Kähler Geometry of the Space of
 $N = 2$ Superconformal Field Theories

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ABSTRACT: The moduli space, $\mathcal{M}$, of $(0,2)$ superconformal field theories with integral $U(1)$ charges is studied, adapting results from previous work on string theory and the Calabi-Yau moduli space. It is shown that $\mathcal{M}$ is Kähler and that the Kähler potential is the norm of the unique top weight right-chiral primary state. Furthermore, the operator which creates this state is shown to be a holomorphic section of a vector bundle on $\mathcal{M}$. Some aspects of the duality transformation which flips the $U(1)$ charge are discussed in this context.

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I. INTRODUCTION AND SUMMARY

Given a $c = 9, N = (0, 2)$ superconformal field theory, one can construct a four dimensional string compactification with $N = 1$ spacetime supersymmetry[1,2,3,4,5,6]. These are phenomenologically the most interesting solutions of string theory. It is the purpose of this paper to study the *local* geometry of the moduli space, $\mathcal{M}$, of these $N = 2$ superconformal theories for general values of c . (In general, $\mathcal{M}$ may have a number of components of different dimension and *a priori*, there is no reason to expect that $\mathcal{M}$ is even connected.) Our results will be derived and phrased in the language of conformal field theory, but the analysis is heavily influenced by insights gained from string theory and closely related studies of the Calabi-Yau moduli space.

The geometry and topology of $\mathcal{M}$ is of interest both for string theory and for the classification program in conformal field theory. While a complete description (*e.g.*, an explicit uniformization) of $\mathcal{M}$ is an ambitious goal, general properties of $\mathcal{M}$ can be fruitfully investigated. Indeed, recent studies of the special case of $(2,2)$ supersymmetry have revealed a rich and intriguing structure[7,8,9,10,11,12]. The geometry of $\mathcal{M}$ near nodes[10] is relevant to the problem of transitions between string vacua characterized by differing topological invariants. The related problem of supersymmetry breaking and the selection of a preferred string vacuum involves the study of line bundles over $\mathcal{M}$.

$\mathcal{M}$ also provides, along with its covering space, the proper framework for study of the fascinating duality transformations which generalize the $R \leftrightarrow 1/R$ duality transformation of toroidal compactifications to general string vacua (see, *e.g.*, [13,14,15,16,17,7,18]). These transformations form the modular group associated to $\mathcal{M}$.

At long distances, a four dimensional string compactification is described by a supersymmetric field theory. An effective way to study $\mathcal{M}$ for the special case $c = 9$ is in terms of this low energy field theory [8,11,9]. This field theory contains massless scalar 'moduli fields' that are coordinates on $\mathcal{M}$. The kinetic term of these fields defines a metric, $\mathcal{G}$, on $\mathcal{M}$. Remarkably, it turns out that this low energy field theory is entirely determined (at least for $(2,2)$ theories) by a locally holomorphic function $\mathcal{F}$ on $\mathcal{M}$ [19]. In particular, the metric $\mathcal{G}$ is determined from $\mathcal{F}$.

As explained in [12], just as the notion of a Kähler potential exists only in the context of *complex* coordinates, the function $\mathcal{F}$ exists only in the context of certain *special* coordinates on $\mathcal{M}$. In those special coordinates, the derivatives of $\mathcal{F}$ are the components of a section

Ω of a vector bundle on $\mathcal{M}$. Properties of this section imply the existence of $\mathcal{F}$ in special coordinates (just as properties of a metric can imply the existence of a Kähler potential in complex coordinates), but the section Ω can be defined independently of special coordinates. Thus a central problem in the study of $c = 9$, $(2, 2)$ theories is to understand and determine Ω .

Considerable progress has been made toward determining $\mathcal{F}$ or Ω locally in the weak-coupling limit for which the conformal field theory can be described as a sigma model with a Calabi-Yau target space. (This corresponds to a subset of $\mathcal{M}$.) It was shown in [20,21,22] that the entire low energy field theory (in particular $\mathcal{F}$ or Ω) can be computed from topological data of the Calabi-Yau space. Through a series of non-renormalization arguments[2,23,24,9], it is now known that most of those results are valid to all orders in perturbation theory and some of them are exact non-perturbatively. One would, of course, like to understand the geometry of the full moduli space, $\mathcal{M}$ without restriction to the weak-coupling limit. It appears that the time is now ripe to tackle this problem.

A priori, the parameters of the low energy field theory or the correlations of the conformal field theory are functions of the *real* coordinates on $\mathcal{M}$. It is quite striking that the essential information is contained within a single *holomorphic* function or section on $\mathcal{M}$. Ultimately, this simplification is due to spacetime supersymmetry, a powerful symmetry which is hidden in the usual formulations of conformal field theory. This holomorphic structure is obviously a key to understanding the geometry of $\mathcal{M}$ and $N = 2$ conformal field theories in general. The situation is reminiscent of the study of conformal field theory correlations as functions on the moduli space, M , of the Riemann surface on which they are defined. A priori, the correlators are real functions on M . However, it was realized[25] several years ago that they are composed of simpler building blocks, called conformal blocks, which vary holomorphically on M . We suspect that $N = 2$ conformal field theories are similarly composed of building blocks which vary holomorphically on $\mathcal{M}$.

In this paper one such holomorphic object, the top weight* chiral primary field Ω^+ , will be discussed. It is our belief that this field defines globally a holomorphic section of the bundle of cohomology classes of chiral fields over $\mathcal{M}$ which can be identified with the aforementioned section Ω . The results of this paper are consistent with this conjecture.

A first step in studying the geometry of $\mathcal{M}$ is to obtain conformal field theoretic expres-

* A note regarding the term ‘top weight’: ‘top’ means that the weight of this field is the largest weight of any primary field in the theory; this terminology follows that used for differential forms.

sions for the various geometrical objects on $\mathcal{M}$. An important example is the Zamolodchikov formula for the metric[26]:

$$\mathcal{G}_{I\bar{J}}(Z^K, \bar{Z}^K) = \frac{1}{8} \langle M_I(0) M_{\bar{J}}(1) \rangle_Z, \quad (1)$$

where M_I are truly marginal moduli field vertex operators, Z^K are coordinates on $\mathcal{M}$ and the subscript Z on the expectation value denotes that the correlation function is evaluated in the conformal field theory associated to the point Z^K . This is a natural metric on $\mathcal{M}$ and coincides for $c = 9$ with that of the low energy field theory of string compactification*.

$N = 1$ spacetime supersymmetry of the low energy field theory (which follows from $N = 2$ world sheet supersymmetry) requires for $c = 9$ that $\mathcal{G}$ be Kähler. This suggests that for any value of c $\mathcal{G}$ will be Kähler. In Section II we will verify this by showing that for a general (0,2) conformal field theory with integral $U(1)$ charges and $c > 3$ the Kähler potential is given by:

$$\mathcal{K}(Z^K, \bar{Z}^K) = \ln \langle \Omega^+(0) \Omega^-(1) \rangle_Z, \quad (2)$$

where $\Omega^+(\Omega^-)$ is the top weight $(0, c/6)$ chiral (antichiral) field which exists in every (0,2) conformal field theory. (This formula is not restricted to $c = 9$.) We will further show, in a sense to be made precise, that $\Omega^+(\Omega^-)$ varies holomorphically (antiholomorphically) on $\mathcal{M}$. $e^{\mathcal{K}}$ may then be interpreted as the norm of a holomorphic section of the vector bundle on $\mathcal{M}$ whose fibers are isomorphic to the conformal field theory Hilbert space. Such a representation for $e^{\mathcal{K}}$ was already known to exist for the special case of weakly coupled (2,2) theories [27,28,10].

II. THE KÄHLER POTENTIAL

Before proceeding to the general case, we first consider (2,2) theories in the weak coupling limit where they can be described by sigma models with complex dimension $\tilde{c} \equiv \frac{c}{3}$ Calabi-Yau target spaces. (Integral $U(1)$ charges implies $\tilde{c}$ is an integer.) The top weight chiral primary operator then corresponds to the top rank holomorphic $\tilde{c}$ -form Ω which exists on every Calabi-Yau space. This operator may be expressed

$$\Omega_{\sigma}^+(z) = \frac{i^{\frac{\tilde{c}}{2}(\tilde{c}+1)}}{\tilde{c}!} \Omega_{i_1 \dots i_{\tilde{c}}}(X) \psi^{i_1}(z) \dots \psi^{i_{\tilde{c}}}(z) \quad (3)$$

* One way to see this is to compare the derivative of $\mathcal{G}$ with the string formula for the three point function.

X^i is a complex coordinate on the Calabi-Yau space with worldsheet superpartner ψ^i , z is a complex worldsheet coordinate and the subscript σ denotes that the quantity is defined in sigma model perturbation theory. This is simply the usual expression for the top rank form with the differentials dX^i replaced by worldsheet fermions $\psi^i(z)$. Similarly the top weight antichiral primary field is

$$\Omega_{\sigma}^{-}(z) = \frac{i^{\frac{\tilde{c}}{2}(\tilde{c}+1)}}{\tilde{c}!} \bar{\Omega}_{\bar{i}_1 \dots \bar{i}_{\tilde{c}}}(\bar{X}) \psi^{\bar{i}_1}(z) \dots \psi^{\bar{i}_{\tilde{c}}}(z) \quad (4)$$

Consider the expression:

$$\mathcal{K}_{\sigma} = \ln \langle \Omega_{\sigma}^{+}(0) \Omega_{\sigma}^{-}(1) \rangle_Z . \quad (5)$$

(All matrix elements in this paper are connected and normalized.) To leading order in sigma model perturbation theory this is, up to constants which can be absorbed by a Kähler transformation

$$\mathcal{K}_{\sigma} = \ln \int d^{2\tilde{c}} X (\Omega \wedge \bar{\Omega}) - \ln \int d^{2\tilde{c}} X (J \wedge J \wedge J) \quad (6)$$

where J is the Kähler form on the Calabi-Yau space. The second integral is the the volume of the Calabi-Yau space. This appears because we have divided out the norm of the vacuum in (6), which contains a factor of the volume arising from the integration over X zero modes. A particular choice of Ω specifies a point on the moduli space $\mathcal{M}_{2,1}$ of complex structures, and a choice of J specifies a point on the moduli space $\mathcal{M}_{1,1}$ of Kähler structures. (6) is thus a function on $\mathcal{M} = \mathcal{M}_{1,1} \times \mathcal{M}_{2,1}$ and is known[27,28,10] to be the Kähler form which reproduces the desired metric[21,3], up to a sign. In fact, the signs in (6) give a negative metric on $\mathcal{M}_{2,1}$ and a positive metric on $\mathcal{M}_{1,1}$. This negative sign will be discussed further after the calculation for the general conformal field theory has been described.

The target space geometry for weakly coupled (0,2) theories has been discussed in [2,3]. It is not in general Kähler, but it has been shown[3] that a holomorphic $\tilde{c}$ -form nevertheless always exists. Little is known about this interesting class of manifolds, but we shall see from the following conformal field theoretic arguments that their moduli space should be Kähler.

To proceed to the general case, denote by $M_I(z, \bar{z})$ the weight (1,1) moduli fields of a (0,2) superconformal field theory which generate supersymmetry-preserving deformations. The index I is tangent to $\mathcal{M}$. Invariance of the perturbed action under supersymmetry transformations implies that such operators must obey $[G_{-\frac{1}{2}}^{\pm}, M_I] = \partial P_I^{\pm}$. The graded Jacobi identity for $G_{-\frac{1}{2}}^{+}$, $G_{-\frac{1}{2}}^{-}$ and M_I is

$$\partial([G_{-\frac{1}{2}}^{+}, P_I^{-}] + [G_{-\frac{1}{2}}^{-}, P_I^{+}] - M_I) = 0. \quad (7)$$

This implies $M_I = M_I^+ + M_I^-$, with

$$M_I^\pm = [G_{-\frac{1}{2}}^\pm, P_I^\mp]. \quad (8)$$

Using the fact that $P_I^+(P_I^-)$ is a chiral (antichiral) primary field, it is easily checked that $i(M_I^+ - M_I^-)$ is also a moduli field. (A chiral (antichiral) field commutes with $G_{-\frac{1}{2}}^+$ ($G_{-\frac{1}{2}}^-$).) $M_I^+(M_I^-)$ generates holomorphic (antiholomorphic) deformations of the (0,2) theory, where here and henceforth the complex structure on $\mathcal{M}$ is used to distinguish between barred and unbarred indices. These deformations are described in terms of the connected correlation functions of the theory according to the formula

$$\langle \phi_1 \dots \phi_n \rangle_Z = \langle e^{-\int \frac{d^2z}{2\pi} (Z^I M_I^+ + \bar{Z}^{\bar{I}} M_{\bar{I}}^-)} \phi_1 \dots \phi_n \rangle \quad (9)$$

in addition to possible renormalizations of the operators ϕ_i .

Every (0,2) superconformal field theory has a charge $q = \tilde{c}$, weight $(0, \frac{\tilde{c}}{2})$ chiral primary field $\Omega^+(z)$ and a charge $-\tilde{c}$ weight $(0, \frac{\tilde{c}}{2})$ antichiral primary field $\Omega^-(z)$. Ω^+ is the top weight chiral primary field. In the special case of (2,2) Calabi-Yau models, Ω^+ is related to the top rank holomorphic $(0, \tilde{c})$ form which exists on every Calabi-Yau manifold via equation (4). This form is known to vary holomorphically on the Calabi-Yau moduli space[28,27], *i.e.*, its integrals over a fixed set of $\tilde{c}$ cycles are holomorphic[27]. An analogous statement may be shown for the general (0, 2) conformal field theory. Namely, the integral of Ω^+ ,

$$\Omega_{\frac{2-\tilde{c}}{2}}^+ = \int \frac{dz}{2\pi i} \Omega^+(z), \quad (10)$$

commutes with the antiholomorphic moduli fields $M_{\bar{I}}^-$, while $\Omega_{\frac{2-\tilde{c}}{2}}^-$ commutes with M_I^+ :

$$[\Omega_{\frac{2-\tilde{c}}{2}}^\pm, M^\mp] = 0. \quad (11)$$

It follows that Ω^+ (Ω^-) does not need to be renormalized under holomorphic (antiholomorphic) deformations of the moduli. Ω^+ (Ω^-) may thus be defined as a holomorphic (antiholomorphic) section of the bundle of operators over $\mathcal{M}$.*

To prove (11), define $R_{\bar{I}} = [\Omega_{\frac{2-\tilde{c}}{2}}^+, M_{\bar{I}}^-]$. $R_{\bar{I}}$ is a charge $\tilde{c}$ weight $(1, \frac{\tilde{c}}{2})$ primary field. Since it has $2h = q$, it is a chiral primary field. Using the fact that the boson of the $U(1)$ charge decouples from the rest of the theory, $R_{\bar{I}}$ may be expressed as

$$R_{\bar{I}} = \bar{j}_{\bar{I}} \Omega^+, \quad (12)$$

* The results of [12] suggest that the chiral cohomology class (with respect to $G_{-\frac{1}{2}}^+$) of which Ω^+ is an element can be defined as a holomorphic section of the *finite* dimensional bundle of cohomology classes of chiral operators.

where $\bar{j}_{\bar{I}}$ is a (1,0) field. It then follows from $[\Omega_{\frac{\tilde{c}}{2}}^-, \Omega^+] = 1$ where $\Omega_{\frac{\tilde{c}}{2}}^-$ is the $L_0 = -\frac{\tilde{c}}{2}$ mode of Ω^- , that

$$[\Omega_{\frac{\tilde{c}}{2}}^-, R_{\bar{I}}] = \bar{j}_{\bar{I}} \quad (13)$$

On the other hand, the commutator may be directly evaluated using the Jacobi identity and the definition of $R_{\bar{I}}$. One has

$$[\Omega_{\frac{\tilde{c}}{2}}^-, R_{\bar{I}}] = [\Omega_{\frac{\tilde{c}}{2}}^-, [\Omega_{\frac{2-\tilde{c}}{2}}^+, [G_{-\frac{1}{2}}^-, P_{\bar{I}}^+]]] = 0, \quad (14)$$

where we have used the fact that $\Omega_{\frac{\tilde{c}}{2}}^-$ commutes with both $G_{-\frac{1}{2}}^-$ and $P_{\bar{I}}^+$. The former is the statement that Ω^- is antichiral primary and the latter follows from the absence of $h = \frac{1-\tilde{c}}{2}$ charged fields. Comparing (13) with (14) we find that $\bar{j}_{\bar{I}} = 0$, and finally,

$$[\Omega_{\frac{2-\tilde{c}}{2}}^\pm, M^\mp] = 0. \quad (15)$$

This equation can also be easily shown to hold for the regulated forms of $M^\pm$ to be introduced shortly, except for the marginal case $\tilde{c} = 1$, for which a more detailed analysis than is given here is required.

Now consider the two point function:

$$e^{\mathcal{K}}(u, v) \equiv \langle \Omega^+(u) \Omega^-(v) \rangle_Z. \quad (16)$$

We will show that $\mathcal{K}(0, 1)$ is the Kähler potential for the Zamolodchikov metric on moduli space. The metric derived from $\mathcal{K}(u, v)$ is

$$\begin{aligned} \mathcal{G}_{I\bar{J}}(u, v) &= \partial_I \partial_{\bar{J}} \mathcal{K}(u, v) \\ &= \int \frac{d^2 z_1}{2\pi} \int \frac{d^2 z_2}{2\pi} \langle M_I^+(z_1, \bar{z}_1) M_{\bar{J}}^-(z_2, \bar{z}_2) \Omega^+(u) \Omega^-(v) \rangle (u - v)^{\tilde{c}}, \end{aligned} \quad (17)$$

plus a term involving two matrix elements linear in M which vanishes. This expression is defined by first replacing $M^\pm$ with the nearly marginal, supersymmetric operators of the form

$$M^\pm(k) = [G_{-\frac{1}{2}}^\pm, P^\mp e^{ikX}] = (M^\pm(0) + ik \cdot \psi^\pm P^\mp) e^{ik \cdot X}, \quad (18)$$

where $X(z, \bar{z}), \psi(z)$ is a $\tilde{c} = 1$, $N = (0, 2)$ supermultiplet with $\psi^\pm = [G_{-\frac{1}{2}}^\pm, X]$. k is then taken to zero at the end of the calculation.

One might have also considered the possibility that finite renormalizations of Ω should be included in the expression (17) for perturbations of the correlation function (16) under deformations of the moduli. This is partially a matter of definition of Ω as a function on $\mathcal{M}$,

and a choice of Kähler gauge. Our ‘minimal’ definition, which does not include such renormalizations, is consistent because Ω remains holomorphic in perturbed correlation functions

$$\bar{\partial}_u < \Omega^\pm(u) \phi_1(v, \bar{v}) \dots \int d^2 z M_I^\pm(z, \bar{z}) \dots > = 0 \quad (19)$$

for $k \rightarrow 0$. This identifies Ω as (a complex multiple of) the top rank chiral field.

For non-zero k , $\mathcal{G}_{I\bar{J}}$ may be evaluated by differentiating with respect to $\bar{u}$ and $\bar{v}$ and then integrating. Using

$$\bar{\partial}_u \frac{1}{u - z} = \pi \delta^{(2)}(u - z), \quad (20)$$

and (18) with (17), one finds

$$\bar{\partial}_u \bar{\partial}_v \mathcal{G}_{I\bar{J}}(u, v) = \frac{1}{4} < [\Omega_{\frac{2-\tilde{c}}{2}}^+, M_I^+(u, \bar{u}; k)] [\Omega_{\frac{2-\tilde{c}}{2}}^-, M_{\bar{J}}^-(v, \bar{v}; -k)] > (u - v)^{\tilde{c}}. \quad (21)$$

It is crucial here that $[\Omega_{\frac{2-\tilde{c}}{2}}^\pm, M^\mp] = 0$ even for $k \neq 0$.

Defining

$$[\Omega_{\frac{2-\tilde{c}}{2}}^+, P_I^-(u, \bar{u})] = S_I^+, \quad (22)$$

$$[\Omega_{\frac{2-\tilde{c}}{2}}^-, P_{\bar{J}}^+(v, \bar{v})] = S_{\bar{J}}^-, \quad (23)$$

and using

$$\begin{aligned} < \psi^+(u) \psi^-(v) > = -\frac{1}{u - v}, \\ < e^{ik \cdot X}(u, \bar{u}) e^{-ik \cdot X}(v, \bar{v}) > = \frac{1}{|u - v|^{2k^2}}, \end{aligned} \quad (24)$$

one has

$$\bar{\partial}_u \bar{\partial}_v \mathcal{G}_{I\bar{J}} = \frac{k^2 (-)^{\tilde{c}}}{8(u - v)^{1-\tilde{c}+k^2} (\bar{u} - \bar{v})^{k^2}} < S_I^+(u, \bar{u}) S_{\bar{J}}^-(v, \bar{v}) >. \quad (25)$$

Integrating with respect to $\bar{u}$ and $\bar{v}$, letting $k^2 \rightarrow 0$ and setting $u = 0$ and $v = 1$, one has

$$\mathcal{G}_{I\bar{J}} = \frac{1}{8} < S_I^+(0) S_{\bar{J}}^-(1) >. \quad (26)$$

The integration constant is set to zero by conformal invariance.

The equivalence of this to the Zamolodchikov formula for $\tilde{c} = 3$ can be seen by using the result of [29,15] that the (1,1) operators S are string vertex operators for auxiliary fields which lie in the same spacetime supermultiplet as the moduli fields. Spacetime supersymmetry then requires that their metric equals that of the moduli fields. More explicitly, S may be written as two contour integrals of the spacetime supercurrent around M and by deforming the contour (26) may be reduced to the Zamolodchikov formula.

For general values of $\tilde{c}$, note that the contour integral of $z^{\tilde{c}-2}\Omega^+(z)$ at infinity vanishes in correlation functions since Ω^+ is weight $\frac{\tilde{c}}{2}$. This leads to the identity:

$$\begin{aligned} 0 &= (u-v)^{(2-\tilde{c})} \frac{1}{2\pi i} \oint_{\infty} dz (z-v)^{\tilde{c}-2} \langle \Omega^+(z) P_I^-(u, \bar{u}) [\Omega_{\frac{2-\tilde{c}}{2}}^-(v, \bar{v}), P_{\bar{J}}^+(v, \bar{v})] \rangle \\ &= \langle [\Omega_{\frac{2-\tilde{c}}{2}}^+, P_I^-(u, \bar{u})] [\Omega_{\frac{2-\tilde{c}}{2}}^-, P_{\bar{J}}^+(v, \bar{v})] \rangle \\ &\quad + (v-u)^{(2-\tilde{c})} \langle P_I^-(u, \bar{u}) [J_0, P_{\bar{J}}^+(v, \bar{v})] \rangle. \end{aligned} \quad (27)$$

The $U(1)$ charge of P^+ is $+1$. The two point function (26) may thus be rewritten

$$\mathcal{G}_{I\bar{J}} = -\frac{1}{8} \langle P_I^-(0) P_{\bar{J}}^+(1) \rangle. \quad (28)$$

This is related to the Zamolodchikov formula by worldsheet supersymmetry. By deforming contours of $G^\pm$ one obtains finally

$$\partial_I \partial_{\bar{J}} \ln \langle \Omega^+(0) \Omega^-(1) \rangle = \frac{1}{8} \langle M_I^+(0) M_{\bar{J}}^-(1) \rangle. \quad (29)$$

III. DUALITY

The metric (29) is positive-definite, in apparent contradiction to the sigma-model result, equation (6), which leads to a negative metric for the $(2, 1)$ forms and a positive metric for the $(1, 1)$ forms. The resolution of this apparent sign discrepancy is quite interesting.

The fact that the conformal field theory calculation could not possibly reproduce the sigma model result can be seen without detailed calculation. The conformal field theory analysis was valid for a general $(0, 2)$ model, and thus used only right-moving supersymmetry. $(2, 1)$ and $(1, 1)$ fields can be distinguished only in $(2, 2)$ theories. They differ in the relative signs of the left and right charges. Our calculation was blind to the left charge, so the metric for *all* the moduli had to have the same sign.

Even in a $(2, 2)$ theory the labelling of $(2, 1)$ and $(1, 1)$ fields is a matter of convention. There is a Z_2 duality symmetry which interchanges them[30]:

$$H \xleftrightarrow{*} -H, \quad (30)$$

where H is the right-moving $U(1)$ boson.

The sigma-model calculation was able to give different signs for $(2,1)$ and $(1,1)$ fields because the prescription for finite renormalizations of Ω_σ under motion in $\mathcal{M}$ broke the symmetry (30). In particular, the components $\Omega_{i_1 \dots i_{\tilde{c}}}$ are renormalized under perturbations generated by $(2,1)$ fields but not under those generated by $(1,1)$ fields. Our definition of Ω^+ as a function on $\mathcal{M}$ thus does not correspond exactly to the operator Ω_σ^+ in the weak-coupling limit.

Ω^+ , in fact, reduces to a different ‘dualized’ operator as follows: consider the operator

$$\star\Omega_\sigma^+ \equiv \frac{i^{\tilde{c}(\tilde{c}+1)/2}}{\tilde{c}!} \star\Omega^{i_1 \dots i_{\tilde{c}}} \psi_{i_1} \dots \psi_{i_{\tilde{c}}}, \quad (31)$$

normalized so that

$$\langle \star\Omega_\sigma^+(0) \Omega_\sigma^+(1) \rangle_Z = 1, \quad (32)$$

at every point on moduli space. The object $\star\Omega^{i_1 \dots i_{\tilde{c}}}$ is an antisymmetric rank $\tilde{c}$ contravariant tensor density of weight 1. There is a natural dual cohomology theory defined for such objects. Define $\star\partial\star$ by

$$(\star\partial\star T)^{[bcd\dots]} \equiv \partial_a T^{[abc\dots]}. \quad (33)$$

This definition does not involve a metric. It is easy to see that $(\star\partial\star)^2 = 0$, so that cohomology classes may be defined. $\star\Omega_\sigma^+$ is an element of such a cohomology class, and (32) depends only on the respective cohomology classes of $\star\Omega_\sigma^+$ and Ω_σ^+ .

Furthermore, the Kähler potential

$$\star\mathcal{K}_\sigma = \ln \langle \star\Omega_\sigma^+(0) \star\Omega_\sigma^-(1) \rangle_Z = -\mathcal{K}_\sigma \quad (34)$$

leads to *minus* the metric of (6). There are two more operators that can be defined in this manner by separately dualizing on either the $\mathcal{M}_{2,1}$ or $\mathcal{M}_{1,1}$ pieces of these moduli spaces. One of these will lead to a positive definite metric on $\mathcal{M} = \mathcal{M}_{2,1} \times \mathcal{M}_{1,1}$. This is the operator which coincides with Ω^+ in the weak coupling limit.

The reason such dualized operators arise can possibly be understood as follows. Under the duality transformation (30) the $(2,1)$ forms, or elements of $H^1(T)$ interchange with the $(1,1)$ forms $H^1(T^*)$:

$$H^1(T) \xleftrightarrow{\star} H^1(T^*). \quad (35)$$

If the dualized field theory has a sigma-model representation, the target space is a different manifold

$$X \xleftrightarrow{\star} \star X, \quad (36)$$

with opposite Euler character. Of course, no explicit construction of such a dualized manifold has been given, so this paragraph should be regarded as speculative. (35) implies that the the Kähler potential as defined in (6) changes sign under (30):

$$\ln\langle\Omega_{\sigma}^{+}(0)\Omega_{\sigma}^{-}(1)\rangle_Z|_X = -\ln\langle\Omega_{\sigma}^{+}(0)\Omega_{\sigma}^{-}(1)\rangle_Z|_{\star X}, \quad (37)$$

where the left hand side is defined with respect to sigma model perturbation theory on X and the right hand side is defined with respect to sigma model perturbation theory on $\star X$. Comparing with (34), one is led to the conclusion that

$$\Omega_{\sigma}^{+} \xrightarrow{\star} \star\Omega_{\sigma}^{+}. \quad (38)$$

This might have been expected since $G_{-\frac{1}{2}}^{+}$, which corresponds to ∂ , is interchanged with $G_{-\frac{1}{2}}^{-}$, which corresponds to $\star\partial\star$. Thus cohomology is dualized,

$$\partial \xrightarrow{\star} \star\partial\star. \quad (39)$$

This appears to be consistent with (38).

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