < \sigma(z_1) \sigma(z_2) \psi(z) >_\nu \\ &= \left[\frac{\theta_1(z_1 - z_2)}{\theta_1(0)} \right]^{\frac{1}{2}} \left[\frac{\theta_\nu(\frac{z_1 - z_2}{2})}{\theta_\nu(0)} \right]^{-\frac{1}{2}} \left[\frac{\theta_1(z_1 - z)}{\theta_1(0)} \right]^{-\frac{1}{2}} \left[\frac{\theta_1(z_1 - z_2)}{\theta_1(0)} \right]^{-\frac{1}{2}} \frac{\theta_\nu(z - \frac{z_1 + z_2}{2})}{\theta_\nu(0)} \end{aligned} \quad (2.12)$$

in each of the sectors $\nu = 1, 2, 3$ or 4 (in the first case $\theta_\nu(0)$ should be replaced by $\theta_1(\delta)$).¹ One may verify that eqn.(2.12) is both modular and monodromy covariant and consistent with the usual short-distance expansions:

$$\begin{aligned} \psi(z) \sigma(z') &\sim \frac{1}{(z - z')^{\frac{1}{2}}} \sigma(z') + \dots \\ \psi(z) \psi(z') &\sim \frac{1}{z - z'} \mathbf{1} + \dots \\ \sigma(z) \sigma(z') &\sim \frac{1}{(z - z')^{\frac{1}{2}}} + (z - z')^{\frac{1}{2}} \psi(z') + \dots \end{aligned} \quad (2.13)$$

(Note that we are working with a diagonal combination $\sigma = \frac{\sigma^+ + \sigma^-}{\sqrt{2}}$, where $\sigma^\pm$ are the spin fields that create the degenerate Ramond states from the vacuum.)

Further, it can be checked that for $\nu = 1, 2$, upon degeneration of the torus and an additional conformal transformation of the co-ordinates on the plane $x \rightarrow \frac{z-1}{z}$ and $y \rightarrow \frac{z-1}{y}$, the (unnormalized) five-point function $< \sigma(0) \sigma(1) \sigma(x) \sigma(\infty) \psi(y) >$ obtained in ref.16) by the null vector method is correctly reproduced.

¹As we shall see, although A_1 is not well defined as $\delta \rightarrow 0$, the product of A_1 with the character is, in the sense of being independent of the regulator δ .

It is also clear that one may obtain in an analogous fashion the correlation function of an arbitrary number of ψ 's and an even number of σ 's on the torus by repeated factorization of the $2n$ -point function involving only σ 's.

To return to the conformal block eqn.(2.10), the remaining bosonic contribution is similar to that computed in ref.1) apart from the spin structure dependence in the character formulas eqns.(2.8). There is a contribution arising from the Wick contraction of the exponential factors using the free scalar propagator on the torus. In addition there is a 'solitonic' contribution resulting in a lattice sum in which the charge α_- couples to the punctures on the torus. Putting everything together, the expression for the conformal block $G(r, s)$ is:

$$G_i^\nu(r, s) = \oint_{C_i} dz A_\nu \left[\frac{\theta_1(z_1 - z_2)}{\theta_1(0)} \right]^{\frac{\alpha_-^2}{2}} \left[\frac{\theta_1(z_1 - z) \theta_1(z - z_2)}{\theta_1(0)^2} \right]^{-\alpha_-^2} \left[\Gamma_\lambda - \Gamma_\lambda \right] \quad (2.14)$$

where²

$$\Gamma_\lambda = \frac{\theta_\nu(0)^{\frac{1}{2}}}{\eta(\tau)^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} q^{\frac{(nN+\lambda)^2}{4N}} e^{2\pi i \frac{(nN+\lambda)}{N} (-\frac{\alpha_-^2}{2} \epsilon_1 - \frac{\alpha_-^2}{2} \epsilon_2 + \alpha_- z)} \quad (2.15)$$

The range of (r, s) in this expression is $0 \leq s \leq r \leq p - 1$ in the $N - S$ sector ($\nu = 3, 4$) while in the R sector ($\nu = 1, 2$) it is $0 \leq s \leq r - 1$ for $0 \leq r \leq \left[\frac{p-1}{2} \right]$ and $0 \leq s \leq r + 1$ for $\left[\frac{p+1}{2} \right] \leq r \leq p - 1$. Thus each primary block $\{\phi_{r,s}\}$ in the trace is counted once only. Allowing r or s to be zero implies, as in ref.1), that some of the conformal blocks must be redundant and one has to show that these in fact vanish upon integration.

We now turn to a discussion of the choice of integration contours in eqn.(2.14). To establish what these should be, it is helpful to consider the $q \rightarrow 0$ limit in which the torus degenerates into a twice-punctured sphere. In the present case then, one expects $G(r, s)$ to degenerate into a four-point function on the plane:

$$< V_{\alpha_0 - \beta}(0) R_{1,2}(x_1) R_{1,2}(x_2) V_\beta(\infty) > \quad (2.16)$$

V_β and the conjugate vertex operator $V_{2\alpha_0 - \beta}$ can be $N_\Delta, N_{\Delta+1/2}$ or R_Δ depending on which combination of spin structures in eqn.(2.14) one takes (see below).

After transforming co-ordinates from the torus to the plane viz. $z_i = \frac{\ln z_i}{2\pi i}$, one obtains the following behaviour in a q -expansion for the various factors in

²When $\nu = 4$, there is a phase $e^{2\pi i \frac{(nN+\lambda)^2}{4N}} \dots \frac{1}{N!}$ to be included which arises from $K_\lambda(\tau + 1)$. For $\nu = 1$, the 'regulated' expressions defined previously are to be used instead.

$G(r, s)$:

$$\theta_1(z_1 - z_2) \rightarrow q^{\frac{1}{2}} \left(\frac{x_1 \pm x_2}{\sqrt{x_1 x_2}} \right), \quad \theta_2(z_1 - z_2) \rightarrow 1 \pm q^{\frac{1}{2}} \left(\frac{x_1^2 + x_2^2}{x_1 x_2} \right)$$

etc. The lattice contribution in this limit becomes:

$$\Gamma_\lambda - \Gamma_\lambda \rightarrow q^{\Delta_{r,s} - \frac{1}{2}} [1 + (\delta_{v,3} + \delta_{v,4}) e^{i\pi\delta_{v,4}} q^{\frac{1}{2}}] \left[\left(\frac{x}{\sqrt{x_1 x_2}} \right)^{r+s+\alpha_-} - e^{i\pi(r\delta_{v,1} + s\delta_{v,4})} q^{\frac{r}{2}} \left(\frac{x}{\sqrt{x_1 x_2}} \right)^{(-r\alpha_1 + s\alpha_-)\alpha_-} \right] \quad (2.17)$$

where ³ $\nu = 1, 2, 3$ or 4 and terms of $O(q)$ beyond the leading order contribution in each case are being neglected.

Lastly, the fermionic part of $G(r, s)$, viz. the three point function A_v , becomes in this limit:

$$A_2 \rightarrow \left(\frac{\lim_{\delta \rightarrow 0} q^{\frac{1}{2}} \theta_1^{-\frac{1}{2}}(\delta)}{1} \right) \frac{(x_1 - x_2)^{\frac{1}{2}}}{(x_1 - x)^{\frac{1}{2}} (x - x_2)^{\frac{1}{2}}} \frac{1}{(x_1 x_2)^{\frac{1}{2}}} \frac{(x \mp \sqrt{x_1 x_2})}{(\sqrt{x_1} \mp \sqrt{x_2})^{\frac{1}{2}}} \\ A_4 \rightarrow \frac{(x_1 - x_2)^{\frac{1}{2}}}{(x_1 - x)^{\frac{1}{2}} (x - x_2)^{\frac{1}{2}}} \frac{(x_1 x_2)^{\frac{1}{2}} x^{\frac{1}{2}}}{(x_1 x_2)^{\frac{1}{2}} x^{\frac{1}{2}}} \left[1 \pm \frac{1}{2} q^{\frac{1}{2}} \frac{(x_1 - x)(x_2 - x) + (x - \sqrt{x_1 x_2})^2}{x \sqrt{x_1 x_2}} \right]$$

It is clear from these formulae that by taking linear combinations $G^1 \pm G^2$ one has only Ramond states propagating around the torus, while $G^3 \pm G^4$ has only Neveu-Schwarz states. In the latter case, one has in particular for the $G^3 + G^4$ combination only $N - S$ primary fields $\phi_{r,s}$ surviving in the trace; on the other hand $G^3 - G^4$ projects onto superdescendants $\phi_{r,s}^S$ since only half-integer powers of q occur in the expansion.

For instance, from (2.14) and the subsequent eqns., one obtains for the combination $G^3 + G^4$ (after incorporating a Jacobian factor of $(\frac{\partial x_1}{\partial x_1} \frac{\partial x_2}{\partial x_2})^{\Delta_{1,2}}$ for the transformation from torus to plane co-ordinates of the external lines):

$$G_i^3(r, s) + G_i^4(r, s) \rightarrow q^{\Delta_{r,s} - \frac{1}{2}} (x_1 - x_2)^{\frac{1}{2}} (x_1 - x)^{\frac{1}{2}} \oint_{C_1} dx x^a (x_1 - x)^b (x - x_2)^c \quad (2.18)$$

³When $\nu = 1, 2$ $\Delta_{r,s}$ is given by the formula eqn.(2.2) with $r - s$ odd; whereas for $\nu = 3, 4$ $r - s$ is to be taken to be even. Also, for $\nu = 1$ there is a prefactor of $\lim_{\delta \rightarrow 0} q^{-1/16} \theta_1(\delta)^{1/2}$ to be included.

⁴Equations (2.18), (2.19) and (2.20) are for generic (r, s) ; the cases $(r, s) = 0$ or 1 need to be discussed separately.

where

$$a(r, s) = [(r+1)\alpha_+ + (s+1)\alpha_-]\alpha_-$$

and

$$b = c = -\alpha_-^2 - \frac{1}{2}$$

Comparing this with $< N_{\Delta_{2\alpha_0 - \alpha_r, s}}(0) R_{1,2}(x_1) R_{1,2}(x_2) N_{\Delta_{\alpha_r, s}}(\infty) >$, it is evident that they agree provided that $\alpha_{r,s}$ is given as expected by eqn. (2.5). (To effect the comparison, the correlation function $< \sigma(x_1) \sigma(x_2) \psi(x) >$ is required; this can readily be obtained using the short-distance expansions eqns.(2.13) and noting that $< \psi > = 0$ on the sphere).

The other combinations are a little more complicated but have essentially the same integral structure ⁵:

$$G_i^3(r, s) - G_i^4(r, s) \rightarrow q^{\Delta_{r,s} + \frac{1}{2} - \frac{1}{2}\alpha_-} (x_1 x_2)^{-\frac{1}{2} - \frac{1}{2}\alpha_-} (x_1 - x_2)^{\frac{1}{2} + \frac{\alpha_-^2}{2}} \oint_{C_1} dx x^a (x_1 - x)^b (x - x_2)^c [(x_1 - x)(x_2 - x) + x^2 + x_1 x_2] \quad (2.19)$$

For $G^1 + G^2$ one has the expression

$$G_i^1(r, s) + G_i^2(r, s) \rightarrow q^{\Delta_{r,s} - \frac{1}{2}\alpha_-} (x_1 x_2)^{-\frac{1}{2} - \frac{1}{2}\alpha_-} (x_1 - x_2)^{\frac{1}{2} + \frac{\alpha_-^2}{2}} \oint_{C_1} dx x^a (x_1 - x)^b (x - x_2)^c \left[\frac{x - \sqrt{x_1 x_2}}{\sqrt{x_1} - \sqrt{x_2}} + \frac{x + \sqrt{x_1 x_2}}{\sqrt{x_1} + \sqrt{x_2}} \right] \quad (2.20)$$

Equation (2.19) should (and indeed does) agree with the four-point function

$$< N_{\Delta_{2\alpha_0 - \alpha_r, s}} + \frac{1}{2}(0) R_{1,2}(x_1) R_{1,2}(x_2) N_{\Delta_{\alpha_r, s} + \frac{1}{2}}(\infty) >$$

on the sphere as one may straightforwardly verify, again using the operator product relations (2.13).

The contours C_i in the last three equations should be such that the integrals satisfy the appropriate hypergeometric differential equation. For arbitrary values of a, b and c this will be true provided one takes C_1 to be the Pochhammer contour⁶ encircling 0 and x_1 (fig.2) and C_2 similarly encircling x_2 and ∞ . Alternatively, as in ref.3), by analytically continuing in the parameters a, b and c

⁵For the moment we suppress relative constants in these combinations; they will eventually be fixed by considerations of monodromy.

⁶The essential point is that the contour circumscribes each singular point twice in opposite senses such that monodromy phases cancel. See for instance ref.20).

so that the resulting integrals are convergent along the real line, one may take (on the extended complex plane) C_1 to be $f_0^{z_1}$ and C_2 as $f_{z_2}^\infty$.

On the torus it is more natural to work with closed contours. Representing the torus as an annulus, C_1 is taken to wind around z_1 and the inner radius, and C_2 around z_2 and the outer radius such that upon degeneration one recovers the Pochhammer contours encircling the origin and x_1 , and x_2 and ∞ respectively. C_1 and C_2 are an s -channel basis of contours¹⁾.

To discuss factorization of the two-point functions it is useful to introduce a t -channel basis of closed contours as well: $\tilde{C}_1$ and $\tilde{C}_2$. These wind around z_1 and z_2 , and around the inner and outer radii of the annulus respectively.

Returning to eqn. (2.20), obtained by degenerating the torus when the trace is taken over Ramond states, as a further check one can reproduce the four-point function of the leading magnetic spin operator in the tricritical Ising model, i.e. $p = 3$. Setting $x_1 = 1$ and performing the additional conformal transformations $x_2 = \frac{z-1}{z-1}$, $x = \frac{v}{v-1}$ and taking into account the corresponding Jacobians, one obtains the result:

$$\begin{aligned} & \langle R_{\frac{1}{2}}(0) R_{\frac{1}{2}}(1) R_{\frac{1}{2}}(z) R_{\frac{1}{2}}(\infty) \rangle \\ & \sim z^{\frac{11}{10}} (z-1)^{-\frac{3}{10}} \left[\sqrt{1 + \sqrt{1 - \frac{1}{z}}} \sum_{i=1}^2 A_i L_i \right. \\ & \quad \left. + \sqrt{1 - \sqrt{1 - \frac{1}{z}}} \sum_{i=1}^2 B_i \tilde{L}_i \right] \end{aligned}$$

where the constants A_i and B_i (to be fixed by monodromy) have been reinstated. Here

$$\begin{aligned} & \stackrel{(-)}{L}_1 = \sqrt{z} (1-z)^{\frac{3}{5}} B\left(\frac{7}{5}, \frac{1}{5}\right) F\left(\frac{4}{5}, \frac{1}{5}, \frac{7}{5}, \frac{8}{5}; 1-z\right) \\ & \stackrel{(+)}{=} \sqrt{z-1} (1-z)^{-\frac{2}{5}} B\left(\frac{2}{5}, \frac{1}{5}\right) F\left(-\frac{1}{5}, \frac{2}{5}, \frac{3}{5}; 1-z\right) \\ & \stackrel{(-)}{L}_2 = \sqrt{z} B\left(\frac{1}{5}, \frac{1}{5}\right) F\left(\frac{4}{5}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}; 1-z\right) \\ & \stackrel{(+)}{=} \sqrt{z-1} B\left(\frac{1}{5}, \frac{6}{5}\right) F\left(\frac{4}{5}, \frac{1}{5}, \frac{7}{5}, \frac{1}{5}; 1-z\right) \end{aligned}$$

and

$B(a, b)$ and $F(a, b, c; 1-z)$ are the usual beta and hypergeometric functions (in this case canonical about the point $z = 1$ rather than 0). This expression may be compared with that given in the second reference of ref.16).

We now turn to a discussion of the cases $r, s = 0$ or 1 and the number of conformal blocks. It is obvious from eqn. (2.17) that for $r = 0$ the corresponding conformal blocks $G(0, s)$ vanish. For $s = 0$, r arbitrary the important point is that, at leading order in q , there are now two competing contributions in the lattice difference in each of the sectors. Consider for instance the $G_3 + G_4$ combination; setting $x_1 = 1$ and $x_2 = y$ one has:

$$G_3^2(r, 0) + G_4^2(r, 0) \sim \oint_{C_{i=1,2}} dx x^{a(0,0)} (1-x)^b (x-y)^c \left[\left(\frac{x}{\sqrt{y}} \right)^{-\frac{1}{2}} - \left(\frac{x}{\sqrt{y}} \right)^{+\frac{1}{2}} \right]$$

On changing variables $x \rightarrow y/x$ in the first term, it is seen that its integrand is in fact equal to that of the second term; but the contour is interchanged, $C_1 \leftrightarrow C_2$, under the transformation. However noting that r is even, one sees that (modulo integers) the branch at the origin is equal and opposite to that at y , whereas the branch at 1 is the negative of that at ∞ . Therefore in this situation we can take the contours to be those illustrated in fig.3. It is then clear that C_1 can be deformed into C_2 . Hence the two terms in fact cancel and the corresponding blocks vanish. The situation for $s = 0$, r arbitrary in the $G^3 - G^4$ and $G^1 \pm G^2$ combinations is exactly similar. The preceding argument concerns of course only the leading terms in a q -expansion. However the same conclusion can be arrived at directly on the torus by performing the change of variables $z = z_1 + z_2 - u$ under which the lattice difference terms transform into each other (for $s = 0$). The $A_\nu(z_1, z_2, z)$ part of the conformal block is invariant for $\nu = 2, 3, 4$ and goes to minus itself when⁷ $\nu = 1$. However the contours C_1 and C_2 are interchanged but, as has just been mentioned, they can be continuously deformed into one another, resulting in the blocks vanishing.

Consider next the $s = 1$, r arbitrary conformal blocks. For the $G^3 + G^4$ and $G^1 \pm G^2$ combinations one may easily verify (using the fact that r must be odd in the first case and even in the second) that the respective integrals have no branch or pole at ∞ . Hence the integrals around the C_2 contour vanish and only the blocks corresponding to C_1 survive. The same is true for $G^3(r, 1) - G^4(r, 1)$ unless r is also equal to one. However in the latter case, as can be seen from eqn (2.17), there is an additional contribution of $O(q^{1/2})$ from the lattice part to

⁷This minus sign is compensated for by an extra minus in Γ_λ when $\nu = 1$ and r odd (see eqn. 2.17): to see this one can perform an α -cycle monodromy transformation on the $\nu = 2$ conformal block, keeping track of the phases that arise.

be taken account of. The resulting expression is:

$$G_i^3(1, 1) - G_i^4(1, 1) \sim \oint_{C_i} dx x^{\alpha(1,1)-1} (1-x)^b (x-y)^c [y(1-x) - (x-y)]$$

On changing variables $x \rightarrow y/x$, one may check using for instance the series expansion of the hypergeometric function that this vanishes both for C_1 and C_2 (in the latter case this can be seen more directly by noting that there is no branch or pole at the origin). Incidentally, this result should be viewed also as a consistency check since $\phi_{1,1}^S$ is in fact a null state whose contribution to the conformal blocks must cancel.

Putting these results together, one may compute the surviving number of conformal blocks. When the trace is taken over $N - S$ primaries (i.e. $G^3 + G^4$) this is

$$\frac{1}{2}[p(2p-9) + 15 + \frac{1}{2}(1 + (-)^p)]$$

which coincides with the number expected from the fusion rule:

$$N_{r,s} \times R_{1,2} \sim R_{r,s-1} + R_{r,s+1}. \quad (2.21)$$

For superdescendants the number is obviously one less than the previous case, which is in agreement with

$$N_{r,s}^S \times R_{1,2} \sim R_{r,s-1} + R_{r,s+1}$$

(where r and s should not both be equal to one).

The counting when the trace is over the R -states is somewhat more intricate, the result being

$$\frac{p}{2}(p-1) + \frac{1}{2}(1 + (-)^p).$$

This is what would be deduced from the fusion rule

$$R_{r,s} \times R_{1,2} \sim (N_{r,s-1}) + (N_{r,s+1}) \quad (2.22)$$

where the brackets on the right-hand side denote that the full supermultiplet i.e. both primaries and superdescendants occur.

Let us examine the properties of the conformal blocks under the monodromy transformation $z_1 \rightarrow z_1 + 1$, or in other words, as z_1 is transported once around the a -cycle. Consider the $G^3 \pm G^4$ combinations first. The relevant phase can be isolated by shifting⁸ z_1 in the various factors of eqn. 2.14. Since A_3 and A_4

⁸When the contour encircles z_1 (i.e. C_1) one has in addition to shift z to the second sheet viz. $z \rightarrow z + 1$.

are interchanged under $z_1 \rightarrow z_1 + 1$, one obtains:

$$G_{\frac{3}{2}}^{\nu=3}(r, s) + G_{\frac{1}{2}}^{\nu=4}(r, s) \longrightarrow e^{2\pi i(-\Delta_{r,s}^{N,S} + \Delta_{r,s\pm 1}^R)} [G_{\frac{1}{2}}^{\nu=3}(r, s) + G_{\frac{3}{2}}^{\nu=4}(r, s)] \quad (2.23)$$

and

$$G_{\frac{1}{2}}^{\nu=3}(r, s) - G_{\frac{3}{2}}^{\nu=4}(r, s) \longrightarrow e^{2\pi i(-(\Delta_{r,s}^{N,S} + \frac{1}{2}) + \Delta_{r,s\pm 1}^R)} [G_{\frac{1}{2}}^{\nu=3}(r, s) - G_{\frac{3}{2}}^{\nu=4}(r, s)] \quad (2.24)$$

Thus it is clear that the contour C_1 corresponds to the propagation of the intermediate state $R_{r,s+1}$, whereas C_2 gives the conformal block with $R_{r,s-1}$ as intermediate state (fig.4). One observes from eqn. 2.24 that within the sectors $\nu = 3$ and 4, $G^3 \pm G^4$ are 'eigenstates' of the s channel monodromy operator¹⁾ $\phi^s(a)_{(rr')(ss')}$.

The monodromy phases for $\nu = 1$ and 2 result in the following transformations under $\phi^s(a)$:

$$G_{\frac{1}{2}}^{\nu=1}(r, s) \pm e^{-i\frac{\pi}{2}} G_{\frac{3}{2}}^{\nu=2}(r, s) \longrightarrow e^{2\pi i(-\Delta_{r,s}^R + \Delta_{r,s\pm 1}^{N-S})} [G_{\frac{1}{2}}^{\nu=1}(r, s) \pm e^{-i\frac{\pi}{2}} G_{\frac{3}{2}}^{\nu=2}(r, s)] \quad (2.25)$$

$$G_{\frac{3}{2}}^{\nu=1}(r, s) \mp e^{-i\frac{\pi}{2}} G_{\frac{1}{2}}^{\nu=2}(r, s) \longrightarrow e^{2\pi i(-\Delta_{r,s}^R + \Delta_{r,s\pm 1}^{N-S} + \frac{1}{2})} [G_{\frac{3}{2}}^{\nu=1}(r, s) \mp e^{-i\frac{\pi}{2}} G_{\frac{1}{2}}^{\nu=2}(r, s)] \quad (2.26)$$

which again serves to associate the respective intermediate states with the choice of contour C_1 or C_2 .

The conformal blocks also have the correct properties under b -cycle monodromy i.e. $z_1 \rightarrow z_1 + \tau$. One finds that the primary fields $\phi_{r,s}$ in the torus trace and the intermediate states $\phi_{r,s\pm 1}$ are interchanged by the transformation. We postpone details to the sequel where monodromy and modular invariant combinations of the conformal blocks are constructed.

2.2 Factorization limit of $< \phi_{1,2} \phi_{1,2} >$

As another check on our two-point function, it is instructive to examine the limit $z_1 \rightarrow z_2$ in which one expects the conformal block $G(r, s)$ to factorize. To do this it is convenient to work instead in a t -channel basis of contours $\tilde{C}_1$ and $\tilde{C}_2$ which have been defined previously. Denote the blocks in this basis as

$$\tilde{G}_i^{\nu}(r, s) = \oint_{\tilde{C}_i} I(r, s)$$

where the integrand $I(r, s)$ is that of eqns. (2.14) and (2.15) and as always ν labels the spin structure or sector. In particular one would like to check that the intermediate states that occur in the factorized conformal block are indeed only those dictated by the fusion rules (see fig.5).

Consider the case of even spin structure ($\nu = 2, 3, 4$) first and take the contour to encircle z_1 and z_2 (i.e. $\tilde{C}_1$). To isolate the singularities corresponding to the various intermediate channels, perform the change of variables $z_1 - z_2 = \epsilon$, $z - z_2 = \epsilon u$ and $z_1 - z = \epsilon(1 - u)$. It is straightforward to obtain in the limit $\epsilon \rightarrow 0$:

$$\tilde{G}_1(r, s) \sim \epsilon^{-\Delta_{1,2}} \int_0^1 du \{u(1-u)\}^b \left[\frac{\theta_\epsilon(0)}{\eta(\tau)} \right]^{\frac{1}{2}} [K_\lambda - K_\lambda].$$

Thus one sees that the intermediate state is the identity channel and the residue simply the even characters (up to a z and moduli independent factor).

If however $\nu = 1$, then one obtains a pole $\epsilon^{2\Delta_{1,2}+1/2}$ with residue

$$\int_0^1 du u^b (1-u)^b (2u-1) = 0.$$

This is of course consistent with the fact that the intermediate state in this case is a null state as mentioned before. Keeping terms of next order in ϵ , one finds now a singularity⁹ $\epsilon^{-2\Delta_{1,2}+3/2}$, its residue being the expectation value on the torus of the supercurrent $< G_{-\frac{3}{2}} >$.

The factorized form of $G(r, s)$ when the contour is $\tilde{C}_2$ may be obtained by the change of variables: $z_1 - z_2 = \epsilon$, $z - z_2 = u$ and $z_1 - z = \epsilon - u$. For $\nu = 2, 3$, or 4 one obtains (in the $\epsilon \rightarrow 0$ limit):

$$\tilde{G}_2(r, s) \sim \epsilon^{2\Delta_{1,2}+\Delta_{1,3}+\frac{1}{2}} \cdot \oint_{\tilde{C}_2} du \frac{\theta'_1(0)\theta_\nu(u)}{\theta_1(u)\theta_\nu(0)} \left[\frac{\theta_1(u)}{\theta'_1(0)} \right]^{2ha_2} [\Gamma_\lambda - \Gamma_\lambda]$$

Thus the intermediate state is the superdescendent $\phi_{1,3}^S$, and recognising the first factor in the integrand as the spin- $\frac{1}{2}$ propagator for even spin structure on the torus viz. $< \psi(0)\psi(u) >$, it is easy to verify that the residue is in fact $< \phi_{1,3}^S(0) > = < N_{\Delta_{1,3}+1/2}(0) Q_- >$ on the torus with even spin structure (fig. 6). Similarly, for $\nu = 1$ instead, one obtains a pole at $\epsilon^{-2\Delta_{1,2}+\Delta_{1,3}}$ with residue $< \phi_{1,3}(0) >$.

⁹For the contour $\tilde{C}_1$, $\epsilon \rightarrow 0$ corresponds physically to shrinking the screening contour to a point. Thus there is no insertion of ψ (from the screening operator $Q_\pm$) remaining on the torus. Hence for $\nu = 1$, one will always get 1-point functions of fermionic states.

We next turn our attention to the possibly more interesting case of $N = 2$ superconformal minimal models. Again, we shall be considering some specific examples. Since essentially no further questions of principle involving the Coulomb gas approach enter, wherever possible we simply state results and fill in details only when new features peculiar to $N = 2$ arise.

3 $N = 2$ Superconformal Minimal Theories

In this section we discuss the construction of correlation functions on the torus for the $c \leq 3$ unitary discrete series of the $N = 2$ superconformal algebra. The algebra²¹⁾ is generated by the energy momentum tensor $T(z)$ and its superpartners $G(z)$, $\tilde{G}(z)$ and the $U(1)$ current $J(z)$ satisfying the following operator product expansions:

$$\begin{aligned} T(z)T(z') &\sim \frac{c/2}{(z-z')^4} + \frac{2T(z')}{(z-z')^3} + \frac{T'(z')}{(z-z')^2} \\ T(z)G(z') &\sim \frac{3/2 G(z')}{(z-z')^3} + \frac{G'(z')}{(z-z')^2} \\ T(z)\tilde{G}(z') &\sim \frac{3/2 \tilde{G}(z')}{(z-z')^3} + \frac{\tilde{G}'(z')}{(z-z')^2} \\ T(z)J(z') &\sim \frac{J(z')}{(z-z')^3} + \frac{J'(z')}{(z-z')^2} \\ G(z)\tilde{G}(z') &\sim \frac{2/3 c}{(z-z')^3} + \frac{4J(z')}{(z-z')^2} + \frac{2(T(z') + J'(z'))}{(z-z')} \\ J(z)J(z') &\sim \frac{c/12}{(z-z')^2} \\ J(z)G(z') &\sim \frac{1/2 G(z')}{(z-z')} \\ J(z)\tilde{G}(z') &\sim \frac{-1/2 \tilde{G}(z')}{(z-z')} \end{aligned} \quad (3.1)$$

As is clear from these expressions, the generator of $U(1)$ transformations $J(z)$ has conformal weight 1, $G(z)$ and $\tilde{G}(z)$ have conformal weight $3/2$ and $U(1)$ charge $q = 1/2$ and $q = -1/2$ respectively. When discussing representations of this algebra, there are three sectors which one has to consider: in the Ramond sector all the generators are periodic (on the cylinder $w = \ln z$), in the Neveu-Schwarz sector G and $\tilde{G}$ are anti-periodic, while in the twisted sector $G^1 = \frac{1}{2}(G + \tilde{G})$ is periodic whereas $G^2 = \frac{1}{2}(G - \tilde{G})$ and J are anti-periodic. It was shown by Boucher, Friedan and Kent²²⁾ that the $N = 2$ superconformal algebra for $c \leq 3$

has unitary representations given by the series

$$\tilde{c} = \frac{1}{3}c = 1 - \frac{2}{p} ; \quad p = 3, 4, \dots \quad (3.2)$$

The set of primary fields in each of the three sectors have correspondingly the following conformal dimensions and $U(1)$ charges.

Neveu - Schwarz sector:

$$\begin{aligned} \Delta_{j,k} &= \frac{j^k - 1/4}{p} & j, k \in \mathbb{Z} + \frac{1}{2} \\ 0 < j, k, j+k &\leq p-1 \end{aligned} \quad (3.3a)$$

Ramond sector¹⁰:

$$\begin{aligned} \Delta_{j,k} &= \frac{j^k - 1/4}{p} + \frac{1}{8} & j, k \in \mathbb{Z} \\ 0 \leq j-1, k, j+k &\leq p-1 \end{aligned} \quad (3.3b)$$

$$q_{\pm}(j, k) \mp \frac{1}{4}, \text{ with } q_{\pm}(j, k) = \pm \frac{j-k}{2p}$$

Twisted sector:

$$\begin{aligned} \Delta_r &= \frac{(p-2r)^2 - 4}{16p} + \frac{1}{8} & r \in \mathbb{Z} \\ 1 \leq r &\leq \frac{p}{2}, \end{aligned} \quad (3.3c)$$

while the $U(1)$ quantum number is not defined.

The Coulomb gas description^{16,23)} of $N = 2$ superconformal models is in terms of a pair of bosons $\phi_1(z)$ and $\phi_2(z)$ and a complex fermion $\psi(z)$, in the presence of fixed charges at ∞ . In terms of these fields, the generators of the algebra are given by:

$$\begin{aligned} T(z) &= : -\partial\phi\partial\bar{\phi} + \frac{1}{2}(\bar{\psi}\partial\psi + \psi\partial\bar{\psi}) + i\alpha_0\partial^2\phi + i\alpha_0\partial^2\bar{\phi} : \\ G(z) &= : \sqrt{2}\psi\partial\phi + 2\sqrt{2}i\alpha_0\partial\psi : \\ \bar{G}(z) &= : \sqrt{2}\bar{\psi}\partial\bar{\phi} + 2\sqrt{2}i\alpha_0\partial\bar{\psi} : \\ J(z) &= : \frac{1}{2}\bar{\psi}\psi + i\alpha_0\partial\phi - i\alpha_0\partial\bar{\phi} : \end{aligned} \quad (3.4)$$

where one has chosen $\phi(\equiv \frac{\phi_1 + i\phi_2}{2})$ and $\bar{\phi}(\equiv \frac{\phi_1 - i\phi_2}{2})$ to have the same background charge α_0 which must take the value $\alpha_0^2 = \frac{1}{4p}$ in order to describe the unitary

¹⁰There is a double degeneracy of states with conformal weight Δ , specified by the conditions $\bar{G}_0|\Delta, q_+ - 1/2\rangle_+ = 0$ and $G_0|\Delta, q_- + 1/2\rangle_- = 0$ (the subscript denoting the zero mode operator).

series $\tilde{c} = 1 - 8\alpha_0^2$. Using the fact that the operator $V_{\alpha,\bar{\alpha}} =: e^{i\alpha\phi + i\bar{\alpha}\bar{\phi}}$ has conformal weight $\Delta_{\alpha,\bar{\alpha}} = \alpha\bar{\alpha} - \alpha_0(\alpha + \bar{\alpha})$ and $U(1)$ charge $q_{\alpha,\bar{\alpha}} = \alpha_0(\bar{\alpha} - \alpha)$, one may construct three types of screening operators:

$$V_{\alpha,-,0} =: \bar{\psi} e^{i\alpha-\phi} ;, \quad V_{0,\alpha,-} =: \psi e^{i\bar{\alpha}-\bar{\phi}} ;$$

and

$$V_{\alpha_+, \bar{\alpha}_+} =: [\alpha_+ \bar{\alpha}_+ \psi \bar{\psi} - i\partial(\alpha_+ \phi - \bar{\alpha}_+ \bar{\phi})] e^{i\alpha_+ \phi + i\bar{\alpha}_+ \bar{\phi}} ; \quad (3.5)$$

where $\alpha_- = \bar{\alpha}_- = -\frac{1}{2\alpha_0}$ and $\alpha_+ = \bar{\alpha}_+ = 2\alpha_0$. The contour integrals $Q = \oint dz V$ are then superconformal invariant and have zero charge for each of the three possibilities. We shall also need to know the primary fields in the various sectors. These operators¹⁶⁾, with conformal weights and charges given by eqns. (3.3), are respectively:

$N - S$ sector:

$$N_{j,k} =: e^{i(\frac{1}{2}-j)\alpha_+\phi + i(\frac{1}{2}-k)\alpha_+\bar{\phi}} ; \quad (3.6a)$$

R -sector:

$$R_{j,k}^{\pm} =: \sigma^{\pm}(z) e^{i(\frac{1}{2}-j)\alpha_+\phi + i(\frac{1}{2}-k)\alpha_+\bar{\phi}} ; \quad (3.6b)$$

where $\sigma^{\pm}(z)$ are now the spin fields¹¹⁾ associated with ψ and $\bar{\psi}$, i.e. they satisfy the following o.p.e.²⁶⁾

$$\begin{aligned} \psi(z)\sigma^{-}(z') &\sim \frac{1}{(z-z')^{1/2}}\sigma^{+}(z') \\ \bar{\psi}(z)\sigma^{+}(z') &\sim \frac{1}{(z-z')^{1/2}}\sigma^{-}(z') \end{aligned} \quad (3.7)$$

and

$$\sigma^{+}(z)\sigma^{-}(z') \sim \frac{1}{(z-z')^{1/4}} + \frac{1}{2}(z-z')^{3/4}\bar{\psi}\psi$$

Primary fields in the remaining sector are represented by a twisted vertex operator T_i ; we do not give its explicit form here since it is not needed in what follows.

As an example let us consider a four-point function on the sphere of two $N - S$ primary fields of type $j, k = 1/2, 3/2$ and two with arbitrary j and k . For

¹¹These have conformal dimension $\Delta = 1/8$ and $U(1)$ quantum number $q = \pm 1/4$ respectively.

this case one needs to insert a single screening operator of type V_{α_+, α_+} . Therefore we have the expression:

$$\begin{aligned} & < N_{j,k}(0) N_{1, \frac{1}{2}}(x) N_{\frac{1}{2}, 1}(1) \tilde{N}_{j,k}(\infty) > \sim \oint_{C_1} dy < e^{i(\frac{1}{2}-j)\alpha_+ \phi(0) + i(\frac{1}{2}-k)\alpha_+ \phi(0)} e^{-i\alpha_+ \phi(x)} \\ & \quad \cdot e^{-i\alpha_+ \phi(1)} e^{i(\frac{1}{2}+j)\alpha_+ \phi(\infty) + i(\frac{1}{2}+k)\alpha_+ \phi(\infty)} (\partial\phi(y) - \partial\bar{\phi}(y)) e^{i\alpha_+ \phi(y) + i\alpha_+ \phi(y)} > \\ & \sim x^{(2j-1)\frac{\alpha_+^2}{2}} (1-x)^{\alpha_+^2} \oint_{C_1} dy y^{(1-j-k)\alpha_+^2} (1-y)^{-\alpha_+^2} \left[\frac{j-k}{y} - \frac{1}{1-y} - \frac{1}{y-x} \right] \end{aligned}$$

where, as before, the choice of contour specifies the conformal block. Note that one has used the vertex $\tilde{N}_{j,k} = e^{i(1/2+j)\alpha_+ \phi + i(1/2+k)\alpha_+ \bar{\phi}}$ which is 'conjugate' to $N_{j,k}$, i.e. it has the same conformal weight but opposite charge. Likewise, $N_{1/2, 3/2}$ has the same weight and opposite charge to $\tilde{N}_{3/2, 1/2}$. Since $<: \psi\bar{\psi} :>$ vanishes on the sphere, the corresponding term in the screening operator does not contribute in this particular example. But as we shall see below, on the torus and also in the Ramond sector it does contribute. (We shall encounter examples of the Ramond sector on the sphere when discussing the degeneration of torus correlation functions.)

Before looking at correlation functions on the torus, one also needs the characters for the models we are considering. These have been calculated in refs. (24,25,26). In the $N-S$ and R sectors they are:

$$\chi_{j,k}^{(\nu)}(q) = \frac{\theta_\nu(0)}{\eta(\tau)} \frac{q^{\frac{k}{p}}}{\eta(\tau)^2} \Gamma_{jk}^{(\nu)}(q) \quad (3.8)$$

The first factor can be identified with the contribution of the $\psi, \bar{\psi}$ system to the characters with the four respective spin structures $\nu = 1, 2, 3, 4$ while $\Gamma_{jk}^{(\nu)}(q)$ should be identified with the contribution of the solitonic modes of the $\phi, \bar{\phi}$ system. The latter is given by:

$$\Gamma_{jk}^{(\nu)}(q) = - \sum_{n=-\infty}^{\infty} q^{pn^2 + (j+k)n} \left[1 - \frac{1}{1 + \bar{\nu} q^{pn+j}} - \frac{1}{1 + \bar{\nu} q^{pn+k}} \right] \quad (3.9)$$

where

$$\bar{\nu} = \begin{cases} +1, & \nu = 2, 3 \\ -1, & \nu = 1, 4. \end{cases}$$

Of course, as in the $N=1$ case, the character for $\nu=1$ vanishes because the spin fields σ^+ and σ^- of the Ramond sector contribute equally but oppositely. However, also as before, the $\nu=1$ sector can contribute to correlation functions.

Perhaps, in the form in which it is written, it is not obvious that $\Gamma_{jk}^{(\nu)}$ can be identified with the solitonic contribution. To make this more transparent we expand the denominators in eqn. (3.9) in powers of q :

$$\frac{1}{1 \pm q^{pn+j}} = \sum_{\ell=0}^{\infty} (\mp)^{\ell} q^{\ell pn + j\ell}$$

and get a double sum over n and ℓ . Now let $n_1 = n + \ell/2$ and $n_2 = \ell/2$ and replace the double summation over n and ℓ by one over n_1 and n_2 where $n_1, n_2 \in \frac{1}{2}\mathbb{Z}$ (with $n = n_1 - n_2 \in \mathbb{Z}$) lie inside the cones $|n_1| > |n_2|$. Then one obtains the expression:

$$\sum_{n_1, n_2} \text{sign}(n_1) (\mp)^{2n_2} q^{p(n_1^2 - n_2^2) + (j+k)n_1 + (j-k)n_2} \quad (3.9')$$

Written in this form, it can now be recognised as the solitonic contributions of ϕ_1 and ϕ_2 : $-n_2^2$ (instead of $+n_2^2$) being due to the fact that ϕ_2 is the imaginary part of ϕ and that in vertex operators the latter is always accompanied by a factor of i . Moreover, since $n_1^2 - n_2^2 \geq 0$, there is no problem with convergence of the sums.

Now, as our first example, let us consider the two-point function of an $N-S$ primary field with $j, k = 1/2, 3/2$ on the torus viz., $< N_{3/2, 1/2}(z_1) N_{1/2, 3/2}(z_2) >$. Again, one requires a single operator of the kind $V_{\alpha_+, \alpha_+}(z)$ (eqn.3.5). Since $< \psi\bar{\psi} >$ is non-zero only for $\nu=1$, the first term only contributes in this sector whereas the $(\partial\phi - \partial\bar{\phi})$ term contributes to the sectors $\nu=2, 3, 4$. Therefore it is convenient to give the result for the conformal blocks arising from the two terms separately. For $\nu \neq 1$,

$$\begin{aligned} G_i^{\nu \neq 1} &= \oint_{C_1} dz < e^{-i\alpha_+ \phi(z_1)} e^{-i\alpha_+ \phi(z_2)} (\partial\phi(z) - \partial\bar{\phi}(z)) e^{i\alpha_+ \phi(z) + i\alpha_+ \phi(z)} > \\ &\sim \oint_{C_1} dz \left[\frac{\theta_1(z_1 - z_2)}{\theta_1'(0)} \right]^{\alpha_+^2} \left[\frac{\theta_1(z_1 - z)}{\theta_1'(0)} \right]^{-\alpha_+^2} \left[\frac{\theta_1(z - z_2)}{\theta_1'(0)} \right]^{-\alpha_+^2} \\ &\quad \cdot \frac{\theta_\nu(0)}{\eta(\tau)} \frac{q^{\frac{k}{p}}}{\eta(\tau)^2} \Gamma_{jk}^\nu(w_1, w_2; q) \\ &\quad \cdot \left\{ \frac{\partial}{\partial z} \ln \left(\frac{\theta_1(z_1 - z)}{\theta_1(z - z_2)} \right) + 2p \frac{\partial}{\partial w_2} \ln \Gamma_{jk}^\nu(w_1, w_2; q) \right\} \end{aligned} \quad (3.10)$$

where the lattice contribution

$$\Gamma_{j,k}^{\nu}(w_1, w_2; q) = -e^{2i\pi \frac{1-k}{2p} w_2} \sum_{n=-\infty}^{\infty} q^{pn^2 + (j+k)n} e^{2i\pi(n + \frac{j+k}{2p})w_1} \left[1 - \frac{1}{1 \pm q^{pn + j} e^{i\pi(w_1 - w_2)}} - \frac{1}{1 \pm q^{pn + k} e^{i\pi(w_1 + w_2)}} \right],$$

with $w_1 = z_1 + z_2 - 2z$ and $w_2 = z_2 - z_1$. To see that the solitonic sums should indeed have the form given here, note that the insertion of the operator $e^{-i\frac{\alpha}{2}\phi_1(z_1)} e^{-i\frac{\alpha}{2}\phi_1(z_2)} e^{i\alpha_0 + \phi(z)}$ contributes $e^{2i\pi(n_1 + \frac{j+k}{2p})(z_1 + z_2 - 2z)}$ to each of the soliton modes in (3.9)', similarly $e^{\frac{\alpha}{2}\phi_2(z_1)} e^{\frac{\alpha}{2}\phi_2(z_2)}$ makes a contribution $e^{2i\pi(n_2 + \frac{j+k}{2p})(z_2 - z_1)}$, and one therefore obtains the expression above. The terms in eqn. (3.10) within braces arise from the $\partial\phi_2(z)$ term in the screening operator: $\ln(\frac{\partial\phi(z_2 - z)}{\partial\phi(z_1 - z)})$ is the result of Wick contracting $\partial\phi_2$ with the vertex operators and $\partial \ln \Gamma$ is the soliton mode contribution of $\partial\phi_2$. One way of seeing the latter is to consider, instead of $\partial\phi_2$, the two operators $e^{i\alpha + \phi_2(u)}$ and $e^{-i\alpha + \phi_2(v)}$ and then take the limit $u \rightarrow v$.

From what has been said, it is clear that the remaining sector $\nu = 1$ gives:

$$G_i^{\nu=1} \sim \oint_{C_i} dz \left[\frac{\theta_1(z_1 - z_2)}{\theta_1'(0)} \right]^{+\alpha_1^2} \left[\frac{\theta_1(z_1 - z)}{\theta_1'(0)} \right]^{-\alpha_2^2} \left[\frac{\theta_1(z - z_2)}{\theta_1'(0)} \right]^{-\alpha_3^2} \cdot \frac{\theta_1'(0)}{\eta(\tau)^3} q^{\frac{j}{p}} \Gamma_{j,k}^{\nu}(w_1, w_2; q) \quad (3.11)$$

where we have used (invoking translational invariance on the torus)

$$<: \psi \bar{\psi}(z) :>_{\nu=1} \sim \theta_1'(0)/\eta(\tau).$$

At this point it is appropriate, as in the $N = 1$ case, to discuss some salient properties of the conformal blocks $G_i^{\nu}(z_1, z_2)$. In the degeneration limit $q \rightarrow 0$, for $\nu = 3, 4$ the leading order corresponds to the four-point function $< N_{j,k}(0) N_{1/2,3/2}(x) N_{3/2,1/2}(1) \tilde{N}_{j,k}(\infty) >$ that was given above. On the other hand, for $\nu = 2$ one gets

$$< R_{j,k}^+(0) N_{\frac{j}{2}, \frac{k}{2}}(x) N_{\frac{j}{2}, \frac{k}{2}}(1) \tilde{R}_{j,k}^-(\infty) > + < R_{j,k}^-(0) N_{\frac{j}{2}, \frac{k}{2}}(x) N_{\frac{j}{2}, \frac{k}{2}}(1) \tilde{R}_{j,k}^+(\infty) >$$

whereas for $\nu = 1$ the result is

$$< R_{j,k}^-(0) N_{\frac{j}{2}, \frac{k}{2}}(x) N_{\frac{j}{2}, \frac{k}{2}}(1) \tilde{R}_{j,k}^+(\infty) > - < R_{j,k}^+(0) N_{\frac{j}{2}, \frac{k}{2}}(x) N_{\frac{j}{2}, \frac{k}{2}}(1) \tilde{R}_{j,k}^-(\infty) >.$$

In the last two cases, one should note that when calculating

$$< \tilde{R}^{\pm}(0) N(x) N(1) \tilde{R}^{\mp}(\infty) >,$$

the term $(\partial\phi - \partial\bar{\phi})$ from the screening operator gives the same contribution for $\pm$ and hence they add when taking the sum, while the term $:\psi\bar{\psi}:(z)$ contributes $\pm \frac{1}{z}$ and therefore only survives when taking the difference which corresponds to the degeneration of the $\nu = 1$ expression.

We can also consider the factorization limit $z_1 \rightarrow z_2$ in a t -channel basis of contours: as expected for $\tilde{C}_1$ one obtains the identity channel with the character as residue.

The conformal blocks (in an s -channel basis of contours C_1 and C_2) have the correct monodromy property when the point z_1 , say, is transported around the two homology cycles. For C_2 , under $z_1 \rightarrow z_1 + 1$

$$\Gamma_{j,k}(w_1, w_2; q) \longrightarrow e^{\frac{2i\pi k}{p}} \Gamma_{j,k}(w_1, w_2; q)$$

and for C_1 under $z_1 \rightarrow z_1 + 1, z \rightarrow z + 1$ (since the contour should also be moved in this case with the point z_1) we have

$$\Gamma_{j,k}(w_1, w_2; q) \longrightarrow e^{-\frac{2i\pi j}{p}} \Gamma_{j,k}(w_1, w_2; q).$$

These two phases $e^{2i\pi k/p}$ and $e^{-2i\pi j/p}$ are respectively $e^{2i\pi(-\Delta_{j,k} + \Delta_{j+1,k})}$ and $e^{2i\pi(-\Delta_{j,k} + \Delta_{j,k-1})}$ which one expects from the fusion rules

$$N_{3/2,1/2} \times N_{j,k} \sim N_{j+1,k} + N_{j,k-1}$$

and

$$N_{3/2,1/2} \times R_{j,k} \sim R_{j+1,k} + R_{j,k-1}.$$

Under the b -cycle monodromy $z_1 \rightarrow z_1 + \tau$, for C_2 , we have

$$q^{\frac{j+k-1}{p}} \Gamma_{j,k}(w_1, w_2; q) \longrightarrow e^{-2i\pi \frac{j+k-1}{p}} q^{\frac{(j+k-1)}{p}} \Gamma_{j+1,k}(w_1, w_2; q)$$

and for C_1 , under $z_1 \rightarrow z_1 + \tau$ and $z \rightarrow z + \tau$

$$q^{\frac{j+k-1}{p}} \Gamma_{j,k}(w_1, w_2; q) \longrightarrow e^{2i\pi \frac{j+k-1}{p}} q^{\frac{(j+k-1)}{p}} \Gamma_{j,k-1}(w_1, w_2; q)$$

These transformations show the exchange of (j, k) with the intermediate states $(j+1, k)$ and $(j, k-1)$ for each of the two blocks G_2 and G_1 respectively.

Finally, using the modular transformation properties of $\Gamma_{j,k}^{\nu}(w_1, w_2; q)$ given in the appendix, one sees that $G_i^{\nu}(z_1, z_2)$ transforms correctly.

We next give two examples of torus two-point functions with Ramond primary fields. For the Ramond field $R_{1,0}^\pm = \sigma^\pm e^{-i\frac{\sigma_1}{2}\phi + i\frac{\sigma_2}{2}\bar{\phi}}$ no screening operator is in fact required and one gets a simple expression:

$$\frac{q^{\frac{1}{2}}}{\eta(\tau)^3} \theta_\nu \left(\frac{z_1 - z_2}{2} \right) \left[\frac{\theta_1(z_1 - z_2)}{\theta_1'(0)} \right]^{-\frac{1}{2} + \frac{\sigma_1^2}{2}} \Gamma_{j,k}^\nu(w_1, w_2; q).$$

It has the correct monodromy, which may easily be verified, and we just note the following fusion rules for this:

$$\begin{aligned} R_{1,0} \times N_{j,k} &\sim R_{j+\frac{1}{2}, k-\frac{1}{2}} \\ R_{1,0} \times \tilde{N}_{j,k} &\sim R_{j-\frac{1}{2}, k+\frac{1}{2}} \\ R_{1,0} \times R_{j,k} &\sim N_{j+\frac{1}{2}, k-\frac{1}{2}}. \end{aligned}$$

The other example we consider is for the Ramond field $R_{1,1}^\pm = \sigma^\pm e^{-i\frac{\sigma_1}{2}\phi - i\frac{\sigma_2}{2}\bar{\phi}}$. It requires one screening operator of type $V_{a,a^+}(z)$. For this case both the $\psi\bar{\psi}$ and $(\partial\phi - \partial\bar{\phi})$ terms of the screening operator contribute; the former gives the expression:

$$\frac{q^{\frac{1}{2}}}{\eta(\tau)^3} \oint_{C_i} dz \left[\frac{\theta_1(z_1 - z_2)}{\theta_1'(0)} \right]^{\frac{\sigma_1^2}{2}} \left[\frac{\theta_1(z_1 - z)}{\theta_1'(0)} \right]^{-\sigma_1^2} \left[\frac{\theta_1(z - z_2)}{\theta_1'(0)} \right]^{-\sigma_2^2} F_\nu(z_1, z_2, z) \Gamma_{j,k}^\nu(w_1, 0; q),$$

while the $(\partial\phi - \partial\bar{\phi})$ term results in

$$\frac{q^{\frac{1}{2}}}{\eta(\tau)^3} \oint_{C_i} dz \theta_\nu \left(\frac{z_1 - z_2}{2} \right) \left[\frac{\theta_1(z_1 - z_2)}{\theta_1'(0)} \right]^{-\frac{1}{2} + \frac{\sigma_1^2}{2}} \left[\frac{\theta_1(z_1 - z)}{\theta_1'(0)} \right]^{-\sigma_1^2} \left[\frac{\theta_1(z - z_2)}{\theta_1'(0)} \right]^{-\sigma_2^2} \frac{\partial}{\partial w_2} \Gamma_{j,k}^\nu(w_1, 0; q)$$

where the correlation function

$$F_\nu(z_1, z_2, z) \equiv \langle \sigma^+(z_1) \sigma^-(z_2) : \psi\bar{\psi}(z) : \rangle$$

$$= \frac{1}{\eta(\tau)} \theta_\nu \left(\frac{z_1 - z_2}{2} \right) \left[\frac{\theta_1(z_1 - z_2)}{\theta_1'(0)} \right]^{-\frac{1}{2}} \left[\frac{\partial}{\partial z} \ln \frac{\theta_1(z - z_1)}{\theta_1(z - z_2)} + 4 \frac{\partial}{\partial z_1} \ln \theta_\nu \left(\frac{z_1 - z_2}{2} \right) \right]$$

In this case too, one can check that the conformal blocks have the requisite properties.

4 Concluding Remarks

The construction of various two-point functions on the torus in superconformal minimal theories has been presented using the Coulomb gas formalism. We have

also discussed some of the consistency checks that the conformal blocks must satisfy and examined their modular and monodromy properties.

We have demonstrated in some cases (for instance for the trivial $N = 1$ null state $G_{-\frac{1}{2}}|0\rangle_{N=s}$) that the corresponding conformal blocks do indeed vanish. However, a significant omission is a general proof for all null states: an argument for this in the conformal case has been sketched in ref. 1). The results of ref. 12) as well are likely to be relevant in this context.

It is of considerable interest to extend the results presented here to a Riemann surface of arbitrary genus. In this connection, there has been recent work on multi-genus conformal minimal models²⁷⁾ and on (super)conformal minimal models on hyperelliptic surfaces²⁸⁾ in particular.

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Appendix A

Here we discuss the modular transformation properties of $^{12} \Gamma_{jk}^{(\pm)}(w_1, w_2; q)$, the lattice part of the conformal blocks, introduced in section 3. Under $\tau \rightarrow \tau + 1$, $\Gamma^{(\pm)}$ for $\nu = 1, 2$ does not change but for $\nu = 3, 4$ $\Gamma^{(+)} \rightarrow \Gamma^{(-)}$ and $\Gamma^{(-)} \rightarrow \Gamma^{(+)}$. One expects this since $\nu = 1, 2$ correspond respectively to the (σ, τ) boundary conditions (P,P) and (P,A) for the fermions and are invariant under $\tau \rightarrow \tau + 1$ whereas $\nu = 3, 4$, corresponding to the boundary conditions (A,A) and (A,P), transform as (A,A) $\rightarrow$ (A,P) and (A,P) $\rightarrow$ (A,A).

In obtaining the $\tau \rightarrow -\frac{1}{\tau}$ transformation we shall have, as usual, to do a Poisson resummation of $\Gamma_{jk}^{(\pm)}$. To this end consider the following function:

$$\Gamma(x) = \sum_n q^{n(n+\tau)^2} \left[1 - \frac{1}{1 + q^{p(n+x+\Delta)}} - \frac{1}{1 + q^{p(n+x-\Delta)}} \right] \quad (\text{A.1})$$

Since $\Gamma(x) = \Gamma(x+1)$ we can write $\Gamma(x) = \sum_s e^{2i\pi sx} G(s)$ where

$$G(s) = e^{-2i\pi \frac{s^2}{4pr}} \int_{-\infty}^{\infty} dx q^{p(x-\frac{s}{2pr})^2} \left[1 - \frac{1}{1 + q^{p(x+\Delta)}} - \frac{1}{1 + q^{p(x-\Delta)}} \right] \quad (\text{A.2})$$

We can write this integral as a contour integral in the following manner. First, write the integral as

$$\int_{-\infty}^{\infty} dx q^{p(x+\frac{s}{2pr})^2} \frac{1}{1 + q^{p(x+\Delta)}} - \int_{-\infty}^{\infty} dx q^{p(x-\frac{s}{2pr})^2} \frac{1}{1 + q^{p(x-\Delta)}} \quad (\text{A.3})$$

Next shift the integration variable $x' = x + \frac{s}{2pr}$ in the first term and $x' = x - \frac{s}{2pr}$ in the second. Since $Im x' = \pm \frac{x^2}{2p|r|^2}$ for both these terms, they combine into an integration along the contour illustrated in fig. 7.

Therefore one has

$$G(s) = -e^{-2i\pi \frac{s^2}{4pr}} \text{sign}(s) \oint \frac{q^{px^2}}{1 + q^{p(x-\Delta)}} e^{i\pi s x} \quad (\text{A.4})$$

Now the integral is given by the residues of the poles: these are at $x = \Delta + \frac{s}{2pr}$ where $r \in 2\mathbb{Z} + 1$ when $s \in 2\mathbb{Z}$ and $r \in 2\mathbb{Z}$ when $s \in 2\mathbb{Z} + 1$. Hence

$$\begin{aligned} G(s) &= -\frac{1}{p\tau} e^{-2i\pi \frac{s^2}{4pr}} \text{sign}(s) \sum_{-|s| < r < |s|} q^{p(\Delta + \frac{s}{2pr})^2}, \\ G(0) &= 0 \end{aligned} \quad (\text{A.5})$$

¹²The $\pm$ denotes the signs in the denominator of $\Gamma_{jk}^{(\pm)}$: the positive sign pertains to $\nu = 2, 3$ while the minus is for $\nu = 1, 4$.

from which it follows that

$$\Gamma(x) = -\frac{q^{p\Delta^2}}{p\tau} \sum_{s=-\infty}^{\infty} \sum_{\substack{-|s| < r < |s| \\ s+r \in 2\mathbb{Z}+1}} \text{sign}(s) q^{\frac{1}{4p}(s^2-r^2)} e^{+2i\pi(xs-r\Delta)} \quad (\text{A.6})$$

where $q' = e^{-2i\pi/r}$. Note the factor $\frac{1}{p\tau}$ multiplying the sum as expected, since each boson should give $\frac{1}{p\tau}$ and $\Gamma(x)$ is the solitonic contribution of two bosons. Moreover this double summation has a form similar to that of eqn.(3.9') above.

We would now like to write this in terms of $\Gamma(x, -\frac{1}{r})$. The summation is over the points within the cone $|r| < |s|$. Choose two orthonormal unit vectors $\hat{e}_1$ and $\hat{e}_2$ and let $\vec{N} = s\hat{e}_1 + r\hat{e}_2$. The sum is then simply

$$\Gamma(x) = -\frac{q^{p\Delta^2}}{p\tau} \sum_{\vec{N}} \text{sign}(s) q^{\frac{N^2}{4r}} e^{2i\pi \vec{N} \cdot \vec{x}} \quad (\text{A.7})$$

where $\vec{z} = x\hat{e}_1 + \Delta\hat{e}_2$. It is also convenient to define two light-like vectors $\hat{E}_1 = \frac{1}{2}(\hat{e}_1 + \hat{e}_2)$ and $\hat{E}_2 = \frac{1}{2}(\hat{e}_1 - \hat{e}_2)$ and write $\vec{N} = n_1\hat{E}_1 + n_2\hat{E}_2$. Then $\sum_{s=1}^{\infty} \sum_{-s < r < s} = \sum_{n_1 \geq 1} \sum_{\substack{n_2 \geq 1 \\ n_1, n_2 \in 2\mathbb{Z}+1}}^{\infty}$ and $\sum_{s=-1}^{-\infty} \sum_{s < r < -s} = \sum_{n_1 = -1}^{-\infty} \sum_{\substack{n_2 = -1 \\ n_1, n_2 \in 2\mathbb{Z}+1}}^{\infty}$.

Now let $\vec{N} = 2p\vec{N}' + \vec{A}$ where $\vec{A} = \alpha_1\vec{E}_1 + \alpha_2\vec{E}_2$ with $1 \leq \alpha_1, \alpha_2 \leq 2p-1$. It follows that

$$\begin{aligned} \Gamma(x) &= -\frac{q^{p\Delta^2}}{p\tau} \left\{ \sum_{\alpha_1, \alpha_2} q^{\frac{A^2}{4p}} e^{2i\pi \vec{A} \cdot \vec{x}} \sum_{n_1, n_2 \geq 0} q^{pN'^2 + \vec{N} \cdot \vec{A}} e^{4i\pi p \vec{N}' \cdot \vec{x}} \right. \\ &\quad \left. - \sum_{\alpha_1, \alpha_2} q^{\frac{A^2}{4p}} e^{2i\pi \vec{A} \cdot \vec{x}'} \sum_{n_1, n_2 \geq 0} q^{pN'^2 + \vec{N} \cdot \vec{A}} e^{4i\pi p \vec{N}' \cdot \vec{x}'} \right\} \end{aligned} \quad (\text{A.8})$$

where $\vec{x}' = -x\hat{e}_1 + \Delta\hat{e}_2$. In the above sums, in order to be able to write the expression in terms of $\Gamma_{j'q'}$, we need to identify j', k' with $\frac{\alpha_1}{2}, \frac{\alpha_2}{2}$. However one cannot do this yet since $2 \leq \alpha_1 + \alpha_2 \leq 4p-2$: we need to perform a reflection about the line $\alpha_1 + \alpha_2 = 2p$, that is to say, $\alpha_1 + \alpha_2 \mapsto 4p - \alpha_1 - \alpha_2$ for $\alpha_1 + \alpha_2 > 2p$. Thus by letting $\alpha_1 = 2p - \alpha'_1$ and $\alpha_2 = 2p - \alpha'_2$ one gets

$$\begin{aligned} \Gamma(x) &= -\frac{q^{p\Delta^2}}{p\tau} \left\{ \sum_{\alpha_1, \alpha_2} q^{\frac{A^2}{4p}} e^{2i\pi(x\frac{\alpha_1 + \alpha_2}{2} - \Delta\frac{\alpha_1 - \alpha_2}{2})} \right. \\ &\quad \left. \left[\sum_{n_1, n_2 \geq 0} - \sum_{n_1, n_2 \leq -1} \right] q^{p n_1 n_2 + n_1 \frac{\alpha_2^2}{2} + n_2 \frac{\alpha_1^2}{2}} \right. \\ &\quad \left. e^{2i\pi p(n_2(x+\Delta) + n_1(x-\Delta))} - (x \rightarrow -x) \right\} \end{aligned} \quad (\text{A.9})$$

where now the range of $j' \equiv \frac{s_1}{2}$ and $k' \equiv \frac{s_2}{2}$ are $0 < j', k', j' + k' \leq p-1$ ($j', k' \in \mathbf{Z} + \frac{1}{2}$). Setting $x = \frac{j+k}{2p} + \frac{w_1}{2pr}$, $\Delta = \frac{j-k}{2p} - \frac{w_2}{2pr}$ with $j, k \in \mathbf{Z} + \frac{1}{2}$

$$q^{\frac{1}{2}} \Gamma_{jk}^{(+)}(w_1, w_2; q) \sim -\frac{1}{p\tau} e^{i\pi \frac{w_2^2 - w_1^2}{2pr}} \sum_{j', k'} 2i \sin\left(\frac{\pi(j+k)(j'+k')}{p}\right) e^{-2i\pi(\frac{j'-k'}{2p} \frac{w_1}{p} \Gamma_{j'k'}^{(+)}(\frac{w_1}{p}, \frac{w_2}{p}, q'))} \quad (\text{A.10})$$

In deriving this result from eqn. (A.9), in the second term $x \rightarrow -x$, we have made a change of the (integration) variable $z \rightarrow z_1 + z_2 - z$. If we take instead $j, k \in \mathbf{Z}$ on the right-hand side one gets $\Gamma_{jk}^{(-)}(\frac{w_1}{p}, \frac{w_2}{p}, q')$.

The modular transformation of $\Gamma_{jk}^{(-)}$ can be obtained similarly, the difference being that in eqn. (A.6) the summation over s and r is such that $s \pm r \in 2\mathbf{Z}$ and therefore in (A.7) the summation is over $n_1, n_2 \in 2\mathbf{Z}$ (both not simultaneously zero), from which it follows that in eqn. (A.10) $\frac{s_1}{2}, \frac{s_2}{2} \in \mathbf{Z}$ and one has obtained the Ramond sector after the modular transformation. Again for $j, k \in \mathbf{Z} + 1/2$ one gets $\Gamma^{(+)}$ and for $j, k \in \mathbf{Z}$, $\Gamma^{(-)}$.

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Fig.1

Fig 2

Fig.3

Fig. 4 $< R_{1,2} R_{1,2} >$ in an s-channel basis. The $\times$ denotes the trace over primaries or superdescendants propagating around the torus.

Fig. 5 Factorization limit in a t-channel basis.

Fig. 6 The dot denotes a ψ arising from an insertion of the screening operator Q_- .

Fig. 7