PREPRINT

132

A.G. SAVINKOV, A.B. RYZHOV

SCATTERING, GREEN'S FUNCTIONS AND SYMMETRIES IN A TOTAL SPACE OF DIRAC MONOPOLE FIBRE BUNDLE

CERN LIBRARIES, GENEVA

CM-P00067842

Moscow - 1989
Препринты Физического института имени П.Н. Лебедева АН СССР являются самостоятельными научными публикациями и издаются по следующим направлениям исследований Института:

- физика высоких энергий и космических лучей
- оптика и спектроскопия
- квантовая радиофизика
- физика твердого тела
- физика космоса
- физика плазмы

В библиографических ссылках на препринты Физического института имени П.Н. Лебедева мы рекомендуем указывать: инициалы и фамилию автора, номер препринта, место издания, сокращенное наименование Института-издателя, год издания.
Пример библиографической ссылки:
И.И. Иванов. Препринт 125, Москва, ФИАН, 1986.

Preprints of the P.N. Lebedev Physical Institute of the Academy of Sciences of the USSR are its independent publications and are issued in the Institute's following fields of research:

- high energy and cosmic ray Physics
- optics and spectroscopy
- quantum Radiophysics
- solid state Physics
- cosmophysics
- plasma Physics

In bibliographical references to the P.N. Lebedev Physical Institute's preprints we recommend to indicate: the author's initials and name, preprint number, place of the publication, abbreviation of the Institute-publisher, year of the publication:

Example of a bibliographical reference:

© Физический институт им. П.Н. Лебедева АН СССР, 1989.
ACADEMY OF SCIENCES OF THE USSR
P. N. LEBEDEV PHYSICAL INSTITUTE

High energy physics and cosmic rays

Department of Nuclear Theoretical Physics

Preprint I32

A.G. Savinkov, A.B. Ryzhov

SCATTERING, GREEN'S FUNCTIONS AND
SYMMETRIES IN A TOTAL SPACE OF
DIRAC MONOPOLE FIBRE BUNDLE

Moscow - 1989
A B S T R A C T

The scattering wave functions and Green's functions are found in a total space of Dirac monopole principal bundle. Also there were found hidden symmetries of a charge - Dirac monopole system and those connecting the states relating to different topological charges \( n = 2\pi g \).
1. INTRODUCTION

It is well known that the most natural description of a charged particle in the Dirac monopole [1,2] field is given by combining position coordinates with gauge coordinate by virtue of principal U(1)-bundle \( P \) having total space \( C^2 \setminus \{0\} \) [3-10]. The combining rises from the intention to avoid difficulties related to the string of potential singularities. Wu, Yang [3] and Greub and Petry [4] have demonstrated that in quantum mechanics one can get rid of the string when defining monopole potential locally, the potentials \( \hat{A}(a), \hat{A}(b) \) are finite on their domains and are joint by gauge transformation on the overlap. Under this consideration wave functions become local sections, and single-valuedness condition for their transition functions leads to the Dirac quantization condition \( Z \varepsilon g = n \in \mathbb{Z} \) [1,3] (\( e \) - electric charge of the particle in the monopole field, \( g \) - magnetic charge of the monopole). However one had to use locally defined operators \( \frac{1}{2\mu} (\hat{p} - e\hat{A}(a))^2 \) and \( \frac{1}{2\mu} (\hat{p} - e\hat{A}(b))^2 \) instead of the Hamiltonian.

To have the global Hamiltonian dynamics one should pass from \( \mathbb{R}^3 \setminus \{0\} \) to space joining gauge and spatial coordinates. Such transfer have been considered by Solov'ev [5] and Petry [6].

As the result it was shown that the charge in the monopole field can be described in terms of nontrivial principal bundle with the total space \( P = C^2 \setminus \{0\} \) and the fibre (structure group) \( U(1) \) [5,7,8]. (This principal bundle refers to as Kustaanheimo - Stiefel bundle. It corresponds to the Hopf fibre bundle \( S^3 \xrightarrow{U(1)} S^2 \) and line bundle associated with Hopf bundle refers to as Wu-Yang bundle in the physical texts [3,9]).
orbit. Such Lagrangian is given by the following expression

\[ L = \frac{\mu}{2} \left( 4 \langle ZZ \rangle (\dot{Z} \dot{Z}) + (\dot{Z} Z - \dot{Z} Z)^2 \right) + i \frac{\dot{Z} Z - \dot{Z}}{Z Z} \]

and the Euler–Lagrange equation ruling the motion over gauge coordinate has identical form \( 0 = 0 \), i.e. is completely undetermined. The other equations being projected on to base \( R^3 \setminus \{0\} \) give ordinary equations of motion for the charge in the Dirac monopole field. Lagrangian \( L \) defines action which is a singlevalued functional on the space \( \mathcal{A} \) of closed paths with fixed initial and final point \( Z_0 \). It was shown by Solov'ev [5] that if trajectories \( Z(t) \) and \( Z'(t) \) have the same image \( X(t) \) on the base and the map of interval \( [0,1] \) to \( S^1 : t \rightarrow \exp[i_1 t + i_2 t] \in S^1 \) (here \( i_1 \) and \( i_2 \) are arguments of \( Z^1 \) and \( Z^2 \)) corresponding to them covers the circle \( S^1 \) equal number of times, then \( S(Z) = S(Z') \) (\( S \) denotes the action). Factor space of \( \mathcal{A} \) with respect to this equivalence relation is simple connected, i.e. is universal covering for the space of closed trajectories \( X(t) \). Its meaning is similar to that of Riemannian surface in the holomorphic functions theory — it turns initial multiplevalued functional on the base into singlevalued one [11].

After Legendre transformation in which canonical momenta are defined by the expression \( n = \partial L / \partial \dot{Z} = 2 \mu (Z Z) \dot{Z} \), \( \bar{n} = \partial L / \partial \dot{Z} = 2 \mu (Z Z) \dot{Z} \) one obtains the Hamiltonian \( H = \frac{1}{4 \mu Z Z} \left( n \bar{n} - \frac{1}{z \bar{z}} \right) \)
determined on the constraint surface \( Z n - \bar{n} Z = i \). The charge \( n \in Z \) being arbitrary, the Hamiltonian and constraint surface have the form
The group $U(1)$ acts on $P$ by the rule:

$$(Z^1, Z^2) \mapsto (Z^1 \exp(i\alpha), Z^2 \exp(i\alpha))$$  \hspace{1cm} (1.1)$$

which corresponds to the charge $n = 2\pi g = 1$. Description of monopoles with other $n \in \mathbb{Z}$ is given by factorization of $P$ over action of discrete subgroup $Z_n \subset U(1)$ with generator $\exp(2\pi i/n)$. Respective bundle spaces are equivalent to lens spaces $L_n$ [4, 10]. Fibre bundle $P$ with $n = 1$ is universal for monopole problem since it contains information on interaction charge and monopole for any $n \in \mathbb{Z}$, so we shall mainly concern $P$. Projection $p$ on the basic manifold $\mathbb{R}^3 \setminus \{0\}$ is determined by the simple formula

$$p : (Z, Z) \mapsto Z_{a_i}Z = X_i$$  \hspace{1cm} (1.2)$$

here $Z = (Z^1, Z^2)$, $\sigma_i$ - Pauli matrices.

Connection form on this bundle is given [5] by

$$W = \frac{Zdz - Zd\bar{z}}{2\bar{z}z}$$  \hspace{1cm} (1.3)$$

In classical mechanics such an approach permits us to consider dynamics on the base as a projection of degenerated Lagrangian dynamics on $P$. Note, that on the base dynamics is not Lagrangian one. The function $\frac{\mu}{2} \dot{X}^2 + eA^X$ is used instead of Lagrangian. The action which it defines is multiple-valued functional of path [11] and it has to regard corresponding variation principle only locally. In the paper [5] Solov'ev proved that on the total space of bundle there exists well defined Lagrangian which is degenerated throughout each gauge
\[ H = \frac{1}{2\mu ZX} \left( \mu \bar{Z} - \frac{n^2}{4Z^2} \right) \]

and

\[ \bar{Z}_j - \bar{Z} = i n \]

respectively [5]. In this form evident is the invariance with regard to the action of group SU(2) on the total space \( Z \rightarrow gZ, g \in SU(2) \) (this fact was pointed out by Horvathy [8]), which commutes with gauge group action. In other words, transformations \( g \in SU(2) \) are automorphisms of fibre bundle \( P \), which induce rotations \( SO(3) \) on the base \( \mathbb{R}^3 \setminus \{0\} \) by (1.2). This symmetry defines a conserved vector \( J_j = \frac{1}{2Z^2} (\bar{Z}_j Z - Z_j \bar{Z}) \) having the commonly used form \( J_i = \mu \varepsilon_{ijk} \left( \frac{n}{2} \frac{\partial}{\partial x^j} \frac{x^k}{Z} \right) \) on the base.

Quantization consists of substituting \( \mu \) for \(-i\partial/\partial Z\) and \( \bar{Z} \) for \(-i\partial/\partial \bar{Z}\), the wave function satisfying an equivariant condition

\[ \Psi(Ze^{i\alpha}, Ze^{-i\alpha}) = e^{in\alpha} \Psi(Z, \bar{Z}) \quad (1.5) \]

in accordance to the constraint condition (1.4).

A total space of fibre bundle formalism is useful in many cases when problems on Dirac monopole are considered. Since wave functions and Green’s functions were defined globally on \( P \), it is more convenient to deal with the bundle space than to work in local terms on the base. Besides, it is possible to set correctly a problem of scattering by monopole in the total space only, since only there the Hamiltonian can be defined globally. Except for Petry work [6] researching scattering problem in total
space in nonstationary setting all preceding works [12-18] considered it on the base.

In Section 2 the solution of scattering problem for spinless particle is obtained in the total space of principal bundle \( P \). The main difficulty is that free and interaction dynamics are related with entirely different spaces \( N_0 \) and \( N_n \) of physical states for different charges \( n \). In stationary scattering theory this leads to the difficulties in setting the boundary condition. In nonstationary one this leads to the difficulties in a choice of the Hamiltonian for free dynamics. Thus, in Petry's work [6] the Hamiltonian \( H_0 = -\frac{1}{2\mu z^2} \delta^2 \) considered as a Hamiltonian of free dynamics includes much part of the interaction with monopole and, besides, its choice is not singlevalued. In Section 2 in the stationary case wave functions, which can be interpreted as scattering states, will be found in the total space \( P \). They are function defined on a Cartesian product of two corresponding bundle spaces and satisfying some symmetry conditions.

Technique developed by solving scattering problem permits us to obtain an explicit expressions for Green's functions in the total space (see section 3). Note, that here we have a much more simple expressions then ones received by Kleinert in [19] by means of path integral calculation.

An important consequence is a finding of representation, the Hamiltonian being written in an extremely simple form. This gives an opportunity to reveal symmetries of the system (section 4). Along with conventional Lie symmetries, \( \text{su}(2) \) Lie algebra of rotations in the total space \( P \) [5,8] and \( \text{so}(2,1) \) [20] time dependent algebra) there was suggested in [21] \( \text{so}(3,1) \) algebra containing afore-mentioned \( \text{su}(2) \) as a subalgebra. The \( \text{so}(3,1) \)
hidden symmetry algebra constructed in classical case was subjected to an examination by Feher [22]. In this paper it was confirmed that so(3,1) symmetry of a charge–Dirac monopole system is impossible. In our representation (in quantum case) symmetries become explicit and we may say that the so(3,1) symmetry algebra does not exist in quantum case. Also we point out some new symmetry algebras.

In the last part of introduction we would like to concern some differential-geometric ingredients being a basis of the subject. Having action of the group $U(1)$ it is easy to find tangent to fibre vector:

$$\frac{d}{d\alpha} \Psi(Ze^{i\alpha}, Ze^{-i\alpha})_{\alpha=0} = i\nu \Psi(Z, Z),$$

$$V = Z\delta - Z\delta.$$ (1.6)

Value $W(i\nu) = i$ is the $U(1)$-generator. Using (1.3) one can find the horizontal vector fields $h_j$ ($j = 1, 2, 3$), i.e. those that $W.h_j = 0$ and which being projected are equal to $\delta \frac{\partial}{\partial Z_j}$:

$$h_j = \frac{1}{2\pi} (Z\sigma_j + \partial \sigma_j Z)$$ (1.7)

$$W_j = \partial \sigma_j, \quad \partial_x = \partial / \partial Z_x, \quad \delta_x = \partial / \partial Z_x.$$

Operators $h_j, V$ satisfy to commutation relations

$$[h_j, V] = 0, \quad [h_j, h_k] = i\Omega_{jkl} V,$$

where $V\Omega_{ij} = 0$, $\Omega_{ij} = -\frac{1}{2} \epsilon_{ijk} \frac{Z_k Z}{(Z^2)^3}$ is the strength of monopole field, $\Omega = dW = \Omega_{ij} \rho^\nu (dX^i, dX^j)$.

The volume element on $\mathbb{C}^2$ which is invariant with respect
to the $U(1)$-action is $dV(Z, Z) = \frac{Z}{n} \, dz_1^1 \wedge dz_2^2 \wedge dz_3^1 \wedge dz_4^2$. It defines inner product [6,23,24]

$$(\Psi_1, \Psi_2) = \int_P \Psi_1^* \Psi_2 \, dV(Z, Z) \quad (1.8)$$

of wave functions on the bundle $P$. If inner product (1.8) is used for variation principle in a total space [23], then we can obtain on the one hand the same Hamiltonian and proper variation principle in a base on the other.

The group $U(1)$ acts on wave functions $\Psi(Z, Z)$ by shifts

$$\Psi(Z, Z) \longrightarrow \Psi(Z\exp(i\alpha), Z\exp(-i\alpha)).$$

Every wave function $\Psi(Z, Z)$ expands into irreducible representations $N_n$ of the group $U(1)$

$$\Psi(Z, Z) = \sum_{n=-\infty}^{\infty} \Psi_n(Z, Z) \quad (1.9)$$

where the wave functions $\Psi_n(Z, Z)$ are defined with the aid of projection operators $P_n$:

$$\Psi_n(Z, Z) = P_n \Psi(Z, Z) = \int_{0}^{2\pi} e^{-i\alpha} \Psi(Z e^{i\alpha}, Z e^{-i\alpha}) \, d\alpha \quad (1.10)$$

It is evident that $\Psi_n(Z, Z)$ satisfies equivariant condition (1.5) and consequently $\Psi_n(Z, Z) \in N_n$ where $N_n$ consists of $n$-equivariant functions. Thus, space of states decomposes in a direct sum of subspaces numbered by the topological charge $n = 2 \text{eg} \in \mathbb{Z}$.

From the $U(1)$-invariance of $dV(Z, Z)$ it follows the orthogonality
of the wave functions $\Psi_n(Z, Z)$, $\Psi_m(Z, Z)$:

$$\int \Psi^*_n \Psi_m dV(Z, Z) = 0, \quad n \neq m$$

It is obvious that

$$[V, P_n] = [h_i, P_n] = 0, \quad VP_0 = P_0 V = 0, \quad V \Psi_n(Z, Z) = n \Psi_n(Z, Z)$$

The function of the form $P_0 \Psi(Z, Z)$ is constant along the fibers or, in other words, can be considered as the function defined on the base $\mathbb{R}^2 \setminus \{0\}$.

Besides, the operators $P_n$, $V$ and $-ih_j$ are hermitian on the functions on $P$ decreasing sufficiently quickly when $Z \to \infty$.

The Shreidinger equation on the bundle space $P$:

$$i\partial_t \Psi(Z, Z) = -\frac{1}{2\mu} h_jh_j \Psi(Z, Z), \quad \Psi(Z, Z) \in \Theta \, N_n, \quad (1.11)$$

here $h_jh_j$ can be calculated by means of identity $(a\bar{a}, b)(c\bar{c}, d) = 2(ad)(\bar{c}b) - (\bar{a}b)(\bar{c}d)$:

$$h_jh_j = \frac{1}{2\mu ZZ} (\partial \bar{\partial} + \frac{v^2}{4ZZ}).$$

It follows that for $\Psi^*(Z, Z) = \Psi_n^*(Z, Z) \in N_n$ we have

$$i\partial_t \Psi_n^*(Z, Z) = -\frac{1}{2\mu ZZ} (\partial \bar{\partial} + \frac{n^2}{4ZZ}) \Psi_n^*(Z, Z). \quad (1.12)$$

It coincides with the equation obtained in [5, 6]. The equation (1.11) for $\Psi(Z, Z) \in \Theta \, N_n$ is rather interesting since there exist symmetries mixing the states with different charges $n$. 
2. SCATTERING BY DIRAC MONOPOLE IN A TOTAL SPACE OF PRINCIPAL FIBRE BUNDLE

In the bundle space $P$ the wave function describing the scattering of charged particle having incident momentum $K^\alpha$ by the Dirac monopole may be found as follows. For momenta $K^\alpha$ one can introduce fibre bundle (we denote it $P^*$) just as it was done for the position coordinates: $K_i = \bar{q}^i q^i$ where $q = (q^1, q^2)$ are variables in $P^*$, and the group $U(1)$ action is $q e^{i\alpha} = (q^1 e^{i\alpha}, q^2 e^{i\alpha})$.

Taking into account gauge invariance and $SU(2)$-rotations of charge - Dirac monopole system [5,8], it is easy to enable that scattering wave function $\Psi_{qq}(Z, \bar{Z})$ defined on $P^* \times P$ is the function of the variables $q Z, Zq$ and $qqZ \bar{Z}$ only. (The groups $SU(2)$ and $U(1)$ may be considered here acting on $P^* \times P$ by the diagonal form: $g : P^* \times P \longrightarrow g P^* \times g P$, $g \in U(1)$ or $g \in SU(2)$.)

One can explain this approach as follows. If we have the scattering in the rotation-symmetrical field, the wave function describing the scattering should be invariant on simultaneous rotation of vectors $K^\alpha$ and $\bar{K}^\alpha$ in $\Psi_{K^\alpha}(\bar{K}^\alpha)$ on the same angle. In the case of Dirac monopole we have a similar situation, in fibre bundle terms it implies $SU(2)$- and $U(1)$-invariance of connection $W (1.3)$ in total space. So it is convenient to think of scattering wave functions as functions of $(q^1, q^2) \in P^*$ connected with $K^\alpha$ by projection map.

It follows that stationary Schrödinger equation

$$(E + \frac{1}{2} \mu \hbar^2 \gamma^i ) \Psi(Z, \bar{Z}) = 0 , \quad \Psi(Z, \bar{Z}) \in \mathcal{G}_n N_n$$
\[ E = \frac{1}{2\mu} K^2 = \frac{1}{2\mu} (\vec{q}q)^2, \] reduces to the equation

\[ i\hbar^2 \frac{\partial^2}{\partial \sigma^2} + \frac{1}{\sigma^2} \left( (1 - \xi\xi) \partial^2 \sigma - \frac{1}{2} (\xi\xi + \xi\xi) \right) + 1) \Psi = 0, \quad (2.1) \]

where

\[ \sigma = \frac{\vec{q}q}{\sqrt{\xi}}, \quad \xi = \frac{Zq}{\sqrt{\sigma}}. \quad (2.2) \]

It is shown in Appendix, that solutions of eq. (2.1) consist

a complete system of functions in \( \bigotimes_n N_n \). One makes choose the

solutions of equation (2.1) as follows. For \( n \geq 0 \)

\[ \Psi_{n\overline{n}q\overline{q}}(Z, \overline{Z}) = \sum_{k=0}^{\infty} e^{ia(k,n)\xi^2} \left[ k + \frac{n + 1}{2} \frac{2\pi}{\sqrt{\sigma}} J_n(k,n) \right] P_{p(0,n)}(2\xi\xi - 1) \quad (2.3) \]

For \( n < 0 \) it should substitute \( \xi \) for \( \xi \) and \( \ln \) for \( n \),

\[ s(k,n) = (k + \frac{n + 1}{2} - \frac{n^2}{4})^{1/2}, \quad J_n(k,n) \quad \text{are Bessel functions}, \]

\[ P_{p(0,n)}(\theta) \quad \text{are Jacobi polynomials}. \]

It is easy to check (see Appendix)

\[ \int \Psi_{n\overline{n}q\overline{q}}^*(Z, \overline{Z}) \Psi_{n\overline{q}'q'}(Z, \overline{Z}) d\overline{V}(Z, \overline{Z}) = (2\pi)^3 s_{n\overline{n}q\overline{q}}(K^2 - K'^2) \]

for wave functions (2.3), when \( q \) and \( q' \) belong to the differ-

tent fibers of bundle \( P^* \). (It is the case of \( \Psi_{n\overline{n}q\overline{q}}(Z, \overline{Z}) \) and

\( \Psi_{n\overline{q}'q'}(Z, \overline{Z}) \) corresponding to different physical states). If \( q \)

and \( q' \) (\( q \neq q' \)) lie on the same fiber over the point \( K_i = \vec{q}_i q = \vec{q}'_i q' \), then \( \Psi_{n\overline{n}q\overline{q}}(Z, \overline{Z}) \) and \( \Psi_{n\overline{q}'q'}(Z, \overline{Z}) \) differ on a

phase factor. For arbitrary \( q, q' \in P^* \) the orthogonality

condition have the form
\[ \int_{\Psi_n\Psi_{\alpha}}(Z,\bar{Z})d\Psi(Z,\bar{Z}) = (2\pi)^{3}\delta_{nm}\delta(K^* - K') \cdot \frac{(\bar{q}'q)^n}{((\bar{q}'q')(\bar{q}q))^n/2} \]  

and we see that the last factor in right handed side is phase factor \( \exp(i\alpha) \) when \( K^* = K' \), \( q = q'\exp(i\alpha) \).

At last, if we assume

\[ \alpha(k, n) = \frac{n}{2} [2k - s(k, n) + \frac{1}{2}] \]

in the formula (2.3), then (it is easy to see)

\[ \Psi_{nq}(Z, \bar{Z}) = \frac{(\bar{q}Z)^n}{(qqZZ)^{n/2}} i^{q\alpha} qZ qZ qZ + \frac{(\bar{q}Z)^n}{(qqZZ)^{n/2}} \sum_{k=0}^{\infty} \left[ k + \frac{n + 1}{2} \right] \exp(i\frac{qZZ}{iq}) \]

when \( r = ZZ \rightarrow \infty \).

The first item is the product of nontrivial factor \( \frac{(\bar{q}Z)^n}{(qqZZ)^{n/2}} \) and function being constant on fibers of bundle \( P \), which can be thought as plane wave on the base. The second term includes the spherical wave, being considered similarly.

From (2.5) one can extract scattering amplitude

\[ f(\theta) = \sum_{n} \left[ k + \frac{n + 1}{2} \right] \frac{1}{iK} \left[ \sin(k - s(k, n) + 1/2) - 1 \right] \cos^{n/2} \cdot P_{k}^{(0,n)}(\cos\theta). \]

Thus, the scattering wave function of charged particle in the field of Dirac monopole in the total space \( P \) is
\[ \Psi_{nqq}^{(Z, Z)} = \frac{(\bar{q}z)^n}{(\bar{q}qZ)_{n/2}} \sum_{k=0}^{n} \left[ \frac{1}{2} (2k - s(k, n)) + \frac{1}{2} \right] \cdot \frac{1}{(2\pi/\bar{q}qZ)_{1/2}} \cdot (2\pi/\bar{q}qZ)^{1/2} \mathcal{M}(k, n)(\bar{q}qZ) P^{(0, n)}_k \left( \frac{Zq}{\bar{q}qZ} - 1 \right). \quad (2.6) \]

In a free case, i.e. when \( n = 0 \), (2.6) coincides with plane wave.

The system of functions (2.3), including the scattering states (2.6), \( \{ \Psi_{nqq}^{(Z, Z)} | (q^1, q^2) \in \mathfrak{P}^* \} \) is complete in \( N_n \) (see Appendix). A completeness condition in \( N_n \) has the following form

\[ \int_{\mathfrak{P}^*} \Psi_{nqq}^{(Z, Z)} \Psi_{nqq}^{(Y, Y)*} d\mathcal{V}(q, \bar{q}) = E_n(Z, \bar{Z}|Y, \bar{Y}) \quad (2.7) \]

where \( d\mathcal{V}(q, \bar{q}) = \frac{\bar{q}q}{n} dq_1^1 dq_2 dq_3 dq_4^1 dq_5^2 dq_6 \), and

\[ E_n = \prod_j \delta(Z\sigma_j Z - Y\sigma_j Y) \frac{(Yz)^n}{(YYZZ)^{n/2}} \quad (n \geq 0), \quad (2.8) \]

\[ E_n = \prod_j \delta(Z\sigma_j Z - Y\sigma_j Y) \frac{(ZY)^{1\bar{n}}}{(YYZZ)^{1\bar{n}/2}} \quad (n < 0) \quad (2.8') \]

are the kernels of identical operators \( \hat{E}_n : \hat{E}_n \Psi = \Psi \) in the space of operators \( N_n \rightarrow N_n \) conserving equivariance condition (1.5). The system of functions \( \{ \Psi_{nqq}^{(Z, Z)} | (q^1, q^2) \in \mathfrak{P}^* \) for all \( n \in \mathbb{Z} \) \} is complete also in \( \bigotimes_n N_n \).

3. Green's Functions in the Total Space \( \mathfrak{P} \).
with help of auxiliary bundle $F^*$ it is easy to obtain convenient form for Green's function of equation (1.7) on the bundle space. Using the results of Section 2 one can write the following integral representation:

$$G_n(t - t'; Z, Z| Y, Y) = \frac{1}{(2\pi)^3} \int \frac{\exp[-i(t - t') \cdot (\vec{\mathbf{q}})^2 / 2\mu]}{p^*}$$

(3.1)

$$\psi_{nq}(Z, Z) \psi^*_{nq}(Y, Y) d\nu(q, \bar{q}) \quad t - t' > 0, \quad (t - t' < 0, \quad G_n = 0)$$

It is evident that

$$\lim_{t \to t'} G_n(t - t'; Z, Z| Y, Y) = E_n(Z, Z| Y, Y)$$

where $E_n(Z, Z| Y, Y)$ - kernel (2.8) of identical operator $E_n$ in the space $N_n$. The integral (3.1) over $F^*$ is easy computable (see Appendix). Thus,

$$G_n(t - t'; Z, Z| Y, Y) = \frac{(2\pi)^2}{4(t-t')/\mu} \frac{(YZ)^n}{(ZZYY)^{(n+1)/2}} \exp \left[ \frac{i((Z\bar{Z})^2 + (YY)^2)}{2(t-t')/\mu} \right]$$

$$\sum_{k=0}^n e^{-i\pi s(k,n)/2} \left[ k + \frac{n + 1}{2} \right] s(k,n) \frac{ZZYY}{(t-t')/\mu} p_{n,k} \frac{(2(YZ)(Y\bar{Z}) - 1)}{YYZZ}$$

(3.2)

For $n < 0$ $YZ$ substitutes for $ZY$ and $n$ for $l$ in $l$.

For Green's function $G_n(Z, Z| Y, Y)$:

$$\frac{1}{ZZ} (\delta^2 + \frac{v^2}{4ZZ}) G_n(Z, Z| Y, Y) = E_n(Z, Z| Y, Y)$$

of the operator $h_j h_j$ on the functions $\psi_n(Z, Z) \in N_n$ it is easy
to produce an explicit formula on the bundle space. It is (see Appendix)

$$G_n(Z, Z| Y, Y) = -\frac{1}{4^n} \left( \frac{Y^2}{YY} \right)^{n+1/2} \tag{3.3}$$

$$\sum_{k=0}^{\infty} \frac{1}{s(k, n)} \left( k + \frac{n+1}{2} \right) P_k(0, n) \left( \frac{ZY}{Y^2} \frac{Z}{YY} \right)^{n+1/2} \left( \frac{YY/ZZ}{YY} \right)^{s(k, n)}, \quad YY \leq ZZ \quad (Z \leq YY) \tag{3.4}$$

Note, concerning (3.2), that Green's functions (3.2) were found in a more simple representation here, then it was done by Kleinert in [19], where path integrals were calculated.

4. SYMMETRIES OF DIRAC MONOPOLE - CHARGE SYSTEM

An important consequence of completeness of scattering wave functions considered in section 2 is an opportunity to reveal symmetries of the system. It is because of the Hamiltonian (see 1.9) has a very simple form in q-representation, and transition to it is given by the formulas:

$$\Psi(Z, Z) = \int P^* F(q, \bar{q}) \Psi_{qq}(Z, Z) dV(q, \bar{q}) \tag{4.1}$$

$$F(q, \bar{q}) = \int P^* \Psi(Z, Z) \Psi^*(qq)(Z, Z) dV(Z, Z) \tag{4.2}$$

where the system of functions of the form

$$\Psi_{qq}(Z, Z) = \frac{1}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \Psi_{nqq}(Z, Z) \tag{4.3}$$

is complete in $\Theta_n N_n$. 

\[ \int_{F^*} \Psi_{qq}^* (Z,\bar{Z}) \Psi_{q\bar{q}}^* (Y,\bar{Y}) dV(q,\bar{q}) = \mathcal{O}_{n=-e} \ E_n (Z,\bar{Z}|Y,\bar{Y}) \quad (4.4) \]

where \( E_n (Z,\bar{Z}|Y,\bar{Y}) \) is given by (2.8).

Transformation into \( q \)-representation, as it follows from (4.2), conserves equivariant condition, i.e. if \( \Psi^r (Z,\bar{Z}) \in N_n^* \), then \( F(q,\bar{q}) \in N_n^* \) (\( N_n^* \) is the set of \( n \)-equivariant functions in the \( q \)-representation). It is easy to show that in the \( q \)-representation the Hamiltonian becomes diagonalised

\[ H \Psi^r (Z,\bar{Z}) = \int_{F^*} \frac{(q\bar{q})^2}{2\mu} F(q,\bar{q}) \Psi_{qq}^r (Z,\bar{Z}) dV(q,\bar{q}) \] 

i.e.

\[ H = \frac{1}{2\mu} (q\bar{q})^2 \quad (4.5) \]

in the \( q \)-representation.

Since on any of 3-dimensional spheres \( q\bar{q} = \text{const} \) the bundle \( P^* \) is Hopf fibre bundle \( S^3 \to S^2 \) \((s^3 \subset P^*)\) which will be denoted here as \( P^*_H \), then consider as the first example the group of its automorphisms \( \text{Aut}^*_H = \{ u \mid u \in \text{Diff}(S^3), u(qe^{i\alpha},qe^{-i\alpha}) = u(q,\bar{q})e^{i\alpha} \text{ for any } e^{i\alpha} \in U(1), q \in S^3 \subset P^*, \bar{q} = \text{const} \} \). It does not change (4.5) and leaves \( F(q,\bar{q}) = F_n (q,\bar{q}) \in N_n^* \) in the same sector \( N_n^* \). Let us find Lie algebra \( \text{aut}^*_H \) of vector fields corresponding to this group. Due to the group homomorphism

\[ r : \text{Aut}^*_H \longrightarrow \text{Diff}(S^2) \]

defined by \((ru)(x) = p(u(q,\bar{q}),\bar{u}(q,\bar{q})), \ x \in S^2, \ (q^1, q^2) \in S^3, \ p \) is projection in bundle \( P^* \) there exists exact sequence

\[ i : \longrightarrow \text{Aut}^*_H \longrightarrow \text{Aut}^*_H \longrightarrow \text{Diff}(S^2) \] 

\( (4.7) \)
Here $\text{Aut}_{v} P_{H}^{*}$ denotes the subgroup in $\text{Aut} P_{H}^{*}$ which consists of the elements $u$ inducing the identity map on base $S^{2}$. Also, if $\text{Diff}_{0}(S^{2})$ is identity component in $\text{Diff}(S^{2})$ and $\text{Aut}_{0} P_{H}^{*} = r^{-1}(\text{Diff}_{0}(S^{2}))$, then

$$1 \longrightarrow \text{Aut}_{V} P_{H}^{*} \longrightarrow \text{Aut}_{0} P_{H}^{*} \longrightarrow \text{Diff}_{0}(S^{2}) \longrightarrow 1 \quad (4.8)$$

is exact sequence. Passing from (4.8) to Lie algebras of vector fields we have appropriate exact sequence

$$1 \longrightarrow \text{aut}_{v} P_{H}^{*} \longrightarrow \text{aut}_{0} P_{H}^{*} \longrightarrow \text{diff}_{0}(S^{2}) \longrightarrow 1 \quad (4.9)$$

where $\text{diff}(S^{2})$ is Lie algebra of vector fields on $S^{2}$, consisting of every possible

$$A_{ij}(q, \bar{q}) \partial_{vi} q \partial_{vj}, A_{ij} \in N_{0}^{*}, A_{ji} = - A_{ij} \quad (i, j = 1, 2, 3)$$

and $\text{aut}_{v} P_{H}^{*}$ is commutative algebra of vector fields

$$ig(q, \bar{q}; V^{*}, g \in N_{0}^{*}, V^{*} = q \partial_{q} - \bar{q} \partial_{\bar{q}}).$$

The exactness of sequence (4.9) means that $\text{aut}_{v} P_{H}^{*}$ is an extension of Lie algebra of vector fields on $S^{2}$ with help of commutative vertical vector fields Lie algebra.

One may find his extension by fixing some connection on $P^{*}$. Thus we can use (1.3): $\mathbf{W}^{*} = \frac{1}{2qq}(q \partial_{q} - q \partial_{\bar{q}})$. The commutators of corresponding to $\mathbf{W}^{*}$ horizontal vector fields $h_{j}^{*}$: $[h_{j}^{*}, h_{k}^{*}] = \Omega_{jk}^{*} V^{*}$

$h_{j} = \frac{1}{2qq}(\bar{q} \partial_{j} \bar{q} + \partial_{q} q)$ determine cocycle $\alpha_{2}(\mu_{1}, \mu_{2}) \in H^{2}(\text{diff}(P^{*}/U(1)), \text{aut}_{v} P^{*})$ of the extension

$$0 \longrightarrow \text{aut}_{v} P^{*} \longrightarrow \text{aut} P^{*} \longrightarrow \text{diff}(P^{*}/U(1)) \longrightarrow 0$$

By the lift
\[ \tau: \mu_j(q, \bar{q}) \partial / \partial k_j \longrightarrow \mu_{1j}(q, \bar{q}) h^*_j, \mu_{2j}(q, \bar{q}) \partial / \partial k_j \longrightarrow \mu_{2j}(q, \bar{q}) h^*_j, \]
\[ \mu_{1j}, \mu_{2j} \in \mathbb{N}_0^* \]
we have \[ \{\tau(\mu_1), \tau(\mu_2)\} = \tau([\mu_1, \mu_2]) + \alpha^2(\mu_1, \mu_2) \]
where \[ \alpha^2(\mu_1, \mu_2) = i\Omega^*(\mu_1, \mu_2)V^* \] and \[ \partial \alpha^2 = 0 \] since \[ d\Omega^* = 0. \]

The 2-cocycle \[ \alpha^2(\mu_1, \mu_2) \] is nontrivial element in cohomology group \[ H^2(\text{diff}(\mathbb{P}^*/U(1), \text{aut}_P \mathbb{P}^*)) \] of Lie algebra of vector fields on \[ \mathbb{P}^*/U(1): \alpha^2(\mu_1, \mu_2) / (\delta \alpha^1)(\mu_1, \mu_2) \]
for any element \[ \alpha^1(\mu) \]
with image in \[ \text{aut}_P \mathbb{P}^* \], since the form \[ \Omega^* \]
realizes nontrivial element in De Rham cohomology group \[ H^2(\mathbb{P}^*/U(1)). \]

If we take other connection \[ \tilde{\nabla}^* \] then corresponding lift leads to an extension which is equivalent to the afore-mentioned. This fact follows from that corresponding to \[ \tilde{\nabla}^* \] vector fields \[ \tilde{h}^*_j \] are \[ \tilde{h}^*_j = h^*_j + ig_j(q, \bar{q})V^* \]
with some \[ g_j \in \mathbb{N}_0^* \] (it is due to \[ ph^*_j = ph^*_j = \partial_j, \]
\[ pv^* = 0 \] and \[ \tau(\mu) = \tau(\mu) + ig(\mu)V^* \], so \[ \{\tilde{\tau}(\mu_1), \tilde{\tau}(\mu_2)\} = \tilde{\tau}([\mu_1, \mu_2]) + i\Omega^*(\mu_1, \mu_2)V^* + i(\mu_1 g(\mu_2) - \mu_2 g(\mu_1) - g([\mu_1, \mu_2]))V^* = \tilde{\tau}([\mu_1, \mu_2]) + i\alpha^2(\mu_1, \mu_2) + (\delta \alpha^1)(\mu_1, \mu_2) \]
with \[ \alpha^1(\mu) = ig(\mu)V^* \].

I.e. the cohomology class \[ \alpha^2(\mu_1, \mu_2) \] is the same, hence [25]
corresponding extensions are equivalent. As the result we may say that Lie algebra of vector fields \[ \text{aut}_P \mathbb{P}^* \]
consists of the elements
\[ \mu_j(q, \bar{q}) h^*_j + ig(q, \bar{q})V^*, \]
where \[ \mu_j, g \in \mathbb{N}_0^* \].

Returning to \[ \mathbb{P}_H^* \subset \mathbb{P}^* \] and to (4.9) we obtain that \[ \text{aut}_P \mathbb{P}_H^* \]
consists of the fields
\[ A = A_{jk}(q, \bar{q}) q g_j q h_k^* + ig(q, \bar{q})V^*, \]  
(4.10)

where \[ A_{jk}, g \in \mathbb{N}_0^* \], \[ A_{jk} = -A_{kj} \].

Operators \[ A \] commute with projection operators (1.10) \[ [A, P^n] = 0 \] and with Hamiltonian (4.5) \[ [A, H] = 0. \] Among the \[ \text{aut}_P \mathbb{P}_H^* \] operators there are more interesting ones, which are Hermitian in regard to inner product (1.8) \[ \langle \Psi_1, \Psi_2 \rangle = \int_{\mathbb{P}_H^*} \Psi_1^* \Psi_2 dV(q, \bar{q}) \]
of the wave functions in \( q \)-representation. The finite-dimensional sub-algebra in \[ \text{aut}_P \mathbb{P}_H^* \] with generators
of SU(2) rotations in the bundle \( P^s \) satisfies this condition. In \( Z \)-representation the generators

\[ J_k = \frac{1}{2} (\mathbb{Z}_k \delta - \delta_k \mathbb{Z}) \]

of SU(2) rotations in \( P \) are corresponding to (4.11). This symmetry is well known. As the next example, let us point out Lie algebra of vector fields which are tangent to fibres \( S^1 \) of the bundle \( P^s \). Basis in this space may be given by the vector fields:

\[ e_n = \frac{\delta q^n}{\left[\delta q(\bar{q} 0)\right]^{n/2}} v^s \quad (n \geq 0), \quad e_n = \frac{\left(\bar{q} 0\right)^{\left|n\right|}}{\left[\delta q(\bar{q} 0)\right]^{\left|n\right|/2}} v^s \quad (n < 0) \]

(\( q \) parametrises different choices of the basis) with commutation relations

\[ [e_n, e_p] = (p - n)e_n + p, \quad [e_n, H] = 0. \]

These operators represent a nontrivial kind of transformations connecting states with different topological charges \( n \)

\[ e_p : N^s \rightarrow N^s, \quad (\text{ker } e_p = N^0). \]

The example (4.12) - (4.14) is in some analogy with results (26) concerning a loop group connecting instanton solutions with different topological charges.

Other Lie algebra (now finite-dimensional) changing \( n \) is seen directly from the form (4.5) of Hamiltonian in q-representation. It is \( su(2) \) algebra with generators
\[ W_1^* = \frac{1}{2} (\bar{q} \sigma_2 \sigma - 5 \sigma_2 q), \quad W_2^* = \frac{1}{2} (\bar{q} \sigma_2 \sigma + 5 \sigma_2 q), \quad (4.15) \]
\[ W_3^* = \frac{1}{2} (q \sigma - \bar{q} \sigma). \]

They are Hermitian in regard to inner product (1.8). Only the last of these generators has a simple form in Z-representation:
\[ W_3 = \frac{1}{2} V. \]

Besides, evident but not less interesting algebra is one formed by the elements
\[ Q_\alpha^* = q_\alpha, \quad \bar{Q}_\alpha^* = \bar{q}_\alpha \quad (\alpha = 1, 2) \quad (4.16) \]
given in Z-representation by integral operators with the kernels
\[ Q_\alpha = \int_{P} \Psi_{qq}^*(Z, Z)q_\alpha \Psi_{qq}(Y, \bar{Y})dV(q, \bar{q}) \quad (4.17) \]

which are generalized functions on P x P, the functions \( \Psi_{qq}^*(Z, Z) \)
(4.3) being their eigenfunctions:
\[ (\hat{q}_\alpha \Psi_{qq}^*)(Z, Z) = q_\alpha \Psi_{qq}^*(Z, Z), \quad (\hat{\bar{q}}_\alpha \Psi_{qq}^*)(Z, Z) = \bar{q}_\alpha \Psi_{qq}^*(Z, Z), \quad (4.18) \]

\[ [\hat{q}_\alpha, H] = [\hat{\bar{q}}_\alpha, H] = 0. \]

It follows from (4.16) that a commutative symmetry algebra formed by the operators
\[ \hat{k}_j = \hat{\tilde{q}}_j \hat{q} \quad (4.19) \]
conserves equivariant condition or in other words does not change the charge \( n \). In accordance with (4.16) and (4.18) the wave functions (2.3) \( \Psi_{qq}^*(Z, Z) \) describing scattering by the Dirac monopole are eigen-functions of (4.19)
\[
\langle \hat{K}_j \psi_{nq}^q(Z, Z) \rangle = \tilde{q}q_j \bar{q} \psi_{nq}^q(Z, Z). 
\]

(4.20)

In conclusion of this section we may point out, as it follows from (4.10) - (4.11), that in quantum case there is no so(3,1) [21] algebra of symmetry (with Hermitian generators in regard to inner product \( \langle \Psi_1, \Psi_2 \rangle = \int p^* \Psi_{1*} \Psi_2 \delta q(q, \bar{q}) \)) which conserves the charge \( n \) and contains (4.11) as subalgebra.

Acknowledgments.

The authors would like to thank Professor I.S. Shapiro and H.A. Solov'ev for helpful discussions.

APPENDIX

To solve equation (2.1) it is necessary first of all to find eigenfunctions of operator

\[
D(\varepsilon, \bar{\varepsilon}) = (1 - \varepsilon \bar{\varepsilon}) \delta_{\varepsilon} \delta_{\bar{\varepsilon}} - \frac{1}{2} (\bar{\varepsilon} \partial_{\varepsilon} + \varepsilon \partial_{\bar{\varepsilon}}). 
\]

(A.1)

For this let us establish the set of values for variables \( \varepsilon, \bar{\varepsilon} \). Since \( \langle qZ \rangle \varepsilon \bar{q}q \bar{Z} \) by Cauchy inequality, then (in accordance with definition (2.2)) \( \varepsilon \) satisfies to inequality \( |\varepsilon| \leq 1 \). Thus, our subject is eigenfunctions

\[
\psi_{\lambda}(\varepsilon, \bar{\varepsilon}) : D\psi_{\lambda}(\varepsilon, \bar{\varepsilon}) = \lambda \psi_{\lambda}(\varepsilon, \bar{\varepsilon})
\]

(A.2)
of operator (A.1), which are defined at the unit circle of the complex variable \( \zeta \) plane.

Because of all functions \( f(\sigma) \) of variable \( \sigma \) are eigenfunctions for projection operator \( P_0 \) (1.5), the expansion on to eigen spaces of projectors \( P_n \) may be carried to the space made up by functions \( u(\varepsilon, \zeta) \) (A.2). Therefore any \( u_\lambda(\varepsilon, \zeta) \in \mathbb{N}_n \), \( n \in \mathbb{Z} \) can be represented as

\[
    u_\lambda^{(n)}(\varepsilon, \zeta) = \varepsilon^n y_n^{(\lambda)}(\varepsilon \zeta), \quad n \geq 0,
\]

\[
    u_\lambda^{(n)}(\varepsilon, \zeta) = \varepsilon^{n+1} y_{\lambda+1}(\varepsilon \zeta), \quad n < 0.
\]

To have a variable on which functions \( y_n^{(\lambda)}(\varepsilon \zeta) \) depend, let us take \( \beta = 2\varepsilon \zeta - 1 \).

As \( \varepsilon, \zeta \) are specified on the unit circle, the variable \( \beta \) is defined on the segment \([-1, 1]\). Since

\[
    D(\varepsilon, \zeta)\varepsilon^n y_n^{(\lambda)}(\beta) = (1 - \zeta \varepsilon)2(n + 1)\varepsilon^n \partial_\beta y_n^{(\lambda)}(\beta) + (1 - \zeta \varepsilon)4\varepsilon \zeta \varepsilon^n
\]

\[
    \partial_\beta^2 y_n^{(\lambda)}(\beta) - (n/2)\varepsilon^n y_n^{(\lambda)}(\beta) - 2\varepsilon \zeta \varepsilon^n \partial_\beta y_n^{(\lambda)}(\beta) =
\]

\[
    \varepsilon^n[(n - (n + 2)\beta)\partial_\beta y_n^{(\lambda)}(\beta) - (n/2) y_n^{(\lambda)}(\beta) + (1 - \beta^2)\partial_\beta^2 y_n^{(\lambda)}(\beta)];
\]

\[
    \varepsilon^n D_n(\beta) y_n^{(\lambda)}(\beta), \quad n \geq 0.
\]

Likewise, for \( n < 0 \) one obtains

\[
    D(\varepsilon, \zeta)\varepsilon^{n+1} y_{n+1}^{(\lambda)}(\beta) = \varepsilon^{n+1} D_{n+1}(\beta) y_{n+1}^{(\lambda)}(\beta)
\]

where for both of cases there is the same operator

\[
    D_{n+1}(\beta) = (1 - \beta^2)\partial_\beta^2 + [(n+1)(n+2)\beta] \partial_\beta - \text{In} / 2.
\]

By this way, the equation (A.2) for eigenfunctions and eigenvalues reduces to

\[
    D_{n+1}(\beta) y_{n+1}^{(\lambda)}(\beta) = y_{n+1}^{(\lambda)}(\beta).
\]
Further, we will concern the case \( n > 0 \), taking into account that \( \varepsilon \) and \( n \) to be replaced by \( \varepsilon \) and \( |n| \) respectively when \( n < 0 \).

With values of \( \lambda : \lambda(k, n) = -nk - k(k + 1) - n/2 \), \( k = 0, 1, 2, \ldots \)

the equation (A.6) provides an equation for the complete orthogonal system of Jacobi polynomials \( p_k^{(0, n)}(\beta) \):

\[
((1 - \beta^2)\beta^2 + ((n - (n + 2))\beta)\beta + nk + k(k + 1))y_n^{(k(k,n))}(\beta) = 0.
\]

Jacobi polynomials

\[
y_n^{(k(k,n))}(\beta) = p_k^{(0, n)}(\beta) = \frac{(-1)^k}{2^k k!} (1 + \beta)^{-n} (1 - \beta)^k (1 + \beta)^{k + n} d^k \beta
\]

are orthogonal with the weight \((1 + \beta)^n\):

\[
\int_{-1}^{1} (1 + \beta)^n p_k^{(0, n)}(\beta)p_{k'}^{(0, n)}(\beta) d\beta = \delta_{kk'} \frac{2^n}{k + (n + 1)/2}.
\]

As the result we obtain the system of orthogonal functions on the unit circle of the plane of complex variable \( \varepsilon \)

\[
u_k^{(n)}(\varepsilon, \varepsilon) = \varepsilon^n p_k^{(0, n)}(2\varepsilon^2 - 1), \quad k = 0, 1, 2, \ldots.
\]

for which (bearing in mind (A.7))

\[
\int u_k^{(n)}(\varepsilon, \varepsilon) u_{k'}^{(n)}(\varepsilon, \varepsilon) (1/2\pi) d\varepsilon d\varepsilon = \delta_{kk'}/(k + (n + 1)/2).
\]

Let us denote \( s(k, n)^2 = [k + \frac{n + 1}{2}]^2 - \frac{n^2}{4} \). Then we can conclude that wave function \( \psi_n(Z, Z) = \psi_n(\sigma, \varepsilon, \varepsilon) \in N_n \) has the expansion

\[
\psi_n(Z, Z) = \sum_{k=0}^{\infty} C(k, n) e^{i\sigma(k, n)Z} J_{s(k, n)}(\sigma) p_k^{(0, n)}(2\varepsilon^2 - 1).
\]

Let us show that solutions (2.3) of the equation (2.1) defined by

\[
\psi_{nq}(Z, Z) = \sum_{k=0}^{\infty} e^{i\sigma(k, n)} e^{i(k - \frac{n + 1}{2})Z} J_{s(k, n)}(\sigma) p_k^{(0, n)}(2\varepsilon^2 - 1)
\]

(A.10)
as \( n \to 0 \) \( a_{m} \) of \( \Psi_{m,q} (Z, \bar{Z}) = \sum_{k=0}^{\infty} e^{i\alpha(k, m)} \epsilon^{m} \left[ k + \frac{n + \frac{1}{2}}{2} \right] \sqrt{2 \epsilon} J_{\frac{1}{2}}(k, m) \epsilon P_{k}(0, m) (2\bar{\epsilon} \epsilon - 1) ^{A.10'} \) \( \) as \( n < 0 \) obey the orthogonality condition (2.4). (Here \( \alpha(k, n) \) are certain real numbers). First of all, evident is faithfulness of the formula

\[
\sum_{k=0}^{\infty} \left[ k + \frac{n + \frac{1}{2}}{2} \right] P_{k}(0, n) \frac{(\epsilon - 1)(\epsilon' - 1)}{\epsilon q q' q'} = 2 \epsilon (1 - \cos(K, K')) \]  \( A.11' \)

in which \( K_{1} = \epsilon q q' q', K_{1}' = \epsilon q q' q' \) and \( (K, K') \) is the angle between vectors \( K \) and \( K' \). It is seen due to completeness of orthogonal system \(( (1 + \beta)/2) \epsilon P_{k}(0, n)(\beta), k = 0, 1, 2, \ldots \) on the segment \([-1, 1]\) and the validity of \( \epsilon P_{k}(0, n)(1) = 1 \).

Now let us concern the inner product (1.8) of the functions \( \Psi_{m,q}(Z, \bar{Z}) \) and \( \Psi_{m,q}(Z, \bar{Z}) \):

\[
\int_{\mathbb{P}} \Psi_{m,q}(Z, \bar{Z}) \Psi_{m,q}(Z, \bar{Z}) dV(Z, \bar{Z}) = 2n_{n} m \epsilon e^{-\alpha(k, n)} e^{\alpha(k', n)} \]

\[
\left[ k + \frac{n + \frac{1}{2}}{2} \right] \left[ k' + \frac{n + \frac{1}{2}}{2} \right]
\]

\[
\int_{\mathbb{P}} J_{s(k, n)}(\epsilon q q' q') J_{s(k', n)}(\epsilon q q' q') \epsilon P_{k}(0, n)(2\bar{\epsilon} \epsilon - 1) \epsilon' P_{k'}(0, n)(2\bar{\epsilon}' \epsilon' - 1) dV
\]

(here \( \epsilon' = \epsilon q \sqrt{q' q' q' Z} \), \( r = Z Z \)). Since the product \( \epsilon P_{k}(0, n) \)

\[
\epsilon P_{k}(0, n)(2\bar{\epsilon} \epsilon - 1) P_{k'}(0, n)(2\bar{\epsilon}' \epsilon' - 1) \in \mathbb{N}_{0}
\]

as function of variables \( Z, \bar{Z} \), i.e. it may considered as function on a base, or more correctly, as function on sphere \( S^{2} \), we can pass to the integrating over base. Then the inner product contains the integral

\[
\int_{\mathbb{P}} J_{s(k, n)}(\epsilon q q' q') J_{s(k', n)}(\epsilon q q' q') \epsilon P_{k}(0, n)(2\bar{\epsilon} \epsilon - 1) \epsilon' P_{k'}(0, n)(2\bar{\epsilon}' \epsilon' - 1) d^{2} X_{n} \]

\[
2\pi \]

\[
\]
\[
\begin{align*}
\mathcal{J}(\xi, \xi') & = \frac{1}{\mathcal{A}_{2}} \int (\xi', \xi) \frac{(2\xi - 1) \mathcal{P}_{k}(\xi, \xi')}{\xi} d\xi d\xi' \\
& = \frac{1}{\mathcal{A}_{2}} \int \frac{2n^{\delta}}{\xi} \frac{(\xi', \xi - 1)}{\xi} d\xi d\xi' \\
& = \frac{1}{\mathcal{A}_{2}} \int \frac{2n^{\delta}}{\xi} \frac{(\xi', \xi - 1)}{\xi} d\xi d\xi'.
\end{align*}
\]

Taking into account the last expression and formula (A.11) (it is need to show only that

\[
\mathcal{J}_{nkk'}(q, q', q', q') = \int (\xi', \xi) \frac{(2\xi - 1) \mathcal{P}_{k}(\xi, \xi')}{\xi} d\xi d\xi' =
\]

\[
\frac{2n^{\delta}}{\xi} \frac{(\xi', \xi - 1)}{\xi} d\xi d\xi' =
\]

\[
\frac{2n^{\delta}}{\xi} \frac{(\xi', \xi - 1)}{\xi} d\xi d\xi'.
\]

and it to be achieved at the end of Appendix), we obtain the orthogonality condition (2.4)

\[
\int \Psi_{nqq}(z, \bar{z}) \Psi_{mq', \bar{z}}(z, \bar{z}) d\nu(z, \bar{z}) = (2n)^{3} \delta_{nm} \delta(\bar{K} - \bar{K}')
\]

for the functions of the form (2.3), provided there is a local section of bundle \( P^* \), joining \( q \) and \( q' \), i.e. when \( q \) and \( q' \) are different physically. It is that because of the factor

\[
\frac{\bar{q}'q'}{\sqrt{(\bar{q}(q')(q'q')}} \rightarrow 1 \quad \text{as} \quad \bar{K} \rightarrow \bar{K}',
\]

when there exists the local section passing through \( q \) and \( q' \).

Let us discuss now the completeness condition in the space \( N_{n} \) of functions (2.3)

\[
\int \Psi_{nqq}(z, \bar{z}), (q^{1}, q^{2}) \in P^* \}
\]

(A.13)
If \((q^1, q^2)\) and \((q'^1, q'^2)\) are the arbitrary points in \(P^*\), then

\[
\int_{P_{nqq'}} \psi_{nqq'}^* (Z, Z) d\nu (Z, Z) = (2\pi)^{3} \delta (R^{*} - R^{*'}) \frac{(\bar{q}' q'^2)^n}{[(\bar{q}' q'^2)(\bar{q} q^2)]^{n/2}} \tag{A.14}
\]

when \(n > 0\), \(K_{A} = \bar{q}' q'^2\), \(K_{A}' = \bar{q}' q'^2\), and

\[
\int_{P_{nqq'}} \psi_{nqq'}^* (Z, Z) d\nu (Z, Z) = (2\pi)^{3} \delta (R^{*} - R^{*'}) \frac{(\bar{q}' q'^2)^{1/n}}{[(\bar{q}' q'^2)(\bar{q} q^2)]^{1/n/2}} \tag{A.14'}
\]

when \(n < 0\). Besides, for functions (2.3) the following equality

\[
\psi_{nqq'}^* (Z, Z) = \psi_{-n2Z} (q, \bar{q}) \tag{A.15}
\]

is valid. (The symmetry condition (A.15) means that if \(\psi_{nqq'} (Z, Z) \in \mathbb{N}_n\) as a function of variables \(Z, Z\) on \(P\), then regarding it as function of variables \(q, \bar{q}\) on \(P^*\) we can write \(\psi_{nqq'} (Z, Z) \in \mathbb{N}_n^*\), where the star in \(\mathbb{N}_n^*\) labels that \(\psi\) regards as function on \(P^*\).) The formulas (A.14) and (A.15) result the completeness condition for system (A.13) in \(\mathbb{N}_n^* :\)

\[
\int_{P_{nqq'}} \psi_{nqq'}^* (Z, Z) \psi_{nqq'}^* (Y, Y) d\nu (q, \bar{q}) = E_n (Z, Z | Y, Y)
\]

where \(d\nu (q, \bar{q}) = \frac{\bar{q} q^2}{n} dq^1 dq^2 \cdot dq'^1 dq'^2\), and

\[
E_n = \prod_{j} \delta (Z_{q_j} Z - Y_{q_j} Y) \left(\frac{YZ}{YZZ}\right)^{n/2} \quad \text{when } n > 0 \quad \tag{A.16}
\]

\[
E_n = \prod_{j} \delta (Z_{q_j} Z - Y_{q_j} Y) \left(\frac{Z}{YZZ}\right)^{1/n} \quad \text{when } n < 0 \quad \tag{A.16'}
\]

Indeed, the operator \(\hat{E}_n\) with kernel (A.16) ((A.16')) is identity operator in the set of operators \(T_n = \{ \tau_n \in T_n \mid \tau_n : \mathbb{N}_n \rightarrow \mathbb{N}_n \}\) which conserve the equivariant condition
\[ (\hat{E}_n \chi_n)(Z, Z) = \int \delta(Z - Y) \frac{(YZ)^{\frac{n}{2}}}{(YZ)^{\frac{n}{2}}} \chi_n(Y, Y) dV(Y, Y), \quad (A.17) \]

\[ \chi_n \in \mathbb{N}_n. \]

It is evident that:

1) \( (\hat{E}_n \chi_n)(Ze^{i\alpha}, Ze^{-i\alpha}) = e^{-i\alpha}(\hat{E}_n \chi_n)(Z, Z) \). it means \( \hat{E}_n \chi_n \in \mathbb{N}_n \)

2) since

\[ \frac{(YZ)^{\frac{n}{2}}}{(YZ)^{\frac{n}{2}}} \chi_n(Y, Y) \in \mathbb{N}_0 \text{ as function of variables } Y, Y, \]

i.e. may be considered as function on the base, the result of integrating \((A.17)\) can be written as

\[ \frac{(Z'Z)^{\frac{n}{2}}}{((Z'Z')(ZZ'))^{\frac{n}{2}}} \chi_n(Z', Z') \tag{A.18} \]

where \( Z \) and \( Z' \) have the same projection \( X \) on the base, \( X_q = Z_q Z = u_q Z' \). Since \( Z \) and \( Z' \) belong to the same fibre over \( X \), we have \( Z' = Z e^{i\alpha} \), and \((A.18)\) reduces to

\[ \exp(-i\alpha) \frac{(ZZ')}{\sqrt{(ZZ')(ZZ')}} \chi_n(Z, Z) = \chi_n(Z, Z). \]

As a result \( \hat{E}_n \chi_n = \chi_n \).  

Finally we may say also that system of functions

\[ \chi_{nqq}(Z, Z), \quad (q^1, q^2) \in \mathbb{P}, \quad n \in \mathbb{Z} \] \tag{A.19}

is complete on \( \mathbb{C}_n \mathbb{N}_n \).

When \( a(k, n) = n(2k - a(k, n) + 1/2)/2 \) functions \( \chi_{nqq}(Z, Z) \)

may be interpreted as scattering states. And so the scattering states \((2.7)\) \( \chi_{nqq}(Z, Z), \quad (q^1, q^2) \in \mathbb{P} \) make up complete system.

Let us now prove the formula \((A.12)\). It is quite evident that

\[ I_{nkk'}(q, q', q, q') = I_{nkk'} \left( \frac{q'q}{((q)(q'q'))^{1/2}}, \frac{qq'}{((q)(q'q'))^{1/2}} \right) \]

Therefore
\[
\frac{1}{qq} (\delta q q + \frac{n^2}{4qq}) I_{n k k', (q, q') q', q') = \\
\frac{1}{(qq)^2} \left( (1 - \bar{\alpha} \alpha) \delta \alpha - \frac{1}{\alpha} (\bar{\alpha} \delta + \bar{\alpha} \epsilon) \right) I_{n k k', (\alpha, \bar{\alpha})} = \\
\frac{1}{(qq)^2} D(\alpha, \bar{\alpha}) I_{n k k', (\alpha, \bar{\alpha})} \\
\]

where \( \alpha = \frac{\bar{\alpha} q}{(\bar{\alpha} q) (\bar{\alpha} q')^{1/2}} \), \( \bar{\alpha} = \frac{\bar{\alpha} q'}{(\bar{\alpha} q) (\bar{\alpha} q')^{1/2}} \). On the other hand, bringing the operator \( \frac{1}{qq} (\delta q q + \frac{n^2}{4qq}) \) under the integral sign, we see

\[
\frac{1}{qq} (\delta q q + \frac{n^2}{4qq}) I_{n k k', (q, q') q', q') = - \frac{1}{(qq)^2} [k(k + 1) + nk + n/2] I_{n k k', (q, q') q', q'}.
\]

Thus, we have

\[
I_{n k k', (q, q') q', q') = 0_k k' c_{n k} \left( \frac{q q'}{(q' q)(q q')} \right)^n \frac{1}{(q' q)(q q')} \frac{(q' q)^n}{(q' q)(q q')} \frac{1}{(q' q)(q q')} \frac{1}{(q' q)(q q')} - 1).
\]

One can find the constant \( C_{n k} \) by calculation the integral

\[
I_{n k k', (q, q') q', q')} \text{ when } q = q' \text{, which is}
\]

\[
I_{n k k', (q, q') q', q')} = 0_k k' c_{n k} = 2n \int [1 + A/2] P_0(0, n)(A) P_0(0, n)(A) dA
\]

\[
= \frac{2n}{k + (n + 1)/2} \text{ in this case. It follows } C_{n k} = \frac{2n}{k + (n + 1)/2}.
\]

Let us consider Green's functions now. For Green's function of equation (1.12):

\[
G_n(t-t'; z, z'; Y, Y') = \frac{1}{(2\pi)^3} \int_0^\beta \exp \left[ -i(t-t') (\bar{q} q') \right] I a(k, n) e^{-i a(k', n)}.
\]

\[
= \frac{2n}{k + (n + 1)/2} \text{ in this case. It follows } C_{n k} = \frac{2n}{k + (n + 1)/2}.
\]

Let us consider Green's functions now. For Green's function of equation (1.12):

\[
G_n(t-t'; z, z'; Y, Y') = \frac{1}{(2\pi)^3} \int_0^\beta \exp \left[ -i(t-t') (\bar{q} q') \right] I a(k, n) e^{-i a(k', n)}.
\]

\[
= \frac{2n}{k + (n + 1)/2} \text{ in this case. It follows } C_{n k} = \frac{2n}{k + (n + 1)/2}.
\]
\[ \frac{(qZ)^n}{(qqZZ)^{n/2}} \frac{(Zq)^n}{(qqYY)^{n/2}} P_{k}^{(0,n)} \left( \frac{2(qZ)(Zq)}{ZqZZ} - 1 \right) P_{k'}^{(0,n)} \left( \frac{2(qY)(Yq)}{qYYY} - 1 \right) dq, dq \]

\[ \frac{1}{2\pi} \frac{1}{(ZqZZ)^{n/2}} \frac{1}{(ZqYY)^{n/2}} \sum_{k=0}^{n} \frac{1}{2} P_{k}^{(0,n)} \left( \frac{2(qY)(YZ)}{ZqYY} - 1 \right) \]

The last integral is familiar

\[ \int_{0}^{\infty} e^{-iak^2} J_{0}(bk) J_{0}(ck) dk = \frac{i}{2} \int_{0}^{\infty} J_{0}(\frac{bc}{2a}) \exp\left( i \frac{b^2 + c^2}{4a} - \frac{i\\mu}{2} \right) \]

Thus, at the result we obtain

\[ G_n(t-t';Z,ZY,Y) = \frac{(2\pi)^2}{1(t-t')/\mu} \frac{(YZ)^n}{(ZYY)^{(n+1)/2}} \exp\left[ i(\frac{ZZ}{ZYY})^{2} + (\frac{YY}{YY})^{2} \right] \]

\[ \sum_{k=0}^{\infty} e^{-\text{im}(k,n)/2} \frac{1}{2} \frac{(2\pi)^2}{1(t-t')/\mu} \frac{(ZZ)^{n+1}}{(ZYY)^{(n+1)+1/2}} \exp\left[ i(\frac{ZZ}{ZYY})^{2} + (\frac{YY}{YY})^{2} \right] \]

for \( n > 0 \) and

\[ G_n(t-t';Z,ZY,Y) = \frac{(2\pi)^2}{1(t-t')/\mu} \frac{(YZ)^{1n}}{(ZYY)^{(1n+1)/2}} \exp\left[ i(\frac{ZZ}{ZYY})^{2} + (\frac{YY}{YY})^{2} \right] \]

\[ \sum_{k=0}^{\infty} e^{-\text{im}(k,1n)/2} \frac{1}{2} \frac{(2\pi)^2}{1(t-t')/\mu} \frac{(ZZ)^{1n+1}}{(ZYY)^{(1n+1)+1/2}} \exp\left[ i(\frac{ZZ}{ZYY})^{2} + (\frac{YY}{YY})^{2} \right] \]

for \( n < 0 \).
The function

\[ G(t - t'; Z, Z| Y, Y) = \sum_{n=-\infty}^{\infty} G_n(t - t'; Z, Z| Y, Y) \]

provides Green's function of equation (1.11). Like in proceeding case the integral representation

\[ G_n(Z, Z| Y, Y) = -\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{1}{p^{n}q^{n}} \psi_{n}(Z, Z) \psi^{*}_{n}(Y, Y) dV(q, \bar{q}) \]

reduces to the expression

\[ G_n(Z, Z| Y, Y) = -\frac{1}{2\pi} \frac{(YZ)^{n}}{(ZZYY)^{(n+1)/2}} \sum_{k=0}^{\infty} \frac{(2Z^{2})^{-1}}{Y^{2}Z^{2}} \int_{0}^{\infty} J_{s(k, n)}(KZZ) J_{s(k, n)}(KY) dK \]

As

\[ \int_{0}^{\infty} J_{s}(aX) J_{s}(bX) dX = \frac{1}{2\pi} \begin{cases} (a/b)^{s} , & \text{when } a < b \\ (b/a)^{s} , & \text{when } b < a \end{cases} \]

we have finally the formula (3.3)

\[ G_n(Z, Z| Y, Y) = -\frac{1}{4\pi} \frac{(YZ)^{n}}{(ZZYY)^{(n+1)/2}} \]

\[ \sum_{k=0}^{\infty} \frac{1}{2} p(0, n) \frac{(2Z^{2})^{-1}}{Y^{2}Z^{2}} \left\{ \begin{array}{l} (YY/ZZ)^{s(k, n)}, YY < ZZ \\ (ZZ/YY)^{s(k, n)}, ZZ < YY \end{array} \right\} \]

for \( n \geq 0 \) and

\[ G_n(Z, Z| Y, Y) = -\frac{1}{4\pi} \frac{(YZ)^{ln!}}{(ZZYY)^{(ln!+1)/2}} \]
for \( n < 0 \). And the function

\[
G(Z, Z | Y, Y) = \sum_{n=-\infty}^{\infty} G_n(Z, Z | Y, Y)
\]

is Green’s function of operator

\[
h_j h_j = \frac{1}{4\pi} \left( \partial \delta + \frac{v^2}{4\pi z^2} \right)
\]
on functions \( \Psi(Z, Z) \in \Omega_n N_n \).
REFERENCES


Препринты Физического института имени П.Н. Лебедева АН СССР рассылаются научным организациям на основе взаимного обмена.

Наш адрес: 117924, Москва В-333, Ленинский проспект, 53

Preprints of the P.N. Lebedev Physical Institute of the Academy of Sciences of the USSR are distributed by scientific organizations on the basis of mutual exchange.

Our address is: USSR, 117924, Moscow В-333, Leninsky prospect, 53