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TWISTED $N=2$ COSET MODELS , DISCRETE TORSION AND ASYMMETRIC HETEROTIC STRING COMPACTIFICATIONS

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Abstract

We consider the theories obtained by modding tensor products of $N = 2$ superconformal coset theories by discrete symmetries. A large class of $(2,2)$ and $(0,2)$ heterotic string compactifications is obtained by modding by products of Z_N and permutation symmetries and by adding discrete torsion and/or background gauge fields. The presence of discrete torsion modifies the usual generalized GSO projection and gives rise to $(2,2)$, $(0,2)$ or non-supersymmetric compactifications depending on its (quantized) value. We present results from a systematic scan for new $(2,2)$ compactifications obtained through these moddings. We also construct left-right asymmetric compactifications by twisting differently left- and right-movers of the $N = 2$ blocks. Some of these constructions provide a generalization of the concept of asymmetric orbifolds to non-toroidal (Calabi-Yau) compactification varieties. We prove that all these models can be interpreted as left-right symmetric compactifications in the presence of discrete torsion. As an application of the above ideas we also build three-generation $SU(3) \times SU(2) \times U(1)$ string models by appropriately modding the 1^9 heterotic string compactification.

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1. Introduction

String vacua can be described by 2-d conformal field theories (CFT's) with appropriate central charge. Whereas the CFT corresponding to the space-time degrees of freedom is fixed the internal CFT is free up to the requirements of unitarity and modular invariance. One possibility for the internal CFT is to use tensor products of free bosons ($c = 1$) or free fermions ($c = 1/2$) and allow twisted boundary conditions. The free bosons may have an interpretation as coordinates in a target space and twisting them leads to orbifold compactifications [1]. The internal CFT can also be built from tensor products of minimal models [2] that, although usually corresponding to interacting theories, are completely solvable. Minimal models and more general solvable CFT's can in turn be obtained through the GKO coset construction [3].

Large classes of 4-dimensional string models constructed in terms of free bosons and fermions have been thoroughly analyzed [1], [4], [5], [6]. In particular, it is well understood how to build modular invariant partition functions providing consistent vacua. However, 4-d strings based on coset models have not been investigated in full detail. The main motivation of this work is to present a systematic analysis of these constructions in order to bring them at the same level of understanding as the ones based on free fermions and bosons. In particular, coset models have several discrete symmetries (the number depending on the number of tensored blocks). Thus we can also allow for twisted boundary conditions to generate new classes of vacua, exactly in the same way as it is done in the orbifold approach. In principle this construction looks richer since there are infinite series of fundamental blocks, and not just $c = 1$ or $c = 1/2$ blocks, that can be tensored together to give different string vacua. However, orbifold compactifications have the advantage of having free parameters (moduli) that can be varied continuously so that every model represents in fact a continuous class of models. In any other construction, including cosets, every model is only a particular point in the space of CFT's and to explore their moduli space, marginal perturbations need to be performed.

The 4-D heterotic string vacua can be obtained from a $(c = 15)_R$ superconformal theory on the right moving sector, decomposed into a spacetime part $(c = 6)_R$ and the internal one, $(c = 9)_R$. The non supersymmetric $(c = 26)_L$ left moving CFT splits into $(c = 4)_L$ for spacetime and a $(c = 22)_L$ internal CFT. The simplest way to guarantee consistency is to decompose the internal part as $(c = 9)_L \otimes (c = 13)_L$ and set $(c = 9)_L \equiv (c = 9)_R$.

The $(c = 13)_L$ CFT is constructed out of free fields and modular invariance restricts it to correspond either to $SO(10) \otimes E_8$ or $SO(26)$ [2], in the same way as $E_8 \otimes E_8$ and $SO(32)$ appear in the 10-D heterotic string. The internal $(c = 9)$ theory is therefore the one that parametrizes the different vacua.

Spacetime supersymmetry in four dimensions requires that the internal right moving CFT have $N = 2$ supersymmetry. Therefore we will restrict ourselves in this paper to the $N = 2$ coset models recently introduced by Kazama and Suzuki [7]. Notice that since the left moving theory $(c=9)_L$ is identical to the right moving one, it inherits the $N = 2$ structure and then these vacua are usually called (2,2). A preliminary study of these models has been presented in reference [8].

Twisted boundary conditions have been included in models constructed from tensor products of $N = 2$ minimal models with the limitation of maintaining the (2,2) structure by twisting identically the left- and right-movers.* Here we explore these issues in a broader setting. First we extend the formalism to include twists in more general coset models. Second, we consider the case of multiple twists and, as in the case of orbifold compactifications, we find that there are some free phases, the so-called discrete torsion⁺, in the partition function that are not determined by modular invariance thus acting as independent parameters.

As a third generalization, we consider independent twists for left and right movers. This, as in the case of asymmetric orbifolds [6], is a natural possibility in heterotic strings, due to the original left-right asymmetry of the theory. Taking into account the relationship of these constructions with manifold compactification, the obtained models would correspond to strings in which right and left movers are compactified on different Calabi-Yau varieties! As a particular case one can recover well known asymmetric orbifolds. Interestingly enough, we have been able to prove in this construction that all of these asymmetric models can also be interpreted as symmetric models with multiple twists and discrete torsion. This result is analogous to the toroidal compactification case [11], where left-right asymmetric models were finally understood as symmetric compactifications in the presence of background

* (0,2) minimal models have been discussed in reference [9] by embedding the internal CFT twists in the gauge degrees of freedom, i.e. Wilson lines. We generalize this approach to the general Kazama-Suzuki models in the present work.

+ Their interpretation as background antisymmetric tensor fields is clear for toroidal compactifications [10]

fields. This could be an indication that probably all 4-D strings could still be understood as compactifications with non-trivial backgrounds.

Finally, we have also studied models obtained by modding out by permutations of the conformal factors. This symmetry exists whenever there are several identical models tensored together. The case when the symmetries are freely acting is straightforward to analyze, whereas in the generic case there are “fixed points” which we discuss in some detail, although we only consider abelian subgroups of the original permutation symmetry.

The motivation underlying any study of string vacua has to be mainly phenomenological and, in our opinion, it is twofold. An exploration of the most general possible construction can give, besides some unification of formalisms, generic features of the models which could lead us to discover model independent properties of the vacua. Secondly, knowing all the parameters that we have at hand allow us to look for realistic string models in a more effective way. Any serious search for phenomenological consequences of string theory requires, at this moment, this scan of interesting models.

To this order we have studied many examples, which also illustrate our formalism. For example, one can use the simplest $c = 9$ coset model, the one obtained from nine copies of the level $k=1$ minimal model, as a laboratory to generate new vacua after the twists are performed. It is known that this model is equivalent to the $Z_3 \times Z_3$ orbifold at the critical point [5], [12]. However, since the discrete symmetries are more explicit in the present formalism, we discovered many $(2,2)$ vacua which were not known in the orbifold approach. We found also that the subset of our models corresponding to $(2,2)$ constructions using only minimal models (and no background gauge fields) seem to correspond to models constructed recently using a different formalism [13], which allow us to interpret those constructions as twisted versions of coset models. Analogous to reference [13], we find that, for a given model, its different twisted versions have a quantized number of families. We discovered that this happens not only for products of minimal models but also for nonminimal cosets. This feature is also preserved in all moddings by permutations that we have performed. This quantization was also known to be present in toroidal orbifolds [14], [5], [12], and so far lacks an explanation.

We have looked for models with three families and standard model-like gauge groups, which we can obtain by considering nontrivial Wilson lines and discrete torsion. We want to emphasize that for phenomenologically interesting models, it is not enough to have three

generations (there are several examples in the literature with three generations and E_6 or $SO(10)$ symmetry), since we have to find a way to break the gauge group and usually, the same mechanism that breaks the gauge group also changes the number of families. We succeeded in finding standard model-like vacua from the 1^9 model.

The paper is organized as follows. In chapter 2, we provide a brief review of $N = 2$ coset models and the general formalism for twisting them is presented. We also consider the inclusion of discrete torsion (which plays an important role in all the discussion) and background gauge fields. In chapter 3 we present the results of a scan for $(2,2)$ models using the techniques discussed in the previous chapter. The general features of these compactifications are discussed. We show some relevant examples, including Kazama-Suzuki models where the number of families is a multiple of three and some four-generation models. The left-right asymmetric moddings are discussed in chapter 4. The asymmetric Z_3 orbifold is reproduced in this approach and here we prove the equivalence of these models with multiply twisted symmetric ones with nontrivial discrete torsion. We also discuss the possibility of obtaining $SO(10)$ models for which the left moving twist is an R -symmetry but the right moving twist is not, thus preserving spacetime supersymmetry. Again, this is also understood in terms of symmetric twists and discrete torsion. A brief discussion of non-supersymmetric models, which are related to supersymmetric ones via discrete torsion is given at the end of the chapter. In chapter 5 we present some results on models obtained modding by permutation symmetries. In chapter 6 we construct phenomenologically interesting models, i.e. three generations and standard model-like gauge group, and discuss some phenomenological perspectives related to this construction. We present our final comments and conclusions in chapter 7.

2. Twisted $N = 2$ coset theories and string compactifications

In this chapter, after briefly reviewing some facts about $N = 2$ coset theories and the construction of the associated 4-dimensional heterotic strings, we generalize the formalism in several ways. We show that the coset theories possess discrete symmetries that can be used to mod them in order to get new string models. In particular, we describe the partition function associated to multiple moddings of the original theories and show how modular invariance leaves undetermined certain relative phases (“discrete torsion”). They play an important role in all the discussion and different choices of this (quantized) phases lead to

completely different 4-dimensional strings. Some choices correspond to (2,2) models, others to (0,2) and yet others give rise to models with broken space-time supersymmetry. We will further discuss the implications of these results in chapter 4. Finally, we briefly recall the modifications which have to be done in the presence of gauge background fields.

2.1 THE $N = 2$ COSET THEORIES

The $N = 2$ coset theories of Kazama and Suzuki [7] are found by applying the GKO construction to the coset $G \times SO(2D)/H$ where $D = \frac{1}{2} \dim G/H$. To obtain a realization of the $N = 2$ superconformal algebra (SCA) the condition that $G/H \times U(1)^{2l}$, $\text{rank } G = \text{rank } H = 2l$, be a Kählerian space must be satisfied. A particular solution is given when $\text{rank } G = \text{rank } H$ and G/H is a hermitian symmetric space (HSS). In this case H has the form $H = \hat{H} \times U(1)$ where $\hat{H}$ is a simple group.

The unitary highest weight representations of the $N = 2$ SCA can be obtained from the representations of the Kac-Moody algebras of G , $SO(2D)$ and H at levels k , 1 and $k + g_G - g_H$ respectively. Here g_G (g_H) is the dual Coxeter number of G (H). To construct primary states of the $N = 2$ SCA we first take highest weight states $|\Lambda\rangle$ of G and $|\tilde{\Lambda}\rangle$ of $SO(2D)$. The possible highest weights (or allowed representations) are restricted by unitarity. Next we decompose the $G \times SO(2D)$ representations into irreducible representations of H to find the possible highest weights $|\lambda\rangle$ of the H algebra. The highest weights or primary states of the $N = 2$ SCA thus obtained are denoted by $|\Lambda, \tilde{\Lambda}, \lambda\rangle$. The above steps are expressed through a relation among characters given schematically by

$$\chi_{\Lambda, k}^G \chi_{\tilde{\Lambda}, 1}^{SO(2D)} = \sum_{\lambda} \chi_{\Lambda, \tilde{\Lambda}, \lambda}^{N=2} \chi_{\lambda, k+g_G-g_H}^H \quad (2.1)$$

The conformal dimension of a $|\Lambda, \tilde{\Lambda}, \lambda\rangle$ state is given by

$$h = \frac{1}{2(k+g_G)}[(\Lambda^2 + 2\alpha_G \cdot \Lambda) - (\lambda^2 + 2\alpha_H \cdot \lambda)] + \frac{1}{2}\tilde{\Lambda}^2 \quad (2.2)$$

where α_G (α_H) is half the sum of the positive roots of G (H). The $U(1)$ superconformal charge of the primary states is given by

$$Q = -\frac{2\alpha^+ \cdot \lambda}{k + g_G} + \sum_{i=1}^D \tilde{\Lambda}^i \quad (2.3)$$

where α^+ is half the sum of the positive roots of G/H . Similar expressions hold for the

primary states $|\tilde{\Lambda}, \tilde{\Lambda}, \tilde{\lambda}\rangle$ of the left SCA.

In the following we will deal mostly with the projective cosets $CP_m = SU(m+1)/SU(m) \times U(1)$. In this case it is convenient to split λ into an $SU(m)$ weight $\hat{\lambda}$ and a $U(1)$ charge q as

$$\lambda = \hat{\lambda} + \frac{\alpha^+}{\eta} q \quad (2.4)$$

where $\eta = 2(\alpha^+ \cdot \alpha^+)$. By definition $\hat{\lambda} \cdot \alpha^+ = 0$. α^+ is given by

$$\alpha^+ = \frac{m+1}{2} w_m \quad (2.5)$$

where w_m is the m^{th} fundamental weight of $SU(m+1)$. Since $w_i \cdot w_m = i/(m+1)$ we find $\eta = m(m+1)/2$. q is defined $\text{mod } 2\eta(k+g_G)$ where $g_G = m+1$. This identification follows from the existence of certain relations among the coset theory fields [15]. Finally, the central charge of a CP_m coset model at level k is given by $c = 3km/(k+m+1)$. From now on we set $g_G \equiv g$.

The $N = 2$ minimal models are obtained as CP_1 cosets. To recover the usual expressions for h and Q we simply set $\Lambda = l/\sqrt{2}$, $\lambda = l/\sqrt{2}$, $\tilde{\Lambda} = s/2$, $\alpha_G = \alpha^+ = 1/\sqrt{2}$ and $\alpha_H = 0$ in eqs. (2.2) and (2.3). Unitarity requires $l \leq k$. s takes the values $s = 0, \pm 1, 2$ and it is defined $\text{mod } 4$. The allowed values of q ($U(1)$ 'representations') must verify $|q-s| \leq l$, $l+q+s = 0 \text{ mod } 2$. A number of states of the $N = 2$ algebra are identified under the Z_2 automorphism $l \rightarrow k-l$, $q \rightarrow q+k+2$, $s \rightarrow s+2$.

2.2 DISCRETE SYMMETRIES AND TORSION

It is known that $N = 2$ minimal models have a Z_{k+2} discrete symmetry [2] [16]. In this section we wish to prove that in general the CP_m cosets have a Z_{k+g} ($g = m+1$) symmetry under which the primary states acquire a phase $e^{2\pi i z/(k+g)}$ where $z = (q+\tilde{q})/2$ is the Z_{k+g} charge. The first step is to construct a partition function in which Z_{k+g} twisted boundary conditions are allowed. Then we show that this partition function is modular invariant. The general result for the partition function of a tensor product of modded CP_m models is then given. In this case there are undetermined relative phases between the moddings ('discrete torsion') which modify the generalized GSO projection.

Eq. (2.1) shows that the modular transformations of $\chi^{N=2}$ factorize into $SU(m+1)$, $SO(2m)$ and $SU(m) \times U(1)$ pieces. Furthermore, this last piece factorizes into $SU(m)$ and $U(1)$ contributions. Since the Z_{k+g} symmetry we want to study is only related to the $U(1)$ charge q we only need to focus on the modular transformations of $\chi_q^{U(1)} \equiv \chi_q$. Under $S: \tau \rightarrow -1/\tau$ and $T: \tau \rightarrow \tau + 1$ we find

$$\begin{aligned}\chi_q|T &= \exp\left(-\frac{i\pi q^2}{2\eta(k+g)}\right) \chi_q \\ \chi_q|S &= \frac{1}{\sqrt{2\eta(k+g)}} \sum_{q'} \exp\left(\frac{2i\pi qq'}{2\eta(k+g)}\right) \chi_{q'}\end{aligned}\quad (2.6)$$

where the T transformation is up to a q independent phase. Here we have substituted eq. (2.4) into the general formulae for transformations of affine characters [17]. We have used the relation $\alpha^+ \cdot \alpha_H = 0$. Also, since χ^H appears in the r.h.s. in eq. (2.1) we have really considered the transformation properties of $\chi_{\lambda, k+1}^H$ in deriving eqs. (2.6). The sum in the second eq. in (2.6) runs from 0 up to $[2\eta(k+g) - 1]$ since the charges are defined $\text{mod } 2\eta(k+g)$.

The partition function we want to consider has the general form

$$Z = \frac{1}{M} \sum_{z, y=0, \dots, M-1} Z(x, y) \quad (2.7)$$

where (x, y) refers to boundary conditions closed up to $e^{2\pi i z \tau / (k+g)}$ and $e^{2\pi i y \tau / (k+g)}$ in the σ and t world-sheet directions respectively. Here $\gamma, M \in Z$ and M is the least integer for which $M\gamma = 0 \text{ mod } (k+g)$. The modding γ defines a Z_M subgroup of Z_{k+g} .

The contributions $Z(x, y)$ in eq. (2.7) are given by

$$Z(x, y) = \sum_q e^{\frac{2\pi i}{k+g} y \gamma (q+z\eta\gamma)} \chi_q^* \chi_{q+2z\eta\gamma} \quad (2.8)$$

Under modular transformations $\tau \rightarrow (a\tau + b)/(c\tau + d)$, $ad - bc = 1$, the boundary conditions change as $(x, y) \rightarrow (dx + cy, bx + ay)$. Hence, modular invariance requires $Z(x, y) \rightarrow Z(dx + cy, bx + ay)$. It is enough to check that $Z(x, y)$ transforms correctly under the generators S and T . This follows immediately using eqs. (2.6). It is also necessary

to verify that $Z(x, y)$ is invariant under modular transformations that leave the boundary conditions invariant. For $\tau \rightarrow \tau + M$ we find that $Z(x, y)$ transforms into itself. We thus conclude that the coset model has a Z_{k+g} symmetry as explained above.

Eq. (2.7) defines the partition function of a Z_M orbifold of the original coset theory. From the form of $Z(x, y)$ we see that in the twisted sectors the right and left $U(1)$ charges are related by

$$\bar{q} = q + 2x\eta\gamma \quad (2.9)$$

Keeping x fixed in (2.7) and summing over y effectively implements a generalized GSO projection into invariant states in the x -twisted sector. Since the conformal dimensions of the states are independent of y we find that the projector is given by

$$\begin{aligned}P(x) &= \frac{1}{M} \left\{ 1 + e^{2\pi i \Delta(x)} + \dots + e^{2\pi i (M-1)\Delta(x)} \right\} \\ \Delta(x) &= \frac{\gamma}{k+g} (q + x\eta\gamma)\end{aligned}\quad (2.10)$$

Hence, the invariance condition is $\Delta(x) = \text{integer}$.

The above analysis generalizes to the case of tensor products of r coset models. The tensor product theory has a discrete symmetry group $\mathcal{G} = Z_{k_1+g_1} \times \dots \times Z_{k_r+g_r}$. We can obtain an orbifoldized theory by modding by a subgroup of $\mathcal{G}$. In this case we have a modding $\gamma = (\gamma_1, \dots, \gamma_r)$ of order M , i.e. M is the least integer for which $M\gamma = (0, \dots, 0) \text{ mod } (k_1 + g_1, \dots, k_r + g_r)$. The partition function for the Z_M orbifold theory has the same form as eq. (2.7) where $Z(x, y)$ is now the generalization of eq. (2.8). In the twisted sectors we now have

$$\bar{q}_i = q_i + 2x\eta_i\gamma_i \quad (2.11)$$

The projection on invariant states is given by

$$\Delta(x) = \sum_{i=1}^r \frac{\gamma_i}{k_i + g_i} (q_i + x\eta_i\gamma_i) = \text{integer} \quad (2.12)$$

It is also possible to construct orbifolds by other abelian subgroups of $\mathcal{G}$. The most general abelian subgroup is of the form $Z_{M_1} \times \dots \times Z_{M_r}$ with M_α dividing M_β for $\alpha < \beta$.

Each Z_{M_a} is generated by a modding γ_a . The partition function can be written as

$$Z = \frac{1}{M_1 \times \dots \times M_p} \sum_{x_a, y_a = 0, \dots, M_a - 1} \epsilon(x_1, \dots, x_p; y_1, \dots, y_p) Z(x_1, \dots, x_p; y_1, \dots, y_p) \quad (2.13)$$

The contributions $Z(x_1, \dots, x_p; y_1, \dots, y_p)$ are given by the generalization of eq. (2.8). The coefficients $\epsilon(x_1, \dots, x_p; y_1, \dots, y_p)$ are arbitrary phases fixed by requiring modular invariance of Z . In the case of a single modding 1-loop modular invariance forces $\epsilon(x, y) = 1$ as we have seen before. In the case of multiple moddings, at 1-loop there might be sectors not mixed by modular transformations thus leaving some $\epsilon(x_1, \dots, x_p; y_1, \dots, y_p)$ arbitrary. However, higher loop modular invariance and factorization imply the following restrictions on ϵ [10]

$$\begin{aligned} \epsilon(X; Y + Y') &= \epsilon(X; Y) \epsilon(X; Y') \\ \epsilon(X; Y) \epsilon(Y; X) &= 1 \\ \epsilon(X; X) &= 1 \end{aligned} \quad (2.14)$$

where $(X; Y) \equiv (x_1, \dots, x_p; y_1, \dots, y_p)$. The phases $\epsilon(X; Y)$ are called discrete torsion.

As an example, let us take a double modding. We find

$$\epsilon(x_1, x_2; y_1, y_2) = e^{2\pi i \xi (x_1 y_2 - x_2 y_1)} \quad (2.15)$$

Since $\epsilon(x_1, x_2; M_1 y_1, y_2) = \epsilon(x_1, x_2; 0, y_2)$ from eq. (2.15) we obtain the condition $e^{2\pi i \xi M_1} = 1$ and similarly $e^{2\pi i \xi M_2} = 1$. Thus, $\xi = e/M_1$, $e = 0, \dots, M_1 - 1$, since M_2 is a multiple of M_1 . Hence, the discrete torsion is quantized. Discrete torsion has the effect of changing the generalized GSO projection. For instance, for a double modding we obtain

$$\begin{aligned} \Delta_1(x_1, x_2) &= \sum_{i=1}^r \frac{\gamma_{1i}}{k_i + g_i} (q_i + x_1 \eta_i \gamma_{1i} + x_2 \eta_i \gamma_{2i}) - \frac{x_2 e}{M_1} = \text{integer} \\ \Delta_2(x_1, x_2) &= \sum_{i=1}^r \frac{\gamma_{2i}}{k_i + g_i} (q_i + x_1 \eta_i \gamma_{1i} + x_2 \eta_i \gamma_{2i}) + \frac{x_1 e}{M_1} = \text{integer} \end{aligned} \quad (2.16)$$

Before leaving this section we would like to comment on another application of the ideas we have discussed. This application concerns the characters χ_ρ of the $SU(2)$ affine algebra

at level k . Here $\rho = l + 1$ and unitarity requires $0 \leq l \leq k$. The modular transformations of χ_ρ are similar to (2.6), with $\eta = 1$, $g = 2$, when the properties

$$\chi_\rho = -\chi_{-\rho} \quad ; \quad \chi_\rho = \chi_{\rho+2(k+2)} \quad (2.17)$$

are taken into account. Therefore, we can obtain modular invariant combinations of the χ_ρ characters by taking orbifolds of the diagonal invariant. This is a well-known fact in the literature [18][16] and it has been proposed as a way to find modular invariants for other affine algebras [19]. Here we just want to point out that using (2.9) and (2.10) (with $q = \rho$) is perhaps the simplest way to generate $SU(2)$ affine invariants. In general, this procedure leads to invariants that can be written as combinations of the known A , D or E invariants at the given level. Typically one obtains invariants of the form $(A - D)$ or $(D - E)$. However, this need not be the case. For example, for $k = 28$, taking the order $M = 3$ modding $\gamma = 10$ yields the invariant*

$$\begin{aligned} R &= |x_3|^2 + |x_6|^2 + |x_{10}|^2 + |x_{12}|^2 + |x_{15}|^2 + |x_{18}|^2 - |x_{20}|^2 + |x_{21}|^2 + |x_{24}|^2 + |x_{27}|^2 \\ &\quad + [(\chi_2 \chi_{22}^* + \chi_5 \chi_{25}^* + \chi_8 \chi_{28}^* - \chi_1 \chi_{19}^* - \chi_4 \chi_{16}^* - \chi_7 \chi_{13}^* - \chi_{11} \chi_{29}^* - \chi_{14} \chi_{26}^* - \chi_{17} \chi_{23}^*) + \text{c.c.}] \end{aligned} \quad (2.18)$$

which cannot be cast as a linear combination of the A , D and E invariants at level $k = 28$. The appearance of negative multiplicities is a general feature of the orbifold procedure.

2.3 STRING COMPACTIFICATIONS

For 4-dimensional strings with space-time supersymmetry (SUSY) the internal right-moving degrees of freedom must provide a realization of the $N = 2$ SCA with $c_R = 9$. The internal left-movers must give a realization of the $N = 0$ conformal algebra with $c_L = 22$. In addition, the requirements of modular invariance and unitarity must be satisfied. To achieve modular invariance it is convenient to split the internal left theory into two pieces as $(c_L = 9) \oplus (c_L = 13)$ and take the $c_L = 9$ theory equal to the $c_R = 9$ theory whereas the $c_L = 13$ piece is realized by 13 free bosons compactified on the $SO(10) \times E_8$ torus. In ref. [2] Gepner explicitly showed how to obtain modular invariant, space-time supersymmetric 4-d heterotic strings when the internal $c = 9$, $N = 2$ theory is given by a tensor product of $N = 2$ minimal models. Kazama and Suzuki have in turn shown how to generalize Gepner's method

* The same invariant may be obtained from 2+1 Chern-Simons theory [20].

to the case of $N = 2$ coset models [7]. A survey of possible compactifications obtainable in that way was presented in [8]. We now wish to describe the modding procedure as applied to heterotic string compactifications based on general CP_m models. We want to particularly emphasize the effects of discrete torsion on the generalized GSO projection. We will show (eq.(2.35)) that the usual projections are modified in the presence of those phases and that $(2,2)$, $(0,2)$ and non-supersymmetric models may be obtained depending on the phases.

Let us start by recalling some relevant notation. Consider a tensor product of r $N = 2$ cosets with primary states given by $\otimes_{i=1}^r |\Lambda_i, \tilde{\Lambda}_i, \tilde{\lambda}_i, q_i\rangle$. To obtain a heterotic compactification we add the necessary right and left extra degrees of freedom. In the right sector we add the uncompactified space-time bosons and fermions. The mass formula (in the light cone) for the right states is given by

$$\frac{m_R^2}{8} = N_R + \frac{1}{2}w^2 + h_{int} - \frac{1}{2} \quad (2.19)$$

where h_{int} is the conformal dimension associated to the internal states, N_R is the oscillator number. w is a vector in the $SO(2)$ weight lattice that carries the space-time properties of the state. For massless states w can be either a scalar (0), vector (v), spinor (s) or anti-spinor ($\bar{s}$) weight. Similarly the left mass formula is given by

$$\frac{m_L^2}{8} = N_L + \frac{1}{2}\bar{w}^2 + \frac{1}{2}p^2 + \bar{h}_{int} - 1 \quad (2.20)$$

where $\bar{w}$ is a vector in the $SO(10)$ weight lattice and p is a vector in the E_8 root lattice. To achieve modular invariance it is necessary to interchange the $SO(10)$ characters as $\chi_0 \leftrightarrow \chi_v$, $\chi_s \leftrightarrow -\chi_{\bar{s}}$. This is known as the heterotic condition.

It is useful to encode information about a right or left state into a vector

$$V = (\tilde{\Lambda}_0; q_1, \dots, q_r; \tilde{\Lambda}_1, \dots, \tilde{\Lambda}_r) \quad (2.21)$$

where $\tilde{\Lambda}_0$ is the $SO(2)$ ($SO(10)$) weight for the right (left) sector. We also define the scalar product

$$V \cdot V' = \sum_{i=0}^r \tilde{\Lambda}_i \tilde{\Lambda}'_i - \sum_{i=1}^r \frac{q_i q'_i}{2\eta_i(k_i + g_i)} \quad (2.22)$$

and the vectors

$$\begin{aligned} \beta_0 &= (\bar{s}; \eta_1, \dots, \eta_r; s_1, \dots, s_r) \\ \beta_i &= (v; 0, \dots, 0; \dots, v_i, \dots) \quad (v \text{ in the } i^{\text{th}} \text{ position}) \end{aligned} \quad (2.23)$$

where $s, \bar{s}$ and v represent the spinor, anti-spinor and vector weights of the corresponding SO groups. Notice that $\beta_0^2 = 1$ when the total internal central charge is $c = 9$. Also, $\beta_i^2 = 2$.

To guarantee space-time SUSY it is necessary to project out all right states not having odd-integral total superconformal $U(1)$ charge (including the $SO(2)$ contribution). This projection consists of imposing

$$Q_{TOT}(V_R) = 2\beta_0 \cdot V_R = \text{odd integer} \quad (2.24)$$

Notice that Q_{TOT} and the total Q_{int} are related by $Q_{TOT} = Q_{int} + 2\tilde{\Lambda}_0 \cdot \bar{s}$. To align the boundary conditions of the fermions in the various sectors it is necessary to impose the constraint

$$Q_i(V_R) = 2\beta_i \cdot V_R = \text{even integer} \quad (2.25)$$

The left vector V_L differs from V_R by an element of the lattice generated by β_0 and the β_i , this is

$$V_L = V_R + n_0\beta_0 + n_i\beta_i \quad (2.26)$$

The heterotic condition is implemented by adding a vector $(v; 0, \dots, 0; 0, \dots, 0)$ to V_L . The $SU(m+1)$ and $SU(m)$ left and right weights are in general correlated through the diagonal invariant since a general classification of other modular invariants is at present lacking. For $SU(2)$ the D or E invariants can also be used.

Notice that conditions (2.24) and (2.25) are also satisfied by V_L . This guarantees the existence of $SO(10)$ spinorials $(16 + \overline{16})$ gauge bosons completing the E_8 generators. This situation changes when modding by symmetries Γ . We will shortly see that in this case it is possible to maintain space-time SUSY while breaking E_8 to $SO(10)$.

Applying the rules and projections described above we can find the massless spectrum of the heterotic compactification corresponding to a given $c = 9$ tensor product. For instance, to find the fields transforming as 27 and $\overline{27}$ of E_6 we can concentrate on the scalar component

so that $m_R = 0$ requires $h_{int} = \frac{1}{2}$. Since $Q_{int} (= Q_{TOT}$ in this case) must be an odd integer and $|Q_{int}| \leq 2h_{int}$ from unitarity we have $Q_{int} = \pm 1$. For definiteness we choose the minus sign (the plus sign will then give the antiparticles). For the left part we look at the 10-dimensional (vector) representation of $SO(10) \times U(1)$ (this is the internal total left superconformal $U(1)$) included in an E_6 generation ($27 = 16_1 + 10_{-1} + 1_2$). For 27 's ($\bar{27}$'s) we then need $\bar{h}_{int} = \frac{1}{2}$ and $\bar{Q}_{int} = -1$ ($\bar{Q}_{int} = 1$).

The Gepner construction can be understood in terms of the orbifold ideas presented in the previous section. To explain this we define

$$\chi_V = \chi_{\tilde{h}_n}^{SO(2)} \prod_{i=1}^r \chi_{q_i}^{U(1)} \chi_{\tilde{\lambda}_i}^{SO(m_i)} \quad (2.27)$$

Under S and T modular transformations we find

$$\begin{aligned} \chi_V|T &= e^{i\pi V \cdot V} \chi_V \\ \chi_V|S &= \sum_{V'} e^{-2\pi i V \cdot V'} \chi_{V'} \end{aligned} \quad (2.28)$$

where the T transformation is up to a V -independent phase and the S transformation is up to a numerical factor. The scalar product used in (2.28) is that defined in (2.22). The full character for the right sector includes the contribution of the transverse, uncompactified coordinates as well as the $SU(m+1)$ and $SU(m)$ pieces. For our subsequent analysis these contributions can be neglected.

We are interested in the partition function obtained by modding the tensor product of $N = 2$ SCFT (and associated $SO(2)$ and $SO(10)$ Kac-Moody algebras) by the symmetries generated by β_0 and β_1 . Let us consider a partition function of the form

$$Z \sim \sum Z(n_0, n_i; m_0, m_i) \quad (2.29)$$

where for the untwisted sector we have

$$Z(0, 0; m_0, m_i) = \sum_V (-1)^{m_0} e^{-2\pi i V \cdot (m_0 \beta_0 + m_i \beta_i)} \chi_V \chi_V^* \quad (2.30)$$

The $(-1)^{m_0}$ factor accounts for the correct spin statistics. From (2.30) we immediately derive the generalized GSO projections given in (2.24) and (2.25). Notice $V = V_R$ and

$V_L = V_R$ in the untwisted sector. Performing modular transformations we find twisted sectors in which V_L is given by (2.26). Notice that in (2.29) n_0, m_0 run from 0 up to $N_0 - 1$ where N_0 is the least integer for which $N_0 \beta_0 \cdot V = \text{integer} \forall V$ and similarly for n_i, m_i . Strictly speaking, the above analysis is only valid for the type II superstring. However, a type II superstring compactification can be converted into a heterotic one by imposing the heterotic condition.

It is clear that we can also use as internal theory an orbifolded tensor product as explained in the previous section. In this case (2.30) changes to

$$Z(0, 0, 0; m_0, m_i, y) = \sum_V (-1)^{m_0} e^{-2\pi i V \cdot (m_0 \beta_0 + m_i \beta_i)} e^{-2\pi i y \Gamma(2V)} \chi_V \chi_V^* \quad (2.31)$$

where in the notation of (2.21) the modding Γ is given by

$$\Gamma = (0; \gamma_1 \eta_1, \dots, \gamma_r \eta_r; 0, \dots, 0) \quad (2.32)$$

Notice that the order of Γ is given by the least integer M for which $M(\gamma_1, \dots, \gamma_r) = 0 \pmod{k_1 + g_1, \dots, k_r + g_r}$. Performing modular transformations we derive the general structure

$$\begin{aligned} Z(n_0, n_i, z; m_0, m_i, y) &= e^{2\pi i \mu(m_0 z - n_0 y)} \sum_V (-1)^{m_0 + n_0} e^{-2\pi i V \cdot (m_0 \beta_0 + m_i \beta_i)} \\ &\quad \times e^{-2\pi i y \Gamma(2V + 2x\Gamma + 2n_0 \beta_0)} \chi_V \chi_V^* e^{2\pi i V \cdot (n_0 \beta_0 + n_i \beta_i + 2x\Gamma)} \end{aligned} \quad (2.33)$$

In eq. (2.33) there is a phase freedom that is conveniently parametrized by a discrete torsion phase μ . The latter obeys a quantization condition $e^{2\pi i \mu} = 1$.

Eq. (2.33) indicates the appearance of twisted sectors in which

$$V_L = V_R + n_0 \beta_0 + n_i \beta_i + 2x\Gamma \quad (2.34)$$

We can also derive the generalized GSO projections

$$\begin{aligned} 2\beta_0 \cdot V_R + 2x\mu &= \text{odd} \\ \Delta^0 &= -\Gamma \cdot (2V_R + 2x\Gamma) - n_0(2\beta_0 \cdot \Gamma - \mu) = \text{integer} \end{aligned} \quad (2.35)$$

together with (2.25).

Eq. (2.35) is the main result of this section. Notice that in order to maintain space-time SUSY we need to impose the charge odd-integrality condition (2.24). Hence, SUSY implies $\mu = 0$. We can check that the Δ -projection is always compatible with SUSY. For instance, if a state $V_R \otimes V_L$ is allowed the SUSY partner $(V_R + \beta_0) \otimes V_L$ will also be allowed since for the latter $n_0 = -1$. However, in general E_8 will be broken to $SO(10)$. To see this, suppose that $V_R \otimes V_L$ is allowed, then the E_6 partner $V_R \otimes (V_L + \beta_0)$ will also be allowed only if $2\beta_0 \cdot \Gamma = \text{integer}$. Notice that modding by a symmetry Γ with $2\beta_0 \cdot \Gamma \neq \text{integer}$ does not necessarily break space-time SUSY provided the discrete torsion is appropriately chosen ($\mu = 0$ with our phase conventions). Notice also the inherent left-right asymmetric nature of discrete torsion. These issues will be further developed in chapter 4.

Up to now we have left untouched the gauge degrees of freedom realized by the 13 free bosons compactified on the $SO(10) \times E_8$ torus. It is possible to embed the discrete symmetries of the internal tensor product into these degrees of freedom [9]. The embedding is realized through shifts in the $SO(10) \times E_8$ lattice Ω . Thus we associate a shift A to a Z_M symmetry parametrized by a modding Γ . Then, the original model is orbifoldized by the combined action $\Gamma \otimes A$. Consistency requires $MA \in \Omega$. One-loop modular invariance implies the level matching constraint

$$MA^2 = 0 \text{ mod } 2 \quad (2.36)$$

As usual, there appear twisted sectors and the surviving states are selected by generalized GSO projections. The second projection in (2.35) now takes the form

$$\Delta = \Delta^0 - [(P + xA)A - \frac{1}{2}xA^2] = \text{integer} \quad (2.37)$$

where Δ^0 is defined in (2.35) and P is a vector in the $SO(10) \times E_8$ weight lattice.

These constructions lead to a variety of (0,2) models with many possibilities for the gauge group. Naively one would expect the $E_6 \times E_8$ symmetry to be broken down to some subgroup as a result of the projection of the $P \in SO(10) \times E_8$ weights onto $P \cdot A = \text{integer}$ (set $x = 0$ in eq. (2.37)). However there are often extra gauge bosons coming from twisted sectors which may enlarge the gauge symmetry. We will see an example in which an $E_6 \times SU(3)$ symmetry is transformed into $SU(9)$ due to some modding in chapter 6. Also the massless chiral superfields no longer transform as 27 's of E_6 but as essentially arbitrary

(lowest dimensional) representations of the unbroken group. Notice also that the above procedure includes as a particular case the addition of "Wilson line" backgrounds on the underlying variety. A (0,2) three generation $SU(3) \times SU(2) \times U(1)$ example constructed using the above techniques will be discussed in some detail in chapter 6.

3. Results for (2,2) compactifications

We have described in the previous section how to obtain a large class of (2,2) and (0,2) compactifications by modding $N = 2$ superconformal field theories and adding background fields. We are going to concentrate in this chapter on the simplest case of (2,2) compactifications. We would like to describe the results of a scan for models based on coset SCFT. We think this is an interesting thing to do at least for two reasons. First, it gives an idea of the kind of string vacua (e.g. massless spectrum) which is to be expected from this type of compactifications. Secondly, it can provide us with some interesting compactification examples which may be phenomenologically relevant. In particular, (2,2) compactifications can be taken as starting points in order to obtain more interesting (0,2) theories closer to the observed physics.

Given a (2,2) heterotic string compactification obtained by tensoring $N = 2$ coset theories one can obtain a new compactification modding by a discrete symmetry generated by Γ . There is a finite number of inequivalent Γ 's for each given parent model. Two moddings Γ_1 and Γ_2 are equivalent if

$$\Gamma_1 = \Gamma_2 + 2\pi(0; \eta_1, \dots, \eta_r; 0, \dots, 0) \quad (3.1)$$

for some integer n . This follows because a modding $2(0; \eta_1, \dots, \eta_r; 0, \dots, 0)$ acts trivially on the spectrum. In some cases (prime order symmetries) the coefficient of Γ_2 in eq. (3.1) can be different from one. This is not true in general as, for instance, a modding $\Gamma' = 2\Gamma$ gives rise generically to a model different from the one generated by Γ .

We will also require of the Γ 's to verify

$$2\beta_0 \cdot \Gamma = - \sum_{i=1}^r \frac{\gamma_i}{(k_i + g_i)} = \text{integer} \quad (3.2)$$

and we choose a trivial discrete torsion as explained in the previous chapter. In this way both SUSY and E_8 gauge invariance are preserved and the resulting model is a (2,2) compactification. Relaxing condition (3.2) leads to a (0,2) $SO(10)$ model. The number of

allowed possibilities for Γ 's in this case substantially increases. We have limited our scan to (2,2) theories so that it becomes more tractable. Anyway, the space of (0,2) models is much bigger than the one of $SO(10)$ models (recall the discussion at the end of previous chapter) so that a general scan of (0,2) models is certainly out of reach. Notice that by an appropriate choice of discrete torsion Γ 's not obeying eq. (3.2) can be interpreted as corresponding to R-symmetries.

As is well known, there are 168 (2,2)-compactifications [21] obtained by using minimal models and the diagonal modular invariant (both for the $SU(2)$ and $U(1)$ characters). There are 190 further compactifications [8] which use at least one $N = 2$ coset theory of the $SU(m+1)/SU(m) \times U(1)$ type and the diagonal modular invariant. Starting from these 358 parent models we have studied the massless spectra of models obtained by a single modding by a Γ . We have not attempted a complete analysis of all models but just a representative scan and we have in general only studied models with levels $k \leq 16$. We have also studied examples with moddings by two Γ 's and possible relative discrete torsion. We describe some general features of the obtained models in what follows.

By modding an original (2,2) compactification involving only $N = 2$ minimal models one gets typically a factor ~ 10 new models depending on the particular case. An interesting property is that there is a "quantization" on the net number of E_6 generations within a given parent model and its descendants*. The number of generations is always a multiple of a "quantum" within a "family" of models. This quantum is, most of the time, 12 and only sometimes it is 6 or 4. A similar quantization was known to occur in orbifold models [14], [5], [12] and seems to be a general property of (2,2) compactifications. We have not found models with quantum = 3 and hence we have not found new (2,2) models with three generations. The model of ref. [22] has quantum = 3 and gives rise to a 3-generation compactification after modding by a Γ -symmetry and a permutation symmetry. This model, however, makes use of the $k = 16$ E_7 modular invariant whereas only the diagonal $SU(2)$ modular invariant has been considered in our analysis.

The simplest example of "family" of models is the one using as parent model nine copies of the $k = 1$ $N = 2$ minimal model (i.e. the $(1)^9$ model). The standard diagonal modular invariant gives us a model with 84 $2\bar{7}$'s and no antigenerations. This model corresponds to the multicritical point (rank=22 gauge group) of the $Z_3 \times Z_3$ orbifold [5], [12]. By doing

* This fact has also been independently remarked by the authors of ref. [13] who use a different approach.

single or double moddings of $(1)^9$ one gets the compactifications shown in table 1. The quantum of generations is 12. Some of the examples in table 1 can be easily associated to multicritical points of known orbifold compactifications. Thus # 5 corresponds to the Z_3 orbifold and #9 and #6 correspond to the $SU(3)^3$ -torus and $K_3 \times SU(3)$ -torus compactifications. Model # 10, which includes torsion, corresponds to a $Z_3 \times Z_3$ orbifold model with torsion constructed in [12]. Table 1 shows, however, some models which were previously unknown. This might look irrelevant since after all the number of E_6 generations is far from realistic. However, as we will show in chapter 5, (0,2) compactifications based on the models of table 1 could be constructed with three quark-lepton generations and interesting gauge groups. The existence of previously unknown (2,2) compactifications could be useful in order to construct new (0,2) models with interesting phenomenological properties (e.g. Yukawa coupling structure).

Something similar happens with other examples. The $(2)^6$ family has its parent model associated to the multicritical point of the $Z_4 \times Z_4$ orbifold and the standard Z_4 orbifold is obtained modding by $\gamma = (1, 1, -1, -1, 0, 0)$. In this case the family quantum is 6. Of course, most of the models are not associated to orbifolds but to more generic weighted projective spaces [23]. A standard example is the $(3)^5$ family of models. Some of its members are displayed in table 2. The original model [24] is associated to a certain point in the moduli space of the Calabi-Yau variety defined by the quintic polynomial in CP_4 . A 4-generation model was known to exist if one divides the model # 2 in the table by an order 5 cyclic permutation (see chapter 5). We have found two new 4-generation models in this family (models # 4 and # 5 in the table) which are obtained without the need of further permutation modding.

Another interesting feature which appears for some moddings is that sometimes the effect of them is to exchange the number of $2\bar{7}$'s and $2\bar{7}$'s so that they act as an effective charge conjugation (see examples # 3 and # 7 in table 2). This happens quite often. From the point of view of the associated compactifying variety, this implies that quite often if a Calabi-Yau variety with Betti numbers $b_{1,1}$ and $b_{1,2}$ exists, also a manifold with the reversed Betti numbers must exist.

Our results concerning (2,2) models making only use of $N = 2$ minimal models seem to be consistent with some recent independent work which uses different techniques [13]

. It seems that the new modular invariants described in that work can be described in

our language in terms of Γ moddings and discrete torsion. We have, however, also studied models involving non-minimal $N = 2$ SCFT.

We have made a scan of models involving at least one $N = 2$ $SU(m+1)/SU(m) \times U(1)$ SCFT with $m \geq 2$. A level k theory corresponding to that coset will be denoted by (m, k) . We have computed essentially all massless spectra of $(2, 2)$ compactifications of that type which can be obtained by a single modding Γ up to level $k = 16$ (and also some cases with higher k 's). Only models which did not correspond to deformations (in the Landau-Ginzburg sense [23]) of some Gepner type model were considered. The general features of the $(2, 2)$ compactifications found are very similar to those obtained just from minimal models. The number of possible Γ moddings is much smaller. This is because the number of conformal factors for each $c = 9$ model is smaller (since the central charge of these $N = 2$ coset models is $c \geq 3$) and when the number of factors increases, most of the time the compactification is a deformation of a Gepner type model. An extreme case is provided by the three irreducible models with $(m, k) = (6, 7), (5, 9)$ and $(4, 15)$. They have by themselves $c = 9$ and hence have no extra factors. No non-trivial modding (apart from the one generated by β_0) can be performed in these models. Thus they seem to correspond to points of moduli space of Calabi-Yau varieties with no discrete isometries.

As was the case with $(2, 2)$ compactifications using only $N = 2$ minimal models, the number of generations is quantized and again 12 is the most frequent quantum, 6 and 4 being more rare. There are altogether four models with an odd number of generations (see table 3). The irreducible model $(5, 9)$ and its deformed equivalent $(4, 10)(1)$ have $N_{27} = 66$, $N_{\overline{27}} = 3$. None of them admits further modding. The model $(2, 4)^2(5)$ has $N_{27} = 95$ and $N_{\overline{27}} = 2$. It admits a modding by a $\gamma = (0, 1, 4)$ yielding $N_{27} = 11$, $N_{\overline{27}} = 32$. The last two models have quantum = 3 although it does not seem possible to further reduce the number of generations. We have, however, made use only of the diagonal modular invariant for the $SU(m+1)$ characters and it could well turn out that new quantum=3 families of models appeared if one used more general invariants. However, a general classification of those is still lacking. The closest to "realistic" model one finds is a 4-generation model based on $(3, 3)(12)^2$ (# 4 in table 3).

Some Kazama-Suzuki compactifications correspond to orbifolds, like $(2, 3)^3$ which is associated to the rank=16 $Z_6 \times Z_6$ orbifold. Modding by appropriate Γ symmetries one gets the two versions of the Z_6 orbifold (# 6 and # 7 in table 3). By multiple modding and

discrete torsion one obtains compactifications (# 8 and # 9) previously obtained in terms of orbifolds [12]. Similar models can be obtained from the $(1)^3(4)^3$ Gepner model although they would correspond to more symmetric (rank=19) points in the moduli space.

As happened with the models which used only minimal $N = 2$ SCFT, there are some moddings whose effect is to reverse the number of generations and antigerations (e.g. # 10 and # 11 in table 3).

As a general conclusion of the analysis of the $(2, 2)$ vacua which can be obtained using both minimal and other $N = 2$ $SU(m+1)/SU(m) \times U(1)$ cosets one observes the following facts: i) There is a quantization of the net number of generations and the quantum is most of the time 12 and less frequently 6 and 4. ii) Using the diagonal $SU(m+1)$ modular invariant there are only a few models with odd numbers of generations and none with three. iii) The typical number of generations for the "parent" (unmodded) models is large ($\sim 72 - 84$) and it is reduced ($\sim 12 - 24$) for their descendants. iv) Very often compactifications with reversed number of generations and antigerations exist. One goes from one of those compactifications to the other by a certain modding.

In closing this chapter we would like to make a remark. The quantization property described above disappears generically in the presence of background gauge fields. Since the presence of those backgrounds seems to be unavoidable if we want to get a realistic gauge symmetry, one should not overemphasize the importance of that property.

4. Left-right asymmetric twists

We have seen in chapter 2 that coset models have a Z_{k+g} symmetry. The holomorphic and antiholomorphic primary states were symmetrically twisted by $e^{\frac{2\pi i z}{k+g}}$ with the Z_{k+g} charge being defined as $z = \frac{(q+\bar{q})}{2}$. We will see in this chapter that q and $\bar{q}$ can be independently seen as the charges for two Z_{k+g} symmetries corresponding to the holomorphic and antiholomorphic (right and left) CFT's respectively. These twists will then define asymmetric CFT's. One can then proceed and construct left-right asymmetric heterotic string compactifications by using those asymmetric CFT as internal $c = 9$ sector. The models obtained provide a generalization of asymmetric orbifolds to more complicated (non-toroidal) varieties.

4.1 THE BASIC FORMALISM

The procedure to follow again is to define modular invariant partition functions reflecting the symmetries. We will consider twists generated by γ_R and γ_L of order M_R and M_L respectively. The coset model partition function, restricted to the $U(1)$ part can be written as

$$Z = \frac{1}{M_L M_R} \sum_{\substack{x_L, x_R = 0, \dots, M_L-1 \\ y_L, y_R = 0, \dots, M_R-1}} Z(x_L, x_R; y_L, y_R) \quad (4.1)$$

where the contributions from the different sectors are

$$Z(x_L, x_R; y_L, y_R) = \alpha \sum_q e^{\frac{2\pi i}{M_L M_R} (y_R \gamma_R q + y_L \gamma_L q)} \chi_{q-2x_R \gamma_R} \chi_{q+2x_L \gamma_L}^* \quad (4.2)$$

and the q -independent phase is restricted from modular invariance to be

$$\alpha(x_L, x_R; y_L, y_R) = e^{-\frac{2\pi i}{M_L M_R} (x_R y_R \gamma_R^2 - x_L y_L \gamma_L^2)} e^{2\pi i c (x_R y_L - x_L y_R)} \quad (4.3)$$

The constant c is not fixed by one-loop modular invariance and plays a role analogous to the discrete torsion introduced in chapter 2. Notice that the relation between the left and right charges is now ($q_R \equiv q - 2x_R \gamma_R$, $q_L \equiv q + 2x_L \gamma_L$)

$$q_L = q_R + 2\eta x_R \gamma_R + 2\eta x_L \gamma_L \quad (4.4)$$

Invariance of (4.1) under the $\tau \rightarrow \tau + (k+g)$ transformation requires $\eta(\gamma_R^2 x_R - \gamma_L^2 x_L) =$ integer, which is always fulfilled since η is an integer.

The symmetric case of chapter 2 can be reproduced from the present formalism by setting $\gamma_L = \gamma, \gamma_R = 0$ or equivalently $\gamma_R = \gamma, \gamma_L = 0$. The naive symmetric situation, $\gamma_L = \gamma_R$ has to be treated with care by restricting also the sectors ($x_R = x_L$, $y_R = y_L$), the resulting partition function does not look identical to (2.7) but it can be seen to describe the same physical system since, for instance, the conditions for invariant states are indeed the same.

For the general asymmetric situation that we are considering, we can obtain the generalized GSO projection for invariant states of the x_R (x_L) twisted sector by summing over y_R (y_L) in (4.1). We find

$$\begin{aligned} \Delta_R &= \frac{\gamma_R}{(k+g)} [q_R + \eta x_R \gamma_R] - c x_L = \text{integer} \\ \Delta_L &= \frac{\gamma_L}{(k+g)} [q_L - \eta x_L \gamma_L] + c x_R = \text{integer} \end{aligned} \quad (4.5)$$

As in chapter 2, this analysis can be extended in a straightforward way for tensor products and for multiple moddings.

Finally, for heterotic string vacua we can extend the analysis of chapter 2 to left-right asymmetric twists Γ_L and Γ_R , defined as in (2.32). We assume β_0 and β_i to act symmetrically and to simplify the discussion we further take $2\beta_0 \Gamma_R = \text{integer}$ and $2\beta_0 \Gamma_L = \text{integer}$. We also disregard possible discrete torsion phases between β_0 and Γ_R , and between β_0 and Γ_L . Then we can write the contributions to the partition function as,

$$\begin{aligned} Z(n_0, n_i, x_L, x_R; m_0, m_i, y_L, y_R) &= \alpha \sum_V (-1)^{m_0 + n_0} e^{-2\pi i V (m_0 \beta_0 + m_i \beta_i + 2y_R \Gamma_R + 2y_L \Gamma_L)} \\ &\quad \times \chi_{V-2x_R \Gamma_R} \chi_{V+2x_L \Gamma_L + n_0 \beta_0 + n_i \beta_i}^* \end{aligned} \quad (4.6)$$

with

$$\alpha = e^{4\pi i c (x_R y_R \Gamma_R^2 - x_L y_L \Gamma_L^2)} e^{2\pi i c (x_R y_L - x_L y_R)} \quad (4.7)$$

From (4.7) we can find the relation between V_L and V_R as,

$$V_L = V_R + 2x_R \Gamma_R + 2x_L \Gamma_L + n_0 \beta_0 + n_i \beta_i \quad (4.8)$$

and also the generalized GSO projections

$$\begin{aligned} \Delta_R^0 &\equiv -\Gamma_R (V_R + x_R \Gamma_R) - c x_L = \text{integer} \\ \Delta_L^0 &\equiv -\Gamma_L (V_L - x_L \Gamma_L) + c x_R = \text{integer} \end{aligned} \quad (4.9)$$

The left-right asymmetric twists can also be embedded in the gauge degrees of freedom by shifts A_R and A_L in the $E_8 \times SO(10)$ lattice, corresponding to Γ_R and Γ_L respectively.

With this consideration, the total GSO projections read

$$\begin{aligned}\Delta_R &\equiv \Delta_R^0 - [(p + x_R A_R + x_L A_L)A_R - \frac{1}{2}(x_R A_R + x_L A_L)A_R] = \text{integer} \\ \Delta_L &\equiv \Delta_L^0 - [(p + x_R A_R + x_L A_L)A_L - \frac{1}{2}(x_R A_R + x_L A_L)A_L] = \text{integer}\end{aligned}\quad (4.10)$$

Imposing the modular invariance of the partition function under $\tau \rightarrow \tau + M$, where M is the minimum common multiple of M_L and M_R , we obtain the modular invariance constraints for A_R and A_L to be:

$$M(x_R A_R + x_L A_L)^2 = \text{even integer} \quad (4.11)$$

Again, the previous analysis can be generalized in a straightforward way when several twists are present on the right and left sectors.

4.2 EXAMPLE: THE ASYMMETRIC Z_3 ORBIFOLD

As an illustration of the previous formalism, we will reproduce a well known example, the asymmetric Z_3 orbifold [6]. Recall that the 1^9 model is equivalent to the $Z_3 \times Z_3$ orbifold. To obtain the symmetric Z_3 orbifold, a twist of the type $\gamma = (1^3, -1^3, 0^3)$ must be performed [8]. The asymmetric Z_3 orbifold however, is in the right moving sector, like the symmetric one with standard embedding, whereas in the left moving sector it is like a torus. It can be seen that two symmetric twists of the type $(1^3, 0^6)$ generate the $N = 4$, $SU(3)^3$ torus compactification (see table 1) from the 1^9 model. Therefore, a natural choice for obtaining the asymmetric Z_3 orbifold is the following (the non-q entries are taken equal to zero):

$$\begin{aligned}\Gamma_R &= (1, 1, -1, -1, -1, 0, 0, 1), \\ \Gamma_{L1} &= (1, 1, 1, 0, 0, 0, 0, 0), \\ \Gamma_{L2} &= (0, 0, 0, 1, 1, 1, 0, 0, 0),\end{aligned}\quad (4.12)$$

with the embeddings

$$\begin{aligned}A_R &= \frac{1}{3}(0, 0, 0, 0, 0)(1, 1, -2, 0, 0, 0, 0, 0) \\ A_{L1} &= A_{L2} = 0\end{aligned}\quad (4.13)$$

With these twists, we can see that the original gauge group $E_8 \times E_6 \times U(1)^8$ gets broken

to $E_6 \times SU(3) \times E_6 \times U(1)^8$ because the states $^*(0, 0, 0)_R^9 \otimes (0, 0, 0)_L^9 \otimes P$, with $P^2 = 2$, are restricted from the projections (4.10) to satisfy $PA_R = \text{integer}$. However, new states are generated in the “twisted” sectors which enhance the gauge group. The states,

$$(0, 0, 0)_R^9 \otimes \begin{cases} (0, -1, 1)^3(0, 1, 1)_L^6 \otimes P \ (P^2 = 5/4) \\ (0, -2, 2)^3(0, 0, 0)_L^6 \otimes P \ (P^2 = 1) \\ (0, 2, 2)^6(0, 0, 0)_L^3 \otimes P \ (P = 0) \end{cases} \quad (4.14)$$

enhance the second (old) E_6 to E_8 like in torus compactifications. Also, we can see that $(0, 0, 0)_R^9 \otimes (0, 2, 2)^3(0, -2, 2)_L^3$, enhance two of the $U(1)$'s to $SU(3)$, like in the Z_3 orbifold. Finally $(0, 0, 0)_R^9 \otimes (1, 1, 2)(1, 1, 0)^2(0, 0, 0)_L^6$ and permutations promote the 6 $U(1)$'s to $SU(3)^3$. We are then left with the gauge group, $E_8 \times E_6 \times SU(3)^4$ which is the same one as for the asymmetric Z_3 orbifold.

For the matter sector, we have the states $V_R \otimes V_L \otimes P$. Here $V_L = 0$ implies $P^2 = 2$ and the projector gives $PA = \text{integer}$, which defines a state transforming like the $(27, 3, 1, 1, 1)$ under $E_6 \times SU(3)^4$. We can see from (4.8) and the requirement $h = \frac{1}{2}$ that there are three possible values for V_R , i.e. $2\Gamma_{L1}$, $2\Gamma_{L2}$ and $2\Gamma_{L1} + 2\Gamma_{L2} - 2\beta_0$. So there are three copies of the $(27, 3, 1, 1, 1)$ states.

Similarly, we find states of the form $V_R \otimes V_L \otimes (P + A)$:

$$V_L = 0, \quad V_R = 2\Gamma_R + 2\Gamma_{L2} - 2\Gamma_{L1}, \quad (P + A)^2 = 2$$

implying the existence of one 4-D state transforming as $(\overline{27}, 3, 1, 1, 1)$ under $E_6 \times SU(3)^4$,

$$V_L = (0, 0, 0, 0, 0, 2, 0, -2), \quad V_R = (0, 0, 2, 0, 0, 2, 0, 2, 0), \quad (P + A)^2 = 4/3$$

defining one copy of the states $(27, 1, 3, 1, 1) + (27, 1, \overline{3}, 1, 1)$.

$$V_L = (2, -2, 0, 2, -2, 0, 0, 0, 0), \quad V_R = (2, 0, 0, 2, 0, 0, 0, 2, 0), \quad (P + A)^2 = 2/3$$

which is one copy of $(1, \overline{3}, 3, 3, 1) + (1, \overline{3}, \overline{3}, 3, 1) + (1, \overline{3}, 3, \overline{3}, 1)$.

* For more clarity, we are displaying the whole (l, q, s) labels for the minimal model states.

It is straightforward to check that the projection conditions (4.10) are satisfied for these states. Therefore, we can see that the spectrum that we just found coincides with the massless spectrum of the asymmetric Z_3 orbifold. This together with the Z_3 symmetries shared by both models is a good evidence for considering them equivalent. More convincing arguments could be found by investigating the pattern of amplitudes for the two models.

4.3 DISCRETE TORSION AND ASYMMETRIC TWISTS

Many more new models can be constructed in a similar way, we can find for instance, asymmetric versions of the $(2, 2)$ model obtained from five copies of the level three minimal model (3^5) , which is known to correspond to a point in a Calabi-Yau manifold, and thus we can say that these new models are asymmetric versions of Calabi-Yau compactifications. The same can be said about any other Kazama-Suzuki model like the ones discussed in chapter 3. Moreover we can construct new models which break the original E_6 gauge invariance but preserving 4-D supersymmetry. This can be obtained e.g. by choosing Γ_L such that the condition $2\Gamma_L\beta_0 = \text{integer}$ is not satisfied but $2\Gamma_R\beta_0 = \text{integer}$ is fulfilled. This generates new models where the gauge symmetry is $SO(10) \times E_8$ which could be related to those discussed in reference [13]. These are possibly the simplest $(0, 2)$ models one can build with the present formalism. A much more general class can be obtained by embedding Γ_R and Γ_L into the gauge degrees of freedom as described in the previous sections.

We have seen how the existence of left-right asymmetric CFT's appear after performing different twists on the left and right moving sectors of the theory. Note, however, that the original symmetric partition function (2.7) looks in fact asymmetric since the arguments of the characters are shifted differently for the left and the right. According to section 4.1, that partition function could also be interpreted as the one corresponding to a shift $\gamma_L = \gamma$ and $\gamma_R = 0$. A more symmetric partition function would be

$$Z(x, y) = \sum_q e^{\frac{2\pi i}{k+g}(\gamma q)} \chi_{q - x\gamma\gamma} \chi_{q + x\gamma\gamma}^* \quad (4.15)$$

which is also modular invariant. We can see that the left and right charges for this partition function are related by

$$q_L = q_R + 2x\gamma\gamma \quad (4.16)$$

which is identical to (2.9). Moreover, the GSO projection is in this case

$$\Delta \equiv \frac{1}{(k+g)} q\gamma = \frac{1}{2(k+g)} (q_L + q_R)\gamma \quad (4.17)$$

coinciding with (2.10). In fact this equivalence can be seen explicitly by the change of variables $q \rightarrow q + x\gamma\gamma$ in (4.15), so the two partition functions describe identical theories.

Similarly, we could wonder whether the left-right asymmetric models described in the previous sections could actually be understood in terms of symmetric models. After all a similar situation happened in the toroidal compactification case, where left-right asymmetric string vacua were found and subsequently interpreted as symmetric compactifications in the presence of background fields [11].

Let us rewrite the GSO projections in (2.16), redefining $\gamma_{Li} \equiv \gamma_{Ri}$ and $\gamma_{2i} \equiv \gamma_{Li}$ and using the relation between left and right charges (2.11)

$$q_{Li} \equiv \tilde{q}_i = q_i + 2x_1\eta_i\gamma_{Ri} + 2x_2\eta_i\gamma_{Li} \equiv q_{Ri} + 2x_1\eta_i\gamma_{Ri} + 2x_2\eta_i\gamma_{Li} \quad (4.18)$$

we then find

$$\begin{aligned} \Delta_1(x_R, x_L) &= \sum_i \frac{\gamma_{Ri}}{(k_i + g_i)} (q_{Ri} + x_R\eta_i\gamma_{Ri} + x_L\eta_i\gamma_{Li}) - x_L\delta = \text{integer} \\ \Delta_2(x_R, x_L) &= \sum_i \frac{\gamma_{Li}}{(k_i + g_i)} (q_{Li} - x_R\eta_i\gamma_{Ri} - x_L\eta_i\gamma_{Li}) + x_R\delta = \text{integer} \end{aligned} \quad (4.19)$$

we can see that these are identical to the conditions $\Delta_L = \text{integer}$, $\Delta_R = \text{integer}$, for asymmetric twists with the discrete torsion chosen to be

$$\delta = -\Gamma_1 \cdot \Gamma_2 + c \quad (4.20)$$

and c is the arbitrary constant of equation (4.3). This equivalence can also be understood as redefinitions of the summation variables in the partition function ($q_i \rightarrow q_i + x_R\eta_i\gamma_{Ri}$).

Therefore we have seen that for a single coset model the asymmetric twist is equivalent to a symmetric one and for products of coset models, the asymmetric cases can be understood as symmetric twists with non-trivial discrete torsion. In particular the asymmetric Z_3 orbifold of the previous section can be obtained by the symmetric twists $\Gamma_1, \Gamma_2, \Gamma_3$ identical

to the ones in equation (4.12) with the same embedding $A_1 = A_R$ for one of the twists and also turning on the discrete torsions $\delta_1 = \Gamma_1 \Gamma_2$, $\delta_2 = \Gamma_1 \Gamma_3$. In ref. [6] it is remarked that the asymmetric Z_3 orbifold could be understood as a symmetric one by acting on the gauge degrees of freedom through Weyl rotations instead of shifts A_1 and reinterpreting the gauge coordinates as compactified space dimensions. Our statement here goes beyond that observation. Even if there is no embedding at all into the gauge degrees of freedom one can always interpret the modding as symmetric by choosing appropriate discrete torsion.

It is certainly very interesting to see how in this approach the asymmetric Z_3 orbifold can be obtained from purely symmetric twists. It is not known if all the asymmetric orbifolds could be understood in this way, although we do not see any reason to prevent it to happen, especially since practically all the $Z_N \times Z_M$ models can be obtained from tensor products of coset theories. This seems to indicate that maybe all presently known 4-D strings could still be interpreted as compactifications of the 10-dimensional heterotic string in the presence of appropriate backgrounds although, of course, we cannot claim at the moment that we have proved that in all generality.

Let us end this chapter by making some comments about the construction of non-supersymmetric compactifications. As can be observed from eq. (2.35) the discrete torsion, apart of introducing a left-right asymmetry as discussed in this chapter, can also break supersymmetry. Take as an example any of the 1⁹ models of table 1 (all of which verify $2\beta_0 \cdot \Gamma = \text{integer}$) and add some relative torsion μ between the modding Γ and β_0 . The projections look now as follows

$$2\beta_0 \cdot V_R + 2x\mu = \text{odd} \quad \Delta^0 = -\Gamma \cdot (2V_R + 2x\Gamma) + n_0\mu = \text{integer}. \quad (4.21)$$

Notice that for $\mu \neq 0$ the gravitino is not massless since for the latter one has $\Gamma \cdot V_R = \text{integer}$, $x = 0$, $n_0 = -1$ and thus $\Delta^0 \neq 0$. The same happens with the gauginos since also for them $n_0 = -1$ but not, of course, with the graviton and $SO(10) \times E_8$ gauge bosons since in this case $n_0 = 0$. The structure of the spectrum of non-singlet particles depends on the particular Γ modding. Notice that given a particular space-time supersymmetric model (like e.g. the 3-generation model of chapter 6) one can associate to it a finite number of non-supersymmetric models with the same gauge group and similar (but non-SUSY)

particle spectrum. They are obtained by assigning some discrete torsion phases μ_i to each of the Γ_i moddings. It is tempting to speculate that these models could correspond to spontaneously broken phases of the original model. Of course the scale of SUSY breaking is the string scale and hence these models as they stand would not be able to cope with the "hierarchy" problem. However perhaps one could learn something about the process of space-time supersymmetry breaking in strings by studying this interesting class of models and their relationship with the corresponding parent supersymmetric model.

5. Some remarks on moddings by permutations

$N = 2$ tensor products also possess symmetries under permutations of identical blocks. Then, it is also possible to find new string models by modding by a sub-group of the full permutation group. We will only consider abelian subgroups of the full permutation group. Generically this subgroup will not act freely thus leading to the appearance of twisted sectors. The study of those would require in general the introduction of new conformal fields in the theory. We will not address the general problem of these twisted conformal fields but will just show how in some cases the contribution of these sectors to the massless spectrum can be inferred. Our aim is to get some idea of how the permutation modding acts in generic models as compared to the Γ moddings described in previous sections.

To illustrate the ideas we take as an example the model obtained by tensoring five copies of the $N = 2$ minimal model at level $k = 3$, i.e. 3^5 . This model corresponds to compactification on a Calabi-Yau manifold described by a hypersurface in CP^4 as observed in [2]. The hypersurface is defined by the quintic polynomial

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0 \quad (5.1)$$

where the z_i are the homogeneous coordinates of CP^4 . We will denote this manifold by K . In ref. [24] K was first proposed as a string compactification variety. The Euler characteristic of K was computed by algebraic methods giving $\chi(K) = -200$.

As shown in [24] $\chi(K) = 2(N_{27} - N_{\overline{27}})$. Hence, from the point of view of the tensor product model we can compute $\chi(K)$ by determining the number of massless fields transforming as 27 and $\overline{27}$ of E_6 . For the 3^5 model $N_{27} = 101$ and $N_{\overline{27}} = 1$. The 101 internal

states corresponding to scalars transforming as 10_{-1} of $SO(10) \times U(1) \subset E_6$ are given by

$$\begin{aligned} & \left(\begin{smallmatrix} 110 \\ 110 \end{smallmatrix} \right)^5 ; \left(\begin{smallmatrix} 110 \\ 110 \end{smallmatrix} \right)^3 \left(\begin{smallmatrix} 220 \\ 220 \end{smallmatrix} \right) \left(\begin{smallmatrix} 000 \\ 000 \end{smallmatrix} \right) ; \left(\begin{smallmatrix} 110 \\ 110 \end{smallmatrix} \right)^2 \left(\begin{smallmatrix} 330 \\ 330 \end{smallmatrix} \right) \left(\begin{smallmatrix} 000 \\ 000 \end{smallmatrix} \right) \\ & \left(\begin{smallmatrix} 110 \\ 110 \end{smallmatrix} \right) \left(\begin{smallmatrix} 220 \\ 220 \end{smallmatrix} \right)^2 \left(\begin{smallmatrix} 000 \\ 000 \end{smallmatrix} \right) ; \left(\begin{smallmatrix} 220 \\ 220 \end{smallmatrix} \right) \left(\begin{smallmatrix} 330 \\ 330 \end{smallmatrix} \right) \left(\begin{smallmatrix} 000 \\ 000 \end{smallmatrix} \right)^3 \end{aligned} \quad (5.2)$$

where underline means permutations. The fields of each sub-theory are represented by their $(\frac{1}{2}q_3)$ labels. For the lone $\overline{27}$ the internal state is $(\frac{110}{-10})^5$.

Permuting the $k = 3$ blocks is equivalent to permuting the z_i variables in (5.1). Some permutations are R-symmetries because they act differently in the Neveu-Schwarz and Ramond sectors. This happens for instance for $P_2 : z_1 \leftrightarrow z_2$ since it involves an odd number of exchanges. We can see that P_2 is an R-symmetry in another way. The Calabi-Yau condition implies the existence of a $(3,0)$ form ω on K [25]. In turn, the existence of ω is required by space-time SUSY [24]. For the K manifold ω is given by

$$\omega = \frac{dy_1 \wedge dy_2 \wedge dy_3}{y_4^4} \quad (5.3)$$

where $y_i = z_i/z_0$. Since P_2 does not leave ω invariant it does not commute with SUSY. By performing the appropriate changes of variables we can convince ourselves that only an order 5 cyclic permutation P_5 and order 3 permutation P_3 are good symmetries of ω . There are other symmetries of K that leave ω invariant such as multiplication of the z_i 's by phases $e^{2\pi i n/5}$. These clearly correspond to the moddings explained in previous chapters.

Let us first study the P_5 symmetry defined by $P_5 : z_i \rightarrow z_{i(\text{mod } 5)}$. It is easy to check that P_5 leaves no fixed points on K . Therefore, $\chi(K/P_5) = \chi(K)/|P_5| = -40$. This means that modding the 3^5 theory by P_5 yields a new model in which the net number of generations is $N_{\text{gen}} = 20$. From (5.2) we see that we can take 21 combinations that are singlets under P_5 . Since the $\overline{27}$ is also a permutation singlet we recover the result for N_{gen} .

The P_3 symmetry can be defined as a cyclic permutation of z_2, z_3, z_4 while z_0, z_1 do not transform. To investigate whether there are fixed points we set

$$(z_0, z_1, z_2, z_3, z_4) \rightarrow (z_0, z_1, z_4, z_2, z_3) \quad (5.4)$$

since in CP spaces $z_i \sim \lambda z_i$. We find $\lambda^3 = 1$ and substituting $z_3 = \lambda z_2, z_4 = \lambda^2 z_2$ in (5.1)

yields

$$z_0^5 + z_1^5 + z_2^5(1 + \lambda^2 + \lambda) = 0 \quad (5.5)$$

For $\lambda = e^{2\pi i/3}$ necessarily $z_0 = z_1 = 0$ from (5.4). From (5.5) we see that we can have $z_2 \neq 0$. In this way we find two fixed points $(0, 0, 1, e^{2\pi i/3}, e^{-2\pi i/3})$, $(0, 0, 1, e^{-2\pi i/3}, e^{2\pi i/3})$ (up to an overall factor). Now, we can also take $\lambda = 1$ showing that P_3 also leaves a fixed surface $S : z_0^5 + z_1^5 + 3z_2^5$. $\chi(S)$ can be computed, for instance, as explained in [24], giving $\chi(S) = -10$. We have thus seen that the surface K/P_3 has singularities. In principle, these singularities can be repaired ('blown up') to give a smooth manifold.

In ref. [26] it is suggested that singularities in a quotient space $\mathcal{M}/\mathcal{G}$, where $\mathcal{M}$ is a Calabi-Yau manifold, can be repaired when the $(3,0)$ form ω is left invariant by the symmetry $\mathcal{G}$. In ref. [1] a formula for the Euler characteristic of the repaired manifold $\mathcal{M}^*/\mathcal{G}$ was proposed. This formula is given by

$$\chi(\mathcal{M}^*/\mathcal{G}) = \frac{1}{|\mathcal{G}|} \sum_{[x,y]=0} \chi(x,y) \quad (5.6)$$

where $|\mathcal{G}|$ is the order of $\mathcal{G}$. Here x, y are elements of $\mathcal{G}$ and $\chi(x, y)$ is the Euler characteristic of the sub-manifold left simultaneously fixed by x and y . By definition $\chi(1, 1) = \chi(\mathcal{M})$. Eq. (5.6) is understood in terms of orbifolds. It shows the appearance of twisted sectors in which we keep x fixed and sum over y . In particular, the contribution of the untwisted sector is found as

$$\chi_{\text{unt}} = \frac{1}{|\mathcal{G}|} \sum_y \chi(1, y) \quad (5.7)$$

We can imagine deriving (5.6) from (5.7) by performing modular transformations. Thus modular invariance implies that all twisted sectors must be included in order to consistently describe the string propagation on the quotient space. Finally notice that when $\mathcal{G}$ acts freely we have $\chi(\mathcal{M}^*/\mathcal{G}) = \chi(\mathcal{M})/|\mathcal{G}| = \chi_{\text{unt}}$. Coming back to our P_3 example, let us denote by θ the permutation generator. Our previous calculation shows that $\chi(1, \theta) = \chi(\theta, 1) = 2 + (-10) = -8$. In this case $\chi(1, 1) = -200$ and clearly all other elements have $\chi(x, y) = \chi(1, \theta)$. Thus we readily obtain $\chi(K^*/P_3) = \frac{1}{3}[-200 + 8 \times (-8)] = -88$. This means that modding the 3^5 theory by P_3 yields a new model with $N_{\text{gen}} = 44$.

From the point of view of $N = 2$ tensor product we can always find the untwisted states by forming combinations of the original states that are singlets under $\mathcal{G}$. In particular we can compute χ_{unt} from N_{27}^{unt} and N_{27}^{unt} . For example, in the $5^5/P_3$ model we find $N_{27}^{unt} = 37$ since of the states in (5.2) there are 37 combinations that are singlets under P_3 . Likewise $N_{27}^{unt} = 1$. We thus obtain $\chi_{unt} = -72$ in agreement with (5.7). To find the states in the twisted sectors requires a knowledge of the structure of these sectors and of the generalized GSO projection needed to determine the invariant states. Alternatively, if the corresponding compactification variety is known, the contribution of the twisted sectors to N_{gen} might be inferred from (5.6). When the $\mathcal{G}$ generator θ has prime order the situation is simplified because $\chi(x, y) = \chi(1, \theta)$ (for all x, y not simultaneously $= 1$) and $\chi(1, \theta)$ can be deduced from χ_{unt} since $\chi(1, 1)$ is also known. We can thus work out many examples with prime order permutations. Some results are indicated in table 4. In this table N_{gen}^0 stands for the number of generations of the original model and N_{gen} for the number of generations of the model obtained by modding by $\mathcal{G}$. We have found no counterexample to the quantization rule of the number of generations that we discussed in chapter three.

The main motivation behind modding a given string model by discrete symmetries is to reduce the number of fields in the spectrum while the gauge group is broken by an appropriate embedding. However, unless the discrete symmetries act freely, a priori there is no guarantee that the resulting spectrum will have the desired features. As we have seen permutation symmetries often leave fixed points. Furthermore, they usually do not commute with phase rotation symmetries (Γ moddings). An exception to this is the 3-generation model [22] obtained from the $N = 2$ tensor product 16^3_E by modding by a $P_3 \times Z_3$ symmetry where P_3 is a freely acting cyclic permutation of the $k = 16$ blocks whereas Z_3 is a Γ modding (also non-freely acting) and both symmetries commute. In this case it is possible to embed the permutation symmetry into the gauge degrees of freedom with the result of breaking the gauge group without altering the number of generations. For example, if we take the order 3 shift

$$A = \frac{1}{3}(1, 1, -2, 0, 0, \dots, 0)$$

E_6 will be broken to $[SU(3)]^3$. Of the original states there are combinations that are singlets under P_3 and combinations that acquire phases under P_3 . For instance, if we have three states

$|a\rangle, |b\rangle, |c\rangle$ transforming cyclically under P_3 we form 3 combinations

$$|0\rangle = \frac{1}{3}(|a\rangle + |b\rangle + |c\rangle)$$

$$|+\rangle = \frac{1}{3}(|a\rangle + \delta^2|b\rangle + \delta|c\rangle)$$

$$|-\rangle = \frac{1}{3}(|a\rangle + \delta|b\rangle + \delta^2|c\rangle)$$

where $\delta = e^{2\pi i/3}$. Clearly $P_3| \pm \rangle = e^{\pm 2\pi i/3}| \pm \rangle$ and $P_3|0\rangle = |0\rangle$. Singlet states will have $SO(10) \times E_8$ weight P such that $P \cdot A = \text{integer}$ so that the full states (including the contribution of the $SO(10) \times E_8$ left moving bosons) are invariant under the combined action of P_3 and the embedding. $| \pm \rangle$ states will have $P \cdot A = \pm 1/3 \pmod{\text{integer}}$. Altogether we find that $|0\rangle, | \pm \rangle$ states corresponding to $(1, 3, \bar{3}) \oplus (\bar{3}, 1, 3) \oplus (3, \bar{3}, 1)$ of $[SU(3)]^3$ once we mod by the combined action of $P_3 \otimes A$. For $\bar{27}$'s we have a similar result. Hence the effect of the $P_3 \otimes A$ modding is to reduce the number of generations from 9 to 3 while breaking the gauge group.

6. (0,2) compactifications : a three generation $SU(3) \times SU(2) \times U(1)$ model

The compactification and modding techniques described in previous chapters can be used in order to search for 4-dimensional classical string solutions resembling the observed particle physics. In view of the enormous number of possibilities available one might feel discouraged since the problem looks like searching for a needle in a haystack. However, in spite of this fact, it is not that easy to construct "realistic" three-generation models and very few are available in the literature. Since the chances that any of these string vacua had been chosen by nature seem very remote, we believe that pushing too far any of these models is probably a waste of time. On the other hand we believe it is useful to build realistic models (as realistic as possible) in order to study how likely is for generic 4-dimensional string vacua to look like the standard model. Furthermore one should look for generic features of as large a class of compactifications as possible in order to draw some general conclusions (and maybe predictions) coming from the string framework and not from the structure of a particular model.

In that perspective it seems sensible to start with the simplest $N = 2$ superconformal systems and, by using the techniques (moddings, torsion, gauge background fields) described in the previous chapters, see if it is possible to build realistic 4-dimensional string vacua out of them. From this point of view the simplest $N = 2$, $c = 9$ SCFT is the $(1)^9$ model which involves only level $k = 1$ $N = 2$ minimal models. One would then try models including higher k only if this class of simple models was not sufficient to accommodate the observed physics.

As we remarked in chapter 3, the $(1)^9$ family of models has a generation quantum = 12 so that no $(2,2)$ compactification with a small number of generations is obtained. However we will explicitly see how the addition of gauge background fields modifies this fact and three generation models are obtained. The number three for the generations is not fortuitous and is related to the fact that the $(1)^9$ class of models has abundant Z_3 symmetries and R-symmetries.

We will now construct a three generation model with a gauge group of the type $SU(3) \times SU(2) \times U(1) \times G$. We will be guided by our experience in building realistic models based on Z_3 and $Z_3 \times Z_3$ orbifolds [27], [28]. One first starts with model 5 in table 1 which is obtained by modding $(1)^9$ by (we only display the q-part of the modding vectors, the rest of the entries are zero)

$$\Gamma = (1, 1, 1, -1, -1, -1, 0, 0, 0) \quad (6.1)$$

In order to break the gauge symmetry and reduce the number of generations we twist in a symmetric fashion by $\Gamma_1 \otimes A_1$ with

$$\Gamma_1 = (1, 1, 1, 0, 0, 0, 0, 0) , \quad A_1 = \frac{1}{3}(1, 1, 0, 0, 0, 0) \times (-2, 0, \dots, 0) \quad (6.2)$$

following the ideas described at the end of chapter 2. In this way, after applying the corresponding projections, the original $E_6 \times SU(3) \times E_6' \times U(1)^6$ symmetry is broken down to $SU(4) \times SU(2) \times SU(3) \times SO(14)' \times U(1)^9$. However, there are extra gauge bosons coming from Γ_1 -twisted sectors which enlarge the gauge symmetry to $SU(9) \times SO(14)' \times U(1)^7$. The next step is to break the $SU(9)$ gauge symmetry down to $SU(5) \times SU(2) \times U(1)^3$. This is achieved through a $\Gamma_2 \otimes A_2$ modding with

$$\Gamma_2 = (0, 0, 0, 1, -1, 0, 0, 0, 0) , \quad A_2 = \frac{1}{3}(0, 0, 0, 0, -2) \times (0, -1, -1, 0, \dots, 0). \quad (6.3)$$

At this level one can check that the gauge group is $SU(5) \times SU(2)^2 \times SO(10)' \times U(1)^{11}$ and

that the massless particle content includes three $SU(5)$ generations of quarks and leptons. The final gauge symmetry breaking is achieved by the modding $\Gamma_3 \otimes A_3$ with

$$\Gamma_3 = (1, 1, 0, 1, 1, 0, 1, 1, 0) , \quad A_3 = \frac{1}{3}(-1, -1, 0, 1, 1) \times (-1, -1, 0, \dots, 0) \quad (6.4)$$

The gauge group is now

$$SU(3) \times SU(2) \times U(1) \times [SO(10)' \times U(1)^{13}] \quad (6.5)$$

In order to get these results one just makes use of the mass formulae of eq.(2.19), (2.20) (conveniently generalized for the case of background gauge fields, as explained in ref. [9]) and the projectors discussed in chapter 2. All the Γ_i symmetries used verify $2\beta_0 \cdot \Gamma_i = \text{integer}$ and no relative torsion between β_0 and the moddings is chosen. In this way space-time supersymmetry remains unbroken. We choose all relative discrete torsions to vanish except for a relative phase between the Γ_1 and Γ_3 moddings which we choose $e = 3(\Gamma_1 \cdot \Gamma_3) = -1$. The projectors associated to the Γ_i moddings have the form

$$\begin{aligned} \Delta_i = & -\Gamma_i \cdot (2V_R + 2 \sum_j x_j \Gamma_j + 2x\Gamma) \\ & - [(p + \bar{w} + \sum_j x_j A_j) \cdot A_i - \frac{1}{2} (\sum_j x_j A_j) \cdot A_i] = \text{integer} \end{aligned} \quad (6.6)$$

Due to the above-mentioned torsion, the Δ_1 and Δ_3 projectors have additional pieces

$$\begin{aligned} \Delta_1 & \rightarrow \Delta_1 + x_3(\Gamma_1 \cdot \Gamma_3) \\ \Delta_3 & \rightarrow \Delta_3 - x_1(\Gamma_1 \cdot \Gamma_3) \end{aligned} \quad (6.7)$$

This relative phase is added to avoid that the $SU(3) \times SU(2)$ gauge symmetry gets enlarged to $SU(5)$ through some extra gauge bosons appearing in the $x_1 = 1$ twisted sector. Notice that $x_i = 0, \pm 1$ label the different twisted sectors.

To give some intuition of the structure of this model let us display some of the massless states. We will denote a massless state by $\Phi_R \otimes \Phi_L$ where

$$\begin{aligned}\Phi_R &= (w) \otimes (l_1, q_1, s_1) \cdots (l_r, q_r, s_r) \\ \Phi_L &= (\bar{l}_1, \bar{q}_1, \bar{s}_1) \cdots (\bar{l}_r, \bar{q}_r, \bar{s}_r) \otimes (p + \bar{w})\end{aligned}\quad (6.8)$$

When adding background fields A_i the $SO(10) \times E_8$ weight lattice gets a shift by $\sum_j x_j A_j$. The $SU(3)$ strong interactions do not come from the $SO(10) \times E_8$ gauge symmetry but from the enlarged $SU(3)$ symmetry of the nine internal $k = 1$ factors. In particular, the charged $SU(3)$ gauged bosons result from the Γ -twisted states

$$\begin{aligned}\Phi_R &= (w = v) \otimes (0, 0, 0)^9 \\ \Phi_L &= (0, 2, 2)^3 (0, -2, 2)^3 (0, 0, 0)^3 \otimes (p, \bar{w} = 0)\end{aligned}\quad (6.9)$$

where the underlining means permutations of the three sets of states. The $SU(2)$ weak gauge bosons correspond to

$$\begin{aligned}\Phi_R &= (w = v) \otimes (0, 0, 0)^9 \\ \Phi_L &= (0, 0, 0)^9 \otimes (\bar{w} = (1, -1, 0, 0, 0); p = 0).\end{aligned}\quad (6.10)$$

In order to find out the number of generations in an $SU(3) \times SU(2) \times U(1)$ model the most convenient way is to search for the fields with the quantum numbers of left-handed quarks $((3, 2)$ reps. under $SU(3) \times SU(2)$). In the present model they are easy to identify. They come from the states (we display the squarks for convenience)

$$\Phi_L = (0, 2, 2)^3 (0, 0, 0)^3 (0, 0, 0)^3 \otimes (p + \bar{w} - A_1 = \frac{1}{3}(2, -1, 0, 0, 0) \times (2, \dots, 0)).\quad (6.11)$$

These states correspond to the $x_1 = -1, x_2 = x_3 = 0, x = 0, \pm 1$ twisted sectors. It is easy to check that these states are massless and also that they survive projection. Due to the three possible permutations in the right factors this model has three left-handed quarks. The

permutations in the left factors just tell us that they are $SU(3)$ triplets and $SU(2)$ doublets. One can also check that there are no further states with the same quantum numbers and opposite chirality so that the model only has three $SU(3) \times SU(2)$ generations. Cancellation of $SU(3) \times SU(2) \times U(1)$ anomalies will then necessarily imply that the rest of the standard quarks and leptons have to be present in the massless spectrum (since the theory is modular invariant). Of course some extra vector-like quarks and leptons may also be (and are in this case) present. It is a extremely tedious but straightforward task to find the rest of the massless spectrum. Thus, for example, candidates for three (either the charge $-1/3$ or $2/3$) right-handed quarks are provided by

$$\begin{aligned}\Phi_R &= (w = 0) \otimes (0, -2, 2)^3 (0, 0, 0)^3 \otimes (0, 0, 0)^3 \\ \Phi_L &= (0, -2, 2)^3 (0, 0, 0)^3 \otimes (p = 0, \bar{w} = (0, 0, 0, 1, 0))\end{aligned}\quad (6.12)$$

from the $x_1 = x_2 = x_3 = 0, x = 0, \pm 1$ sectors. Candidates for Higgs doublets (since a coupling with the above left- and right-handed quarks is allowed by gauge invariance) are

$$\begin{aligned}\Phi_R &= (w = 0) \otimes (0, -2, 2)^3 (0, 0, 0)^3 \\ \Phi_L &= (0, 0, 0)^9 \otimes (p + \bar{w} + A_1 = \frac{1}{3}(-2, 1, 0, -3, 0) \times (-2, 0, \dots, 0))\end{aligned}\quad (6.13)$$

An analogous model [27], [28], [29] can be constructed in terms of the standard Z_3 orbifold if we appropriately choose the tori radi and background fields to enlarge the gauge symmetry of the orbifold by 6 $U(1)$ factors. It has some interesting phenomenological features as explained in ref. [28], [30]. It should be possible to find other interesting 3-generation models starting from the different moddings in table 1 and adding appropriate backgrounds. Although the number of possibilities is very large, physical constraints like the number of generations, gauge group or existence of adequate Yukawa couplings for all quarks and leptons can be rather severe. Perhaps one can eliminate a large number of possibilities by using these constraints without studying the details of each model.

Let us finally make a comment on the gauge group. In the particular model presented in this section, the "hidden" $SO(10)$ gauge group does not couple to quarks and leptons. They, however, couple generically to the $U(1)^{13}$ gauge interactions. Typically there are

massless scalar singlets which are charged with respect to the $U(1)$'s. These singlets have flat directions in their scalar potential (marginal deformations in the CFT language) in such a way that when they get a vev, the $U(1)^3$ symmetry is spontaneously broken to a much smaller subgroup. Thus the "observable" group may be just the standard model group. This is in fact what happens in the orbifold model [28] which is analogous to the model discussed in in this chapter and this is what one expects for the latter.

7. Final comments and conclusions

In the present paper we have generalized the tensoring of $N = 2$ minimal models compactification method in several ways. We made use of general $N = 2$ SCFT based on CP_m spaces and considered arbitrary (left-right symmetric and asymmetric) abelian moddings. We also included the effect of discrete torsion and possible gauge background fields. The results for a scan for new $(2,2)$ models are also presented and the general features of these compactifications discussed. We find that there is a "quantization" of the net number of generations and that quantum is most of the times 12 and less frequently 6 and 4. Very rarely one finds examples with an odd number of generations. This quantization property disappears when background gauge fields are added.

The class of models studied by Gepner (a few hundred $(2,2)$ compactifications) is extended in an enormous way (probably millions of models) by including new $(2,2)$, $(0,2)$ and non-supersymmetric compactifications. A large class of asymmetric and symmetric orbifolds seem to be obtainable with this approach (at least at particular points in their moduli space). Furthermore it also includes left-right asymmetric (and symmetric) constructions related to more general (non-toroidal) varieties.

We have shown that all the above left-right asymmetric constructions can be understood as left-right symmetric compactifications in the presence of discrete torsion. An example of this are the standard asymmetric orbifolds [6] (or at least those which can be obtained in terms of tensored $N = 2$ SCFT). Of course it is well known that discrete torsion has an intrinsic left-right asymmetric character. The point here is that one can always convert an asymmetric modding into a symmetric one by adding an appropriate discrete torsion. Then theories (like asymmetric orbifolds) which look inherently 4-dimensional could be understood as compactifications of 10-dimensional heterotic strings in the presence of background fields

(discrete torsion). This is certainly conceptually attractive. It remains to be seen if indeed this is true for all possible 4-dimensional strings.

The addition of discrete torsion phases can also lead to non-supersymmetric models. To any given supersymmetric model one can associate a finite number of non-SUSY models with the same gauge group and general characteristics by adding appropriate phases in the partition function. It could be useful to study the relationship between each parent SUSY model and its non-SUSY relatives in order to shed some light on the problem of SUSY breaking in string models. Using all the above techniques we were able to obtain three generation models with gauge group close to the standard model. These are the first models of this type ever obtained with the tensoring of SCFT method. They are obtained by appropriately modding the 1^9 model. We believe one should not emphasize the search for three-generation $(2,2)$ compactifications. After all those models have an E_8 gauge symmetry which has to be eventually broken to something looking like the standard model. The result will in general be a $(0,2)$ compactification and hence, why not considering $(0,2)$ models to start with? The methods discussed in this paper should be very useful in order to analyze general classes of realistic 4-dimensional strings and trying to extract general properties of the string scheme and not of a particular model.

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Table 1

Moddings in 1^9 model

#	moddings	torsion e	N_{27}	$N_{\overline{27}}$
1	$(0, \dots, 0)$	0	84	0
2	$(1, -1, 0, 0, 0, 0, 0, 0, 0)$	0	84	0
3	$(1, -1, 0, 1, -1, 0, 0, 0, 0)$	0	40	4
4	$(1, 1, 1, 1, -1, 0, 0, 0, 0)$	0	24	12
5	$(1, 1, 1, -1, -1, -1, 0, 0, 0)$	0	36	0
6	$(1, 1, 1, 0, 0, 0, 0, 0, 0)$	0	21	21
7	$(1, 1, 1, -1, -1, -1, 0, 0, 0)$ $(1, 0, 0, -1, 0, 0, 1, -1)$	0	16	4
8	$(1, -1, 0, 0, 1, -1, 0, 0, 0)$ $(1, 1, 0, 1, -1, 0, 0, -1, -1)$	0	13	13
9	$(1, 1, 1, 0, 0, 0, 0, 0, 0)$ $(0, 0, 0, 1, 1, 1, 0, 0, 0)$	0	9	9
10	$(1, 1, 1, 0, 0, 0, 0, 0, 0)$ $(0, 0, 0, 1, 1, 1, 0, 0, 0)$	1	3	27

Table 2

moddings in 3^5 model

#	moddings	torsion e	N_{27}	$N_{\overline{27}}$
1	$(0, \dots, 0)$	0	101	1
2	$(0, 1, 2, 3, 4)$	0	21	1
3	$(1, -1, 0, 0, 0)$	0	49	5
4	$(1, 1, -2, 0, 0)$	0	17	21
5	$(0, 1, 2, 3, 4)$ $(0, 0, 1, 1, -2)$	0	13	9
6	$(0, 0, 0, 1, -1)$ $(0, 0, 1, 1, -2)$	0	23	7
7	$(0, 0, 1, 1, -2)$ $(1, -1, 0, 0, 0)$	0	5	49
8	$(0, 1, 2, 3, 4)$ $(0, 0, 0, 1, -1)$	1	13	9

Table 3

Moddings in models built using $N = 2 SU(m+1)/SU(m) \times U(1)$ cosets

#	coset model	moddings	torsion e	N_{27}	$N_{\overline{27}}$
1	$(4, 10)1$	$(0, \dots, 0)$	0	66	3
2	$(2, 4)^2 5$	$(0, 0, 0)$	0	95	2
3	$(2, 4)^2 5$	$(0, 1, 4)$	0	11	32
4	$(3, 3)12^2$	$(0, 1, 13)$	0	31	35
5	$(2, 3)^3$	$(0, 0, 0)$	0	84	0
6	$(2, 3)^3$	$(3, 5, 2)$	0	35	11
7	$(2, 3)^3$	$(3, 5, 4)$	0	29	5
8	$(2, 3)^3$	$(3, 2, 3)$ $(3, 5, 2)$	2	9	9
9	$(2, 3)^3$	$(3, 2, 3)$ $(3, 5, 2)$	1	3	27
10	$(2, 6)10^2$	$(0, 0, 0)$	0	59	11
11	$(2, 6)10^2$	$(0, 1, 11)$	0	11	59

Table 4

Examples of moddings by permutation symmetries

model	G	N_{gen}^{10}	N_{gen}^{ant}	N_{gen}
$1 \cdot 4^4$	I_3	102	36	42
6^4	I_3	148	50	52
$1 \cdot 3^3 13$	I_3	72	21	21
$(2, 3)^3$	I_3	81	30	36
$(2, 2)^3 3$	I_3	81	28	28
1^9	I_5	81	20	36
2^6	I_5	90	18	18