$G_2(2)$ AS THE AUTOMORPHISM GROUP OF THE OCTONIONIC ROOT SYSTEM OF E_7

F.KARSCH

Theory Division

CERN, 1211 Geneva 23, Switzerland

and

M.KOCA*

Fachbereich Mathematik

Universitat Frankfurt

Robert-Mayer-Str. 6-10

6000 Frankfurt am Main 11

F.R.Germany

ABSTRACT

A simple method is suggested for the construction of the seven-dimensional representation of the adjoint Chevalley group $G_2(2)$, the automorphism group of the octonionic root system of E_7 . The maximal subgroups of $G_2(2)$ preserving the octonionic root systems of the maximal subgroups of E_7 are identified. Possible implications in physics are discussed.

*Alexander von Humboldt fellow, on leave of absence from: Cukurova University,
Department of Physics, Adana, Turkey.

1. INTRODUCTION

There is growing interest in the problem of deformations of the conformal field theories [1]. Recently Zamolodchikov [2] has proved that the Ising model perturbed with a non-zero magnetic field remains integrable. He has also shown that the theory contains exactly eight massive particles and conjectured that the associated S-matrix describes the scaling limit of the Ising model at the critical point with non-zero magnetic field. Later, it has been shown [3] that this theory can be identified with the $\widehat{E}_8$ Toda field theory [4]. The masses of the eight particles of the $\widehat{E}_8$ Toda field theory turn out to be exactly those masses calculated by Zamolodchikov [2]. A similar calculation has been done for the $\widehat{E}_7$ Toda field theory which describes the perturbed tricritical Ising model [5].

Recently one of us has shown that, within the context of the octonionic description of the E_8 root system, the simple roots (indeed all roots) of E_8 can be generated, by multiplication, from its three simple roots which can be associated with its $SU(4)$ subgroup whose Coxeter-Dynkin (CD) diagram is the incidence diagram of the Ising model [6-7]. This suggests that there may arise a close connection between the octonionic presentation of the E_8 root system and the Ising model with non-zero magnetic field. If this conjecture can be justified, then octonions may play a prominent role in the formulation of the relevant field theories.

In this paper we study the automorphism group of the octonionic root system of E_7 represented by pure imaginary octonions. This group is known as the adjoint Chevalley group $G_2(2)$ [8] of order 12096 which is the automorphic extension of the finite simple group $G_2'(2)$ of order 6048, also known as the derived Chevalley group [9]. $G_2'(2) \approx U_3(3)$ was first discovered by Dickson in 1901 [10]. Here we give a 7×7 matrix representation of $G_2(2)$ using a simple method. The paper is organised as follows.

In section 2 we give a brief description of the octonionic presentation of the root systems of E_8 and E_7 and define the automorphism of the E_7 root system with the actions of the elements from the coset space $E_8/E_7 \times SU(2)$. This transformation is used to define the 7×7 matrices, acting on the imaginary units of octonions, which preserve the octonion algebra [7]. We show that the derived Chevalley group $G_2'(2)$ can be generated by three matrices associated with the simple roots describing the CD diagram of $SU(4)$. In Section 3 we study the extension of $G_2'(2)$ to $G_2(2)$ by the outer automorphism of $G_2'(2)$. In Section 4 we identify the maximal subgroups of $G_2(2)$ leaving the octonionic root sys-

tems of the maximal subgroups of E_7 invariant. We give explicit expressions of the matrices generating the maximal subgroups of G_2 . Finally in Section 5 we discuss our results and make remarks concerning the relations of $G_2(2)$ with the Hall-Janko group and the Weyl group of E_8 .

2. E_7 ROOT SYSTEM WITH PURE IMAGINARY INTEGRAL OCTONIONS

The root system of E_8 can be described by integral octonions [11] which form a closed non-associative algebra of order 240. There is a natural classification of the octonionic roots where ± 1 , pure imaginary integral octonions, and the octonions with non-zero scalar parts respectively describe the roots of $SU(2)$, E_7 and the coset space $E_8/E_7 \times SU(2)$:

$$\begin{array}{ccc} SU(2) & E_7 & E_8/E_7 \times SU(2) \\ \hline \pm 1 & \pm e_i, 1/2(\pm e_i \pm e_k \pm e_l \pm e_m) & 1/2(\pm 1 \pm e_n \pm e_p \pm e_q) \end{array} \quad (2.1)$$

where the indices take the values

$$(i = 1, \dots, 7)$$

$$(jklm : 1246, 1257, 1345, 1367, 2356, 2347, 4567) \quad (2.2)$$

$$(npq : 123, 147, 165, 245, 267, 346, 357)$$

(2.1) can be obtained from the CD diagram of E_8 (Fig.1). The set of roots of E_8 in (2.1) is closed under the octonionic multiplication of the roots.

Let A , B , and R be arbitrary octonionic roots of E_8 provided R belongs to the set of roots of the coset space $E_8/E_7 \times SU(2)$. It has been proved in [7] that the transformations

$$A' = RA\bar{R}, \quad B' = RB\bar{R} \quad (\bar{R}: \text{octonionic conjugate of } R) \quad (2.3)$$

preserve the octonionic multiplication AB :

$$A'B' = (AB)' = R(AB)\bar{R} \quad (2.4)$$

(2.3) leaves the scalar parts of the octonions unchanged so that the roots remain in their own sectors under this transformation. That is, if A and B are pure imaginary octonions they remain pure imaginary after the transformation. Therefore (2.3) not only preserves the root system of E_7 described by pure imaginary integral octonions but also keep their algebraic relations. Let us define the matrix R_{ij} by the transformation

$$e_i' = 2R_{ij}e_j' = (\pm R)e_i(\pm \bar{R}) \quad i, j = 1, \dots, 7 \quad (2.5)$$

where e_i' satisfy the octonion algebra described by e_i . It is known that the automorphism group of the octonion algebra is the exceptional group G_2 [12]. Therefore the matrices defined by (2.5) should be the elements of some finite subgroup of G_2 . The matrices in (2.5) satisfy the relation $R^3 = I$. As they are the elements of the coset space there exist 28 pairs of the form $(\pm R, \pm \bar{R})$. We use the same notation for the root and the matrix which we associate with (2.5). We hope that the distinction between them is clear. By (2.5) we can define at most 28 matrices. Nevertheless, these matrices generate under multiplication new matrices which cannot be obtained by the method of (2.5). In fact, three matrices are sufficient to generate whole group elements. For this purpose we have chosen three roots, from the coset space, which describe the simple roots of $SU(4)$ in E_7 . Let us choose them as follows:

$$P = -1/2(1 - e_1 - e_3 + e_6), \quad Q = 1/2(1 - e_1 + e_2 + e_3), \quad R = -1/2(1 + e_1 + e_2 + e_3) \quad (2.6)$$

It is clear that they describe the simple roots of $SU(4)$ where Q is the root in the middle of its CD diagram. The matrices corresponding to the transformations of these roots with their conjugates can be calculated using (2.5):

$$\begin{aligned} P = 1/2 & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & -1 \end{bmatrix} & Q = 1/2 \begin{bmatrix} 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{bmatrix} & R = 1/2 \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix} \quad (2.7) \end{aligned}$$

They satisfy the relations

$$P^3 = Q^3 = R^3 = (PQR)^7 = I \quad (2.8)$$

where I is the 7×7 unit matrix. It is straightforward to check that these matrices leave the root system of E_7 invariant. Note that the pair of roots in (2.6) (P, Q) , (Q, R) and (P, R) each generate a quaternionic root system of $SO(8)$ which form the binary tetrahedral group. Analogously the 7×7 matrices in (2.7) associated with these roots pairwise generate a group of order 24 isomorphic to the binary tetrahedral group [7]. It has been shown that, for a particular choice of a root from the coset space, the

root system of E_7 can be arranged in such a way that there exist 9 different constructions of the binary tetrahedral groups out of the roots of E_7 [7]. As we have 28 choices, the order of the group turns out to be $24 \times 9 \times 28 = 6048$. This is in accord with the order of the derived Chevalley group $G_2'(2)$. A computer calculation justifies this fact that the group generated by the matrices of (2.7) is of the order 6048. We have displayed the period-trace correlations of the elements of $G_2'(2)$ in Table 1 and compared with the result obtained by Dickson [13]. They are in agreement.

3. $G_2(2)$ AS THE AUTOMORPHIC EXTENSION OF $G_2'(2)$

The derived Chevalley group $G_2'(2)$ involves three diagonal 7×7 matrices other than the unit matrix. They are simply given by

$$A_1 = (1, 1, 1, -1, -1, -1, -1), \quad A_2 = (1, -1, -1, 1, -1, -1, 1), \quad A_3 = (1, -1, -1, 1, 1, -1, -1) \quad (3.1)$$

where the elements corresponding to the respective sets of indices 123, 147, 156 are positive. These matrices are in one-to-one correspondence with the occurrence of the Hurwitz integers [14] in (2.1) described by the three sets of quaternionic units 123, 147, 156. The matrices in (3.1) satisfy the relations

$$A_1 A_2 = A_2 A_1 = A_3 \quad (\text{cyclic permutations of } 1, 2, 3) \quad (3.2)$$

One can readily check that there are 4 more diagonal matrices whose positive elements correspond to the set of indices 246, 257, 345, 367. They also leave the octonion algebra and the octonionic root system of E_7 invariant. But none of these latter matrices is an element of $G_2'(2)$.

Let $A_4 = (-1, 1, -1, 1, -1, 1, -1)$ denote the matrix with positive elements corresponding to the indices 246. The remaining diagonal matrices can be obtained as the products of A_4 with A_1, A_2 , and $A_1 A_2 = A_3$ just in the manner the octonionic units are obtained from the quaternionic imaginary units by multiplying them with an independent imaginary unit of octonion. Let us define the remaining diagonal matrices by

$$A_7 = A_4 A_1, \quad A_8 = A_4 A_2, \quad A_5 = A_4 A_3 \quad (3.3)$$

The seven diagonal matrices satisfy, under multiplication, the same algebraic structure of octonions except they are both associative and commutative. Their multiplicative structure can be written in a compact form

$$A_i A_j = A_j A_i = A_k$$

where ijk take the values of the sets of indices 123, 165, 147, 246, 257, 435, 367. The outer automorphism of $G_2'(2)$ can be obtained by the action of A_4 , $A_4^2 = I$. Let $h \in G_2'(2)$ denote an arbitrary element of $G_2'(2)$. One can always find an element $g \in G_2'(2)$ such that

$$A_4 h A_4^{-1} = g \quad (3.4)$$

A_5 , A_6 , and A_7 also constitute the outer automorphisms of $G_2'(2)$ but because of the relations of (3.3) their outer automorphisms are related to (3.4). Thus the automorphic extension of $G_2'(2)$ can be made with A_4 . The products of the elements of $G_2'(2)$ with A_4 from left or right define the new sets of elements of the extended group $G_2(2)$, of order 12096, which admits $G_2'(2)$ as the normal subgroup.

4. MAXIMAL SUBGROUPS OF $G_2(2)$

In this section we study the subgroups preserving the octonionic root systems of the maximal subgroups of E_7 . The maximal subgroups of E_7 are given by

$$\begin{aligned} E_7 &\supset E_6 \times U(1) \\ E_7 &\supset SU(8) \\ E_7 &\supset SU(2) \times SO(12) \\ E_7 &\supset SU(3) \times SU(6) \end{aligned} \quad (4.1)$$

$$\underline{E_6 \times U(1)}$$

$U(1)$ being in the Cartan subalgebra of E_7 , it is represented by a zero root. Therefore the roots of E_6 in E_7 are the root system of $E_6 \times U(1)$. Using Fig.1 we obtain the roots of E_6 and the weights corresponding to its $\underline{27} + \underline{27}^*$ representations as follows:

E_6 roots:

$$\begin{aligned} &\pm e_1, \pm e_2, \pm e_4, \pm e_6, && 1/2(\pm e_1 \pm e_2 \pm e_4 \pm e_6) \\ &1/2(\pm e_1 \pm e_2 - e_3 + e_7), && 1/2(\pm e_1 \pm e_2 + e_3 - e_7) \\ &1/2(\pm e_1 \pm e_4 - e_3 + e_5), && 1/2(\pm e_1 \pm e_4 + e_3 - e_5) \\ &1/2(\pm e_1 \pm e_6 - e_7 + e_3), && 1/2(\pm e_1 \pm e_6 + e_7 - e_3) \end{aligned} \quad (4.2)$$

The Weights of $\underline{27} + \underline{27}^*$

$$\begin{aligned} &\pm e_3, \pm e_5, \pm e_7 \\ &1/2(\pm e_1 \pm e_2 + e_3 + e_7), && 1/2(\pm e_1 \pm e_2 - e_3 - e_7) \\ &1/2(\pm e_1 \pm e_4 + e_3 + e_5), && 1/2(\pm e_1 \pm e_4 - e_3 - e_5) \\ &1/2(\pm e_1 \pm e_6 + e_7 + e_3), && 1/2(\pm e_1 \pm e_6 - e_7 - e_3) \end{aligned} \quad (4.3)$$

$$\begin{aligned} &1/2(\pm e_4 \pm e_6 + e_3 + e_7), && 1/2(\pm e_4 \pm e_6 - e_3 - e_7) \\ &1/2(\pm e_2 \pm e_6 + e_3 + e_5), && 1/2(\pm e_2 \pm e_6 - e_3 - e_5) \\ &1/2(\pm e_2 \pm e_4 + e_7 + e_3), && 1/2(\pm e_2 \pm e_4 - e_7 - e_3) \end{aligned}$$

To search for all the transformations leaving (4.2) and (4.3) invariant is rather lengthy. In principle, this could have been investigated with a computer calculation. We have rather used an intuitive method to find out a number of matrices preserving (4.2) and (4.3) separately invariant. Using them we have generated all the rest. A simpler method can be described as follows.

When we decompose the root system of E_6 under $E_6 \times SU(3)$ (see Fig.1) we observe that the $SU(3)$ roots, the orthogonal vectors to those in (4.2), are given by

$$C : \pm 1, \pm 1/2(1 + e_3 + e_5 + e_7), \pm 1/2(1 - e_3 - e_5 - e_7) \quad (4.4)$$

Any transformation on the imaginary octonionic units leaving the set of $SU(3)$ roots in (4.4) invariant also preserves the set of roots in (4.2). The imaginary units e_3 , e_5 , and e_7 constitute a non-associative

triad. The transformation of these octonionic units determines the complete transformation matrix acting on seven imaginary units. We refer the reader to reference [7] for the details of the procedure. By selecting certain matrices from $G_2(2)$ preserving (4.4) we have generated the maximal number of elements of $G_2(2)$ leaving the root systems in (4.2), (4.3), and (4.4) unchanged. We have checked that they form a group, of order 216, which is one of the maximal subgroups of $G_2(2)$ [9]. The transformation of the octonionic imaginary units by the matrix A_4 leaves the root systems (4.2), (4.3) and (4.4) invariant, too. Therefore the automorphic extension of the group of order 216 can be made with A_4 . The extended group, of order 432, is one of the maximal subgroups of the adjoint Chevalley group $G_2(2)$. This is the maximal group which preserves the octonionic root system of E_6 . The group of order 432 is generated by the matrices

$$S = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix} \quad T = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.5)$$

They satisfy the relation

$$T^6 = S^8 = (TS)^{12} = I \quad (4.6)$$

We have calculated the periods and the traces of the 432 matrices which can be found in Table 2.

SU(8)

The root system of E_7 splits under its maximal subgroup $SU(8)$ as follows

$$126 = 56 + 70 \quad (4.7)$$

where 56 represents the non-zero roots of $SU(8)$ and 70 stands for the weights of its 70 dimensional representation. The roots of $SU(8)$ are given by the set

$$\begin{aligned} & \pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm 1/2(e_1 \pm e_2 \pm e_3 + e_7), \pm 1/2(e_1 \pm e_2 + e_4 \pm e_6) \\ & \pm 1/2(e_1 \pm e_3 - e_4 \pm e_5), \pm 1/2(e_1 \pm e_3 \pm e_6 - e_7), \pm 1/2(\pm e_2 \pm e_3 + e_4 + e_7), \pm 1/2(e_4 \pm e_5 \pm e_6 - e_7) \end{aligned} \quad (4.8)$$

One can guess several transformations which preserve (4.8) and, at the same time, leave the octonion algebra invariant. The matrices which we have selected from $G_2(2)$ generated a group of order 168 isomorphic to $L_2(7)$ [9]. This group corresponds to one of the other maximal subgroups of $G_2(2)$. There exist several other notations for $L_2(7)$ used in mathematical literature. One of the viable notations is $PSL(2,7)$ [15]. The automorphic extension of $L_2(7)$ which is denoted by $SL(2,7)$ can, in principle, be made with A_4 . However, as A_4 does not leave the $SU(8)$ root system in (4.8) invariant the automorphic extension of $L_2(7)$ by A_4 is not the maximal subgroup of $G_2(2)$ preserving the set in (4.8). The following matrix which neither belongs to $L_2(7)$ nor to $G_2(2)$ leaves the root system of $SU(8)$ unchanged:

$$U = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \quad U^2 = I \quad (4.9)$$

It can be checked that U constitutes the outer automorphism of $L_2(7)$. Therefore $L_2(7)$ can be automorphically extended by U where the extended group $SL(2,7)$, of order 336, is the maximal subgroup of $G_2(2)$, which also preserves the octonionic root system of $SU(8)$. $SL(2,7)$ is one of the modular groups. A general definition of this class of finite groups can be made as follows: $SL(2,p)$ ($p = \text{prime}$) is the special linear homogeneous group of 2×2 matrices, of modulo p , of order $p(p^2 - 1)$. $SL(2,7)$ has the subgroup of the binary octahedral group of index 7. The group preserving the octonionic root system of $SU(8)$ is isomorphic to $SL(2,7)$ and can be generated by the matrices

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad X = 1/2 \begin{bmatrix} 0 & -1 & 1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \end{bmatrix} \quad (4.10)$$

They satisfy the relations

$$V^4 = X^6 = (VX)^3 = I \quad (4.11)$$

The period-trace correlations of the matrices of the group of order 336 is given in Table 3.

SU(2) × SO(12)

From Fig.1 we obtain the following roots.

$$\begin{aligned} \text{SU}(2) : & \pm e_3 \\ \text{SO}(12) : & \pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_6, \pm e_7, 1/2(\pm e_1 \pm e_2 \pm e_4 \pm e_6) \\ & 1/2(\pm e_1 \pm e_3 \pm e_6 \pm e_7), 1/2(\pm e_2 \pm e_3 \pm e_4 \pm e_7) \end{aligned} \quad (4.12)$$

It is clear from an inspection of (4.12) that the matrices satisfying $R_{33} = \pm 1$ may leave this system invariant. If we select these matrices from the elements of $G_2(2)$ we observe that they indeed preserve the root systems in (4.12) separately invariant. Moreover, they generate a group, of order 96, which is also one of the maximal subgroups of $G_2(2)$. The matrix A_4 preserves the root system of $\text{SU}(2) \times \text{SO}(12)$. Therefore the automorphic extension of the group of order 96 can be made by A_4 . Then the extended group, of order 192, is one of the maximal subgroups of $G_2(2)$. It is obvious from these discussions that the set of matrices with $R_{33} = \pm 1$ in $G_2(2)$ form the automorphism group of the root system of $\text{SU}(2) \times \text{SO}(12)$. One can check that the other sets of matrices with $R_{ii} = \pm 1$ ($i = 1, \dots, 7$) in $G_2(2)$ also form a group of order 192. However there is one amusing distinction between the sets of matrices with $R_{11} = \pm 1$ and the others. The matrix elements of the set of matrices with $R_{11} = \pm 1$ are either 0 or ± 1 in contrast to the other sets with $R_{ii} = \pm 1$ ($i \neq 1$) where the matrix elements of the matrices also include the numbers $\pm 1/2$. The reason is clear. In the root system we have chosen in (2.1), e_1 plays a special role as it occurs repeatedly in the quaternionic units 123, 147, and 156. The group of order 192 is generated by the matrices

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad Y = 1/2 \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad (4.13)$$

We have the relations

$$A_1^2 = Y^6 = (A_1 Y)^3 = I \quad (4.14)$$

The trace-period correlation of this group is displayed in Table 4.

SU(3) × SU(6)

In this case Fig.1 leads to the following root systems

$$\begin{aligned} \text{SU}(3) : & \pm e_3, \pm 1/2(e_1 - e_2 - e_3 + e_7), \pm 1/2(e_1 - e_2 + e_3 + e_7) \\ \text{SU}(6) : & \pm e_3, \pm e_4, \pm e_6, \pm 1/2(e_1 + e_2 \pm e_4 \pm e_6), \pm 1/2(e_1 \pm e_3 \pm e_6 - e_7) \\ & \pm 1/2(e_2 \pm e_3 \pm e_4 + e_7) \end{aligned} \quad (4.15)$$

The subgroup of $G_2(2)$ preserving this system is of order 18. The diagonal matrix A_4 does not preserve this root system. It can be checked that A_6 leaves them invariant. The outer automorphism of the group of order 18 can be made by A_6 . Then the maximal group preserving (4.15) is a subgroup of $G_2(2)$ of order 36. It can be generated by the matrices

$$Z = 1/2 \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & -1 & -1 \end{bmatrix} \quad W = 1/2 \begin{bmatrix} 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \quad (4.16)$$

which satisfy the relations

$$Z^2 = W^6 = (ZW)^6 = I \quad (4.17)$$

The trace-period correlation of this group is displayed in Table 5. It seems that the groups we have generated in this case are not maximal in $G_2'(2)$ and correspondingly in $G_2(2)$ [9], a surprising fact which we have not understood.

Before we conclude this section we would like to note the following interesting relations between the groups preserving the octonionic root systems and the Weyl groups of the relevant groups. The adjoint Chevalley group $G_2(2)$ is a subgroup of the Weyl group $W(E_7)$ with index 240 which is equal to the index of $W(E_7)$ in $W(E_8)$:

$$\frac{|W(E_8)|}{|W(E_7)|} = \frac{|W(E_7)|}{|G_2(2)|} = 240 \quad (4.18)$$

We have similar relations to (4.18) for the other groups

$$\frac{|W(E_6)|}{|W(E_8)|} = \frac{|W(SU(8))|}{|W(SO(12))|} = \frac{|W(SU(2) \times SO(12))|}{|W(SU(3) \times SU(6))|} = \frac{216}{168} = \frac{96}{192} = \frac{18}{240} \quad (4.19)$$

5. DISCUSSIONS AND CONCLUSION

As stated in the introduction, the importance of the adjoint Chevalley group $G_2(2)$ and its normal subgroup $G_2'(2)$ may turn out to be relevant if any sort of relations between the octonions and the statistical mechanical models associated with E_8 , E_7 , E_6 and their maximal subgroups are obtained. In this paper we have given the explicit 7×7 matrix realisation of $G_2(2)$ and its maximal subgroups. The $G_2'(2)$ is one of the maximal subgroups of the sporadic Hall-Janko group J_2 of order 604800 [9]. Analogously $G_2(2)$ is the maximal subgroup of the covering group of J_2 . The Hall-Janko group J_2 is characterized by three sets of icosians establishing also its connection with the Leech lattice [16]. The relations between the icosians and the octonions have been recently obtained [17]. Since $G_2'(2)$ preserves the scalar parts of octonions it is the point-preserving subgroup of the Weyl group of E_8 which is the automorphism group of the 240 integral octonions in (1.1). It is obvious from this discussion that the extension of the automorphism group of the octonionic root system of E_7 in various directions is possible. Perhaps the generalizations of the statistical mechanical models associated with E_8

and E_7 to the models which can be associated with the Leech lattice can be made via this connection. The $G_2(2)$ is nowhere discussed at length. We hope that the material discussed here may shed some light on the problems relevant to physics and mathematics.

6. ACKNOWLEDGEMENTS

We would like to thank Professor Murat Gunaydin for illuminating discussions. One of us (M.K.) is grateful to Professor John Ellis for hospitality at CERN, where part of the work was carried out.

References

- [1] A.B.Zamolodchikov, JETP Lett. 43(1986)730; Sov.J.Nucl.Phys. 46(1988)1090; A.W.W.Ludwig and J.L.Cardy, Nucl.Phys. B285 [FS19](1987)687; E.J.Martinec, Phys.Lett.B217 (1989)431; C.Vafa and N.Warner, Phys.Lett. B218 (1989)51.
- [2] A.B.Zamolodchikov, Int.J.Mod.Phys. A4 (1989)4235; "Integrable field theory from conformal field theory", Proc. of the Taniguchi Symposium, Kyoto 1988, to appear in Advanced Studies in Pure Mathematics.
- [3] T.Eguchi and S.Yang, Phys.Lett. B224 (1989)373.
- [4] A.V.Mikhailov, M.A.Olshanetsky and A.M.Perelomov, Commun.Math.Phys.79 (1981)473.
- [5] P.Christie and G.Mussardo, "Integrable systems away from criticality: the Toda field theory and S-matrix of the tricritical Ising model", UCSBTH-89-19; P.Christie, "S-matrices of the tricritical Ising model and Toda system", UCSBTH-89-29; G.Mussardo, "Away from criticality: some results from the S-matrix approach", UCSBTH-89-24.
- [6] M.Koca, Phys.Lett. B228 (1989)200
- [7] M.Koca, "Symmetries of the octonionic root system of E_8 ", submitted to J.Phys.A:Mathematical and General.
- [8] C.Chevalley, Tohoku Math.J. 7(1955)14; Amer.J.Math. 77(1955)778.
- [9] J.H.Conway, R.T.Curtis, S.P.Norton, R.A.Parker and R.A.Wilson, ATLAS of Finite Groups (Oxford Univ. Press 1985).
- [10] L.E.Dickson, Trans.Amer.Math.Soc. 2(1901)363; Math.Ann. 60(1905)137; Linear Groups with an Exposition of the Galois Field Theory, Dover, New York(1958)
- [11] H.S.M.Coxeter, Duke Math. J. 13(1946)561; L.E.Dickson, Ann.Math. 220(1919)155; F.Gursey, Mod.Phys.Lett. A2(1987)967; A3(1988)1155; "Supergroups in critical dimensions and division algebras", invited lecture presented at the 5th Capri symposium on symmetry in fundamental interactions (Yale preprint); M.Koca, 1986 ICTP preprint IC/86/224 unpublished; M.Koca and N.Ozdes, J.Phys. A22 (1989)1469
- [12] M.Gunaydin and F.Gursey, J.Math.Phys. 14(1973)1651.

- [13] L.E.Dickson, Mathematical papers Vol.3 (1975)337, Ed.A.Adrian Albert, CHELSEA, New York
- [14] A.Hurwitz, Math.Werke Vol.2 (1933)303
- [15] H.S.M.Coxeter and W.O.J.Moser, Generators and Relations for Discrete Groups, Springer-Verlag, 2nd edition, Berlin, 1965.
- [16] Robert A.Wilson, Geometriae Dedicata 20 (1986)157.
- [17] M.Koca, J.Phys. A22 (1989)1947; J.Phys. A22 (1989)4125.

Table 1: Trace-Period Correlation of the Elements of the Derived Chevalley group $G_2'(2)$ (period of a matrix M is defined by $M^n = I$)

n	1	2	3	4	6	7	8	12
Trace	-2		56					
-1		63		378			1512	
0						1728		1008
1			672					
2					504			
3				126				
7								1

Table 3: Trace-Period Correlation of the Elements of the Group of order 336 (Automorphism Group of the Octonionic Root System of $SU(8)$)

n	1	2	3	4	6	7	8	12
Trace	-2							
-1		49		42	56			
0						48		
1			56				84	
2								
3								
7								1

Table 2: Trace-Period Correlation of the Elements of the Group of order 432 (Automorphism Group of the Octonionic Root System of E_6)

n	1	2	3	4	6	7	8	12
Trace	-2		2					
-1		45			72		108	
0								108
1			24					
2					18			
3				54				
7								1

Table 4: Trace-Period Correlation of the Elements of the Group of order 192 (Automorphism Group of the Octonionic Root System of $SO(12) \times SU(2)$)

n	1	2	3	4	6	7	8	12
Trace	-2							
-1		43		18	32		24	
0								
1			32				24	
2								
3				18				
7								1

Table 5: Trace-Period Correlation of the Elements of the Group of order 36 (Automorphism Group of the Octonionic Root System of $SU(6) \times SU(3)$)

		n											
Trace	n	1	2	3	4	6	7	8	12				
				2		6							
-2													
-1			15										
0													
1				6									
2						6							
3													
7	1

Figure 1: Extended Coxeter-Dynkin Diagram of E_8 Leading to the Representation of the Root System of E_7 with Pure Imaginary Integral Octonions