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A CONNECTION BETWEEN CONNECTIONS:
TODA CONFORMAL FIELD THEORIES
AND TOPOLOGICAL GAUGE THEORIES

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ABSTRACT

The connection between Toda Conformal Field Theories and three-dimensional Topological Gauge Theories is considered. In the two-dimensional theory the correlation functions of chiral vertex operators are given a topological interpretation, by exploiting the fact that the Lax-Pair, which underlies the integrability of the theory, defines a flat two-dimensional connection. In this picture, the screening charges of the Coulomb gas formalism are simply encoded in the parallel transport of the connection.

1. Introduction

The most remarkable development in the sphere of two-dimensional conformal field theory (CFT) is its connection with topological gauge theories in three dimensions [1–3]. The crux of the relationship is the observation that the space of conformal blocks is identified with the Hilbert space of the three-dimensional theory. The topological nature of such gauge theories is manifested in the fact that the action is the integral of a topological density, like the Chern–Simons form of a gauge connection A . The equations of motion imply the vanishing of the field strength $F[A] = 0$. As A can be gauged away there is no local dynamics, however, it has become clear that such theories yield a wealth of topological information.

It seems to be a general feature of Rational Conformal Field Theories, that they can be related to some three-dimensional topological gauge theory [4]. From a two-dimensional point of view CFTs are examples of integrable field theories, in that they have an infinite number of commuting integrals of motion. In CFT these are provided by the chiral algebra. In the more general case of field theories in two dimensions, the delicate question of integrability seems to rest on the existence of the Lax–Pair, from which the Quantum Inverse Scattering techniques follow [5]. The Lax–Pair is a two-dimensional gauge connection $\mathcal{A}(\phi)$, which depends on the dynamical fields ϕ of the theory in such a way that the equations of motion of ϕ imply the vanishing of the field strength of $\mathcal{A}$:

$$\mathcal{F}[\mathcal{A}] = 0.$$

The integrals of motion are constructed out of the parallel transport of $\mathcal{A}$ which defines the monodromy matrix. For example, the non-affine Toda theories (Toda Conformal Field Theories) have such a connection with components

$$\begin{aligned}\mathcal{A} &= \sum_{i=1}^r e^{\beta\alpha_i \cdot \phi} E_{\alpha_i} \\ \overline{\mathcal{A}} &= -\beta\bar{\partial}\phi \cdot H - \sum_{i=1}^r E_{-\alpha_i}.\end{aligned}\tag{1.1}$$

Here, $\{E_{\pm\alpha_i}, H\}$ are generators of a Lie algebra g of rank r [†], (see ref.[6] for more details. In particular, $\mathcal{A}$ receives quantum corrections.)

[†] For simplicity we are taking g to be simply-laced.

It is tantalizing to ponder the possible connection between the connections. Can the two-dimensional flat connection $\mathcal{A}$ be identified with the three-dimensional connection A , of the topological theory which associates itself to the CFT, on some embedded two-surface? As a step towards understanding this question we shall consider these Toda CFTs, which have been much studied recently, (see refs.[6–9] and references therein.) Such theories are defined for each semi-simple Lie algebra g (which is simply-laced in our case), and are based on the action

$$S[\phi] = \frac{1}{4\pi} \int d^2z \left(\frac{1}{2} \partial\phi \cdot \bar{\partial}\phi - \frac{1}{\beta^2} \sum_{i=1}^r e^{\beta\alpha_i \cdot \phi} \right). \quad (1.2)$$

Here, $\{\alpha_i\}$ is the set of simple roots of g . It may now easily be verified that the equations of motion imply the flatness of the gauge connection $\mathcal{A}$ defined above in (1.1). For particular values of the coupling constant β ,

$$\beta^2 = -\frac{h+k}{h+k+1}, \quad k \in \mathbf{Z} > 0, \quad (1.3)$$

where h is the Coxeter number of g , the Toda CFTs describe the coset models $(g_1 \times g_k)/g_{k+1}$ [7,8], and, in fact, exhibit a chiral algebra which is larger than the Virasoro algebra alone. These extended currents are related to the Casimirs of g [11], the case when $g = A_2$, giving the W algebra of ref.[10], being the classic paradigm.

Our investigation begins with the observation of ref.[6] that the chiral vertex operators of Toda CFT can be written in terms of the parallel transport of $\mathcal{A}$. Arbitrary conformal blocks can then be written as an expectation value of the parallel transport along contours on the two dimensional (Riemann) surface, with certain insertions. This suggests that these expectation values can be interpreted as ‘squashed’ versions of some three-dimensional extension.

One reason for pursuing this connection between Toda CFT and topological gauge theories is the fact that Toda CFTs allow integrable deformations [7,8]. The idea posited in refs.[7,8] involves promoting g to its affine extension $\hat{g}$. It would be interesting, indeed, if such integrable field theories, which are no longer CFTs, have some three-dimensional interpretation. We shall discuss this possibility briefly later.

2. Chiral Vertex Operators

In ref.[6] we found that the chiral vertex operators in the Toda conformal field theory, based on the Lie algebra g , are constructed from unitary highest-weight representations of g . The holomorphic vertex operators $\chi^\omega(z)$, and their anti-holomorphic partners, are built out of the flat two-dimensional connection $\mathcal{A}$ (the Lax-Pair) taking values in g . In fact, the form of the chiral vertex operators is very simple:

$$\chi^\omega(z) \equiv (\omega| : e^{-\beta H \cdot \phi(z, \bar{z})} : g(z, \bar{z}), \quad \bar{\chi}^\omega(\bar{z}) \equiv g^{-1}(z, \bar{z})|\omega) \quad (2.1)$$

In the above $g(z, \bar{z})$ is the parallel transport of the connection $\mathcal{A}$ between an arbitrary reference point and the point $(z, \bar{z})$. $(\omega|$ is a highest weight state of g^* . Chirality follows from the integrability condition $\mathcal{A} = g dg^{-1}$ and the fact that the connection has a very particular form in terms of the Lie algebra g , (i.e. the fact that $\mathcal{A}$ is a raising operator in g). We remark that the full operator spectrum of the theory, should include operators which have no classical limit and correspond to χ with β replaced by $1/\beta$. We shall not consider these operators here, this amounts to considering a generalization of the $(1, n)$ -subalgebra in the minimal series [12].

The conformal fields constructed from the chiral vertex operators in (2.1), for a diagonal theory, are

$$: e^{-\beta \omega \cdot \phi(z, \bar{z})} := \chi^\omega(z) \bar{\chi}^\omega(\bar{z}).$$

We shall now concentrate solely on the holomorphic sector of the theory. In [6] we considered the theory in Minkowski space, in which case $z, \bar{z}$ are light-cone coordinates. In this letter, in order to make contact with Euclidean conformal field theory, we take z to be some holomorphic local coordinate of a compact Riemann surface $\Sigma^\dagger$. In [6] we found that the chiral vertex operator could be written in terms of a free chiral scalar field $\psi(z)^\ddagger$ acting in the Hilbert space $\mathcal{H}_P$ ($P \in \Sigma$, and z is a local coordinate with

★ Our notation is such that $|\cdot\rangle$ refers to a vector in a representation space of g , while $(\cdot|)$ will be reserved for states in the Hilbert space of ϕ .

† We therefore replace the time ordering in equation (4.1) of [6], by path ordering, denoted by $\mathcal{P}$.

‡ With normalization $\langle \psi(z)\psi(y) \rangle = -\ln(z-y)$.

$z(P) = 0$):

$$\chi^\omega(z) = (\omega | : e^{-\beta\omega \cdot \psi(z)} : \mathcal{P} \exp \left\{ \int_0^z dy \sum_{i=1}^r : e^{\beta\alpha_i \cdot \psi(y)} : E_{\alpha_i} \right\} . \quad (2.2)$$

The chiral vertex operator at a point Q , acting in the Hilbert space at a point P , therefore, has the simple interpretation of the action of the parallel transport of $\mathcal{A}$ from P to Q on the operator $(\omega | : \exp -\beta\omega \cdot \psi(Q) : (to the left))$. More precisely,

$$\chi^\omega(Q) = (\omega | : e^{-\beta\omega \cdot \psi(Q)} : \mathcal{P} \exp \int_P^Q \mathcal{A} . \quad (2.3)$$

The connection with topological gauge theories can now be glimpsed, however, we shall delay a discussion of this until section 4. We also remark that we can construct concrete representations of the ‘Verlinde operators’, which represent the monodromy of the chiral vertex operators around homotopically non-trivial cycles in Σ . Using the notation of ref.[13] we have

$$\chi^\omega(z) \phi_\omega(\mathbf{c}) = \chi^\omega(z) \mathcal{P} \exp \oint_{\mathbf{c}} \mathcal{A} ,$$

where $\mathcal{A}$ is taken in the representation V^ω of g , and $\mathbf{c}$ is a non-trivial cycle in Σ .

We now consider the construction of the Hilbert space $\mathcal{H}_P$. From ref.[7] we know that the holomorphic component of the quantum stress tensor has the following form:

$$\Theta(z) = -\frac{1}{2} : \partial\psi \cdot \partial\psi : + \left(\frac{1}{\beta} + \beta \right) \rho \cdot \partial^2\psi , \quad (2.4)$$

where ρ is the sum of the fundamental weights of g . The central term in the Virasoro algebra is

$$c = r \left(1 + h(h+1) \left(\frac{1}{\beta} + \beta \right)^2 \right) ,$$

and the chiral vertex operator are primary fields of (holomorphic) conformal dimension

$$\Delta(\omega) = -\frac{\beta\omega}{2} \cdot \left(\beta\omega + 2\rho \left(\frac{1}{\beta} + \beta \right) \right) .$$

In the neighbourhood of P we can expand the field ψ in terms of creation and annihilation

lation operators in the usual way

$$\psi(z) = q - ia_0 \ln z + i \sum_{\substack{n \in \mathbf{Z} \\ \neq 0}} a_n z^{-n}/n, \quad (2.5)$$

where the oscillators have the non-zero commutation relations

$$[a_n, a_m] = n\delta_{n+m,0}, \quad [q, a_0] = i.$$

The space $\mathcal{H}_P$ is spanned by the states created from the oscillator vacuum $|0\rangle$ by the action of the creation operators $\{a_{-n}, n > 0\}$ and the ‘momentum’ operators $\{e^{i\alpha \cdot q}\}$. The modes of a conformal field $\Phi(z)$, of dimension Δ , are defined via

$$\Phi(z) = \sum \Phi_n z^{-n-\Delta}.$$

In particular, the modes of the stress tensor are

$$L_n = \frac{1}{2} \sum_{m \in \mathbf{Z}} : a_{n+m} \cdot a_{-m} : + i(n+1) \left(\frac{1}{\beta} + \beta \right) \rho \cdot a_n. \quad (2.6)$$

Recall that β is the coupling constant of the Toda theory. To describe the minimal theories, and their extended generalizations, we must allow for the possibility of β being imaginary. In this case the representation of the Virasoro algebra is unitary if we impose

$$a_n^\dagger = -a_{-n} + 2\Lambda \delta_{n,0},$$

where $\Lambda = -i(\beta^{-1} + \beta)\rho$ is real[†]. So, although the Fock space includes negative norm states, the Verma module (of the extended algebra) defined on a highest weight state is unitary, at least when β takes the certain special values of equation (1.3) corresponding to the unitary series of ref.[11]. Notice that the zero-mode is anomalous. This affects the hermiticity of the states; to see this consider the states $|p\rangle = \exp ip \cdot q |0\rangle$, and $\langle p| = \langle 0| \exp -ip \cdot q$. (Here, $|0\rangle$ and $\langle 0|$ are the oscillator vacuums.) Notice that when β is

[†] When β is real we must impose instead $a_n^\dagger = a_{-n} - 2\Lambda \delta_{n,0}$

imaginary the ‘momentum’ eigenvalues are real. By construction $\langle k|p\rangle = \delta_{k,p}$, however, $(a_0|p)\rangle^\dagger = |p\rangle^\dagger(-a_0 + 2\Lambda)$, therefore

$$|p\rangle^\dagger = \langle -p + 2\Lambda|$$

Using the representation in equation (2.6) allows for the determination of the $sl(2, \mathbb{C})$ vacuum, which we denote $|\mathbf{v}\rangle$. We find

$$|\mathbf{v}\rangle = |0\rangle, \quad \langle \mathbf{v}| = \langle 2\Lambda|,$$

and so we are in the position of having a ‘charge at infinity’. In our application to Toda theory we shall discover that the correct specification of vacuum states is

$$|\Omega\rangle\rangle = |2\Lambda\rangle = Z(0)|\mathbf{v}\rangle, \quad \langle\langle\Omega| = \langle 2\Lambda| = \langle \mathbf{v}|. \quad (2.7)$$

Here, $Z(z) =: e^{i2\Lambda\cdot\psi(z)}$ is the conformal field of dimension zero. Notice that the $sl(2, \mathbb{C})$ vacuum itself could not be the true vacuum because $\langle \mathbf{v}|\mathbf{v}\rangle = 0$, due to the zero-mode anomaly.

Our main conjecture is that the correlation function[†]

$$\mathcal{F}_{\{\omega_i\}}(\Sigma - \{P_i\}) = \langle \chi^{\omega_1}(P_1) \chi^{\omega_2}(P_2) \cdots \chi^{\omega_n}(P_n) Z(P) \rangle_\Sigma, \quad (2.8)$$

which is a section of a $V^{\omega_1} \otimes V^{\omega_2} \otimes \cdots \otimes V^{\omega_n}$ valued bundle over the moduli space of the punctured Riemann surface $\Sigma - \{P_i\}$, spans the space of conformal blocks $\mathcal{M}_{\{\omega_i\}}(\Sigma - \{P_i\})$ of the g -extended CFT. These correlation functions have the simple interpretation of the expectation value of the parallel transport of the connection $\mathcal{A}$ along the contours emanating from the point P to a set of points $\{P_i\}$, taken in representations $\{V^{\omega_i}\}$ of the Lie algebra g , with operators sitting on the ends, and the operator Z at the vertex P , (see figure 1). We can define the ‘star’ basis for $\mathcal{M}_{\{\omega_i\}}(\Sigma - \{P_i\})$, as the set vectors

$$\Psi(\{\omega'_i\}) = \mathcal{F}_{\{\omega_i\}}(\Sigma - \{P_i\})|\omega'_1\rangle \otimes |\omega'_2\rangle \cdots \otimes |\omega'_n\rangle, \quad (2.9)$$

where the vectors $\otimes_i |\omega'_i\rangle$ form a basis for the dual space $(\otimes_i V^{\omega_i})^*$.

[†] In this letter we shall just consider the case when Σ has genus zero. At higher genus we must take account of closed Wilson lines which can exploit the non-trivial cycles in Σ .

To avoid confusion we should like to point out that, although the chiral vertex operators (in the sense of [14]) are V^ω -valued, the highest-weight states of the Virasoro algebra are not (unlike in WZW theories). They are simply

$$: e^{-\beta\omega\cdot\psi(0)} : |\mathbf{v}\rangle = |i\beta\omega\rangle.$$

This is to be compared with the situation in WZW models, where the elements of the space of conformal blocks are themselves $\otimes_i V^{\omega_i}$ valued; here they are not.

Written in terms of operators acting in $\mathcal{H}_P$, the correlation functions are

$$\mathcal{F}_{\{\omega_i\}}(z_1, \dots, z_n) = \langle \langle \Omega | \chi^{\omega_1}(z_1) \chi^{\omega_2}(z_2) \cdots \chi^{\omega_n}(z_n) | \Omega \rangle \rangle. \quad (2.10)$$

In the next section we justify our conjecture by considering the three-point functions for the case $g = A_1$.

In [6] we argued that the chiral vertex operators satisfied the following braid relation:

$$\chi^{\omega_1}(z) \otimes \chi^{\omega_2}(y) = \chi^{\omega_2}(y) \otimes \chi^{\omega_1}(z) R^{\omega_1, \omega_2}, \quad |z| > |y|, \quad (2.11)$$

where R is the, suitably normalized, quantum R -matrix of the quantum group associated with the q -deformed Lie algebra of g , where the deforming parameter is

$$q = e^{-2\pi i \beta^2}. \quad (2.12)$$

Associativity of operator multiplication implies that R satisfies the Yang-Baxter equation. The above facts imply that the correlation functions above are a particularly convenient basis for displaying how the conformal blocks transform under representations of the Braid Group.

The full operators of the theory involve a combination of chiral vertex operators, with β and $1/\beta$ as coupling constants. That is

$$\chi^{\omega_1, \omega_2}(z) = (\omega_1 | \otimes (\omega_2 | : e^{-\beta\omega_1\cdot\psi(z) - (1/\beta)\omega_2\cdot\psi(z)} : \mathcal{P} \exp \int_0^z \mathcal{A}^\beta \otimes 1 + 1 \otimes \mathcal{A}^{1/\beta}. \quad (2.13)$$

The ordering of $\mathcal{A}^\beta$ and $\mathcal{A}^{1/\beta}$ is not relevant as they are mutually local.

3. Three-Point Correlation Functions for $g = A_1$

The goal of this section will be to indicate how one could prove the conjecture that the correlation functions defined in (2.9) are complete. We shall show how, for the case when $g = A_1$ (which corresponds to the Liouville theory with imaginary coupling constant β), the correlation functions defined in equation (2.10) are a linear combination of those of the Coulomb gas representation [15,16]. In fact, we shall just consider the three-point functions, but we see no obstacle in generalizing.

Consider the correlation function of three fields in the representations with spins j_i , $i = 1, 2, 3$, at the points z_i . On expanding the exponentials of the chiral vertex operators we see that a typical term contributing to the correlation function is

$$\langle\langle\Omega| : e^{-\beta j_1 \psi(z_1)} : T_0^{z_1}(n_1) : e^{-\beta j_2 \psi(z_2)} : T_0^{z_2}(n_2) : e^{-\beta j_3 \psi(z_3)} : T_0^{z_3}(n_3) |\Omega\rangle\rangle, \quad (3.1)$$

with $0 \leq n_i \leq 2j_i$. In the above we have changed the normalization of ψ ($\sqrt{2}\psi \rightarrow \psi$) to agree with refs.[15,16]. The T 's are the ordered integrals

$$T_0^z(n) = n! \int_0^z dy_1 : e^{\beta \psi(y_1)} : \int_0^{y_1} dy_2 : e^{\beta \psi(y_2)} : \dots \int_0^{y_{n-1}} dy_n : e^{\beta \psi(y_n)} :,$$

defined in ref.[6]. Due to presence of the zero-mode the expression in (3.1) is only non-zero if

$$j_1 + j_2 + j_3 = n_1 + n_2 + n_3.$$

We now use the theorems of [6] to rewrite the expression. Firstly we use equation (B4) of [6], to write $T_0^{z_2}(n_2) = q^{n_2(n_2-1)/2} \bar{T}_0^{z_2}(n_2)$, where $q = \exp -2\pi i \beta^2$ and $\bar{T}$ denotes the anti-ordered integral. Then we notice that $\bar{T}_0^{z_2}(n_2) = (-)^n T_{z_2}^0(n_2)$. Finally, we can commute $: \exp -\beta j_2 \psi(z_2) :$ through $T_{z_2}^0(n_2)$ at the cost of a phase $q^{-j_2 n_2}$. Therefore, up to a phase only, the expression in (3.1) is equal to

$$\langle\langle\Omega| : e^{-\beta j_1 \psi(z_1)} : T_0^{z_1}(n_1) T_{z_2}^0(n_2) : e^{-\beta j_2 \psi(z_2)} :: e^{-\beta j_3 \psi(z_3)} : T_0^{z_3}(n_3) |\Omega\rangle\rangle.$$

We now consider a particular linear combination of such expressions and use a result

which is theorem 1 of [6], namely

$$\sum_{p=0}^{n_1+n_2} \binom{n_1+n_2}{p} T_0^{z_1}(n_1+n_2-p) T_{z_2}^0(p) = T_{z_2}^{z_1}(n_1+n_2), \quad (3.2)$$

to form

$$\langle \mathbf{v} | : e^{-\beta j_1 \psi(z_1)} T_{z_2}^{z_1}(n_1+n_2) : e^{-\beta j_2 \psi(z_2)} :: e^{-\beta j_3 \psi(z_3)} : T_0^{z_3}(n_3) Z(0) | \mathbf{v} \rangle.$$

Projective invariance may now be used to send $z_1 \rightarrow \infty$ and $z_2 \rightarrow 1$ (the other projective degree of freedom has been used to fix Z at the origin). If $j_1 \leq j_2$ then there are potentially $2j_1 + 1$ functions of the form (3.1). Of course, these functions in general are not linearly independent. For general g and the special values of β in equation (1.3), corresponding to the generalized unitary minimal series, there are extra decouplings in such a way that the fields with $\omega \cdot \theta \leq k + 1$ [†] form a closed subalgebra. We recognize these $2j_1 + 1$ functions as spanning the space of conformal blocks of the form

$$\langle \phi_{1,2j_1+1}(\infty) \phi_{1,2j_2+1}(1) \phi_{1,2j_3+1}(z) \phi_{\overline{1},\overline{1}}(0) \rangle,$$

where we have used the notation of refs.[12,15,16], (in particular, see section 3 of ref.[16]). Notice that $Z = \phi_{\overline{1},\overline{1}}$ is the ‘conjugate’ field of the identity in the Coulomb gas sense. The number of solutions reflects the existence of a null vector in the Verma module $[\chi^j = \phi_{1,2j+1}]$ of order $2j + 1$, and so correlation functions involving χ^j satisfy a $(2j + 1)^{th}$ order differential equation. In the more general case of an arbitrary algebra g , the Verma module $[\chi^\omega]$ contains a null vector of order $\dim V^\omega$, and so a correlation function involving χ^ω will satisfy a $(\dim V^\omega)^{th}$ order differential equation. These nulls vector can be considered as the quantum analogues of the differential equations satisfied by the vertex operators in the classical theory, namely

$$\prod_{\omega' \in W^\omega} (\partial + \beta \omega' \cdot \partial \phi) \chi^\omega(z) = 0 \quad (\text{classically}),$$

where W^ω is the set of weights in the representation with highest weight ω , and the product is ordered with respect to the heights of the weight vectors[‡]; the lower heights being to the right.

[†] Here, θ is the highest root of g , and the representations satisfying this condition correspond to the integrable representations of $\hat{g}$ at level $k + 1$. In fact, there is evidence that the non-zero couplings also correspond to the couplings of $\hat{g}$ at level $k + 1$, we intend to return to this point elsewhere.

[‡] The height of a weight $\omega' = \omega - \sum_{i=1}^r n_i \alpha_i$ is $\sum_{i=1}^r n_i$.

4. Connection With Topological Gauge Theories

In a series of papers [1–3] Witten has shown how a gauge theory in three dimensions, whose action is proportional to the integrated Chern–Simons form:

$$S[A] = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + 2/3 A \wedge A \wedge A), \quad (4.1)$$

(here, $k \in \mathbf{Z}$, in order that $\exp iS[A]$ be gauge invariant), is related to a WZW conformal field theory. In order to make this connection Witten constructs a series of gauge invariant observables from the parallel transport of A in some representation of the gauge group G :

$$W^\omega(P, Q) = \mathcal{P} \exp \int_Q^P A.$$

In the above ω , a highest weight, labels the particular representation involved. In the simplest cases these observables are closed Wilson lines, in which case the path integral with respect to the action in (4.1) gives some functional of the lines, each labelled with a representation of G . The topological nature of the theory manifests itself in the fact that these functionals depend only on the topology of the underlying manifold M and the configuration of the Wilson lines: in this sense the theory yields topological invariants. There are, however, more complicated gauge invariant observables that can be constructed associated to graphs with 3–point vertices corresponding to the possible invariant couplings of $V^{\omega_1} \otimes V^{\omega_2} \otimes V^{\omega_3}$ [2][†]. Consider, now, the functional integral defined by (4.1) over a three–manifold M with boundary Σ in the presence of some observable. Suppose that the observable punctures the surface Σ in points $\{P_i\}$ corresponding to representations $\{V^{\omega_i}\}$, $i = 1, \dots, n$. The crucial point is that the Hilbert space of the three–dimensional theory is identified with the space of conformal blocks of the WZW theory for the surface Σ with the primary fields $\{\Phi^{\omega_i}\}$ inserted at points $\{P_i\}$. The functional integral gives a vector in the space of such conformal blocks.

The question is: can we work this procedure backwards? i.e. starting from a CFT can we determine the nature of its associated topological theory in three dimensions.

[†] Witten argues that one can, in fact, restrict ones attention to the integrable representations of $\hat{g}$ at level k .

We would like to go some way towards answering this question for Toda CFTs by emphasizing the apparent topological character that our construction of the conformal blocks in section 2, enjoys. They were constructed as the expectation value of an ordered product of parallel transports of the connection $\mathcal{A}$ in a set of representations $\{V^{\omega_i}\}$ between a reference point P and the set of points $\{P_i\}$, with the appropriate operators sitting at the ends. That is

$$\mathcal{F}_{\{\omega_i\}}(\Sigma - \{P_i\}) = (\omega_1| \otimes \cdots \otimes (\omega_n| \left\langle \mathcal{P} \left(\bigotimes_{i=1}^n : e^{-\beta \omega_i \cdot \psi(P_i)} : \exp \int_{C_i} \mathcal{A} \right) \right\rangle_{\text{TODA}}, \quad (4.2)$$

where the expectation value $\langle \cdots \rangle_{\text{TODA}}$ reminds us to insert the field Z at the vertex $P^\dagger$. In the above the contour $C = \cup_i C_i$ is the star illustrated in figure 1. If we had taken a vector in $\otimes_i V^{\omega_i}$ other than the product of highest-weight vectors, on the left in (4.2), we do not generate any independent elements of $\mathcal{M}_{\{\omega_i\}}(\Sigma - \{P_i\})$, this is due to the fact that $\mathcal{A}$ is a raising operator in g (i.e. a lowering operator when acting to the left). This means

$$(\omega - \alpha | \mathcal{P} \exp \int \mathcal{A} | \omega') \propto (\omega | \mathcal{P} \exp \int \mathcal{A} | \omega' + \alpha),$$

for some root α of g .

The three-dimensional topological gauge theories that relate to coset models have already been constructed in ref.[4]. These models involve the difference of the Chern-Simons actions for the two groups describing the coset. As conformal field theories, coset models have recently been given a Lagrangian interpretation as gauged WZW models [17,18]. In view of the fact that WZW models have recently been written in a bosonized [19] form, suggests that we regard Toda CFT as the bosonized version of the $(g_k \times g_1)/g_{k+1}$ coset model; and hence related to the topological theories of ref.[4].

Suppose we start with the three-dimensional theory and consider the path integral on M in the presence of some Wilson line observable $\hat{C}$ puncturing $\Sigma = \partial M$. In the case when $\hat{C}$ is homotopic to some configuration of Wilson lines C on Σ (i.e. there exists some continuous map $\hat{C} \rightarrow C$, see figure 2), then it is tempting to suppose that

[†] If we had included the full operator algebra, by introducing the ‘ $1/\beta$ operators’, then we could construct more complicated correlation functions by inserting any of the generalized operators of (2.13) at the points P_i and P .

the residue of the three-dimensional functional integral is an expectation value of the Wilson line of the connection $\mathcal{A}$, in the conformal field theory. As yet, a general proof of this does not exist.

5. Discussion

The main result of this letter is the fact that the correlation functions of Toda CFT can be written in form which suggests that they can be extended to a expectation values of some Wilson line observable in some three-dimensional topological gauge theory. In this scenario the two-dimensional connection which forms the Lax-Pair of the theory would be directly related with some three-dimensional connection, in the topological gauge theory, on some imbedded two-surface. However, the main question posed, namely—what is the connection between the connections?—still remains unanswered at a formal level. We suspect that the answer will involve a careful analysis of the topological theories developed in [4], gauged WZW models and their bosonized forms.

Our formalism gives a rather attractive description of the coulomb gas recipe for constructing the conformal blocks of the minimal models, at least on the plane. It would be a worth while seeing if the formalism extended in a nice way to arbitrary genus.

We now turn our attention to the affine Toda theories. The construction of the (untwisted) theories involves adding in the term

$$\frac{1}{\beta^2} e^{-\beta\theta\cdot\phi},$$

where θ is the highest root of g , to the Lagrangian in equation (1.2). The effect of this is to stabilize the potential which develops a minimum, in contrast to the non-affine case. The theory is no longer scale invariant, so conformal invariance is lost, and the theory describes massive fields. At the classical level we can also achieve this by extending the root space of g to that of its affine extension $\hat{g}$ by including the imaginary root α_0 . The field ϕ now has $r + 1$ components, however, one of the fields decouples classically in the sense that the other fields do not depend on it. The integrability of the theory hinges on the fact that a Lax-Pair can be constructed, taking values in the affine algebra $\hat{g}$, by extending the sums in (1.1) to include the extra root. The equations

of motion imply the the curvature of this connection vanish, as in the non-affine theory. We could go on to construct chiral vertex operators in the same way as in the non-affine theory, however, to do this requires highest-weight representations of $\hat{g}$. Therefore, we cannot simply take $\hat{g}$ to be the loop group extension of g , which allows no highest-weight representations. Classical solutions to the sine-Gordon ($g = A_1$) equation have been constructed in this way [20], however, in the classical theory the extra component of ϕ can be decoupled. In the quantum theory it is not clear whether this would occur. Rather we are led to consider a new type of theory based on the affine algebra, which although it does not look, *prima facie*, like a conformal field theory, it seems to enjoy the property of holomorphic factorization. We think that it would be interesting to see if such theories can be constructed consistently, and whether they relate in any way to theories in three-dimensions. If we pursued the more conventional line and took $\hat{g}$ to be the loop-algebra extension of g , then the Lax-Pair becomes a one-parameter family of flat connections $\mathcal{A}(\lambda)$, where λ is known conventionally as the spectral parameter.

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FIGURE CAPTIONS

Figure 1. The star contour and operator insertions used to define the correlation function in equation (2.8).

Figure 2. An example of a contour in M which projects onto Σ , giving a three point function.

Figure 1.

Figure 2.