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Braids and Quantum Group Symmetry in Chern-Simons Theory

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Abstract

The monodromy matrices defined by the quantum holonomies acting on the physical state space of the Chern-Simons theory are derived. Up to equivalence, these matrices are reconstructed by means of a matrix-valued gauge connection satisfying the Gauss law. In terms of this connection, the relation of the Chern-Simons model with conformal field theory and quantum groups is established. The braid group representation realized on the physical states is obtained. The quantum group symmetry appears as a hidden symmetry of the quantized theory.

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7

1. Introduction

Many of the recent developments in both quantum field theory and the study of the geometry of three-manifolds $\{M^3\}$ concern the three-dimensional Chern-Simons (C-S) quantum field theory

$$S_{CS} = \frac{k}{4\pi} \int_{M^3} d^3x \, \epsilon^{\mu\nu\rho} \text{Tr} \left(A_\mu \partial_\nu A_\rho + i \frac{2}{3} A_\mu A_\nu A_\rho \right), \quad (1.1)$$

where A_μ is a gauge connection associated with a non-Abelian gauge group G . The important role played by the C-S model in the classification of $\{M^3\}$ and in the analysis of their topological properties was suggested firstly by A. S. Schwarz [1] and M. Atiyah [2]. The main ideas exposed in [1,2] concern the use of the C-S model to obtain differential topological invariants of M^3 like the Ray-Singer torsion, the Casson invariant and, more generally, the Floer homology groups. The crucial observation is that the definition of the C-S action does not require any choice of a metric on M^3 , as a consequence, the expectation values of metric-independent observables are characterized only by their "topological" properties. From a physical point of view, the C-S model represents a remarkable generally covariant quantum field theory in which the general covariance is not achieved by integrating over the metrics (as in General Relativity).

E. Witten has found [3] that deep connections exist between the C-S model and the two-dimensional rational conformal field theories (and, in general, the integrable statistical mechanics models [4]). Using these connections he has argued [3] that the expectation values of the Wilson line operators $\langle W(L) \rangle$, computed in the fundamental representation of the gauge group $SU(N)$, are related to the known link polynomials of knot theory. Although the general ideas supporting this remarkable scenario have already been introduced, many details have not been fully explored. In particular, one would like to understand better the relation of the C-S theory and link invariants with conformal field theory (CFT) and quantum groups.

In this paper, we would like to elaborate on these problems by using exclusively the properties of the three-dimensional field theory described by the C-S action (1.1). Our main concern will be the study of the braiding symmetry realized on the quantum state space of the C-S theory.

We start by considering the monodromy representation defined by the quantum holonomies and realized on the physical state space. In order to study this structure, we perform the canonical quantization of the model. We construct the wave functions of the states associated with point-like non-Abelian sources (punctures) on the plane. We compute the monodromy matrix B acting on the state space of the quantized theory and defined by the transport of a puncture along a closed path in the punctured plane. We prove that the elements of B are reproduced up to equivalence by the holonomies associated with a suitable gauge connection $\hat{A}$ satisfying the Gauss law on a relevant reduced space. We discuss then the role of $\hat{A}$ in connecting the C-S model with both CFT and quantum groups. We show that on one hand the connection $\hat{A}$ coincides precisely with the connection found by Knizhnik and Zamolodchikov [5] in analysing the properties of the correlation functions of primary fields in the two-dimensional conformal Wess-Zumino-Witten (WZW) model. On the other hand, in terms of $\hat{A}$ one can easily construct the Kohno connection ω [6]. The holonomies of ω give a linear representation [6] of the braid group which commutes with the diagonal action of the quantum group G_q with deformation parameter q satisfying $q^h = 1$. It turns out that the quantum group symmetry emerges as a *hidden* symmetry of the quantized C-S theory. Thus our construction clarifies also the relation of formal knot theory with quantum groups, a question which has attracted much attention recently [7].

2. Canonical Formalism in the Schrödinger Picture

The three-dimensional C-S action in a $(1+2)$ -decomposition of R^3 reads*

$$S_{CS} = \frac{k}{4\pi} \int d^3x \left(A_2^a \dot{A}_1^a + A_0^a F_{12}^a \right), \quad (2.1)$$

where

$$F_{12}^a = \partial_1 A_2^a - \partial_2 A_1^a - f^{abc} A_1^b A_2^c \quad (2.2)$$

and f^{abc} are the structure constants of a simple group G . Let us briefly recall the structure of the field theory described by the action (2.1). The time-components A_0^a appear in the expression (2.1) without time derivatives; therefore, A_0^a can be understood as Lagrange multipliers enforcing the Gauss constraint (in the absence of sources)

$$F_{12}^a = 0. \quad (2.3)$$

As we shall see, the main part of the non-trivial content of the theory is represented precisely by the Gauss law. Indeed, let us ignore for a while the Gauss constraint and, accordingly, the A_0 -term in the action. In this case the dynamics described by the first term of the action (2.1) would be trivial. For, with an action of the type

$$S_0 = \int dt p \dot{q}, \quad (2.4)$$

one simply gets

$$\langle q', t' | q, t \rangle = \int Dp Dq e^{iS_0} = \delta(q' - q). \quad (2.5)$$

Clearly, in our case the fields $A_i^a(\vec{x})$ play the role of the coordinates, the components $A_0^a(\vec{x})$ are the conjugate momenta and $\frac{4\pi}{k}$ is the counterpart of $\hbar$. This means that the canonical operator commutation relations are

$$[A_1^a(\vec{x}), A_2^b(\vec{y})] = i \delta^{ab} \frac{4\pi}{k} \delta^3(\vec{x} - \vec{y}). \quad (2.6)$$

* In our notations $x^\mu = (x^0, \vec{x}) = (x^0, x^1)$ with $i = 1, 2$. We shall use also the two-dimensional antisymmetric tensors ϵ_{ij} and ϵ^{ij} defined by $\epsilon_{12} = \epsilon^{21} = 1$, $\epsilon_{21} = \epsilon^{12} = -1$.

Without loss of generality, one can assume k to be positive because, as explained in [8], a change in the sign of k is equivalent to reversing the orientation of the three-manifold. In our context, this simply means that negative values of k can be recovered by exchanging the roles of A_1^a and A_2^a as canonical conjugated variables. Following Dunne, Jackiw and Trugenberger [9], we use the standard Schrödinger representation of (2.6). Namely,

$$A_1^a(\vec{x}) \mapsto \phi^a(\vec{x}), \quad A_2^a(\vec{x}) \mapsto -i \frac{4\pi}{k} \frac{\delta}{\delta \phi^a(\vec{x})} \quad (2.7)$$

and the state wave functions $\{\Psi\}$ are functions of the coordinates, that is functionals of $\phi^a(\vec{x})$. By making use of the canonical commutation relations (2.6), one finds that the operators

$$G^a(\vec{x}) = - \left(\frac{k}{4\pi} \right) F_{12}^a(\vec{x}) \quad (2.8)$$

are the generators of time-independent gauge transformations. More precisely, by integrating $G^a(\vec{x})$ with test functions $\Lambda^a(\vec{x})$ vanishing at infinity, i.e.

$$G(\Lambda) \equiv \int d^2x \Lambda^a(\vec{x}) G^a(\vec{x}), \quad (2.9)$$

one obtains

$$i [G(\Lambda), A_j^a(\vec{x})] = \partial_j \Lambda^a(\vec{x}) + f^{abc} \Lambda^b(\vec{x}) A_j^c(\vec{x}) \quad (2.10)$$

and

$$[G(\Lambda), G(\Lambda')] = i f^{abc} \int d^2x \Lambda^a(\vec{x}) \Lambda'^b(\vec{x}) G^c(\vec{x}). \quad (2.11)$$

It is very convenient to introduce now the field variable $U(\vec{x})$ living on the group G and satisfying

$$T^a A_1^a(\vec{x}) = -i U^{-1}(\vec{x}) \partial_1 U(\vec{x}), \quad (2.12)$$

where T^a are the generators of the defining representation of G and their normalization is given by $\text{Tr } T^a T^b = \delta^{ab}/2$. In terms of the coordinates $\phi^a(\vec{x})$, $U(\vec{x})$ can be written as the path-ordered integral

$$U(\vec{x}) = U(x^1, x^2) = P \exp \left(i \int_{-\infty}^{x^1} ds T^a \phi^a(s, x^2) \right). \quad (2.13)$$

Note that, according to eq.(2.10), a gauge transformation generated by $G(\Lambda)$ is realized on $U(\vec{x})$ through the action of the group by right multiplication

$$i[G(\Lambda), U(\vec{x})] = iU(\vec{x})T^a\Lambda^a(\vec{x}) . \quad (2.14)$$

Let us consider now the Gauss constraint (2.3). To find the solution of this constraint means finding the states annihilated by $G^a(\vec{x})$. In quantizing the theory in $R \times R^2$, there is only one state with this property [9]; we call it the vacuum and its wave function Ψ_0 takes a very simple form in terms of $U(\vec{x})$ [9]

$$\begin{aligned} \Psi_0 = \exp \left[-i \frac{k}{12\pi} \int_D d^3y \epsilon^{\alpha\beta\gamma} \text{Tr} (U^{-1} \partial_\alpha U U^{-1} \partial_\beta U U^{-1} \partial_\gamma U) \right. \\ \left. - i \frac{k}{4\pi} \int d^2x \text{Tr} (U^{-1} \partial_1 U U^{-1} \partial_2 U) \right] . \end{aligned} \quad (2.15)$$

In the expression (2.15), the integral of the Wess-Zumino three-form is performed on a three-dimensional manifold D whose boundary is R^2 . Note that different choices for the extension of U on D , i.e. different mappings of D into $G = SU(N)$, lead to the same wave function Ψ_0 provided that k takes integer values. In other words, only for integer values of k is the wave function of the vacuum single-valued. In general, for any simple group G , one can always choose the normalization of the action in such a way that for integer values of k the vacuum wave function is single-valued.

Under a field variation $U(\vec{x}) \mapsto U(\vec{x}) + \Delta U(\vec{x})$, the vacuum wave function transforms as

$$\begin{aligned} \Delta \Psi_0 &= i \frac{k}{2\pi} \int d^2x \text{Tr} [U^{-1} \Delta U \partial_2 (U^{-1} \partial_1 U)] \Psi_0 \\ &= -i \frac{k}{2\pi} \int d^2x \text{Tr} [\Delta (U^{-1} \partial_1 U) U^{-1} \partial_2 U] \Psi_0 \end{aligned} \quad (2.16)$$

and therefore one obtains

$$T^a A_2^a(\vec{x}) \Psi_0 = -i \frac{4\pi}{k} T^a \frac{\delta}{\delta \phi^a(\vec{x})} \Psi_0 = -i U^{-1}(\vec{x}) \partial_2 U(\vec{x}) \Psi_0 . \quad (2.17)$$

Eq.(2.17) combined with the identity

$$T^a A_1^a(\vec{x}) \Psi_0 = -i U^{-1}(\vec{x}) \partial_1 U(\vec{x}) \Psi_0 , \quad (2.18)$$

which follows from the definition (2.12), show that Ψ_0 is indeed the wave function of the vacuum, i.e.

$$G(\Lambda) \Psi_0 = 0 . \quad (2.19)$$

We are interested now in the states corresponding to the situation in which there are some distinguished points (punctures) on the plane R^2 with a definite non-trivial representation of the local gauge group assigned to each puncture. In computing the expectation value $\langle W(L) \rangle$, these states appear as intermediate states at fixed time $t = t_0$. The punctures arise as intersection points of the link L with the plane $t = t_0$ [3]. The construction of the corresponding wave functions is very simple and in what follows we shall illustrate some different possibilities with a few examples. Let us recall first that the matrix valued variable $U(\vec{x})$ takes values in G . So, we can define $U^{(R)}(\vec{x})$ to be the representative of $U(\vec{x})$ in the R -representation of G . Consider then the wave function

$$\Psi[\vec{y}, R]_{IJ} = U_{IJ}^{(R)}(\vec{y}) \Psi_0 , \quad (2.20)$$

where the indices I and J label the matrix elements of $U^{(R)}(\vec{y})$. By making use of eqs.(2.14) and (2.19), one finds

$$(iG(\Lambda)\Psi[\vec{y}, R])_{IJ} = i\Psi[\vec{y}, R]_{IK} R_{KJ}^a \Lambda^a(\vec{y}) . \quad (2.21)$$

Eq.(2.21) shows that $\Psi[\vec{y}, R]_{IJ}$ describes a situation in which there is a single puncture located in $\vec{y}$ with the R -representation of G assigned to it. The different vectors of the R -representation on which $G^a(\vec{y})$ act are labelled by the second index J of $\Psi[\vec{y}, R]_{IJ}$. Similarly, the wave function

$$\Psi[\vec{y}, R]^{IJ} = U_{IJ}^{(R)-1}(\vec{y}) \Psi_0 \quad (2.22)$$

also describes a puncture in $\vec{y}$ associated with the R -representation of the group G . Differently from the wave function (2.20), however, the vectors of the R -representation associated with $G^a(\vec{y})$ are labelled in this case by the first index I of $\Psi[\vec{y}, R]^{IJ}$. Clearly, multiplying Ψ_0 by several matrix elements of the variable U (and/or of its inverse U^{-1})

in different points and in different representations, one obtains a wave function of the type

$$\Psi[\vec{y}_1, R_1]_{IJ}[\vec{y}_2, R_2]^{KL} \dots = U_{IJ}^{(R_1)}(\vec{y}_1) U_{KL}^{(R_2)}(\vec{y}_2) \dots \Psi_0, \quad (2.23)$$

which describes several punctures (located in $\vec{y}_1, \vec{y}_2, \dots$) associated with the representations $R_1, R_2, \dots$ of G respectively. The wave functions (2.23) represent some sort of "building blocks" in terms of which the intermediate states (in the presence of Wilson lines) are constructed.

Observe that, among the indices explicitly displayed in the wave function (2.23), the first lower index and the second upper index are free; they label the states of the charges "at infinity". In general, we call the indices of this type q-indices. By construction, in the wave functions of the type (2.23) there are two indices associated with each puncture: one of them labels the states of the $G(\Lambda)$ -representation and the other one is a q-index. The q-indices do not transform under the action of $G(\Lambda)$ and in this sense they label "hidden" degrees of freedom. As we shall see later, the q-indices are precisely the origin of the degrees of freedom on which the quantum group G_q acts. In the wave functions corresponding to Wilson lines defined in a finite region, all q-indices are contracted in different combinations. In order to see that, let us consider first the wave function $\Psi(\Gamma)$,

$$\Psi_{IJ}(\Gamma) = W_{IJ}(\Gamma) \Psi_0, \quad (2.24)$$

obtained by applying the quantum holonomy operator

$$W(\Gamma) = P \exp \left(i \int_{\Gamma} A_i^a(\vec{x}) R^a dx^i \right) \quad (2.25)$$

on the vacuum. The path-ordered integral in eq.(2.25) is performed along the oriented non-intersecting smooth path Γ in R^2 connecting the points $\vec{y}_1$ with $\vec{y}_2$ (as shown in Fig.1) and parametrized by $0 \leq s \leq 1$. From eqs.(2.10) and (2.19), it follows that

$$(iG^a(\vec{x}) \Psi(\Gamma))_{IJ} = -i\delta(\vec{x} - \vec{y}_1) R_{IK}^a \Psi_{KJ}(\Gamma) + i\delta(\vec{x} - \vec{y}_2) \Psi_{IK}(\Gamma) R_{KJ}^a. \quad (2.26)$$

So $\Psi(\Gamma)$ describes two punctures in $\vec{y}_1$ and $\vec{y}_2$ which are associated with the same representation R of G . Notice that eq.(2.26) is exactly the operator Gauss law constraint

in the presence of (in this case two) punctures. We call *physical* the states satisfying the operator Gauss law. Clearly, if one applies several quantum holonomy operators on the vacuum, one obtains a physical state corresponding to an even number of punctures located at the endpoints of the paths.

Now we would like to write $\Psi(\Gamma)$ explicitly and find the relation between $\Psi(\Gamma)$ and the wave functions of type shown in eq.(2.23). By using eqs.(2.17) and (2.18), one finds

$$W(\Gamma) \Psi_0 = \Psi_0 P \exp \left[i \int_{\Gamma} \left(-i V^{-1}(\vec{x}) \partial_i V(\vec{x}) dx^i - i \frac{4\pi}{k} R^a \frac{\delta}{\delta \phi^a(\vec{x})} dx^2 \right) \right], \quad (2.27)$$

where

$$V(\vec{x}) = U^{(R)}(\vec{x}). \quad (2.28)$$

Note that in eq.(2.27) Ψ_0 has been moved to the left. The "one-form" to be integrated in (2.27) is the $V(\vec{x})$ -gauge transformed of B_i defined by

$$B_1(\vec{x}) = 0, \quad B_2(\vec{x}) = -i \frac{4\pi}{k} V(\vec{x}) R^a \frac{\delta}{\delta \phi^a(\vec{x})} V^{-1}(\vec{x}). \quad (2.29)$$

Therefore, eq.(2.27) can be rewritten as

$$W(\Gamma) \Psi_0 = \Psi_0 V^{-1}(\vec{y}_1) \tilde{W}(\Gamma) V(\vec{y}_2), \quad (2.30)$$

where

$$\tilde{W}(\Gamma) = P \exp \left[\frac{4\pi}{k} \int_0^1 ds \dot{z}^2(s) V(\vec{z}(s)) R^a \frac{\delta}{\delta \phi^a(\vec{z}(s))} V^{-1}(\vec{z}(s)) \right]. \quad (2.31)$$

In the expression (2.30), the functional derivative appearing in $\tilde{W}$ acts on the right. A straightforward computation gives

$$\frac{\delta V(\vec{y})}{\delta \phi^a(\vec{x})} = i\delta(x^2 - y^2) \theta(y^1 - x^1) V(\vec{x}) R^a V^{-1}(\vec{x}) V(\vec{y}). \quad (2.32)$$

The expression (2.32) is well defined for $x^1 \neq y^1$, but presents ambiguities when $x^1 = y^1$ because of the theta function. So, the problem that one finds here is how to compute the action of $\delta/\delta \phi^a(\vec{x}(s))$ on $V^{-1}(\vec{z}(s))$.

Actually, this problem is not completely new and unexpected. In analysing the covariant BRS quantization of the C-S theory [8], we learned that a framing procedure

is necessary to solve the ambiguities appearing in the computation of the expectation values of the Wilson lines. The ambiguities that we found in [8] were related to the short-distance behaviour of the two-point function of the gauge potential. Here again we find an ambiguity due to a short-distance effect between the components of the gauge potential. In fact, the problem of defining the expression (2.32) when $x^1 \rightarrow y^1$ (in relation with eq.(2.31)) is precisely the counterpart, in the operator approach that we are following, of the "self-linking problem" found in the covariant three-dimensional setting. It should be noted that, having to do with open contours, the possible prescriptions that one may adopt to solve such ambiguities actually represent only a part of the solution of the whole framing problem.

For example, a way to give a well-defined meaning to (2.31) is to shift the contour $\vec{x}(s)$ by a constant vector $\varepsilon \vec{n}$ (with $|\vec{n}| = 1$),

$$\vec{x}(s) \mapsto \vec{y}(s) = \vec{x}(s) + \varepsilon \vec{n} \quad , \quad (2.33)$$

in the field variable $V(\vec{x})$ only,

$$V(\vec{x}(s)) \mapsto V(\vec{y}(s)) \quad . \quad (2.34)$$

The $\varepsilon \rightarrow 0$ limit has to be performed after the computations. Then one gets

$$\begin{aligned} V(\vec{y}) R^{\varepsilon} \frac{\delta}{\delta \phi^{\alpha}(\vec{x})} V^{-1}(\vec{y}) &= V(\vec{y}) R^{\varepsilon} V^{-1}(\vec{y}) \frac{\delta}{\delta \phi^{\alpha}(\vec{x})} \\ &\quad - i \delta(x^2 - y^2) \theta(y^1 - x^1) V(\vec{y}) R^{\varepsilon} V^{-1}(\vec{y}) V(\vec{x}) R^{\varepsilon} V^{-1}(\vec{x}) \quad . \end{aligned} \quad (2.35)$$

Let us concentrate on the second term on the RHS of eq.(2.35). This term is proportional to $\delta(\varepsilon n^2) \theta(\varepsilon n^1)$. So it is clear that one can always choose $\vec{n}$ in such a way that this term is identically zero; for instance, $\vec{n} = (-1, 1)/\sqrt{2}$. The above procedure just corresponds to a choice of framing in our two-dimensional context. We call this recipe the "constant framing". In conclusion, in constant framing one has

$$\widetilde{W}(\Gamma) \Big|_{cf} = P \exp \left[\frac{4\pi}{k} \int_0^1 ds \dot{x}^2(s) V(\vec{x}(s)) R^{\varepsilon} V^{-1}(\vec{x}(s)) \frac{\delta}{\delta \phi^{\alpha}(\vec{x}(s))} \right] \quad . \quad (2.36)$$

Consider now eq.(2.30) and the action of the path-ordered operator (2.36) on $V(\vec{y}_2)$. In order to avoid possible ambiguities at the end point $\vec{y}_2$ of Γ , one can consistently

extend the constant framing procedure by introducing also an infinitesimal displacement in $V(\vec{y}_2)$; $V(\vec{y}_2) \rightarrow V(\vec{y}_2 + \varepsilon \vec{n})$. Then, by making use of eq.(2.32), it is easy to see that each functional derivative $\delta/\delta \phi^{\alpha}(\vec{x}(s))$ acting on the right gives zero because every point of the path Γ (see Fig.1) never intersects the horizontal ($x^2 = \text{const.}$) straight line determined by the position of the puncture $\vec{y}_2 + \varepsilon \vec{n}$. Therefore we obtain

$$\Psi_{IJ}(\Gamma) = V_{IK}^{-1}(\vec{y}_1) V_{KJ}(\vec{y}_2) \Psi_0 = \Psi[\vec{y}_1, R]^{IK} [\vec{y}_2, R]_{KJ} \quad . \quad (2.37)$$

Compared with eq.(2.23), the expression (2.37) shows that the q-indices are indeed contracted.

The state described by $\Psi_{IJ}(\Gamma)$ is invariant under smooth deformations of Γ . However, for a general deformation of Γ as well as for different choices of $\vec{n}$, the wave function (2.37) may change by a phase factor $\delta = \exp[i \frac{4\pi}{k} m c_2(R)]$, where $c_2(R) = R^a R^a$ and m is an integer. This factor has its origin in some sort of "self-interaction" of the quantum holonomy operator. The derivation of the value of δ will be given in the next section. One could be tempted to relate δ with the variable $\alpha = \exp[-i \frac{2\pi}{k} c_2(R)]$ measuring the behaviour of $< W_R(C) >$ under a change of framing in the three-dimensional context [8]. We have not found, however, any completely convincing argument in favour of such an interpretation. The point is that the framing of knots in R^3 is an intrinsic and unambiguous three-dimensional concept, whereas any two-dimensional prescription we have examined, suffers from ambiguities. The issue of finding the two-dimensional realization of the well-defined three-dimensional framing will not be considered here because it is irrelevant for the computation of the monodromy matrices.

3. Monodromy Matrices

We want to study now the monodromies obtained by applying quantum holonomy (or Wilson line) operators on the physical states. A similar problem has been investigated by different methods in [10]. The technique that we will use has been employed also in the context of $2 + 1$ -gravity [11]. Our starting point is the wave function Ψ

associated with the four-puncture configuration shown in Fig.2.

$$\Psi = \Psi[\tilde{y}_1]^{IM}[\tilde{y}_2]_{MJ}[\tilde{y}_3]^{KN}[\tilde{y}_4]_{NL} = V^{-1}(\tilde{y}_1)_{IM}V(\tilde{y}_2)_{MJ}V^{-1}(\tilde{y}_3)_{KN}V(\tilde{y}_4)_{NL}\Psi_0. \quad (3.1)$$

For simplicity we consider all the punctures in the R -representation, eq.(2.28); so we omit in the wave function the symbols specifying the representations. In view of our previous result (2.37), the wave function (3.1) can be obtained by applying two Wilson line operators on the vacuum: one of them connects $\tilde{y}_1$ with $\tilde{y}_2$ and the other connects $\tilde{y}_3$ with $\tilde{y}_4$, as shown in Fig.2. The parallel transport, say, of the puncture $\tilde{y}_2$ along the loop C enclosing the puncture $\tilde{y}_4$, as shown in Fig.3, defines a monodromy on the four-puncture physical states. This operation gives the state

$$B\Psi = V^{-1}(\tilde{y}_1)_{IM}V(\tilde{y}_2)_{MJ} \left(P \exp i \int_C dz^i A_i^a(\vec{z}) R^a \right)_{QJ} V^{-1}(\tilde{y}_3)_{KN}V(\tilde{y}_4)_{NL}\Psi_0. \quad (3.2)$$

By using eqs.(2.27,31) and in constant framing, one gets

$$B\Psi = \Psi_0 V^{-1}(\tilde{y}_1)_{IM}\tilde{W}(C)_{MS}V(\tilde{y}_2)_{SJ}V^{-1}(\tilde{y}_3)_{KN}V(\tilde{y}_4)_{NL}, \quad (3.3)$$

where $\tilde{W}$ is given in eq.(2.36). Because of the theta-function in eq.(2.32), the functional derivative present in $\tilde{W}$ in the expression (3.3) acts non-trivially only on $V(\tilde{y}_4)_{NL}$. Then, around the intersection point $\vec{z}(s_0)$ of the contour C with the line $x^2 = y_4^2$ and for $x^1 < y_4^1$, from eq.(2.32) one has [11], at first order in $\frac{1}{k}$,

$$\begin{aligned} d\vec{s} \dot{\vec{z}}^2(s) (V(\vec{z}) R^a V^{-1}(\vec{z}))_{MS} \frac{\delta}{\delta \phi^a(\vec{z})} V(\tilde{y}_4)_{NL} &= \\ &= i d\vec{s} \dot{\vec{z}}^2(s) \delta(\vec{z}^2(s) - y_4^2) (V(\vec{z}) R^a V^{-1}(\vec{z}))_{MS} (V(\vec{z}) R^a V^{-1}(\vec{z}) V(\tilde{y}_4))_{NL} \\ &= i d\vec{s} \dot{\vec{z}}^2(s) \delta(\vec{z}^2(s) - y_4^2) R_{MS}^a R_{NQ}^a V(\tilde{y}_4)_{QL} \\ &= i \text{sign}(\dot{\vec{z}}^2(s_0)) R_{MS}^a R_{NQ}^a V(\tilde{y}_4)_{QL}. \end{aligned} \quad (3.4)$$

For the contour C in Fig.3 one has $\text{sign}(\dot{\vec{z}}^2(s_0)) = 1$ and exponentiating the result (3.4) one finds

$$B\Psi = B_{M,N}^{S,Q} \Psi[\tilde{y}_1]^{IM}[\tilde{y}_2]_{SJ}[\tilde{y}_3]^{KN}[\tilde{y}_4]_{QL}, \quad (3.5)$$

where $B_{M,N}^{S,Q}$ are the matrix elements of

$$B = \exp \left(i \frac{4\pi}{k} R^a \otimes R^a \right). \quad (3.6)$$

In eq.(3.6) the matrix B can be understood as acting on the tensor product of the two R -representations whose vectors are labelled by the q -indices S and Q in the expression $V(\tilde{y}_2)_{SJ}V(\tilde{y}_4)_{QL}$. Note that all the indices of the monodromy matrix B are contracted with q -indices, eq.(3.5).

For punctures associated with different representations R_1 and R_2 of the group G , the monodromy matrix associated with the contour C reads

$$B = \exp \left(i \frac{4\pi}{k} R_1^a \otimes R_2^a \right). \quad (3.7)$$

When the puncture entering in eq.(3.4) is the endpoint of the quantum holonomy operator itself, see Fig.4, the counterpart of the B -matrix is

$$\delta = \exp \left(i \frac{4\pi}{k} c_2(R) \right), \quad (3.8)$$

which is precisely the phase factor mentioned in the previous section.

In order to illustrate the meaning of eqs.(3.5,6), let us consider the case in which R is the fundamental representation of $G = SU(N)$, ($R^a = T^a$). By means of the Fierz-identity

$$T_{IJ}^a T_{KL}^a = \frac{1}{2} \delta_{IL} \delta_{JK} - \frac{1}{2N} \delta_{IJ} \delta_{KL}, \quad (3.9)$$

one gets

$$B_{M,N}^{S,Q} = b_1 \delta_M^S \delta_N^Q + b_2 \delta_N^S \delta_M^Q, \quad (3.10)$$

where

$$b_1 = \exp \left(-\frac{2i\pi}{Nk} \right) \cos \frac{2\pi}{k}, \quad b_2 = i \exp \left(-\frac{2i\pi}{Nk} \right) \sin \frac{2\pi}{k}. \quad (3.11)$$

So, in this case, eq.(3.5) takes the form

$$B\Psi = b_1 \Psi[\tilde{y}_1]^{IM}[\tilde{y}_2]_{MJ}[\tilde{y}_3]^{KN}[\tilde{y}_4]_{NL} + b_2 \Psi[\tilde{y}_1]^{IM}[\tilde{y}_4]_{ML}[\tilde{y}_3]^{KN}[\tilde{y}_2]_{NJ}, \quad (3.12)$$

which has the graphical representation shown in Fig.5.

It is important to note that, apart from the "self-interaction" ambiguities that we have discussed in the previous section, the results (3.5-7) are invariant under smooth deformations of the contour C on the punctured plane. In other words, the action of the operator $\tilde{W}(C)$ on all the punctures, except that one belonging to C , is unambiguous and defines a representation of the fundamental group of the punctured plane. In fact, if C encloses several punctures, the resulting monodromy is the ordered product of the single B -matrices. Moreover, the change of orientation of C in Fig.3 leads to a monodromy matrix which is the inverse of (3.6). Finally, if the loop C encloses both the punctures $\tilde{y}_3$ and $\tilde{y}_4$, the resulting B -matrix is the identity.

Summarizing, the monodromy properties of the physical states of the C-S model are fully described by a set of matrices $\{B\}$ of the type (3.7), which act on the q -indices of the states (2.23). In order to make contact with CFT and quantum groups it is sufficient to concentrate on $\{B\}$.

4. A Flat Connection on the Punctured Plane

As mentioned in the Introduction, our main concern is to derive the braid group representation realized on the physical state space of the C-S theory. The information concerning this representation is somehow contained in the operator approach developed in the previous section and now the question is how to extract it. In order to answer this question, let us recall what we have learned by using the canonical formalism.

A complete understanding of the variable measuring the twisting of the framed knots cannot be fully achieved (presently, at least) in a purely two-dimensional context. This fact does not represent a real problem for the following reason. The braiding of the punctures can always be done without modifying the structure of the framing of the underlying Wilson lines. This means that in our two-dimensional context we can consistently ignore all the "self-interaction" problems. Having reconstructed in this way the braid representation, one can always introduce additional twistings of the framed knots because the associated variable α measuring the twist is already known.

Remember that the twisting of the framed knots is completely understood and under control in the covariant quantization of the theory [8].

We have also learned that one can unambiguously compute the monodromy matrices $\{B\}$. The important point here is the topological character of these matrices. In fact, the B 's are well defined precisely because we consider quantum holonomy operators associated with closed paths. For this reason, all the asymmetries introduced in the canonical quantization (for instance, the choice of A_1^2 and A_2^2 as canonical conjugated coordinate and momentum respectively) are irrelevant. On the contrary, in dealing with open paths the results obtained by using the canonical formalism strongly depend on the conventions adopted. But in constructing the braid representation, one has to consider several quantum holonomy operators associated with open paths and acting simultaneously on the physical states of the theory. So, how can we extract in this case reliable results about the braid representation? The key point is that, by using the information carried by the monodromy matrices, we can reconstruct the braid group representation. Let us explain now the method that we use to solve this problem.

It is useful to introduce first a few definitions. Let $R_n^2 = R^2 \setminus \{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n\}$ be the punctured plane and $X_n = \{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n : \tilde{y}_p \neq \tilde{y}_q \text{ for } p \neq q\}$. The fundamental group of X_n is called the pure braid group; it is well defined also for inequivalent punctures. The Artin braid group B_n , which we are interested in, is defined instead as the fundamental group of the space $Y_n = X_n/S_n$, S_n being the permutation group of n elements (in this case, the punctures must be equivalent).

Our strategy consists of connecting the representation of $\pi_1(R_{n-1}^2)$ (defined by the monodromy matrices) with a representation of $\pi_1(X_n)$ which, in turn, extends to a representation of B_n . The derivation is essentially unique and leads to a B_n representation which turns out to be in agreement with all the known facts and computations performed in the C-S theory.

In order to make contact with the braid group, a specific representation ϱ of $\pi_1(R_{n-1}^2)$ has to be considered. In this section we write down an appropriate set of monodromy matrices $\{B\}$ which define the representation ϱ . Afterwards we derive a flat connection $\hat{A}$ on R_{n-1}^2 with the following property: the holonomies of $\hat{A}$ give rise

to a representation χ of $\pi_1(R_{n-1}^2)$ which is equivalent to ϱ .

In order to construct ϱ , we consider the state

$$\Psi[\vec{x}_1, R_1]^{I_1 K_1} [\vec{y}_1, R_1]_{K_1 I_1} \cdots [\vec{x}_n, R_n]^{I_n K_n} [\vec{y}_n, R_n]_{K_n I_n} \quad (4.1)$$

As we have seen before, the B -matrices can be considered as acting on the tensor product of the R_p -representations associated with the punctures $\{\vec{y}_p\}$ and labelled by the q -indices $\{K_p\}$. We concentrate on the set $\mathcal{M}$ of B -matrices obtained by moving, say $\vec{y}_q$, along loops which do not enclose any of the punctures $\{\vec{x}_1, \dots, \vec{x}_n\}$. $\mathcal{M}$ gives a representation of $\pi_1(R_{n-1}^2)$. Denote by $\{\epsilon_p : p = 1, \dots, q-1, q+1, \dots, n\}$ the generators of $\pi_1(R_{n-1}^2)$; ϵ_p is associated with a clockwise oriented loop winding once around $\vec{y}_p$ and is represented by

$$\varrho(\epsilon_p) = \exp\left(i \frac{4\pi}{k} R_p^a \otimes R_q^a\right) \quad (4.2)$$

Let us turn now to the construction of the representation χ . We consider the punctured plane R_n^2 ; to each puncture $\vec{y}_p$ there is associated a finite dimensional linear space H_p on which a representation R_p of G is defined. The "induced" Gauss constraint reads [3]

$$F_{12}^a(\vec{z}) = -\frac{4\pi}{k} \sum_{p=1}^n \hat{R}_p^a \delta(\vec{z} - \vec{y}_p) \quad (4.3)$$

where $\hat{R}_p^a$ acts on $H = H_1 \otimes \cdots \otimes H_n$ as $1 \otimes \cdots \otimes R_p^a \otimes \cdots \otimes 1$. Eq.(4.3) has to be understood as a constraint induced on H by the Gauss law (2.26). Clearly (4.3) has no c-number solutions, but we will show now that it admits matrix-valued solutions. In terms of the Green function of the plane

$$\partial_i \partial^i \ln |\vec{z} - \vec{y}| = 2\pi \delta(\vec{z} - \vec{y}) \quad (4.4)$$

the gauge connection $\hat{A}_i^a(\vec{z})$, which solves the Gauss constraint (4.3), can be written as

$$\hat{A}_i^a(\vec{z}) = \frac{2}{k} (\epsilon_{ij} + i\delta_{ij}) \sum_{p=1}^n \hat{R}_p^a \partial^j \ln |\vec{z} - \vec{y}_p| \quad (4.5)$$

Indeed, the linear part of F_{12}^a , computed with $\hat{A}_i^a(\vec{z})$, satisfies exactly eq.(4.3) whereas the quadratic part vanishes. The hat in $\hat{A}_i^a(\vec{z})$ has been inserted to remind us that $\hat{A}_i^a(\vec{z})$ is matrix-valued.

Let us study now what happens when the puncture $\vec{y}_q$ is moved along a closed contour C , which does not intersect the remaining $n-1$ punctures. The arising monodromy is described by the holonomy (associated with C) of the connection

$$\hat{A}_{(q)i}(\vec{z}) = \hat{A}_{(q)i}^a(\vec{z}) \hat{R}_q^a = \frac{2}{k} (\epsilon_{ij} + i\delta_{ij}) \sum_{p \neq q} \hat{R}_p^a \partial^j \ln |\vec{z} - \vec{y}_p| \quad (4.6)$$

which is obtained from (4.5) by omitting the term $p = q$ on the right-hand side. As usual, the self-interaction of the puncture $\vec{y}_q$ is ignored. The connection (4.6) is flat in R_{n-1}^2 and the associated holonomy

$$\chi(C) = P \exp\left(i \oint_C \hat{A}_{(q)i} dx^i\right) \quad (4.7)$$

naturally defines a representation of $\pi_1(R_{n-1}^2)$.

In the remaining part of this section we show that the representation χ is equivalent to ϱ . First of all we observe that the curvature $F_{(q)i}(\vec{z})$ of $\hat{A}_{(q)i}(\vec{z})$ is given by

$$F_{(q)i}(\vec{z}) = -\frac{4\pi}{k} \sum_{p \neq q} \hat{R}_p^a \hat{R}_q^a \delta(\vec{z} - \vec{y}_p) \quad (4.8)$$

Eq.(4.8) admits several solutions. Indeed, let $\Omega(\vec{z})$ be an invertible matrix acting on H and such that

$$\Omega^{-1}(\vec{y}_p) \hat{R}_p^a \hat{R}_p^a \Omega(\vec{y}_p) = \hat{R}_p^a \hat{R}_p^a \quad (4.9)$$

for any $p \neq q$. Then

$$\hat{A}_{(q)i}^n(\vec{z}) = \Omega^{-1}(\vec{z}) \hat{A}_{(q)i}(\vec{z}) \Omega(\vec{z}) - i \Omega^{-1}(\vec{z}) \partial_i \Omega(\vec{z}) \quad (4.10)$$

also satisfies eq.(4.8) because $F_{(q)}^n = F_{(q)}$ as a consequence of (4.9). If one denotes by χ^n the holonomy relative to $\hat{A}_{(q)}^n$, then

$$\chi^n(C) = \Omega^{-1}(\vec{y}_q) \chi(C) \Omega(\vec{y}_q) \quad (4.11)$$

i.e. the representations χ and χ^n of $\pi_1(R_{n-1}^2)$ are equivalent. Consequently, if there exists an appropriate Ω_0 such that

$$\chi^{\Omega_0}(C) = \varrho(C) \quad (4.12)$$

then ϱ and χ are equivalent. The argument showing that Ω_0 exists is the following. Since the curvature $F_{(q)}$ vanishes outside the $(n-1)$ punctures $\tilde{y}_p$, it is always possible to find Ω such that $\hat{A}_{(q)}^\Omega$ vanishes on a simply-connected open domain $\mathcal{O}$ of R_{n-1}^2 . The conditions (4.9) can always be satisfied because $\pi_1(SL(N)) = 1$. One can take $\mathcal{O}$ to contain $\tilde{y}_q$ and to be of the form shown in Fig.6. With this choice of Ω , $\hat{A}_{(q)}^\Omega$ may differ from zero only inside the strips $\{s_p\}$. Consider now the limit in which the strips become cuts starting from $\tilde{y}_p$ and going to infinity. In this limit $\hat{A}_{(q)}^\Omega$ converges to $\hat{A}_{(q)}^{\Omega_0}$ which vanishes outside the cuts and still satisfies eq.(4.8):

$$\begin{aligned}\hat{A}_{(q)1}^{\Omega_0}(\tilde{z}) &= 0 \\ \hat{A}_{(q)2}^{\Omega_0}(\tilde{z}) &= \frac{4\pi}{k} \sum_{p \neq q} \hat{R}_p^\alpha \hat{R}_q^\alpha \delta(z^2 - y_p^2) \theta(y_p^1 - x^1)\end{aligned}\quad (4.13)$$

Now eq.(4.12) is an obvious consequence of (4.13) and this concludes the proof of the equivalence between χ and ϱ .

Summarizing, the monodromy matrices $\{B\}$ obtained in the quantized C-S theory are reproduced up to equivalence by the holonomies $\{\chi(C)\}$ associated with the solution of the "induced" Gauss law (4.3). In this way, the field theory problem of the quantum monodromies is reduced to a simple "quantum mechanics" problem with a finite number of degrees of freedom.

Note that Ω_0 actually depends on the positions of all the punctures on the plane and, in the domain $\mathcal{O}$, it satisfies the equation

$$\frac{\partial}{\partial y_q^i} \Omega_0(\tilde{y}_1, \dots, \tilde{y}_n) = i \frac{2}{k} (\epsilon_{ij} + i\delta_{ij}) \sum_{p \neq q} \hat{R}_p^\alpha \hat{R}_q^\alpha \Omega_0(\tilde{y}_1, \dots, \tilde{y}_n) \frac{\partial}{\partial y_q^j} \ln |\tilde{y}_q - \tilde{y}_p| \quad (4.14)$$

We shall come back to this equation, which appears also in the context of the WZW model, later on.

5. Skein Relation and Knizhnik-Zamolodchikov Equation

We illustrate now the important role of the B -matrix by deriving the skein relation satisfied by the link polynomials [12]. The starting point is represented by the so-called "half-monodromies" associated with the exchange of two punctures in the same

representation, as shown in Fig.7. Twice this operation must reproduce the matrix B (eq.(3.6)) and therefore, the matrix associated to the oriented exchange operation $\tilde{y}_1 \leftrightarrow \tilde{y}_2$ is taken to be

$$E = \exp \left(i \frac{2\pi}{k} R^a \otimes R^a \right) \quad (5.1)$$

In a three-dimensional picture, B and E can be interpreted as follows [3]. The B -matrix describes the effect of the double exchange, shown in Fig.8, of the two punctures. The exchange operation of the punctures in Fig.7 corresponds to the replacement of the configuration in Fig.9(a) with the configuration in Fig.9(b). Vice versa, the inverse matrix E^{-1} is associated with the operation shown in Fig.9(c). At the level of states, one has the following situation. The initial state $|L_0\rangle$, Fig.9(a), is

$$|L_0\rangle = |I\rangle > |J\rangle > \quad (5.2)$$

where $|I\rangle > |J\rangle \in H_1 \otimes H_2$. Under the exchange shown in Fig.9(b), one obtains the state

$$|L_-\rangle = \Sigma E |L_0\rangle, \quad (5.3)$$

whereas for the case shown in Fig.7(c), one has

$$|L_+\rangle = \Sigma E^{-1} |L_0\rangle, \quad (5.4)$$

where Σ is the permutation operator: $\Sigma |I\rangle > |J\rangle = |J\rangle > |I\rangle$.

In the fundamental representation of $G = SU(N)$ for which the Fierz-identity (3.9) holds, the above formulae take the simple form

$$|L_0\rangle = |s\rangle > |a\rangle, \quad (5.5)$$

$$|L_-\rangle = \exp \left[\frac{i(N-1)\pi}{Nk} \right] |s\rangle > -\exp \left[\frac{-i(N+1)\pi}{Nk} \right] |a\rangle, \quad (5.6)$$

$$|L_+\rangle = \exp \left[\frac{-i(N-1)\pi}{Nk} \right] |s\rangle > -\exp \left[\frac{i(N+1)\pi}{Nk} \right] |a\rangle, \quad (5.7)$$

where $|s\rangle > (|a\rangle)$ is the symmetric (antisymmetric) component of the state $|L_0\rangle$.

Eqs.(5.5-7) illustrate in a very explicit way the "marvellous property of a two-dimensional space that any three vectors obey a relation of linear dependence". As a consequence of this fact, one obtains the basic formula (skein relation)

$$\beta |L_+ \rangle - \beta^{-1} |L_- \rangle = z |L_0 \rangle, \quad (5.8)$$

where

$$\beta = q^{\frac{1}{2N}}, \quad z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}, \quad (5.9)$$

and

$$q = \exp \left(-i \frac{2\pi}{k} \right). \quad (5.10)$$

Eqs.(5.8-10) are in agreement with our explicit perturbative computation of the skein relation obtained for the fundamental representation of $SU(N)$ and in vertical framing [8,13]. The values of β and q agree also with Witten's result [4] based on the two-dimensional WZW model, if one makes the identification

$$k = \ell + c_0. \quad (5.11)$$

Here ℓ is the level of the WZW model and c_0 is the quadratic Casimir operator in the adjoint representation of G ; $c_0 = N$ for $SU(N)$. Some comments are in order:

- 1) The results (5.8-10) are non-perturbative and follow from the canonical commutation relations (2.6) alone.
- 2) The skein relation (5.8), together with the knowledge of the behaviour under Reidemeister moves of type one and the convention on the overall normalization, uniquely determine [12] the link polynomials and therefore the expectation values of the Wilson line operators (in the fundamental representation of $SU(N)$).
- 3) The skein relation and the values of the parameters β and z are totally independent of any definition of the pairing (scalar or inner product) between the states. This means, in particular, that any formal manipulation of the functional measure defining the pairing cannot modify at all the values (5.9) and (5.10). This excludes any *quantum mechanics induced shift*

$$k \rightarrow k + c_0. \quad (5.12)$$

in the values of the skein parameters.

Since the last point is subject to some controversy in the literature, it is perhaps useful to add some clarifying remarks. One should distinguish between two completely different aspects. The first one concerns the quantization in perturbation theory. The C-S model is finite and therefore the physical (dressed or renormalized) coupling constant k is a finite function of the bare coupling constant k_B . The explicit form of this function depends on both the regularization procedure and renormalization prescription adopted and can be arbitrarily modified by finite local counterterms (including the action itself). This arbitrariness reflects the fact that k_B is unphysical; physics depends on k . So, finding a scheme* in which $k = k_B + c_0$ does not mean that a quantum mechanics induced shift occurs! If present, such a shift can be detected only by comparing at least two physical renormalized quantities. This fact leads us to the second aspect of the problem, which makes sense independently of perturbation theory and is actually the meaningful issue. Once the physical coupling constant k has been fixed, for example by the exact renormalized commutation relation (2.6), the question is if the shift (5.12) is observed in the values of the skein parameters (or in $< W(L) >$). Our claim here and in our previous papers [8,15,16] is that such a shift does not occur.

Let us recall why one is tempted to expect the shift (5.12). Quantizing the C-S model on $R \times \Sigma$ where Σ has a boundary, the commutator (2.11), computed by means of (2.6), acquires now a central term. More precisely, one has

$$[G(\Lambda), G(\Lambda')] = i f^{abc} \int d^2 z \Lambda^a(\vec{z}) \Lambda'^b(\vec{z}) G^c(\vec{z}) + i \frac{k}{4\pi} \int_{\partial\Sigma} dy^\mu \epsilon_{\mu\nu} \Lambda^a(\vec{y}) \partial^\nu \Lambda'^a(\vec{y}), \quad (5.13)$$

which gives rise to a level- k representation of a Kac-Moody (K-M) algebra [17,18,19]. On the other hand, a representation of this kind appears also in the level- k WZW model, where the Sugawara form of the energy-momentum tensor implies (5.12) in the correlation functions. So, if one assumes that the conformal blocks of the level- k WZW model are described by the physical states of the C-S model with coupling constant k ,

* Of course, there are infinitely many schemes giving $k = k_B + c_0$; some explicit examples at one loop can be found in [14].

one would expect the shift (5.12) in the C-S skein parameters. The absence of such a shift does not mean that any relation of the WZW conformal blocks with the C-S physical states is lost; one has simply to identify correctly the skein (or monodromy) parameters according to eq.(5.11). Summarising, one has

K-M central extension Monodromy Parameters			
WZW model	ℓ	$\ell + c_v$	
C-S theory	k	k	

The different behaviour of the WZW and C-S models can easily be understood. In the WZW model the shift (5.12) is due to the Sugawara form of the energy-momentum tensor. On the other hand, the C-S action reduces to the so-called *chiral* WZW model [17] living on the boundary of M^3 . But the chiral WZW model does not have the standard Lagrangian kinetic term and therefore there is no Sugawara construction inducing the shift. Note that the absence of a standard kinetic term for the chiral WZW model is just a consequence of the general covariance of the C-S theory. So the fact that there is no shift in the C-S model is actually a manifestation of general covariance.

The relation (5.11) is confirmed also by the following construction. By introducing complex coordinates $z = x^1 + ix^2$, $z_p = y_p^1 + iy_p^2$ and defining

$$\hat{A} = -\frac{i}{2}(\hat{A}_1 - i\hat{A}_2) \quad , \quad (5.14)$$

from eq.(4.6) it follows that $\hat{A}_{(q)}$ depends only on z and z_p :

$$\hat{A}_{(q)}(z) = \frac{2}{k} \sum_{p \neq q} \frac{\hat{R}_p^a \hat{R}_q^a}{z - z_p} \quad . \quad (5.15)$$

In complex notation the holonomy $\chi(C)$ reads

$$\chi(C) = P \exp \left(- \oint_C \hat{A}_{(q)}(z) dz \right) \quad , \quad (5.16)$$

The expressions (5.15) and (5.16) are well known in conformal field theory. In analysing the properties of the correlation functions of primary fields in the level- ℓ WZW model,

Knizhnik and Zamolodchikov [5] have derived the equation

$$\begin{aligned} \frac{\partial}{\partial z_q} < \Phi_1(z_1) \dots \Phi_q(z_q) \dots \Phi_n(z_n) > = \\ = \frac{2}{\ell + c_v} \sum_{p \neq q} \frac{\hat{R}_p^a \hat{R}_q^a}{z_q - z_p} < \Phi_1(z_1) \dots \Phi_q(z_q) \dots \Phi_n(z_n) > \quad , \end{aligned} \quad (5.17)$$

where $\hat{R}_q^a$ and $\hat{R}_p^a$ are the generators of the representations of the group G under which the primary fields Φ_q and Φ_p transform. Comparing eq.(5.15) with eq.(5.17), one finds that the gauge connection $\hat{A}_{(q)}(z)$ coincides precisely with the Knizhnik-Zamolodchikov connection if eq.(5.11) holds. It should be noted that in eq.(5.17) only the so-called integrable representations can appear [20], whereas in (5.15) there is a priori no such restriction. Another difference between the WZW model and the C-S parameters is that ℓ can assume only positive integer values and therefore $\ell + c_v > N$ for $SU(N)$. In the C-S theory, instead, k can assume arbitrary integer values and in particular k can also be less than or equal to N .

The relation between the C-S theory and the two-dimensional CFT has been discussed recently by several authors [10,17,21]. We would like to add here a few comments. First of all, it is clear that such a connection cannot be realized at the Lagrangian level. In the usual field theory sense, the degrees of freedom, the physical state spaces and, more generally, the physical contents are completely different. As we have seen, the C-S theory provides monodromy representations of the fundamental group of the punctured plane. When the punctures belong to the same representation of G , one gets actually (see the next section) a monodromy representation of the braid group. On the other hand, in any CFT the correlation functions (conformal blocks) also provide monodromy representations by the so-called crossing matrices [22]. At this level, a correspondence between the C-S theory and the CFT can be realized for particular values of the parameters and/or of the representations of G . Of course, the monodromy representation by no means uniquely determines the particular conformal model. As is well known, the knowledge of the central charge and conformal weights of the primary fields is also required. Yet the knowledge of the monodromy representations in CFT is important for determining the structure of the fusion rules [23]. On the other hand, the fusion

algebra is strictly connected with the theory of representations of finite groups [24]. Even more, such a symmetry structure is naturally provided by some quantum groups with a deformation parameter which is a root of the unity [25]. In the next section we show how quantum groups enter into the game in connection with the C-S monodromies (3.7) and (4.7).

We conclude this section by comparing eq.(4.14) with eq.(5.17). In complex notation, eq.(4.14) takes the form

$$\frac{\partial}{\partial z_q} \Omega_0(z_1, \dots, z_n) = \frac{2}{k} \sum_{p \neq q} \frac{\hat{R}_p^* \hat{R}_q^*}{z_q - z_p} \Omega_0(z_1, \dots, z_n), \quad (5.18)$$

which has the structure of eq.(5.17). The solution of eq.(5.18) can be expressed in terms of path-ordered integrals of the Knizhnik-Zamolodchikov connection. It is remarkable that the integrated Knizhnik-Zamolodchikov connection provides the intertwining operator Ω_0 between ϱ and χ . This suggests that the WZW correlation functions may have a similar interpretation.

6. Braids and the Appearance of Quantum Groups

It is known that any oriented link (knot) can be obtained as the closure of an element α of the braid group B_n with suitable n . An example is shown in Fig.10 in which the trefoil knot is represented as the closure of $\alpha = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \in B_3$. Recall that the generators $\{\sigma_i : i = 1, \dots, n-1\}$ of B_n satisfy

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, & |i-j| \geq 2, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & 1 \leq i \leq n-2. \end{aligned} \quad (6.1)$$

In the previous section we have derived the skein parameters by means of the half-monodromy ΣE (see eq.(5.1)) which, in view of the interpretation illustrated by Figs.8 and 9, gives just a representation

$$\sigma_1 \mapsto \Sigma E \quad (6.2)$$

of B_2 . Our aim now is to find the half-monodromies in the general case of $n \geq 2$ punctures. This means that, starting from the representation χ of $\pi_1(R_{n-1}^2)$ for n

equivalent punctures, one has to construct a representation of B_n . The problem is well posed; such a representation certainly exists on the physical space of the C-S model because of general covariance. Differently from what it might appear at first sight however, the construction of this representation is not immediate. For instance, the naive extension of the procedure used for $n = 2$ does not work for $n \geq 3$. The difficulties appear in satisfying the relations (6.1) which become non-trivial for $n \geq 3$. In order to construct the B_n representation realized on the physical space of the C-S model, we shall use some results of Kohno [6] relating monodromy representations of braid groups to the quantum Yang-Baxter equation.

Our first step will be to construct the Kohno connection ω . First of all we observe that eq.(5.15) can be rewritten in the form

$$\hat{A}_{(q)}(z) dz = \frac{2}{k} \sum_{p \neq q} \Omega_{pq} d \ln(z - z_p), \quad (6.3)$$

where $\Omega_{pq} = \hat{R}_p^* \hat{R}_q^*$. Moreover it is easy to verify that

$$\begin{aligned} [\Omega_{pq}, \Omega_{pr} + \Omega_{qr}] &= [\Omega_{pq} + \Omega_{pr}, \Omega_{qr}] = 0, & p < q < r, \\ [\Omega_{pq}, \Omega_{rs}] &= 0 & \text{for distinct } p, q, r \text{ and } s, \end{aligned} \quad (6.4)$$

which are known as infinitesimal pure braid relations. Let us consider now the superposition

$$\omega = \frac{1}{2} \sum_{q=1}^n \hat{A}_{(q)}(z_q) dz_q = \frac{2}{k} \sum_{1 \leq p < q \leq n} \Omega_{pq} d \ln(z_p - z_q), \quad (6.5)$$

which is precisely the Kohno connection. The physical meaning of the expression (6.5) is clear: ω takes into account all the "interactions" between the punctures. In terms of ω , we can compute now the holonomies associated with the simultaneous displacements of several punctures. The one-form (6.5) represents a connection of the trivial vector bundle over

$$X_n = \{(z_1, \dots, z_n) \in C^2 : z_p \neq z_q \text{ if } p \neq q\} \quad (6.6)$$

with fibre H . As a consequence of eqs.(6.4), ω is a flat connection. Let γ be an element of the fundamental group $\pi_1(X_n)$ and let

$$\chi(\gamma) = P \exp \left(- \oint_{\gamma} \omega \right) \quad (6.7)$$

be the associated holonomy. Eq.(6.7) generalizes eq.(5.16) to the case in which several punctures are moving simultaneously. By construction $\chi(\gamma)$ acts on H and one can show [6] that χ defines a linear representation of the pure braid group of n strings $P_n = \pi_1(X_n)$ in H . In the particular case when all the $\{R_p\}$ -representations coincide, ω is symmetric under permutations of the coordinates $\{x_p\}$ and induces a connection over $Y_n = X_n/S_n$, S_n being the permutation group of n elements. Therefore in this case χ extends actually to a representation φ of the braid group $B_n = \pi_1(Y_n)$ on H . This representation, called the monodromy representation of B_n , has been investigated by Kohno [6]. We summarize below Kohno's results concerning φ and its relation with quantum groups.

Let $\{E_\alpha\}$ be the step operators of the $SU(N)$ Lie algebra in the Cartan-Weyl basis and let

$$r = \sum_{\alpha} \text{sgn} \alpha E_{\alpha} \otimes E_{-\alpha} \quad (6.8)$$

Define [26]

$$r(x) = r - t + \frac{2}{1-x}t, \quad (6.9)$$

where x is a parameter, $t = T^a \otimes T^a$ and T^a are the generators in the fundamental representation. One can verify that $r(x)$ satisfies the multiplicative classical Yang-Baxter (YB) equation

$$[r_{12}(x), r_{13}(xy)] + [r_{12}(x), r_{23}(y)] + [r_{13}(xy), r_{23}(y)] = 0, \quad (6.10)$$

where r_{ij} acts as r on the i -th and j -th factors and as the identity on the other factor in $H_1 \otimes H_2 \otimes H_3$. A standard procedure [26,27] leads then to a solution

$$R(x, q) = 1 + (q-1)r(x) + \dots \quad (6.11)$$

of the quantum YB equation

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x), \quad (6.12)$$

q being the deformation parameter. The associated T -matrix is given by

$$T(q) = \lim_{z \rightarrow \infty} z^{-1} \bar{R}(z; q), \quad (6.13)$$

where $\bar{R} = \Sigma R$ and Σ is the permutation operator.

Kohno's theorem states that:

i) Considering $\frac{1}{k}$ as a parameter, the braid group representation φ is equivalent to

$$\theta(\sigma_i) = q^{\nu} [1 \otimes \dots \otimes T(q) \otimes \dots \otimes 1], \quad (6.14)$$

where $\nu = \frac{1-N}{2N}$, $T(q)$ acts on the i -th and $i+1$ -th factors of H by

$$T(q) = \sum_{\alpha} E_{\alpha\alpha} \otimes E_{\alpha\alpha} + \sqrt{q} \sum_{\alpha \neq \beta} E_{\alpha\beta} \otimes E_{\beta\alpha} + (1-q) \sum_{\alpha > \beta} E_{\alpha\alpha} \otimes E_{\beta\beta} \quad (6.15)$$

and q is related to k by eq.(5.10);

ii) θ commutes with the diagonal action of $SU(N)_q$ on H and defines actually a Hecke algebra (see e.g. [12]) representation of B_n , known as Pimsner-Popa-Temperley-Lieb representation.

Some clarifying remarks are in order. From point i) of the theorem it follows that there exists an invertible linear mapping $V : H \rightarrow H$ such that

$$\varphi = V^{-1} \theta V. \quad (6.16)$$

Since φ is an extension of χ , eq.(6.16) relates χ with θ as well. The diagonal action of $SU(N)_q$ on H is defined by the co-product. In the particular case of $SU(2)_q$

$$q^{\frac{1}{2}} X_{\pm} (X_{\pm}) q^{-\frac{1}{2}} X_{\pm} = q^{\pm \frac{1}{2}} (X_{\pm}), \quad (6.17)$$

$$[X_+, X_-] = \frac{q^{X_3} - q^{-X_3}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad (6.18)$$

the co-product Δ is

$$\Delta(q^{\pm \frac{1}{2}} X_{\pm}) = q^{\pm \frac{1}{2}} X_{\pm} \otimes q^{\pm \frac{1}{2}} X_{\pm}, \quad (6.19)$$

$$\Delta(X_{\pm}) = q^{\frac{1}{2}} X_{\pm} \otimes X_{\pm} + X_{\pm} \otimes q^{-\frac{1}{2}} X_{\pm}. \quad (6.20)$$

From eq.(6.16) it is clear that φ commutes with the V -conjugate action of the $SU(N)_q$ representation commuting with θ . For generic q , one can show [28] that $\theta(\sigma_i)$ generate actually the *centralizer* of the quantum group. Finally, the above theorem holds [6,28] also for the fundamental representations of all non-exceptional simple Lie groups and,

with slight modifications, for an arbitrary irreducible finite dimensional representation of $SU(2)$.

Observe that at this stage one can forget about the connection $\hat{A}$ we started with; the final expression (6.14) for θ depends only on k and the representations associated with the punctures. From this point of view, $\hat{A}$ can be considered as a technical device for the construction of the monodromy representation θ of B_n .

The above theorem has deep physical implications and represents a key point in establishing the role of quantum groups in C-S theory. As we have seen, the relevant structure realized on the physical state space of the C-S model is described by the monodromy representation θ . Together with the property of general covariance, the knowledge of θ determines uniquely the Wilson line expectation values $\langle W(L) \rangle$. Consider for instance $\langle W(T) \rangle$ associated with the trefoil knot T shown in Fig.10. The intermediate physical states are determined by intersecting T with the planes $t = \text{const.}$ At $t = 0$ one can choose a standard physical state Φ_0 given for example by

$$\Phi_0 = \delta_{J_1}^{I_1} \delta_{J_2}^{I_2} \delta_{J_3}^{I_3} \Psi[\tilde{x}_1]^{K_1 J_1} [\tilde{x}_2]^{K_2 J_2} [\tilde{x}_3]^{K_3 J_3} [\tilde{y}_1]_{I_1 L_1} [\tilde{y}_2]_{I_2 L_2} [\tilde{y}_3]_{I_3 L_3} \quad (6.21)$$

Now we want to find the wave functions of the intermediate states for $t > 0$; in particular, we are interested in the state Φ_1 at $t = 1$. At this stage, in order to use the θ representation given by eqs.(6.14,15), it is necessary to perform a change of basis in the q -indices. Two steps are involved in this operation. First, one has to relate the B -matrices with the χ representation (6.7). In this case, the intertwining operator $\tilde{\Omega}$ is the natural extension in X_n of the operator Ω_0 defined in sect.4. Then we have to use eq.(6.16) to connect φ with θ . Of course, $\tilde{\Omega}$ and V separately depend on the positions of the punctures. However, in view of the result (6.14,15), the combined intertwining operator

$$U = \tilde{\Omega} V \quad (6.22)$$

only depends on the deformation parameter q . We introduce then the wave functions

$$(U^{-1})_{J_1 J_2 J_3}^{Q_1 Q_2 Q_3} U_{S_1 S_2 S_3}^{I_1 I_2 I_3} \Psi[\tilde{x}_1]^{K_1 J_1} [\tilde{x}_2]^{K_2 J_2} [\tilde{x}_3]^{K_3 J_3} [\tilde{y}_1]_{I_1 L_1} [\tilde{y}_2]_{I_2 L_2} [\tilde{y}_3]_{I_3 L_3} \quad (6.23)$$

The indices $Q_1 Q_2 Q_3$ and $S_1 S_2 S_3$, which originate from the q -indices, represent quantum group (QG) indices in the standard basis, see eqs.(6.17-19). Now eq.(6.21) can be rewritten as

$$\Phi_0 = \delta_{Q_1}^{S_1} \delta_{Q_2}^{S_2} \delta_{Q_3}^{S_3} \Psi_{L_1 L_2 L_3; S_1 S_2 S_3}^{K_1 K_2 K_3; Q_1 Q_2 Q_3} \quad (6.24)$$

The physical state Φ_1 at $t = 1$ is obtained from Φ_0 by replacing the δ -factors in (6.24) by the monodromy $\theta(\alpha)$ associated with the braid element $\alpha = (\sigma_1 \sigma_2)^2$ shown in Fig.10(a), i.e.

$$\Phi_1 = \theta(\alpha)_{Q_1 Q_2 Q_3}^{S_1 S_2 S_3} \Psi_{L_1 L_2 L_3; S_1 S_2 S_3}^{K_1 K_2 K_3; Q_1 Q_2 Q_3} \quad (6.25)$$

Observe now that θ acts on the QG-indices and that these indices appear contracted in the physical states. Therefore, any invertible linear transformation $\mathcal{F}$ acting on the QG-indices (the lower and upper QG-indices transform by $\mathcal{F}$ and $\mathcal{F}^{-1}$ respectively) and commuting with θ represents a hidden symmetry of the quantized C-S theory. The theorem of Kohno states that $G_q \subset \{\mathcal{F}\}$.

The above construction makes manifest the role of quantum groups in the C-S theory. The QG-indices appear quite naturally and are associated with the group indices of the charges "at infinity". Note that the quantum group symmetry is realized in a very peculiar way. Due to the fact that the quantum group action commutes with the monodromy representation θ , any physical state obtained by cutting a Wilson line (defined in a finite region) is a G_q -singlet. In this sense, the quantum group symmetry of the C-S theory is analogous to the colour symmetry of QCD.

Our construction sheds also new light on the quantum group "interpretation" of the fusion rules and on the role of quantum groups in rational CFT.

Summarising, among several statistical models on the lattice [25] and the generalized Toda systems [26], the C-S theory represents a remarkable example of a continuous three-dimensional generally covariant field theory, admitting a hidden quantum group symmetry.

7. Conclusions

We conclude by mentioning some open problems. It would be interesting to have an explicit construction of the generators of G_q in terms of the quantum fields. This would further clarify the physical meaning of the quantum group symmetry. Concerning the link invariants, it is known [12,28,29] that they can be obtained from a Hecke algebra representation of the braid group by means of the Ocneanu trace. This trace is essentially unique (see e.g. [30]) and it is an open problem to derive it directly from the quantized C-S theory. In other words, one would like to implement the closure of the braid in the field theory context, i.e. in terms of physical states of the type (6.25).

Clearly, a generalization of the results of the present paper to a generic three-manifold M^3 would be of great interest for knot theory.

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FIGURE CAPTIONS

Fig.1 The path Γ defining $W(\Gamma)$.

Fig.2 Four-puncture configuration described by the wave function (3.1).

Fig.3 Parallel transport of the puncture 2 along the loop C .

Fig.4 Example of a contour giving a phase factor δ .

Fig.5 Graphical representation of eq.(3.12).

Fig.6 The domain $\mathcal{O}$.

Fig.7 Exchange operation associated with a "half-monodromy".

Fig.8 Action of the B -matrix.

Fig.9 Braiding operation on Wilson lines.

Fig.10 The trefoil as a closure of the braid $(\sigma_1 \sigma_2)^2 \in B_3$.

Fig. 1

Fig. 2

Fig. 3

Fig.4

Fig. 5

q

Fig. 6

Fig. 7

Fig. 9

Fig. 8

(a)

(b)

Fig. 10