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TOWERS OF ALGEBRAS IN RATIONAL CONFORMAL FIELD THEORIES

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A B S T R A C T

Jones fundamental construction is applied to Rational Conformal Field Theories. The Jones algebra which emerges in this application is realized in terms of duality operations. The generators of the algebra are an open version of Verlinde's operators. The polynomial equations appear in this context as sufficient conditions for the existence of Jones algebra. The ADE classification of modular invariant partition function is put in correspondence with Jones classification of subfactors.

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1. - INTRODUCTION

Conformal field theory has become an excellent theoretical laboratory to study the interconnected concepts of duality and integrability. New branches of mathematics as knot theory [1] or Hopf algebras (quantum groups) [2] appear as very appropriate tools to unravel the secrets of this fascinating dynamics.

Two dimensional conformal field theories are in principle completely determined by the chiral algebra and the generic condition, on physical observables, to be invariant under duality and modular transformations [3 - 5]. The chiral algebra - "tensor calculus", i.e., the algebraic relations satisfied by the chiral vertices [6 - 9] (intertwining operators between different irreducible representations of the chiral algebra) share important similarities with the one defined for quasitriangular Hopf algebras [10 - 11]. This connection, already pointed out in reference [8], was established with some generality in reference [12].

In the quantum group approach duality plays the role of the commutant of a quasitriangular Hopf algebra, in other words it is the underline structure of the decomposition rules of tensor products of irreducible representations of the chiral algebra, decomposition which is determined by the fusion rules of the theory. In this framework it is natural to ask about the relationship between fusion algebra and duality. It turns out that the answer is in spirit very close to the relation existing between the rules used to define IRF models [13], the constraints between the allowed values of the statistical variables on adjacent sites of the lattice, and its integrability properties, the Yang - Baxter relations for the corresponding Boltzmann weights. This relation with integrable models is at the origin of recent results connecting braiding properties of conformal field theories and the Boltzmann weights of some IRF models [14 - 15].

The natural mathematical framework to look for the desired connection between fusion rules and duality is the Jones construction of towers of multi - matrix algebras [16]. This is in fact a construction that was already used in the context of lattice models to derive the Temperley - Lieb algebra, a close relative of the braid group, from the rules of construction of the model [17].

Many steps in the Jones construction have a natural correspondence with recent results in conformal field theory in particular with Verlinde's theorem [18] and with the applications and generalizations of knot theory developed by Witten [19].

Jones algebra, an ancestor of duality, is built as a set of projections defined on a tower of multi - matrix algebras. In the case of conformal field theory this tower is completely determined by the fusion rules of the theory. Under very general conditions a trace consistent with the tower of inclusions defined by the fusion algebra exists. This trace satisfies the Markov property, the algebraic parallel of topological invariance, and determines the defining relations of the Jones algebra. This fact potentially opens a common avenue for understanding both the origin of duality and integrability. Three dimensional Chern - Simons physics [19] is a beautiful way to materialize in physical terms the Markov property.

The purpose of the present paper is to use Jones fundamental construction as a way to get from the fusion algebra part of the duality and modular properties of the corresponding conformal field theory. In particular we will present explicit representations of Jones algebra in terms of purely conformal field theory objects. As an important bonus of the construction we find a new way to understand the classification of modular invariants. In fact for each 'simple current' in the language of reference [20] we find a particular realization of Jones algebra as projections on a different tower. Using the results of [20] we can relate Jones classification of subfactors with the one of modular invariants. These results shed light on the origin of the ADF classification [21].

The content of the paper is divided in five sections. The first one is a general introduction to the theory of towers of algebras [22]. It contains the necessary stuff for the rest of the paper. The second section is intended to give the first physical application of Jones and Ocneanu's work in the context of IRF - lattice models. The connection between braiding matrices and Boltzmann weights is studied in this general framework. Sections three and four are the core of the paper. In section three we translate Jones construction to the language of conformal field theories. In particular we find explicit representation of the generators of Jones algebra in terms of an open version of Verlinde's operators. In this context the polynomial equations are sufficient conditions for the algebra to exist. In the last section we attempt to clarify the origin of the ADF classification of modular invariant partition functions in terms of Jones classification of subfactors.

2. – MATHEMATICAL BACKGROUND

This chapter is intended to provide the mathematical tools that will be used in the rest of the paper. The reader familiar with Jones recent work on C^* -algebras and the theory of subfactors can go directly to the next chapter.

Of fundamental importance in Jones approach to the theory of knot invariants is the C^* -algebra

$\Lambda_{\beta,n}$ defined by:

$$\begin{aligned} 1, e_1, e_2, \dots, e_{n-1}, \\ e_i^2 = e_i, \quad e_i^* = e_i \\ \beta e_i e_{i+1} e_i = e_i \\ e_i e_j = e_j e_i \quad |i-j| \geq 2 \end{aligned} \tag{2.1}$$

and equipped with a trace satisfying the Markov property

$$tr(xe_n) = \beta^{-1} tr(x) \tag{2.2}$$

for x any word on $\{e_1, \dots, e_{n-1}\}$. The algebra $\Lambda_{\beta,n}$ is intimately connected with the Hecke algebra $H_n(q)$ defined by:

$$\begin{aligned} g_i^2 &= (q-1)g_i + q \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \\ g_i g_j &= g_j g_i \quad |i-j| \geq 2 \end{aligned} \tag{2.3}$$

The map going from $H_n(q)$ to $\Lambda_{\beta,n}$ is defined by:

$$\begin{aligned} \pi : H_n(q) &\rightarrow \Lambda_{\beta,n} \\ \pi(g_i) &= (q+1)e_i - 1 \end{aligned} \tag{2.4}$$

with $\beta = 2 + q + q^{-1}$ elements in $H_n(q)$ of the form:

$$x_i = g_i g_{i+1} g_i + g_i g_{i+1} + g_{i+1} g_i + g_i + g_{i+1} + 1 \tag{2.5}$$

are in the kernel of the map π . Jones knot invariants are obtained representing the braid group B_n into $\Lambda_{\beta,n}$ by the map (2.4) $\pi(e_i) = (q+1)e_i - 1$ and after that computing the Markov trace of the corresponding element in $\Lambda_{\beta,n}$.

In this chapter we will present the construction of the algebra $\Lambda_{\beta,n}$ in the general framework of towers of algebras. This construction will be essential for understanding the role of these algebras in statistical mechanics and conformal field theory.

Towers of multi – matrix algebras: The fundamental construction

Consider a multi – matrix algebra $M^{(0)} = \bigoplus_{i=1}^n M^{(0)}_{\mu_i}$ with $M^{(0)}_{\mu_i}$ the space of $\mu_i \times \mu_i$ – complex matrices. Let $M^{(1)} = \bigoplus_{j=1}^n M^{(1)}_{\nu_j}$ be a multi – matrix subalgebra of $M^{(0)}$ containing the identity. The inclusion $M^{(1)} \subset M^{(0)}$ is defined by a $n \times m$ matrix Λ such that:

$$\mu_i = \sum_{j=1}^n \nu_j \Lambda_{ji} \tag{2.6}$$

The inclusion defined by Λ maps any element $(y_1 y_2 \dots y_n)$ of $M^{(0)}$ into the element $(x_1 x_2 \dots x_m)$ of $M^{(1)}$ defined by:

$$x_i = \text{diag} (y_1 \dots y_1, \dots, y_n \dots y_n) \tag{2.7}$$

where y_j appears Λ_{ji} times. Associated with this pair of algebras we can define a new one $M^{(2)} = \text{End}_{M^{(0)}}(M^{(1)})$ defined as the space of endomorphisms of $M^{(1)}$ considered as a right $M^{(0)}$ – module. This simply means the space of endomorphisms $\hat{a} : M^{(1)} \rightarrow M^{(1)}$ satisfying $\hat{a}(xy) = \hat{a}(x)y$ for any $x \in M^{(1)}$ and $y \in M^{(0)}$. Notice that the space $M^{(1)}$ itself can be visualized as a subspace of $M^{(2)}$. In fact to any element $x \in M^{(1)}$ we can associate an endomorphism $\hat{x}$ in $M^{(2)}$ defined by left multiplication with x :

$$\hat{x} : z \in M^{(1)} \rightarrow \hat{x}(z) = xz \tag{2.8}$$

The space $M^{(2)}$ is again a multi – matrix algebra: $M^{(2)} = \bigoplus_{j=1}^n M^{(2)}_{f_j}$ with $f_j = \sum_{i=1}^n \mu_i \Lambda_{ji}^T$. The previous construction can be now repeated starting with the pair $M^{(1)} \subset M^{(2)}$ with the inclusion defined by the transposed Λ^T of the inclusion matrix Λ . The new algebra $M^{(3)}$ is defined as $\text{End}_{M^{(2)}}(M^{(1)})$. Continuing this construction we get a tower of algebras:

$$M^{(0)} \subset M^{(1)} \subset M^{(2)} \subset \dots \quad (2.9)$$

with the inclusion matrix $\Lambda \left(M^{(2k)} \subset M^{(2k+1)} \right)$ and its transposed $\Lambda^T \left(M^{(2k+1)} \subset M^{(2k)} \right)$

Bratteli diagrams

Given a pair of multi-matrix algebras $M^{(0)} \subset M^{(1)}$ with inclusion matrix Λ we can associate to them a bicolored diagram, the Bratteli diagram, as follows. For $M^{(0)} = \bigoplus_{j=1}^n M_{j_j}^{(0)}$, $M^{(1)} = \bigoplus_{i=1}^m M_{i_i}^{(1)}$ we define a diagram with $n+m$ points, n of them colored black and the rest white. The j -point black is joined to the i -point white by Λ_{ji} segments. Each point in the diagram is accompanied by the corresponding dimension. As an example consider the pair of multi-matrix algebras:

$$M^{(0)} = \mathbb{C} \oplus M_2^{(0)} \subset M^{(1)} = \mathbb{C} \oplus M_3^{(1)} \oplus M_2^{(1)} \oplus M_2^{(1)} \subset M^{(2)}$$

$$\Lambda = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad (2.10)$$

the corresponding Bratteli diagram is:

Notice that to fix completely the diagram is enough to give the matrix Λ and the set of dimensions appearing in the first row. The interest of these diagrams is that they fix, up to isomorphisms, the inclusion of a pair of multi-matrix algebras. In the rest of this paper we will always consider Bratteli diagrams which are connected. For a tower of algebras $M^{(0)} \subset M^{(1)} \subset M^{(2)} \subset \dots$ the Bratteli diagram is obtained by extending the one associated with the inclusion $M^{(0)} \subset M^{(1)}$ with the one describing the inclusion $M^{(1)} \subset M^{(2)}$ and so on.

The problem of classification of Bratteli diagrams for multi-matrix algebras reduces to the one of classifying finite matrices over the non negative integers. An old result of Kroneker is that for any finite matrix Λ over the integers the norm of Λ , $\|\Lambda\|$ is either ≥ 2 or equal to $2\cos \frac{\pi}{q}$ for some natural $q \geq 2$. The norm of a $(n \times m)$ -matrix $\Lambda: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined as the $\text{Max} \{ \|\Lambda \xi\|; \xi \in \mathbb{R}^n, \|\xi\| \leq 1 \}$. These results provide a complete classification of the finite connected graphs associated with inclusion matrices Λ of norm smaller than 2. All these graphs are given by the Coxeter diagrams of the ADE series.

Given a Bratteli diagram a "model" of the diagram is defined as any tower of algebras whose inclusions are the ones described by the diagram. A model that will be useful in our discussion of statistical mechanics is Ocneanu's path model. The construction of the model is as follows. First you extend the diagram by adding an upper point joined to each vertex of the first row by as many lines as the dimension associated to the vertex (fig 2). Next you define paths on the diagram starting at the upper point and such that you pass only once through each level of the diagram (fig 3). Let us call Ω the space of these infinite paths (fig 3,4). Let us now consider a pair of two paths $(\xi, \eta) \in \Omega$ finishing at the same point in the level $-r$. Associated with this pair we define an operator $T_{(\xi, \eta)(r)}$ acting on elements ω of Ω as follows:

$$T_{(\xi, \eta)(r)} \omega = \delta(\xi, \omega \circ \eta) \eta \circ \omega \circ \xi \quad (2.12)$$

where $\circ$ denotes the usual composition of paths and $\omega \circ \eta, \omega \circ \xi$ are respectively the parts of the path ω finishing and starting at the level r of the diagram (fig 5). The space of operators $T_{(\xi, \eta)(r)}$ define an algebra A_r . The model of the Bratteli diagram is defined by the tower of algebras $A_0 \subset A_1 \subset A_2 \subset \dots$. It is amusing to notice the similarity between the operators $T_{(\xi, \eta)}$ and Witten's vertex in string theory (fig 6).

Traces: The conditional expectation.

Consider again a pair $M^{(0)} \subset M^{(1)}$ of multi-matrix algebras $M^{(0)} = \bigoplus_{j=1}^n M_{j_j}^{(0)}, M^{(1)} = \bigoplus_{i=1}^m M_{i_i}^{(1)}$ with inclusion matrix Λ . Let us now define traces on these algebras in a way consistent with the inclusion defined by Λ . A trace tr on the multi-matrix algebra $M^{(1)}$ is defined by a vector $(t_1^{(1)}, \dots, t_m^{(1)})$ as follows:

$$x \in M^{(1)}; \quad \text{tr}^{(1)}(x) \equiv \sum_{i=1}^m t_i^{(1)} \text{tr}(x_i) \quad (2.13)$$

Two traces $\text{tr}^{(0)}, \text{tr}^{(1)}$ defined on a pair of multi-matrix algebras $M^{(0)} \subset M^{(1)}$ with inclusion matrix defined by Λ are consistent if:

$$\sum_{i=1}^m \Lambda_{ji} t_i^{(1)} = t_j^{(0)} \quad (2.14)$$

which simply means that:

$$t_r^{(1)}(\pi(y)) = t_r^{(0)}(y) \quad \forall y \in M^{(0)} \quad (2.15)$$

with π the embedding map associated to Λ defined by the equation (2.7). The traces are faithful if all the components of the corresponding vector are different from zero. For a pair of algebras $M \subset M^{(1)}$ the trace defined on $M^{(0)}$ by the vector $\vec{t}^{(0)}$ satisfying (2.14) will be called the restriction to $M^{(0)}$ of the trace defined on $M^{(1)}$.

Given a pair of algebras $M \subset M^{(1)}$ equipped with a faithful trace on $M^{(1)}$ such that its restriction to $M^{(0)}$ is also faithful there exist a unique linear map $E: M^{(0)} \rightarrow M^{(0)}$ such that:

$$\begin{aligned} \text{i)} \quad t_r^{(0)}(E(x)) &= t_r^{(1)}(x) & x \in M^{(1)} \\ \text{ii)} \quad E(\gamma_1 x \gamma_2) &= \gamma_1 E(x) \gamma_2 & \gamma_1, \gamma_2 \in M^{(0)}; x \in M^{(1)} \\ \text{iii)} \quad E(\gamma) &= \gamma & \gamma \in M^{(0)} \end{aligned}$$

This map is called a conditional expectation from $M^{(1)}$ to $M^{(0)}$. The map E is simply the orthogonal projection on $M^{(0)}$ defined with respect to the scalar product in $M^{(1)}$ given by $\langle x, y \rangle = \vec{t}^{(1)}(xy)$, $x, y \in M^{(1)}$. Since the trace on $M^{(1)}$ and its restriction to $M^{(0)}$ are both faithful we have:

$$M^{(1)} = M^{(0)} \oplus M^{(0)\perp}$$

The property ii) of the map E implies that E belongs to $\text{End}_{M^{(0)}} M^{(1)}$ and therefore is an element of the algebra $M^{(1)}$ obtained from the pair $M \subset M^{(1)}$ by the fundamental construction. Moreover it is easy to prove that the whole algebra $M^{(1)}$ is, as a vector space, generated by $M^{(1)}$ and elements of the form $x E y$; $x, y \in M^{(1)}$.

Markov trace: The algebra $A_{p,1,n}$.

Our next goal will be to use the fundamental construction to define a family of conditional expectations, one for each pair in the tower, and to look for its algebraic relations. The first step of this construction will be, of course, to characterize the traces defined on the first pair of the tower that can

be extended to the rest of the algebras. Taking into account that the new elements that appear in the algebra $M^{(2)}$ are of the form $x E y$, with $x, y \in M^{(1)}$, the only thing we need to extend the trace on $M^{(1)}$ to a faithful trace on $M^{(2)}$ is to define the trace of elements of the kind $x E y$ with $x, y \in M^{(1)}$.

We say that a trace on $M^{(1)}$ is a Markov trace with modulus β if there exists a trace on $M^{(2)}$ such that:

$$t_r^{(2)}(x E y) = \beta^{-1} t_r^{(1)}(x), \quad x, y \in M^{(1)} \quad (2.16)$$

A trace on $M^{(1)}$ satisfying this property for some β certainly propagates to the whole tower. Considering now the pair $M^{(1)} \subset M^{(2)}$ and using $t_r^{(2)}$ on $M^{(1)}$ we can define a new conditional expectation. Moreover we can associate to the whole tower of algebras $M^{(0)} \subset M^{(1)} \subset \dots$ obtained by the fundamental construction a family of conditional expectations:

$$\begin{aligned} E_1 : M^{(1)} &\rightarrow M^{(0)} \\ E_2 : M^{(2)} &\rightarrow M^{(1)} \\ \vdots & \\ E_i : M^{(i)} &\rightarrow M^{(i-1)} \end{aligned}$$

that will be the candidates for generators of the algebra $A_{p,1,n}$ defined in (2.1).

Before studying the algebraic relations between the conditional expectations associated with a tower of algebras we need to know under what conditions we have a Markov trace and for what values of β . If we consider the case of multi-matrix algebras $M^{(0)} \subset M^{(1)}$ with a connected Bratteli diagram we will always have a Markov trace for the modulus β given by:

$$\beta = \|\Lambda\|^2 \quad (2.17)$$

The value $\|\Lambda\|^2$ have a deep mathematical meaning which is at the origin of the problem of classification of subfactors. In fact $\|\Lambda\|^2$ is the value of the index $[M^{(1)} : M^{(0)}]$. This index is defined as a generalization of the Murray-von Neumann coupling constant as $\dim_{M^{(0)}} L^2(M^{(1)}; \frac{1}{k})$ which is a measure of the "size" of $L^2(M^{(1)}; \frac{1}{k})$ as $M^{(0)}$ module. The space $L^2(M^{(1)}; \frac{1}{k})$ is the Hilbert space defined on $M^{(1)}$ using as scalar product $\langle x, y \rangle = \frac{1}{k} \text{tr}(xy)$. To relate the index with the inclusion matrix allows us to use the classification of finite matrices to get constraints on the value of the index. In the rest of this paper the meaning of β as an index will be of little relevance.

Let us now come back to the problem of determining the algebraic relations satisfied by the conditional expectations.

If the pair $M^{(0)} \subset M^{(1)}$ is equipped with a Markov trace then the conditional expectations satisfy:

$$\begin{aligned} \beta \ E_i \ E_{i+1} \ E_i &= E_i \\ E_i \ E_j &= E_j \ E_i \\ \beta \ \text{tr}(x E_i) &= \text{tr}(x) \quad , \quad x \in M^{(i)} \end{aligned} \quad (2.18)$$

From this relations follows that the algebra generated by $\{1, E_1, \dots, E_{n-1}\}$ is isomorphic to $A_{\beta, n}$ as defined by (2.1) and (2.2). The condition $E_i^2 = E_i$ is automatic, recall that the conditional expectations are defined as projectors.

To prove the relations (2.18) we need to make use of the Markov property of the trace. Consider for instance the first relation in (2.18). Let us prove it for the case $i = 1$. We have the algebras $M^{(0)} \subset M^{(1)} \subset \dots$. The element $\beta \ E_1 \ E_2 \ E_1$ belongs to $M^{(3)}$ and therefore is acting on $M^{(2)}$ as an endomorphism. A generic element of $M^{(3)}$ is of the kind $x \ E_1 \ y$ with x, y elements of $M^{(2)}$. Acting on this element with $\beta \ E_1 \ E_2 \ E_1$ we get:

$$\begin{aligned} \beta \ E_1 \ E_2 \ E_1 [x \ E_1 \ y] &= \beta \ E_1 \ E_2 [E_1(x) \ E_1 \ y] \\ &= E_1 [E_1(x) \ \beta \ E_2(E_1 \ y)] \end{aligned} \quad (2.19)$$

Let us now compute $\beta \ E_2(E_1)$. First it is easy to check that:

$$\begin{aligned} \text{tr}(\beta \ E_1 x - x) &= \beta \ \text{tr}(E_1 x) - \text{tr}(x) \\ &= \beta \ \text{tr}(x E_1) - \text{tr}(x) = 0 \quad , \quad \forall x \in M^{(1)} \end{aligned} \quad (2.20)$$

where we have used the Markov property of the trace. Using the trace to define the scalar product we get the orthogonal decomposition of $\beta \ E_1$:

$$\beta \ E_1 = (\beta \ E_1 - 1) + 1 \in M^{(1)\perp} \oplus M^{(1)} = M^{(2)} \quad (2.21)$$

Now using the fact that E_2 is the orthogonal projection on $M^{(2)}$ we get:

$$\beta \ E_2(E_1) = 1 \quad (2.22)$$

Finally from (2.22) and (2.19) we get:

$$\begin{aligned} \beta \ E_1 \ E_2 \ E_1 [x \ E_1 \ y] &= E_1 [E_1(x) \ y] = E_1(x) \ E_1(y) \\ &= E_1(x \ E_1 \ y) \end{aligned} \quad (2.23)$$

which concludes the proof of the first relation (2.18). Notice that the two essential ingredients of the proof are the Markov property of the trace and the fact that

Diagrammatic Representation of the algebra $A_{\beta, n}$

We will finish this chapter describing a useful pictorial representation of relations (2.18). To each generator E_i of $A_{\beta, n}$ we associate a graph as in fig. 7. Next we define multiplication of these graphs by the following rules. Draw the two graphs one wishes to multiply one below the other, connect the i -th upper vertex with the lower i -th vertex and finally disregard in the resulting graph all the cycles. The graph obtained will be multiplied by $(\beta)^{n/2}$ with n the number of cycles appearing in the process of multiplication. In fig. 8 the relations (2.18) are depicted.

3. - IRF - MODELS DEFINED ON A GRAPH

In this model we will construct solvable IRF models associated to some connected Bratteli diagrams. The construction will be in spirit very close to the one developed by Pasquier [17] for Coxeter diagrams. The recently discovered connection between Boltzmann weights of IRF models and braiding matrices of some conformal field theories [14 - 15] will appear as a natural corollary of the fundamental construction described in chapter II.

A solvable IRF model [1.3] is defined by giving the state variables, the heights

ℓ_i and the constraints that the heights on adjacent sites of the lattice must satisfy. Once this information is given the model is characterized by Boltzmann weights $W(\ell_1, \ell_2, \ell_3, \ell_4 | u)$ associated to each plaquette (fig.9) satisfying the star triangular equation:

$$\sum_c W(bdc|u) W(acfg|u+v) W(cdef|v) = \sum_c W(abc|g|v) W(bdec|u+v) W(cdfg|u) \quad (3.1)$$

with u the spectral parameter. For a graphical representation of equation (3.1) see fig.10. Equation (3.1) simply means the commutativity of transfer matrices. Recall that the transfer matrix is defined as the product of Boltzmann weights for the set of plaquettes enclosed between two consecutive rows of the lattice.

Sometimes the constraints on the values of the state variables on adjacent sites of the lattice can be used to define a graph associated to the model. To each site of the graph we will associate one height. The number of vertices of the graph will be equal to the number of different state variables used to define the model. Two heights can appear in adjacent sites of the lattice if and only if they are associated in the graph to adjacent vertices. IRF models satisfying these constraints will be called graph-IRF models [17-23-24].

IRF models from Bratteli diagrams. -

Let us consider a connected Bratteli diagram such that the vector of dimensions on the first row is $(1,1,\dots,1)$ fig.11. Let us now associate state variables ℓ_i ($i=1,\dots,6$ for the example of fig.11.) to the vertices of the Bratteli diagram (fig.12). Configurations on the lattice will correspond to distributions of heights on the sites of the lattice satisfying the constraints imposed by the graph (fig.12). This configurations will divide the lattice into two sublattices one black and another white corresponding to the bicoloration of the graph (fig.13). To construct a solvable model we need to find the Boltzmann weights satisfying (3.1). In order to do that we will use the fundamental construction and Ocneanu's path model [25].

As we show in chapter I the Bratteli diagram defines , up to isomorphism, a pair of algebras $\mathcal{H}^{(0)} \subset \mathcal{H}^{(1)}$. The fundamental construction provides a unique way to define an infinite tower of algebras $\mathcal{H}^{(0)} \subset \mathcal{H}^{(1)} \subset \mathcal{H}^{(2)} \dots$. An interesting ,but simple fact, is that the space Ω of paths on the extended Bratteli diagram is one to one related to the space of states , defined at "fixed time", of the IRF model given by the Bratteli graph. In fact any path $\omega \in \Omega$ can be parametrized by the heights of the vertices through which it passes the path: $\omega \in \Omega \rightarrow |\ell_1, \ell_2, \dots, \ell_i, \dots\rangle$ and this can be certainly interpreted as a state of the lattice model (fig.14). Taking into account that the Bratteli diagram we start with is connected we can define a Markov trace with $\beta = \|\Lambda\|^2$. Using this trace we can define the family of conditional expectations E_i satisfying (2.18).

The Markov trace can be used to define a functional on the heights . In fact the trace is determined by two vectors $\vec{t}^{(0)}, \vec{t}^{(1)}$ defining the trace on the pair of multi - matrix algebras associate to the diagram (see (2.13)). Once we associate heights to the points in the diagram we can define the functionals $t(\ell_i)$ assigning to ℓ_i the value of $\vec{t}$ on the corresponding point. For the example of fig.12 we will get :

$$t(\ell_1) = t_1^{(0)}, \quad t(\ell_2) = t_2^{(1)}, \quad t(\ell_3) = t_2^{(0)} \dots \quad (3.2)$$

Now we will give the explicit expression of the conditional expectation E_1 using the operators (2.12) defined in the path model:

$$E_1 = \sum_{\substack{\ell, \lambda \in \Omega[0,1] \\ \ell_0 = \lambda_0}} \frac{\sqrt{t^{(1)}(\ell_1)} \sqrt{t^{(1)}(\lambda_1)}}{t^{(0)}(\ell_0)} T_{(\lambda, \lambda^*)}(\ell, \ell^*) \quad (3.3)$$

where the paths in the sum are paths starting at level 0 and finishing at level 1 , and such that they start at the same point. The paths $(\lambda, \lambda^*) \in \Omega_{\ell_0, \ell_1}^{(1)}$ are the ones symmetric with respect to the level 1. For the example of fig.11 the representation of E_1 is depicted in fig.15. On a generic path - state we get:

$$E_1 | \ell_1, \ell_2, \ell_3, \dots \rangle = \sum_{\ell'_1} \frac{\sqrt{t(\ell_2)} \sqrt{t(\ell'_1)}}{t(\ell_1)} \delta(\ell_1, \ell'_1) | \ell_1, \ell'_1, \ell_2, \dots \rangle \quad (3.4)$$

The generalization of (3.4) to any other conditional expectation E_i is straightforward. The conditional expectations defined by (3.4) satisfy relations (2.18) for $\beta = \|\Lambda\|^2$ and Λ the inclusion matrix of the Bratteli diagram. Using now the map (2.4) relating the algebra $A_{\beta, \Lambda}$ to the Hecke algebra we can define operators g_i satisfying the Yang-Baxter relation (2.3). The g_i are defined by :

$$g_i = (q+1) E_i - \mathbb{1} \quad (3.5)$$

The g_i defined by (3.5) are good candidates for Boltzmann weights. In fact if we define the whole transfer matrix of the model in terms of the g_i and we represent that as the value of the Boltzmann weight for the interpolating plaquette we get :

$$\langle e_i^1 \dots | g_n | e_i^0 \dots \rangle = \delta_{e_i^0 e_i^1} \dots \delta_{e_{i,n-1}^1 e_{i,n}^1} W(e_{i,n-1}^1, e_{i,n}^1, e_{i,n}^1, e_{i,n}^1) \delta_{e_{i,n+1}^1 e_{i,n+2}^1} \dots \quad (3.6)$$

with the Boltzmann weight defined by:

$$W(e_{i,n-1}^1, e_{i,n}^1, e_{i,n+1}^1, e_{i,n+2}^1) = \begin{array}{|c|} \hline e_{i,n+1}^1 \\ \hline \square \\ \hline e_{i,n-1}^1 \\ \hline \end{array} = (q+1) \delta_{e_{i,n-1}^1 e_{i,n+1}^1} \frac{\sqrt{t(e_{i,n}^1) t(e_{i,n+2}^1)}}{t(e_{i,n+1}^1)} - \delta_{e_{i,n-1}^1 e_{i,n+2}^1} \quad (3.7)$$

Notice that this Boltzmann weights are completely determined by the Bratteli diagram. The partition function of this models are given by the Markov trace of the transfer matrix defined by the g_i . Notice that in the way we have built our IRF model, using the fundamental construction the trace defining the partition function is forced to be Markov.

IRF models and knot invariants

The IRF models we have constructed from the Bratteli diagram can be automatically used to define statistical models of knot invariants. The rules of the game are very simple. Any knot or link can be described by a word in the braid group B_n for some n . To get the corresponding IRF knot invariant we first translate this word to $A_{\beta, \Lambda}$. For that we use the map :

$$\sigma_i \in B_n ; \pi(\sigma_i) = (q+1) E_i - \mathbb{1} \quad (3.8)$$

The knot invariant is now given in terms of the Markov trace of the corresponding word in $A_{\beta, \Lambda}$ with $\beta = \|\Lambda\|^2$. The representation of the invariant in terms of Boltzmann weights can be easily obtained using (3.6) to compute the trace. The value of the Markov trace will be given by the product of Boltzmann weights, one for each crossing in the knot, summed up over all possible configurations. These ones are defined distributing different heights to the parts of the knot in a way consistent with the constraints imposed by the Bratteli diagram for heights associated to contiguous parts of the knot (see fig.16).

Connection with conformal field theory

We will try now to make contact with conformal field theory. The first thing we can do is to read the heights of the IRF model as labels of different irreducible representations of some chiral algebra. In this interpretation it is natural to consider the constraints imposed on the values of adjacent heights as resulting from the fusion algebra of the model. More explicitly we should look for some field ϕ such that :

$$N_{\phi \phi} \phi = \Lambda \phi \quad (3.9)$$

We denote ϕ_{ℓ_i} the primary fields associated with the irreducible representations of the chiral algebra we have labelled by the heights ℓ_i . In principle there is no a priori reason that for any connected Bratteli diagram we can find a conformal field theory containing a collection of primary fields satisfying (3.9). In general such a RCFT does not exist.

To illustrate the meaning of equation (3.9) let us consider the example of the Λ -graph:

For this graph it is easy to find a RCFT satisfying (3.9) namely the $SU(2)$ level 3 WZW model. The primary fields $\phi_0, \frac{1}{2}, 1, \frac{3}{2}$ can be associated to each point in the graph:

The field ϕ satisfying (3.9) is clearly the field of spin $\frac{1}{2}$. Let us assume for the time being that for a given Bratteli diagram we can find a RCFT satisfying the required conditions. In this case we will see that the braiding properties of this RCFT are given in terms of the Boltzmann weights of the corresponding IRF model.

The information contained in (3.9) can be encoded in the graphical representation of the Boltzmann weight as follows:

Each corner of the square represents a contribution of the fusion algebra of the RCFT. The net result is to impose the constraints on the values of the heights on adjacent sites of the lattice. We must stress that (3.10) is not a different Boltzmann weight, but it must be interpreted as the RCFT representation of the IRF Boltzmann weight. A more pictorial way to understand (3.10) is to consider a crossing of a knot (see fig.16) and imagine the field ϕ running on the knot [14].

The interesting fact is that (3.10) has its own meaning in RCFT as the braiding matrix $\mathcal{B}_{\phi_1, \phi_2}^{\phi_3, \phi_4}$ relating the conformal blocks of fig.17. In fact $\mathcal{B}$ defines a representation of the braid group on the space of conformal blocks corresponding to the $s-u$ crossing transformation. In particular $\mathcal{B}$ satisfies the hexagonal equation. Recall that for the IRF models obtained using the fundamental construction, the Boltzmann weights are given by the matrix representation of the $\mathcal{R}_1^1$ s defined in (3.5). Therefore the Boltzmann weights (3.10) and the braiding matrix relating the conformal blocks of fig.16 are both representations of the braid group on the same vector space. The identification of these two quantities is only possible when the representation of the braid group is given by a Hecke algebra. This happens when the field ϕ in addition to (3.9) satisfies :

$$\phi \times \phi = id + \psi \quad (3.11)$$

for some field ψ . A more detailed discussion on this point is postponed to the end of chapter 4. This concludes the description of the set of conditions a RCFT must satisfy in order to have its braiding matrices given by the Boltzmann weights of some graph IRF model.

4. - RATIONAL CONFORMAL FIELD THEORIES AND TOWERS OF ALGEBRAS

Jones fundamental construction has a direct application to statistical mechanics, but it can also be applied to RCFT's. In doing so we shall find the meaning of the basic objects of the fundamental construction in the language of CFT, giving in this way another interpretation to Jones work in many respect parallel to that of the IRF models.

Let us briefly summarize the content of this chapter. First we construct towers of multi-matrix algebras using the fusion algebra of a RCFT. For each primary field ϕ we shall find one of such an algebra A_ϕ . Next we define a trace on A_ϕ and study the conditions for this trace to be Markov. Then we show that the RCFT provides in a natural way a Markov trace on A_ϕ with the modulus β given by $\beta = \left(\frac{S_{\phi\phi}}{S_{00}}\right)^2$, where S_{ij} is the modular matrix. Finally we give a RCFT interpretation of the projection operators E_i and deduce the Jones algebra from the pentagonal equation.

Towers.

In reference [12] it was shown how to construct towers of multi-matrix algebras starting from a RCFT. Since this construction is essential for our purpose, let us briefly summarize it.

We start from an RCFT with fusion rules:

$$\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k \quad (4.1)$$

The fusion coefficient N_{ij}^k is an integer which describes the number of ways the representations i and j couple to give the representation k . We shall consider for simplicity the case where N_{ij}^k is either zero or one. Moreover we will assume that the theory is selfconjugated. Under these two assumptions the fusion matrix is totally symmetric. All the results we will present can be easily extended to more general cases.

Let us now consider the space $V_{P_n}^n$ generated by the conformal blocks of fig.18. This space is completely characterized by the field ϕ and the "expectator" P_n . The dimension of this space is given in terms of the fusion matrix as follows:

$$d_{P_n}^n \equiv \dim V_{P_n}^n = \sum_{P_0, \dots, P_{n-1}} N_{P_0 P_1}^{\phi} N_{P_1 P_2}^{\phi} \dots N_{P_{n-1} P_n}^{\phi} = (N_\phi)^n \quad (4.2)$$

$$(N_\phi)_{ij} \equiv N_{\phi_i \phi_j}^{\phi}$$

This dimension counts the number of possible internal channels. The spaces $V_{p_n}^n$ are representation spaces for all the duality transformations involving exclusively the $(n+2)$ fields ϕ . This simply means that we can braid or perform operator product expansions (OPE) between the fields ϕ and still get a linear combination of conformal blocks of the same type. Any duality operation $\mathbf{x}$ is represented on $V_{p_n}^n$ by a matrix $\mathbf{x}_{p_n}^n \in \text{End}(V_{p_n}^n) = M_{d_{p_n}^n}(\mathbb{C})$

$$(\times \mathcal{F})_{p_{n+1}, \dots, p_0}^{p_n} = \sum_{p_0' \dots p_{n-1}'} \mathbf{x}_{p_{n-1}}^{p_n} \times_{p_{n-1}}^{p_n} p_{n-1} \dots p_0 \mathcal{F}_{p_{n-1}, \dots, p_0}^{p_n} \quad (4.3)$$

For different expectators p_n we get different representations of the same duality transformations that we can collect into a multi - matrix algebra $\tilde{A}_{\phi}^n$ defined as:

$$\tilde{A}_{\phi}^n = \bigoplus_{p_n} M_{d_{p_n}^n}(\mathbb{C}) \quad (4.4)$$

The multiplication inside $\tilde{A}_{\phi}^n$ is defined as ordinary matrix multiplication:

$$\mathbf{x} \mathbf{y} = \bigoplus_{p_n} \mathbf{x}_{p_n}^{p_n} \mathbf{y}_{p_n}^{p_n} \quad (4.5)$$

A priori not all the elements of $\tilde{A}_{\phi}^n$ are identifiable with duality operations, however we will see that any element in $\tilde{A}_{\phi}^n$ is essentially given as a certain operation involving only the field ϕ . This will be precisely the outcome of the fundamental construction in RCFT's. For this reason we shall call the formal union $A_{\phi} = \bigcup_{n=0}^{\infty} \tilde{A}_{\phi}^n$ the "duality algebra" of the field ϕ .

The duality algebra A_{ϕ} has the structure of an infinite tower of multi - matrix algebras:

$$A_{\phi}^0 \subset A_{\phi}^1 \subset A_{\phi}^2 \subset \dots \quad (4.6)$$

The inclusion maps:

$$\pi^n : A_{\phi}^{n-1} \rightarrow A_{\phi}^n \quad n = 1, 2, \dots \quad (4.7)$$

are defined by promoting any operation $\mathbf{x} \in A_{\phi}^{n-1}$ to an operation $\tilde{\pi}^n(\mathbf{x}) \in \tilde{A}_{\phi}^n$ which leaves untouched the extra ϕ field. Its explicit expression is:

$$\pi^n(\mathbf{x})_{p_{n-1}, \dots, p_0}^{p_n} = \delta_{p_{n-1}, p_{n-1}}^{p_n} N_{\phi}^{p_{n-1}} \times_{p_{n-1}}^{p_n} p_{n-1} \dots p_0 \quad (4.8)$$

The term $\delta_{p_{n-1}, p_{n-1}}^{p_n}$ reflects the fact that $\times$ inside $\tilde{A}_{\phi}^n$ is blind to the last field ϕ . Equation (4.8) is a particular case of the diagonal inclusion given in (2.7). The inclusion $\tilde{A}_{\phi}^n \subset A_{\phi}^{n+1}$ is characterized by a matrix $\tilde{\Lambda}^n$ which relates the dimensionalities $d_{p_n}^n$ and $d_{p_{n-1}}^{n-1}$ as follows:

$$d_{p_n}^n = \sum_{p_{n-1}} d_{p_{n-1}}^{n-1} \tilde{\Lambda}_{p_{n-1}, p_n}^n \quad (4.9)$$

Using eqs. (4.20) and (4.8) we get that the inclusion matrix is given by the fusion rules :

$$\tilde{\Lambda}_{p_{n-1}, p_n}^n = N_{\phi}^{p_{n-1}} p_n \quad (4.10)$$

So we will be tempted to identify $\tilde{\Lambda}^n$ with the fusion matrix N_{ϕ} . This, in spite of eq. (4.10), is not true in general. As an example let us consider the $SU(2)$ WZW model at level 4. For the field ϕ the spin $\frac{1}{2}$ representation we get the tower of algebras of fig.19:

$$\begin{aligned} A_{\psi_2}^0 &= \mathbb{C} \oplus \mathbb{C} \subset A_{\psi_2}^1 = M_2(\mathbb{C}) \oplus \mathbb{C} \subset \\ A_{\psi_2}^2 &= M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \subset A_{\psi_2}^3 = M_5(\mathbb{C}) \oplus M_4(\mathbb{C}) \dots \end{aligned} \quad (4.11)$$

with inclusion matrices given by:

$$\tilde{\Lambda}^1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \tilde{\Lambda}^{2n} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \tilde{\Lambda}^{2n+1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad (4.12)$$

This case illustrates some general features of the towers associated to any RCFT. In particular the existence of a certain level say n_0 (for the previous example $n_0 = 1$), after which the inclusion matrices are consecutively $\tilde{\Lambda}$ and $\tilde{\Lambda}^T$. To prove this result we will need the following lemma: if i is a field which appears as expectator at level n , then then i is also an expectator at level $n+2$. Indeed, look for a field p such that $N_{\phi}^{p,p} = 1$. By the symmetry of $N_{\phi}^{i,j}$ we have $N_{\phi}^{p,i} = 1$ and from fig.20 the desired result follows. A consequence of this lemma is that number of components of $\tilde{A}_{\phi}^{2n}$ (i.e. number of black points in the Bratteli diagram at level $2n$) increases with n . A similar thing occurs

for $\vec{d}^{2n+1}$. The rationality of the RCF means that the number of irreps of the chiral algebra is finite and therefore after a certain level n_0 the lengths of $\vec{d}^{2n}$ and $\vec{d}^{2n+1}$ stabilized at some numbers $N_{\text{even}}, N_{\text{odd}}$. In the example of fig 19 $N_{\text{even}} = 3, N_{\text{odd}} = 2$.

The previous result implies that after a certain level n_0 the RCF tower of algebras is isomorphic to the one obtained using the fundamental construction described in chapter 2 on the pair $A_{\vec{d}}^{n_0} \subset A_{\vec{d}}^{n_0+1}$. Therefore we get the following set of isomorphism:

$$A_{\vec{d}}^{n+2} \cong \text{End}_{A_{\vec{d}}^{n+1}}(A_{\vec{d}}^{n+1}) \quad n \geq n_0 \quad (4.13)$$

To understand the implications of this fact we need to introduce traces and conditional expectations.

Traces: Conditional Expectations

As it was explained in chapter 2 a trace on the multi-matrix algebra $\vec{A}_{\vec{d}}^n$ is defined by a vector $\vec{t}^n$ as follows:

$$\begin{aligned} t_{\vec{d}}^n(x) &= \sum_{p_0, \dots, p_n} t_{\vec{p}_n}^n \times_{\vec{p}_{n-1}}^n \dots \times_{\vec{p}_0}^n p_n \dots p_0 \\ &= \sum_{\vec{p}_n} t_{\vec{p}_n}^n \text{tr}(x^{\vec{p}_n}) \quad x \in \vec{A}_{\vec{d}}^n \end{aligned} \quad (4.14)$$

The trace $t_{\vec{d}}^n$ is faithful if $t_{\vec{p}_n}^n \neq 0 \forall p_n$ and positive if $t_{\vec{p}_n}^n > 0 \forall p_n$. The consistency between the traces $t_{\vec{d}}^n, t_{\vec{d}}^{n+1}$ and the inclusion $\vec{A}_{\vec{d}}^{n+1} \subset \vec{A}_{\vec{d}}^n$ forces the relation:

$$\sum_{\vec{p}_n} N_{\vec{d}} \vec{p}_{n-1} \times_{\vec{p}_n}^n t_{\vec{p}_n}^n = t_{\vec{p}_{n-1}}^{n+1} \quad (4.15)$$

Observe that (4.15) is dual to (4.9). From chapter 2 we know that if $\vec{A}_{\vec{d}}^n$ and $\vec{A}_{\vec{d}}^{n+1}$ have both faithful traces related by (4.15), then there exist a unique linear map $E_n: \vec{A}_{\vec{d}}^n \rightarrow \vec{A}_{\vec{d}}^{n+1}$ such that

$$\begin{aligned} \text{i)} \quad t_{\vec{d}}^{n+1}(E_n(x)y) &= t_{\vec{d}}^n(xy) \quad , \quad x \in \vec{A}_{\vec{d}}^n, y \in \vec{A}_{\vec{d}}^{n+1} \\ \text{ii)} \quad E_n(\gamma_1 \times \gamma_2) &= \gamma_1 E_n(x) \gamma_2 \quad , \quad x \in \vec{A}_{\vec{d}}^n, \gamma_1, \gamma_2 \in \vec{A}_{\vec{d}}^{n+1} \\ \text{iii)} \quad E_n(\gamma) &= \gamma \quad , \quad \gamma \in \vec{A}_{\vec{d}}^{n+1} \end{aligned} \quad (4.16)$$

The way to construct E_n is to make the orthogonal decomposition $\vec{A}_{\vec{d}}^n = \vec{A}_{\vec{d}}^{n-1} \oplus (\vec{A}_{\vec{d}}^n)^{\perp}$ with respect to the scalar product in $\vec{A}_{\vec{d}}^n$ defined by $\langle x, y \rangle = \vec{t}^n(xy)$. Then E_n is simply the orthogonal projection on $\vec{A}_{\vec{d}}^{n-1}$. The explicit formula in our case reads:

$$E_n(x)_{p_{n-2} \dots p_0, p_{n-2} \dots p'_0} = \frac{1}{t_{\vec{p}_{n-1}}^n} \sum_{\vec{p}_{n-1}} t_{\vec{p}_n}^n \times_{\vec{p}_{n-1}}^n p_{n-1} p_{n-2} \dots p'_0 \quad (4.17)$$

We see by (4.17) why $t_{\vec{d}}^{n-1}$ has to be faithful, otherwise the denominator would vanish.

Let us now apply Jones fundamental construction to the pair $\vec{A}_{\vec{d}}^0 \subset \vec{A}_{\vec{d}}^1$. We want to consider the algebra $\text{End}_{\vec{A}_{\vec{d}}^0} \vec{A}_{\vec{d}}^1$ and its relation to $\vec{A}_{\vec{d}}^2$. The inclusion $\vec{A}_{\vec{d}}^1 \subset \text{End}_{\vec{A}_{\vec{d}}^0} \vec{A}_{\vec{d}}^1$ is defined by the map:

$$x \in \vec{A}_{\vec{d}}^1 \longmapsto \hat{x}: y \longmapsto xy \quad , \quad y \in \vec{A}_{\vec{d}}^0 \quad (4.18)$$

or in components:

$$(xy)_{p_0 p_0'} = \sum_{p_0''} x_{p_0 p_0''}^1 y_{p_0'' p_0'}^1 = \sum_{p_0'' p_1'} \hat{x}_{p_0 p_1'}^{p_0''} y_{p_1' p_0'}^1 \quad (4.19)$$

where:

$$\hat{x}_{p_0 p_1'}^{p_0''} \equiv \delta_{p_1 p_1'} \delta_{p_0 p_0''} N_{\vec{d}} \vec{p}_1 \times_{\vec{p}_0 p_0'}^1 \quad (4.20)$$

Recall that the inclusion $\vec{A}_{\vec{d}}^1 \subset \vec{A}_{\vec{d}}^2$ is given by:

$$\pi(x)_{p_1 p_0, p_1' p_0'}^{p_2} = \delta_{p_1 p_1'} \delta_{p_2 p_2} N_{\vec{d}} \vec{p}_1 \times_{\vec{p}_0 p_0'}^1 \quad (4.21)$$

Certainly if the range of values of $\vec{p}_0^H$ and $\vec{p}_2$ agree then $\hat{x}$ can be identified with

$\pi(x)$. This is what happens after the critical level n_0 but not before. Condition ii) of (4.16) tells us that E_n is an element of $\text{End}_{\vec{A}_{\vec{d}}^0} \vec{A}_{\vec{d}}^1$, indeed:

$$\begin{aligned} E_1(x)_{p_0 p_0'}^{p_1} &= \delta_{p_0 p_0''} N_{\vec{d}} \vec{p}_0^{p_1} E_1(x)_{p_0'' p_0'}^{p_0} = \delta_{p_0 p_0''} N_{\vec{d}} \vec{p}_0^{p_1} \sum_{p_1'} \frac{t_{\vec{p}_1'}^1}{t_{\vec{p}_0}^0} \times_{p_0 p_0'}^1 \\ &= \sum_{p_0'' p_1'} (\hat{E}_1)_{p_1 p_0, p_1' p_0'}^{p_0} \times_{p_0'' p_0'}^{p_1'} \end{aligned} \quad (4.22)$$

with:

$$(\hat{E})_{p_1 p_0, p_1' p_0'}^{p_0} = \delta_{p_0 p_0''} \delta_{p_0 p_0'} \frac{t_{\vec{p}_1'}^1}{t_{\vec{p}_0}^0} N_{\vec{d}} \vec{p}_0^{p_1} \quad (4.22)$$

In this formula the index p_0^H can be replaced by p_2 due to the term $\delta_{p_0^H}$. Therefore $\hat{E}_1$ is an element of A_Φ^2 . Formulae (4.20) and (4.22) show that :

$$\text{End}_{A_\Phi^0} A_\Phi^1 \subset A_\Phi^2 \quad (4.23)$$

the equality has been obtained when the expectators of A_Φ^0 coincides with those of A_Φ^2 . The general formula of $\hat{E}_n$ is given by:

$$(\hat{E}_n \mathcal{T})_{p_{m-1} \dots p_0} = \delta_{p_{m+1} p_{n-1}} \sum_{p_n} \frac{t_{p_n}^n}{t_{p_{n-1}}^{n-1}} \quad (4.24)$$

One can check using (4.18) that $\hat{E}_n$ is a projector, i.e. $\hat{E}_n^2 = \hat{E}_n$.

The Markov trace

The trace t_n^x on A_Φ^x is a Markov trace of modulus β if :

$$\beta t_n^{n+1}(x \hat{E}_n) = t_n^x(x) \quad \forall x \in A_\Phi \quad (4.25)$$

which reads in our case:

$$\beta t_{p_{n-1}}^{n+1} = t_{p_{n-1}}^{n-1} \quad (4.26)$$

Notice that if $A_\Phi^2 \text{End}_{A_\Phi^0} A_\Phi^1$ then t_n^x is completely fixed by t_n^{n-1} and the Markov trace propagates up in the tower. Now using (4.10) and (4.15) we deduce the eqs. satisfied by the vectors t_n^x once the tower follows the fundamental construction:

$$\begin{aligned} \beta t_n^{2n} &= \Lambda_\Phi^T \Lambda_\Phi^T t_n^{2n} \\ \beta t_n^{2n+1} &= \Lambda_\Phi^T \Lambda_\Phi^T t_n^{2n+1} \end{aligned} \quad (4.27)$$

A necessary and sufficient condition for the existence of a positive Markov trace of modulus β is that $\beta = \| \Lambda_\Phi \|^2$ and it is unique up to an overall constant.

Fortunately in RCFT's we can find the explicit solution to (4.27). The basic result we need is the so-called Verlinde theorem [18] which relates the fusion matrix $N_{ij;k}$ and the modular S matrix which implements the modular S transformation $\tau \rightarrow -\frac{1}{\tau}$ on the characters $\chi_i(\tau) (\chi_i(-\frac{1}{\tau}) = \sum_j S_{ij} \chi_j(\tau))$:

$$N_{ij;k} = \sum_n \frac{S_{in} S_{jn} S_{kn}}{S_{0n}} \quad (4.28)$$

S is a symmetric and unitary matrix. If the theory is selfconjugated then $S^2 = 1$. Using this theorem the solution of (4.15) and (4.27) is given by:

$$t_i^n = \frac{S_{0i}}{S_{00}} \left(\frac{S_{00}}{S_{0i}} \right)^n \quad (4.29)$$

and

$$\beta = \left(\frac{S_{00}}{S_{0i}} \right)^2 \quad (4.30)$$

Furthermore for unitary RCFT's $\frac{S_{0i}}{S_{00}}$ is a positive quantity, which implies that t^n and t^{2n+1} are Perron - Frobenius eigenvectors of $\Lambda_\Phi \Lambda_\Phi^T$ and $\Lambda_\Phi^T \Lambda_\Phi$ respectively. If these matrices are irreducible then the Perron - Frobenius theory asserts the uniqueness of t^n . The irreducibility of $\Lambda_\Phi \Lambda_\Phi^T$ and $\Lambda_\Phi^T \Lambda_\Phi$ is equivalent to the connectivity of the Bratteli diagram involved.

Given the Markov trace one finds immediately that the conditional expectations $\hat{E}_n$ satisfy the TLJ algebra:

$$\begin{aligned} \beta \hat{E}_i \hat{E}_{i+1} \hat{E}_i &= \hat{E}_i \\ \hat{E}_i \hat{E}_j &= \hat{E}_j \hat{E}_i \quad |i-j| \geq 2 \\ \hat{E}_i^2 &= \hat{E}_i \end{aligned} \quad (4.31)$$

which can be checked using (4.24) and (4.26).

All the defining relations of the conditional expectations $\hat{E}_n$ are invariant under a similarity transformation $\hat{E}_n \rightarrow U \hat{E}_n U^{-1}$. Choosing conveniently U one can obtain the Ocneanu expression [25] (see also [17]):

$$(e_n \mathcal{F})_{p_{n-1} \dots p_0} = \delta_{p_{n+1} p_{n-1}} \sum_{p_n} \frac{(t_{p_n}^n t_{p_n}^{1/2})}{t_{p_{n-1}}} \quad (4.32)$$

$m \geq n+1$

which is the promised RCFT version of the IRF eq (3.4). The factor multiplying the conformal block in the right-hand side of (4.32) is given in terms of the S matrix as:

$$\frac{(t_{p_n}^n t_{p_n}^{1/2})^{1/2}}{t_{p_{n-1}}} = \frac{\left(\frac{S_{0p_n}}{S_{00}} \frac{S_{0p_n}}{S_{00}} \right)^{1/2}}{\frac{S_{0p_{n-1}}}{S_{00}}} \quad (4.33)$$

Our next task is to give a purely RCFT meaning to Jones construction, that is to explain it in terms of duality.

RCFT interpretation of Conditional Expectations

The set of conformal blocks of fig. 18 is just one possible basis for the space $\hat{V}_{p_n}$ of conformal blocks with $n \dots \phi$ legs and one p_n -leg. There is, however, another basis as those depicted in fig. 21.

Any set of basis is related through duality. In this paper we are primarily concerned with the $s-t$ duality which is expressed diagrammatically as:

$$\begin{array}{c} \text{Diagram 1: } \text{A horizontal line with } p \text{ legs labeled } i, j, k, \dots, \ell. \end{array} = \sum_{p'} F_{pp'}^{[j, k]} \begin{array}{c} \text{Diagram 2: } \text{A horizontal line with } \ell \text{ legs labeled } i, j, k, \dots, \ell. \end{array} \quad (4.34)$$

The number of conformal blocks (i, j, k and l kept fixed) is the same in both sides of (4.34) due to the

$$\sum_p N_{jp}^i N_{kl}^p = \sum_{p'} N_{p'l}^i N_{jk}^{p'} \quad (4.35)$$

eq.:

One goes from the s -channel conformal block (left-hand side of (4.34)) to the t -channel conformal block (right-hand side) by taking the OPE of the fields j and k . The duality coefficients $F_{pp'}^{[j, k]}$ satisfy a pentagonal equation ($F_{pp'}^{[j, k]} = F_{p'p}^{[j, k]}$) which reflects the associativity of the OPE [8].

After these preliminaries, we can give the interpretation of the conditional expectations e_n in RCFT's as projectors onto the identity representation in the t -channel basis. In more definite terms the operators e_n , as given in (4.32), can be realized as the product of the following three operations:

- 1) Take the OPE between the two ϕ fields which have in common the s -channel n .
- 2) Project onto the identity representation in the resulting t -channel
- 3) Return to the original s -channel

Steps 1) and 3) are nothing but changes of basis so that the genuine property of e_n is step 2).

The actual computation goes as follows:

$$\begin{aligned} & \begin{array}{c} \text{Diagram 1: } \text{A horizontal line with } p_{n+1} \text{ legs labeled } \phi, \phi, \dots, \phi. \end{array} \\ &= \sum_{\ell} F_{p_n \ell}^{[\phi, \phi]} \begin{array}{c} \text{Diagram 2: } \text{A horizontal line with } p_{n-1} \text{ legs labeled } \phi, \phi, \dots, \phi. \end{array} \\ & \xrightarrow{e_n} F_{p_n 0}^{[\phi, \phi]} \begin{array}{c} \text{Diagram 3: } \text{A horizontal line with } p_{n+1} \text{ legs labeled } \phi, \phi, \dots, \phi. \end{array} \\ &= \delta_{p_{n+1} p_{n-1}} \sum_{p_n} F_{p_n 0}^{[\phi, \phi]} F_{0 p_n}^{[\phi, \phi]} \begin{array}{c} \text{Diagram 4: } \text{A horizontal line with } p_{n-1} \text{ legs labeled } \phi, \phi, \dots, \phi. \end{array} \quad (4.36) \end{aligned}$$

The projection onto the identity is what forces the equality between p_{n+1} and p_{n-1} as in (4.32). To establish the complete identification with e_n we have to show:

$$F_{p_n o} \left[\begin{smallmatrix} \phi & \phi \\ p_{n-1} & p_{n-1} \end{smallmatrix} \right] F_{o p_n} \left[\begin{smallmatrix} \phi & p_{n-1} \\ \phi & p_{n-1} \end{smallmatrix} \right] = \frac{(t_{p_n}^n t_{p_n}^{n-1})^{1/2}}{t_{p_{n-1}}^{n-1}} \quad (4.37)$$

The duality $\bar{\Gamma}$ matrices which appear in (4.37) are the type of matrices involved in the following pentagonal identity:

$$F_{oj} \left[\begin{smallmatrix} \kappa & i \\ \kappa & i \end{smallmatrix} \right] F_{io} \left[\begin{smallmatrix} j & j \\ \kappa & \kappa \end{smallmatrix} \right] = F_{oo} \left[\begin{smallmatrix} \kappa & \kappa \\ \kappa & \kappa \end{smallmatrix} \right] N_{ij\kappa} \quad (4.38)$$

where

$$F_{oo} \left[\begin{smallmatrix} \kappa & \kappa \\ \kappa & \kappa \end{smallmatrix} \right] \equiv F_{\kappa} = \varepsilon_{\kappa\kappa}^o \frac{S_{oo}}{S_{o\kappa}} \quad (4.39)$$

$\varepsilon_{\kappa\kappa}^o = (-1)$ depending on whether the identity representation appears symmetrically or antisymmetrically in the OPE $\kappa \times \kappa = d + \dots$. Notice that F_{κ} defined in (4.39) differs by the sign $\varepsilon_{\kappa\kappa}^o$ with the F_{κ} of reference [8].

Using (4.38) and choosing an appropriate normalization of the chiral vertices one deduces :

$$F_{oi} \left[\begin{smallmatrix} j & \kappa \\ j & \kappa \end{smallmatrix} \right] = F_{io} \left[\begin{smallmatrix} j & j \\ \kappa & \kappa \end{smallmatrix} \right] = \left(\frac{F_j F_{\kappa}}{F_i} \right)^{1/2} N_{ij\kappa} \quad (4.40)$$

from which equation (4.37) follows.

Taking into account the freedom in choosing basis for the space of conformal blocks it may sometimes be convenient to express the action of e_n as:

$$\begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| \xrightarrow{e_n} \delta_{p_{n+1} p_{n-1}} \left(\frac{F_{\phi} F_{p_{n-1}}}{F_{p_n}} \right)^{1/2} \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| \quad (4.41)$$

Let us consider now the RCT meaning of the properties of these operators. The equations $e_n^2 = e_n$ and $e_n e_m = e_m e_n$ for $|n-m| \geq 2$ need no explanation. More interesting is the eq. $e_n e_{n+1} e_n = \bar{\beta} e_n$, whose proof is given below. Making extensive use of $s-t$ duality transformations we get $e_n e_{n+1} e_n = F_{\bar{\beta}}^2 e_n$ which agrees with (4.30):

$$\begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| \xrightarrow{e_n} \delta_{p_{n+1} p_{n-1}} \left(\frac{F_{\phi} F_{p_{n-1}}}{F_{p_n}} \right)^{1/2} \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| \\ = \delta_{p_{n+1} p_{n-1}} \left(\frac{F_{\phi} F_{p_{n-1}}}{F_{p_n}} \right)^{1/2} \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| \\ \xrightarrow{e_{n+1}} \delta_{p_{n+1} p_{n-1}} \left(\frac{F_{\phi} F_{p_{n-1}}}{F_{p_n}} \right)^{1/2} F_{\phi} \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| \\ \xrightarrow{e_n} \delta_{p_{n+1} p_{n-1}} \left(\frac{F_{\phi} F_{p_{n-1}}}{F_{p_n}} \right)^{1/2} F_{\phi}^2 \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| \quad (4.42)$$

F_{ϕ} appears twice because there are two $s-t$ duality transformations involved:

$$\begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| \xleftrightarrow{\leftarrow} \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| = F_{\phi} \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| + \dots \\ \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| \xrightarrow{\rightarrow} \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| = F_{\phi} \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \left| \begin{array}{c} \phi \\ \vdots \\ \phi \end{array} \right| + \dots \quad (4.43)$$

where the arrows show the OPE operations. We have thus proved that the TJJ algebra contains a piece of information of the pentagonal equation.

Let us comment now on another interesting fact about the TJJ operators e_n . In the proof of the Verlinde formula (4.28) one introduces the operator $\phi_{\kappa}(b)$ associated to the cycle b of the torus, whose action on the character χ_i is :

$$\phi_{\kappa}(b) \chi_i = N_{\kappa i}^j \chi_j(b) \quad (4.44)$$

The action of $\phi_{\kappa}(b)$ on the character consists in introducing the identity in the form $\kappa \times \kappa = d + \dots$ then going around the cycle b with one of the fields κ and taking the OPE with the other field κ .

Moore and Seiberg [8] showed that this operation can be realized in terms of the fusion matrices F as follows:

$$\begin{aligned}
 & \text{Diagram: a line with index } i \text{ entering a circle, with two lines exiting labeled } \kappa \text{ and } j. \\
 &= \sum_j F_{0j} \left[\begin{smallmatrix} \kappa & i \\ \kappa & i \end{smallmatrix} \right] \text{Diagram: a circle with two lines entering labeled } \kappa \text{ and } i, \text{ and one line exiting labeled } j. \\
 &\rightarrow \sum_j F_{0j} \left[\begin{smallmatrix} \kappa & i \\ \kappa & i \end{smallmatrix} \right] \text{Diagram: a circle with two lines entering labeled } \kappa \text{ and } i, \text{ and one line exiting labeled } j. \\
 &\xrightarrow{e_i} \sum_j F_{0j} \left[\begin{smallmatrix} \kappa & i \\ \kappa & i \end{smallmatrix} \right] F_{i0} \left[\begin{smallmatrix} \kappa & \kappa \\ j & j \end{smallmatrix} \right] \text{Diagram: a circle with two lines entering labeled } \kappa \text{ and } \kappa, \text{ and one line exiting labeled } j. \\
 &= F_{\kappa} \sum_j N_{ij\kappa} \text{Diagram: a circle with two lines entering labeled } \kappa \text{ and } \kappa, \text{ and one line exiting labeled } j.
 \end{aligned}
 \tag{4.45}$$

where we recognize the action of the conditional expectation e_i in the last step. The Verlinde operator $\phi_{\kappa}(b)$ acting on the characters is therefore a TJJ operator acting on closed loops:

$$\phi_{\kappa}(b) \chi_i = e_i \left(\text{Diagram: a circle with two lines entering labeled } \kappa \text{ and } \kappa, \text{ and one line exiting labeled } i \right)
 \tag{4.46}$$

Other open path's versions of the Verlinde's operators have already been considered in references [26] and [27].

Connection with Braiding

A case where the TJJ algebra appears to be very closely connected to the braiding properties of a RCTF is when the fusion rule of the field ϕ satisfies:

$$\phi \times \phi = id + \psi
 \tag{4.47}$$

for some field ψ . The braiding matrix B_n between two fields ϕ can be related to the corresponding operator e_n (fig.18). The only information we need is the braiding factor for the simple move:

$$\text{Diagram: two lines crossing} = \varepsilon_{\phi\phi}^j e^{i\pi(h_j - 2h\phi)} \text{Diagram: two lines crossing}
 \tag{4.48}$$

where h_j, h_{ϕ} are the conformal weights of the fields j and ϕ respectively and $\varepsilon_{\phi\phi}^j = \pm 1$ is the parity of j inside $\phi \times \phi$. The relation between B_n and e_n is obtained by the following procedure:

$$\begin{aligned}
 & \text{Diagram: two parallel lines} = \text{Diagram: two parallel lines} + \text{Diagram: two parallel lines} \\
 & B_n \rightarrow \varepsilon_{\phi\phi}^0 e^{-2\pi i h\phi} \text{Diagram: two parallel lines} + \varepsilon_{\phi\phi}^{\psi} e^{i\pi(h_{\psi} - 2h\phi)} \text{Diagram: two parallel lines} \\
 &= \left(\varepsilon_{\phi\phi}^0 e^{-2\pi i h\phi} - \varepsilon_{\phi\phi}^{\psi} e^{i\pi(h_{\psi} - 2h\phi)} \right) \text{Diagram: two parallel lines} + \varepsilon_{\phi\phi}^{\psi} e^{i\pi(h_{\psi} - 2h\phi)} \text{Diagram: two parallel lines} \\
 & \tag{4.49}
 \end{aligned}$$

which yields :

$$\begin{aligned}
 B_n &= -\varepsilon_{\phi\phi}^{\psi} e^{i\pi(h_{\psi} - 2h\phi)} g_n \\
 g_n &= (1+q) e_n - 1I
 \end{aligned}
 \tag{4.50}$$

with :

$$\begin{aligned}
 q &= -\varepsilon_{\phi\phi}^0 \varepsilon_{\phi\phi}^{\psi} e^{i\pi h_{\psi}} = e^{i\pi(h_{\psi} + \frac{\varepsilon+1}{2})} \\
 \varepsilon &= \varepsilon_{\phi\phi}^0 \varepsilon_{\phi\phi}^{\psi}
 \end{aligned}
 \tag{4.51}$$

The values of q and β are related by $\beta = 2+q+q^{-1}$, which in the previous case yields:

$$\beta = \left(2 \cos \frac{\pi}{2} \left(h_4 + \frac{\epsilon+1}{2} \right) \right)^2 \quad (4.52)$$

and therefore $\beta < 4$.

As we have shown in chapter 2 the operators $\mathfrak{g}_i$ generate the Hecke algebra $H_n(q)$ subject to the additional condition (2.5), which implies that only representations with Young tableaux of at most two rows can appear in the braiding of the field ϕ . One example of this is the $SU(2)$ WZW model with $\phi = \psi_2$. Any RCTT with the fusion rule (4.47) shares this property.

5. - GENERALIZED TLJ - OPERATORS AND MODULAR INVARIANT PARTITION FUNCTIONS

In the previous chapter we have defined the TLJ operators e_n associated to the internal channel n as the operator which projects onto the identity representation on the corresponding t -channel:

$$\begin{array}{c} j \\ | \\ \hline p_n \end{array} \begin{array}{c} k \\ | \\ \hline l \end{array} \xrightarrow{e_n} \delta_{j,l} \delta_{j,k^*} F_{p,0} \left[\begin{array}{c} k^* \\ l \end{array} \right] \begin{array}{c} k \\ | \\ \hline l \end{array} \quad (5.1)$$

This expression is valid for RCTT's where ϕ^* is not necessarily equal to ϕ . The TLJ algebra has the value of β given by:

$$(\beta \phi^{-1}) = \left(F_{0,0} \left[\begin{array}{c} \phi \ \phi^* \\ \phi \ \phi \end{array} \right] \right)^2 \Rightarrow \beta \phi = \left(\frac{S_{\phi 0}}{S_{00}} \right)^2 \quad (5.2)$$

We shall propose a generalization of the TLJ algebra operators to another set of operators e_n^J where the projection in the t -channel is done on the representation J , namely:

$$\begin{array}{c} j \\ | \\ \hline p_n \end{array} \begin{array}{c} k \\ | \\ \hline l \end{array} \xrightarrow{e_n^J} F_{p,J} \left[\begin{array}{c} j \ k \\ i \ l \end{array} \right] N_{j,k}^J N_{j,l}^J \begin{array}{c} k \\ | \\ \hline l \end{array} \quad (5.3)$$

with the above definition we can prove the following lemma: if J is a primary field with unique fusion rule with all the primary fields in the theory :

$$J \times \phi = \phi' \quad \forall \phi \quad (5.4)$$

then the operators e_n^J satisfy the equations

$$(e_n^J)^2 = e_n^J, \quad e_n^J e_m^J = e_m^J e_n^J \quad |n-m| \geq 2$$

$$e_n^J e_{n+1}^J \left(\begin{array}{c} i \ j \ k \\ | \\ \hline p_n \end{array} \right) = \delta_{i,k} \left(F_{JJ} \left[\begin{array}{c} k \ j k^* \\ j \ k \end{array} \right] \right)^2 e_n^J \left(\begin{array}{c} i \ j \ k \\ | \\ \hline p_n \end{array} \right) \quad (5.5)$$

therefore the operators e_n^J form a generalized TLJ algebra. A field J satisfying (5.4) has already been considered in reference [20] under the name of "simple current" for constructing modular invariant partition functions. We will present another way to understand the connection between simple currents and modular invariants from the point of view of Jones fundamental construction.

Let us first consider the problem of whether the algebra (5.5) follows from the fundamental construction of a tower of algebras, say A_n^J . It is easy to see that this is not the case. First of all notice that the conformal blocks on which A_n^J would act are the ones of fig.26. The inclusion matrix $A_n^J \subset A_{n+1}^J$ is N_{JJ}^J while that for $A_n^J \subset A_{n+1}^J$ is N_{JJ}^J and clearly both matrices are different except for $J=i,l$, which implies that the corresponding Bratteli diagram does not follow from the fundamental construction. Moreover A_n^J is not a good duality algebra since, for example, the braiding of two legs ϕ and $J\phi^*$ takes the conformal blocks outside the family defined in fig.22.

To solve this problem we need to consider a bigger space of conformal blocks closed under duality operations and such that it furnish a representation space of the algebra (5.5). In order to simplify matters we will consider again the selfconjugate theories, which implies that $J=i,l$. Let us now define the algebra A_n^{0+J} as the duality algebra acting on the space of conformal blocks of fig.23

where we allow arbitrary distributions , on the external legs, of fields ϕ and $\mathcal{J}\phi$. The operators $e_n^0 + e_n^{\mathcal{J}}$ are now projections:

$$e_n^0 + e_n^{\mathcal{J}} : A_{\phi}^{0+\mathcal{J},n} \longrightarrow A_{\phi}^{0+\mathcal{J},n-1} \quad (5.6)$$

satisfying the TLIJ algebra iff:

$$(\beta_{\phi}^{-1} = \left(\frac{s_{00}}{s_{\phi 0}} \right)^2 = \left(F_{\mathcal{J}\mathcal{J}} \left[\begin{smallmatrix} \phi & \mathcal{J}\phi \\ \mathcal{J}\phi & \phi \end{smallmatrix} \right] \right)^2 \quad (5.7)$$

and

$$\frac{s_{0\mathcal{J}}}{s_{00}} = 1 \quad , \quad \frac{s_{\phi 0}}{s_{00}} = \frac{s_{\mathcal{J}\phi,0}}{s_{00}} \quad (5.8)$$

Equation (5.8) is always satisfied by a simple current such that $\mathcal{J}^2 = 1$ [20]. We have strong indications that equation (5.7) is also true, however we have only checked explicitly for the case of $SU(2)$ WZW models.

The next question now is whether the operators $e_n^0 + e_n^{\mathcal{J}}$ are the conditional expectations of the tower of algebras $A_{\phi}^{0+\mathcal{J}}$. We know from Jones work that there exists a fundamental tower where the operators $\tilde{e}_n^{\mathcal{J}} = e_n^0 + e_n^{\mathcal{J}}$ are conditional expectations. On the other hand any RCIT defines towers of algebras according to the fundamental construction. It is therefore tempting to conjecture that the towers on which the algebras $\tilde{e}_n^{\mathcal{J}}$ are realized as conditional expectations are in one-to-one correspondence with the non diagonal modular partition functions associated to the simple current $\mathcal{J}$ [20]. More precisely , given a RCIT we first construct the corresponding TLIJ algebra e_n^0 next and for each simple current $\mathcal{J}$ we define the new TLIJ algebra generated by $\tilde{e}_n^{\mathcal{J}}$. Finally to each representation of $\tilde{e}_n^{\mathcal{J}}$ as projections of a fundamental tower we associate the modular invariant given by $\mathcal{J}$. If the value of β of the algebra generated by $\tilde{e}_n^{\mathcal{J}}$ is smaller than 4 then the possible towers on which they can be realized as conditional expectations are classified by the inclusion matrices defined from the ADI: Coxeter graphs and therefore we get strong restrictions on the allowed partition functions coming from simple currents. In summary we find that given a simple current $\mathcal{J}$ we can either construct a new partition function $\tilde{Z}_{\mathcal{J}}$ or a fundamental tower. If $\beta < 4$ as it is the case for $SU(2)$ WZW models both constructions give rise to the same ADI: classification.

To refresh this rather abstract discussion let us now proceed to the explicit construction of the towers, say $\tilde{A}_{\phi}^{0+\mathcal{J}}$, where the generators $\tilde{e}_n^{\mathcal{J}}$ act as conditional expectations. One can obtain $\tilde{A}_{\phi}^{0+\mathcal{J}}$ from $A_{\phi}^{0+\mathcal{J}}$ (see fig.24) by performing an orbifold-like construction.

The tower $\tilde{A}_{\phi}^{0+\mathcal{J}}$ is defined as the duality algebra acting on the orbifold-like conformal blocks $\tilde{\mathcal{F}}_H^{0+\mathcal{J}}$ which are depicted in fig.24. $\tilde{\mathcal{A}}$ denotes the orbit of the field a under the action of $\mathcal{J}$. Since $\mathcal{J}^2 = 1$ the orbits contain one or two elements. The latter case occurs when the field a is a fixed point of $\mathcal{J}$, $\mathcal{J}a = a$. If this happens then one has to make two copies of that field inside $\tilde{\mathcal{F}}_H^{0+\mathcal{J}}$:

$$\begin{array}{c} \tilde{A}_{\phi}^{0+\mathcal{J}} \qquad \qquad \tilde{A}_{\phi}^{0+\mathcal{J}} \\ \begin{array}{c} a \qquad \qquad \mathcal{J}a \\ \diagdown \quad \diagup \\ \tilde{\mathcal{A}} \end{array} \qquad \begin{array}{c} b \qquad \qquad \mathcal{J}b = b \\ \diagdown \quad \diagup \\ \tilde{\mathcal{A}}_2 \end{array} \end{array} \quad (5.9)$$

The orbifold-like conformal blocks $\tilde{\mathcal{F}}_H^{0+\mathcal{J}}$ are families of usual conformal blocks where some of them have been duplicated (orbifold points). The inclusion matrix for the tower $\tilde{A}_{\phi}^{0+\mathcal{J}}$ can be derived from that of $A_{\phi}^{0+\mathcal{J}}$ and has the structure considered in reference [28] in the context of statistical mechanics:

$$\begin{aligned} \tilde{\lambda}_{\tilde{\mathcal{A}}\tilde{\mathcal{B}}} &= N_{\phi,a,b} + N_{\mathcal{J}\phi,a,b} & \mathcal{J}a \neq a, \mathcal{J}b \neq b \\ \tilde{\lambda}_{\tilde{\mathcal{A}}\tilde{\mathcal{B}}_r} &= \tilde{\lambda}_{\tilde{\mathcal{B}}_r\tilde{\mathcal{A}}} = N_{\phi,a,b} & \mathcal{J}a \neq a, \mathcal{J}b = b \\ \tilde{\lambda}_{\tilde{\mathcal{A}}_r\tilde{\mathcal{B}}_s} &= \delta_{rs} N_{\phi,a,b} & \mathcal{J}a = a, \mathcal{J}b = b \end{aligned} \quad (5.10)$$

Clearly $\tilde{A}_{\phi}^{0+\mathcal{J}}$ is a fundamental tower and the operators $\tilde{e}_n^{\mathcal{J}} = e_n^0 + e_n^{\mathcal{J}}$ descend to a TLIJ algebra in $\tilde{A}_{\phi}^{0+\mathcal{J}}$. Let us illustrate this orbifold construction for the WZW model $SU(2)$. The integrable representations of $SU(2)$ are labelled by the spin $j = 0, \frac{1}{2}, \dots, \frac{k}{2}$ and there are $k+1$ of them. The field $\mathcal{J} = \frac{k}{2}$ is a simple current for all values of k . The orbits consist of two elements $\tilde{\mathcal{J}} = (j, \frac{k}{2}-j)$ if $\mathcal{J} \neq k/4$ and there exist a fixed point field if k is even corresponding to the spin $\mathcal{J} = k/4$. Let us choose $\phi = 1/2$ because in this case $\beta_{\phi} = \left(2 \cos \frac{\pi}{k+2} \right)^2 < 4$. The inclusion matrix $\lambda_{\mathcal{H}_2}$ of the tower $\tilde{A}_{\mathcal{H}_2}^0$ is the incidence matrix of the following graph:

which coincides with the Coxeter diagram $\Lambda_{k+1/2}$.

We apply now the orbifold construction to $\Lambda_{1/2}$. Recall that $\tilde{\Lambda}_{1/2}$ is defined on the orbits of $J = \frac{k}{2}$ which in this case can be labelled by the spin $0 \leq j \leq p$. The graphs associated to $\tilde{\Lambda}_{1/2}$ for k even are as follows:

If $k = 0 \text{ mod } (4)$ $\tilde{\Lambda}_{1/2}$ is the incidence matrix of the D -even series, while if $k = 2 \text{ mod } (4)$ $\tilde{\Lambda}_{1/2}$ is the incidence matrix of the D -odd series. Recall that for Coxeter diagrams Γ of the type $\Lambda(D)$, the incidence matrix has norm $2 \cos \frac{\pi}{h_\Gamma}$ where h_Γ is the Coxeter number of Γ . Therefore $\|\tilde{\Lambda}_{1/2}\| = \|\Lambda_{1/2}\|$.

For k odd the allowed representations ρ_α labelling the points in the Bratteli run over integers and half-integers spins so that after the orbifold construction one gets again the incidence matrix of the A series ($\tilde{\Lambda}_{1/2}(\kappa; \text{odd}) = \Lambda_{1/2}$). This simple result can illustrate the power of Jones classification. In fact

from the classification of towers with $\|\Lambda\| < 2$ we know that they can only correspond to the ADE Coxeter graphs. For k odd the value of $\|\Lambda_{k/2}\|$ is odd and thus never equal to the corresponding value of the others Coxeter graphs. In this case we have only one realization of the TLL algebra, namely the one corresponding to the only possible modular invariant.

These results are in one-to-one correspondence with the construction of modular invariant partition functions for $SU(2)$ level k . For $k = 0 \text{ mod } (4)$ the D -even partition functions are a consequence of the extension of the chiral algebra $SU(2)$ with the simple current $J = k/2$ which has integer conformal weight. For $k = 2 \text{ mod } (4)$ the D -odd partition functions are due to an automorphism of the fusion rules. If k is odd no new modular invariant appears. These results, as we have seen, can be obtained in trying to extend the operators $\mathcal{O}_\alpha^0$ to $\mathcal{O}_\alpha^0 + \mathcal{O}_\alpha^J$ maintaining essentially the Markov trace. We observe here a close relationship between Markov traces and modular invariant partition functions. A more detailed study on these topics will be the content of a future work [29].

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FIGURE CAPTIONS

Fig 1. – Bicolored ADI: Coxeter graphs.

Fig 2. – Extended Bratteli diagram.

Fig 3. – Three examples of paths on the diagram of fig 2.

Fig 4. – Example of a path which does not belong to Ω .

Fig 5. – Action of Ocneanu’s operators.

Fig 6. – Analogy with Witten’s vertex.

Fig 7. – Pictorial representation of the elements of $A_{p,n}$.

Fig 8. – Diagrammatic proof of the relations (2.18).

Fig 9. – Boltzmann weight.

Fig 10. – Star – triangular relation for IRF models.

Fig 11. – First two rows of a Bratteli diagram.

Fig 12. – Bicolored graph associated to the Bratteli of fig 11.

Fig 13. – IRF configuration allowed by the graph of fig 12.

Fig 14. – Association between Ocneanu’s paths and IRF states.

Fig 15. – Ocneanu’s operator acting on an IRF state.

Fig 16. – IRF knot.

Fig 17. – Action of a braiding of the legs ϕ in a Conformal block.

Fig 18. – A basis for the space of conformal blocks.

Fig 19. – First levels of the Bratteli diagram associated to $Sl(2)_{k=4}$. On the right – hand side appear the corresponding conformal blocks.

Fig 20. – If i is a p_k expectator then it is also a p_{n+1} .

Fig 21. – Modular – like conformal blocks.

Fig 22. – Conformal blocks where the algebra A_4^3 is defined.

Fig 23. – Conformal blocks where the algebra A_4^{ot3} is defined.

Fig 24. – Orbifold – like conformal blocks where A_4^{ot3} acts.

Fig. 1

Fig. 2

LEVEL
-1
0
1
2
3
$\dots$

LEVEL

LEVEL
-1
0
1
2

Fig. 5

$$= T_{(\xi, \eta)_{(2)}}(\omega)$$

Fig. 4

Fig. 3

Fig. 8

Fig. 6

Fig. 7

A square diagram with vertices labeled q_1 (bottom-right), q_2 (top-right), q_3 (top-left), and q_4 (bottom-left). The center of the square is labeled u .

Fig. 13

Fig. 16

Fig. 17

Fig. 18

Fig. 19

Fig. 20

Fig. 21

Fig. 22

Fig. 23

Fig. 24