CERN – TH.5586/89

PATH-INTEGRAL APPROACH TO CLASSICAL MECHANICS

E.GOZZI AND M.REUTER^{*}

CERN, Geneva

ABSTRACT

In this talk we review the path-integral approach to classical Hamiltonian dynamics (CM) recently proposed. This formulation brings to light a set of hidden (BRS-like) symmetries connected with the symplectic geometry of CM. We also explore some further universal hidden symmetries related to the dynamics of the system. These new invariances have the character of a genuine supersymmetry. We show that dynamical systems with this supersymmetry unbroken are ergodic. KAM systems and systems with few constants of motion beside the energy (and even integrable ones) have this supersymmetry spontaneously broken. In the conclusion we present a new derivation of the KMS conditions based on this hidden supersymmetry.

Talk given by E.G. at the Cargèse workshop on "Probabilistic methods in quantum field theory and quantum gravity" to appear in the proceedings, ed.P.H.Damgaard, Plenum Press Publ.

CERN-TH. 5586/89

November 1989

^{*} Address after October 1st: Inst. for Theoretical Physics, Hannover University, Appelstrasse 2, D-3000 Hannover 1, FRG

1. INTRODUCTION

Just after the introduction of quantum mechanics, Koopman[1] and von Neumann[2] proposed an operatorial approach to classical mechanics (CM). This suggests that CM could also be formulated via path-integrals. A first attempt to provide such a formulation of classical mechanics was made in ref.[3]. Let us now review (but for more details see ref.[4]) the operatorial and the path-integral formulations for classical Hamiltonian mechanics. We start from Hamilton's equations:

$$\dot{\phi}^a(t) = \omega^{ab} \partial_b H(\phi(t)) \quad (1.1)$$

where $\phi^a \equiv (q^1, \dots, q^n, p_1, \dots, p_n)$, $a = 1, \dots, 2n$, is a coordinate on a $2n$ -dimensional phase-space $\mathcal{M}_{2n}$, H is the Hamiltonian and $\omega^{ab} = -\omega^{ba}$ is the symplectic 2-form.

Another important concept we shall need is that of a probability density function $\varrho(\phi^a, t)$ on phase space. The time evolution of these distributions is given by

$$\frac{\partial}{\partial t} \varrho(\phi_a, t) = -\{ \varrho, H \} \equiv -\hat{L} \varrho(\phi^a, t) \quad (1.2)$$

Here we have introduced the Liouville operator $\hat{L} = -\partial_a H \omega^{ab} \partial_b$ which is the central element of the operatorial approach to classical dynamics[1], [2]. Equation (1.2) is formally solved by

$$\varrho(\phi, t) = e^{-\hat{L}t} \varrho(\phi, 0) \quad (1.3)$$

Due to the fact that there is an operatorial approach, there must exist a corresponding path-integral formulation. The simplest idea is to write a classical generating functional of the form:

$$Z_{cm} = \int \mathcal{D}\phi^a \tilde{\delta}[\phi^a - \phi_{cl}^a] \quad (1.4)$$

where ϕ_{cl}^a are the classical solutions of Hamilton's equations. The delta-functional forcing the system on its classical trajectories can be rewritten as:

$$\tilde{\delta}[\phi^a - \phi_{cl}^a] = \tilde{\delta}[\phi^a - \omega^{ab} \partial_b H] \det \left(\partial_t \delta_b^a - \omega^{ac} \partial_c \partial_b H \right) \quad (1.5)$$

We can now Fourier transform the delta function on the RHS of (1.5) and exponentiate

the determinant using anticommuting ghosts. Thus we arrive at

$$Z_{cm} = \int \mathcal{D}\phi^a \mathcal{D}\lambda_a \mathcal{D}c^a \mathcal{D}\bar{c}_a \exp i \int dt \{ \tilde{\mathcal{L}} + \text{source terms} \} \quad (1.6)$$

with the Lagrangian

$$\tilde{\mathcal{L}} = \lambda_a \left[\dot{\phi}^a - \omega^{ab} \partial_b H(\phi) \right] + i \bar{c}_a \left[\partial_t \delta_b^a - \omega^{ac} \partial_c \partial_b H(\phi) \right] c^b \quad (1.7)$$

Note that $\tilde{\mathcal{L}}$ contains only first-order derivatives. Therefore, we can immediately read off the Hamilton function

$$\tilde{\mathcal{H}} = \lambda_a \omega^{ab} \partial_b H + i \bar{c}_a \omega^{ac} \partial_c \partial_b H c^b \quad (1.8)$$

From the path-integral (1.6) we can compute the equal-time (anti-) commutators of ϕ^a, λ_a, c^a and $\bar{c}_a$. Using standard techniques[5] we find that

$$\langle [\phi^a, \lambda_b] \rangle = i \delta_b^a, \langle [\bar{c}_b, c^a] \rangle = \delta_b^a \quad (1.9)$$

and all other commutators vanish. In particular, ϕ^a and ϕ^b commute for all values of the indices a and b . In terms of the q 's and p 's (which were combined into ϕ^a) this means $\langle [q^i, p_j] \rangle = 0$ for all i and j . This shows very clearly that we are doing classical mechanics. The operator algebra (1.9) can be realized by differential operators

$$\lambda_a = -i \frac{\partial}{\partial \phi^a}, \quad \bar{c}_a = \frac{\partial}{\partial c^a} \quad (1.10)$$

and multiplicative operators ϕ^a and c^a acting on functions $\tilde{\varrho}(\phi^a, c^a, t)$. Inserting the above operators into $\tilde{\mathcal{H}}$ we obtain:

$$\tilde{\mathcal{H}} = -i \omega^{ab} \partial_b H \partial_a + i \frac{\partial}{\partial c^a} \omega^{ac} \partial_c \partial_b H c^b \quad (1.11)$$

Looking at (1.8) with (1.10), it is clear that the fermionic part of $\tilde{\mathcal{H}}$ is absent in the sector with ghost charge zero and that the bosonic part is $(-i)$ times the Liouvillean: $\tilde{\mathcal{H}}|_{p=0} = -i\hat{L}$. This confirms that our path-integral is the right one to reproduce the operatorial approach of Koopman and von Neumann. Many features of a dynamical system are encoded in the spectrum of $\hat{L}$. The one we are most interested in here is that of ergodicity. It is characterized in the following way[6]: a

system is ergodic if, at fixed energy, the eigenstate ρ_0 of $\hat{L}$ with eigenvalue zero is non-degenerate. This means that, for a given energy, the equation $\hat{L}\rho_0 = 0$ has only one solution. Because of eq. (1.2), this solution is time-independent. Generally speaking, the solutions of the above equation are functions of the constants of motion $I_i(\phi^a)$, $i = 1, 2, \dots$, i.e., $\rho_0 = F(I_i)$ where $\{H, I_i\} = 0$. If the system is completely integrable, for instance, it has n constants of motion (including the energy). If, on the other side, the system is ergodic, the only constant of motion is the energy $E = H(\phi^a)$, so that $\rho_0 = \rho_0(E)$. For a fixed energy this is a non-degenerate function: it is just a constant [6] on the energy hypersurface. If there exists a further constant of motion $I(\phi^a)$, we may set $\rho_0 = F(E, I(\phi^a))$, which is in general not constant for fixed E . Hence any F of the form above, normalizable on the energy hypersurface, is a zero-eigenstate of the Liouvillian and therefore the system is not ergodic.

2. HIDDEN SUPERSYMMETRY AND OTHER GRADED SYMMETRIES

Let us now discuss the symmetries of $\tilde{S} = \int \tilde{L} dt$. In [4] we found that $\tilde{S}$ is invariant under the transformations generated by the $ISp(2)$ charges:

$$\begin{aligned} Q &= i\epsilon^a \lambda_a, \\ \tilde{Q} &= i\tilde{\epsilon}_a \omega^{ab} \lambda_b \\ Q_g &= \epsilon^a \tilde{\epsilon}_a \\ \tilde{K} &= \frac{1}{2} \omega_{ab} \epsilon^a \tilde{\epsilon}^b, \\ \tilde{K} &= \frac{1}{2} \omega^{ab} \tilde{\epsilon}_a \tilde{\epsilon}_b \end{aligned} \quad (2.1)$$

We also remind the reader of the basic (graded) commutation relations on which the operatorial formalism of our theory is based:

$$\begin{aligned} [\phi^a, \lambda_b] &= i\delta_b^a \\ [\tilde{\epsilon}_b, \epsilon^a] &= \delta_b^a \end{aligned} \quad (2.2)$$

The generators (2.1) are not the only generators commuting with $\tilde{H}$. Let us consider the quantities

$$\begin{aligned} N &= \epsilon^a \partial_a H \\ \tilde{N} &= \tilde{\epsilon}_a \omega^{ab} \partial_b H \end{aligned} \quad (2.3)$$

Using (2.2) it is easy to verify that they are conserved: $[\tilde{H}, N] = [\tilde{H}, \tilde{N}] = 0$. Similarly,

for all conserved quantities I_i we can define

$$\begin{aligned} N_i &= \epsilon^a \partial_a I_i \\ \tilde{N}_i &= \tilde{\epsilon}_a \omega^{ab} \partial_b I_i \end{aligned} \quad (2.4)$$

and we again find that these new charges are conserved: $[\tilde{H}, N_i] = [\tilde{H}, \tilde{N}_i] = 0$. If we furthermore assume that the I_i 's are in involution, then the N_i and $\tilde{N}_j$ commute among themselves and with N and $\tilde{N}$.

Next let us look at the algebra of the following linear combinations of our symmetry generators:

$$\begin{aligned} Q_H &\equiv Q - \beta N, & Q_i &\equiv Q - N_i \\ \tilde{Q}_H &\equiv \tilde{Q} + \beta \tilde{N}, & \tilde{Q}_i &\equiv \tilde{Q} + \tilde{N}_i \end{aligned} \quad (2.5)$$

with β an arbitrary parameter. In the differential operator representation of eqn. (1.10), they can be written as*

$$\begin{aligned} Q_H &= e^{\beta H} Q e^{-\beta H}, & Q_i &= e^{I_i} Q e^{-I_i} \\ \tilde{Q}_H &= e^{-\beta H} \tilde{Q} e^{\beta H}, & \tilde{Q}_i &= e^{-I_i} \tilde{Q} e^{I_i} \end{aligned} \quad (2.6)$$

Using (2.2) one can easily verify that

$$[Q_H, Q_H] = [\tilde{Q}_H, \tilde{Q}_H] = [Q_i, Q_j] = [\tilde{Q}_i, \tilde{Q}_j] = 0 \quad (2.7)$$

$$[Q_H, Q_i] = [\tilde{Q}_H, \tilde{Q}_i] = 0 \quad (2.8)$$

$$[Q_H, \tilde{Q}_H] = 2i\beta \tilde{H} \equiv 2\beta l_{(dH)^*} \quad (2.9)$$

$$[Q_i, \tilde{Q}_j] = 2(l_{(dI_i)^*} + l_{(dI_j)^*}) = [Q_j, \tilde{Q}_i] \quad (2.10)$$

$$[Q_H, \tilde{Q}_i] = (\beta l_{(dH)^*} + l_{(dI_i)^*}) = [Q_i, \tilde{Q}_H] \quad (2.11)$$

$$[l_{(dH)^*}, l_{(dI_i)^*}] = [l_{(dI_i)^*}, l_{(dI_j)^*}] = 0 \quad (2.12)$$

Here we have used the notation $(dH)^* \equiv \omega^{ab} \partial_b H \partial_a$ and $(dI_i)^* \equiv \omega^{ab} \partial_b I_i \partial_a$ for the Hamiltonian vector fields generated by the gradients of H and I_i , respectively. Furthermore

* See also Witten[7]

l_a denotes the Lie derivative along some vector field v [8] [4]. The most interesting of these relations is eq. (2.9). It shows that the anticommutator of Q_H and $\tilde{Q}_H$ is the super-Hamiltonian and thus these operators, unlike Q and $\tilde{Q}$, are genuine supersymmetry generators. Because all the $Q_i, \tilde{Q}_i$ and $l_{(A_i)}$, defined above, commute with the Hamiltonian $\tilde{\mathcal{H}}$, as it is easy to check:

$$[Q_i, \tilde{\mathcal{H}}] = [\tilde{Q}_i, \tilde{\mathcal{H}}] = [l_{(A_i)}, \tilde{\mathcal{H}}] = 0 \quad (2.13)$$

we find that each constant of motion I_i of the original Hamiltonian $H(\phi)$ gives rise to a graded symmetry of the super-Hamiltonian $\tilde{\mathcal{H}}$. The three generators of this graded symmetry $Q_i, \tilde{Q}_i$, and $l_{(A_i)}$ form an $S(2)$ superalgebra in the classification of De Crombrugghe and Rittenberg [9]. The fact that the I_i 's have vanishing Poisson brackets among themselves (they are in involution [4]) and with $H(\phi)$ implies that different Q_i 's commute (the same holds for different $\tilde{Q}_i$'s and different $l_{(A_i)}$'s), see eqs. (2.7), (2.10) and (2.12). Hence for integrable systems[†] the symmetry of $\tilde{\mathcal{H}}$ is maximal and is $S(2)^n$. For systems that are not integrable, but have some $m < n$ constants of motion in involution, the symmetry is only $S(2)^m$. If (as in most of the cases) they are not in involution, then each of them makes an $S(2)$ symmetry, but the overall symmetry is not the direct sum of $S(2)$'s. For ergodic systems, i.e., those that have only $H(\phi)$ as a constant of motion, the only symmetry is the supersymmetry of eq. (2.9).

3. ERGODIC SYSTEMS

Up to now we have only shown that the action $\tilde{S}$ is invariant under the supersymmetry and under the graded symmetries mentioned above, but we have not asked the question whether these symmetries can be spontaneously broken [10]. The question to ask is: how does the ground state transform? Is it invariant like the action? Let us examine here only the supersymmetry (2.9) and not the other graded symmetries. For simplicity, let us assume that we are only interested in such observables A which do not depend on the ghosts: $A = A(\phi)$. Then in order to obtain a non-zero expectation value,

$$\langle A \rangle_t = \int d^{2n} \phi d^{2n} c A(\phi^a) \tilde{\rho}(\phi^a, c^a, t) \quad (3.1)$$

the density $\tilde{\rho}$ has to contain $2n$ ghosts: $\tilde{\rho}(\phi^a, c^a, t) = \rho(\phi^a, t) c^1 \dots c^{2n}$. The classical

[†] They have n -constants of motion I_i in involution.

average then has the standard form (without ghosts):

$$\langle A \rangle_t = \int d^{2n} \phi A(\phi) \varrho(\phi, t) \quad (3.2)$$

Since the above $\tilde{\rho}$ contains a maximum number of ghosts it is trivially annihilated by Q_H : $Q_H \tilde{\rho} = c^a (\partial_a - \beta \partial_a H) \tilde{\rho}(\phi, c) = 0$. Invariance under $\tilde{Q}_H$ requires

$$Q_H \tilde{\rho} = \frac{\partial}{\partial c^a} \omega^{ab} (\partial_a + \beta \partial_a H) \tilde{\rho}(\phi, c) = 0 \quad (3.3)$$

This equation can be satisfied only if

$$\varrho(\phi^a) = k e^{-\beta H(\phi^a)} \quad (3.4)$$

where k is a constant. This result tells us that the supersymmetric invariant state has to be a Gibbs state [11], i.e., a state that depends only on the Hamiltonian. The parameter β naturally plays the role of $1/T$ with T the temperature and k the Boltzmann constant. We feel that this result is a real striking surprise: it tells us that the canonical, or rather the Gibbs distribution, is selected among all other possible states by the requirement of being invariant under the universal supersymmetry present for any dynamical system. It has been a challenge [11] for many years to find the physical reasons behind the Gibbs states, and many concepts like stability, locality, ergodicity, KMS conditions, have been explored. It might be that this supersymmetry embodies all these features. Let us now proceed and go on to the proof that, if the supersymmetry (2.9) is unbroken, then the system is ergodic, while for KAM and integrable systems the supersymmetry is spontaneously broken. The proof goes as follows: let us assume that our system has another constant of motion $I(\phi)$ besides the energy $H(\phi)$. So the system is not ergodic. Then, besides $\varrho_a(H)$, any normalizable function of $I(\phi)$, $\varrho'_a(I)$, would be an eigenfunction of $\tilde{L}$ with zero eigenvalue and the corresponding $\tilde{\rho}_a(\phi, c) = \varrho'_a c^1 \dots c^{2n}$ would be a "vacuum" of the super-Hamiltonian: $\tilde{\mathcal{H}} \tilde{\rho}_a = 0$. Having both $\tilde{\rho}_0(H)$ and $\tilde{\rho}_0(I)$ as ground states, not only ergodicity is spoiled but also supersymmetry. In fact, $\tilde{\rho}_0(I)$ is not supersymmetric invariant, i.e., $Q_H \tilde{\rho}_0 \neq 0$ just because the solutions of the equation $Q_H \tilde{\rho} = 0$ are only states of the form $\tilde{\rho} = k e^{-\beta H} c^1 \dots c^{2n}$ and $\tilde{\rho}_a(I)$ is not of this form. So for systems with more than one constant of motion, the supersymmetry is broken. In fact the ground state could be any

normalizable function of them^{*}

$$\tilde{\rho}_0'' = F[\tilde{\rho}_0, \tilde{\rho}_0'] = \tilde{F}[H, I, c] \quad (3.5)$$

and this is not supersymmetric invariant! Therefore integrable systems, or systems with few integrals of motion besides the energy, have this supersymmetry spontaneously broken. Furthermore the system has no "stability" reasons to choose the supersymmetric vacuum instead of the most general state (3.5): both of them are ground states of the "effective" theory because both of them satisfy the operatorial condition $\tilde{H}\tilde{\rho} = 0$. In general it might be that we are not able to find analytic constants of motion, but still the system is non-ergodic presenting some invariant[†] surfaces, of measure different from zero, on the energy manifold: KAM[6] systems are of this kind. In this case the ρ'_0 associated to these surfaces could be built [12], and it is easy to see that they are not supersymmetric invariant. So far we have argued that unbroken supersymmetry implies ergodicity. It is important to note that the converse of this statement is not true. In fact, since the energy is not fixed in Z_{em} , we could take $e^{-\beta H}$, as well as $F(H), G(H) \dots$ as acceptable ground states for an ergodic system, where $F(H), G(H) \dots$ are arbitrary normalizable functions. Clearly all these functions are constants on the energy hypersurface and thus proportional to each other, but not outside. Since in general $F, G \dots$ are not supersymmetric invariant, we conclude that ergodicity is possible even if our supersymmetry (2.9) is spontaneously broken, i.e., unbroken supersymmetry is not necessary for ergodicity, but only sufficient. We will analyze, in a forthcoming publication, the symmetries of Z_{em} with a $\delta(H - E)$ inserted and we will discuss which unbroken symmetry is a necessary condition for ergodicity.

4. KMS CONDITION

As we said before, the Gibbs nature of equilibrium states has been studied very extensively in the literature [11]. In particular, a lot of work has gone into the study of the deep physical reasons for the Gibbs states on one hand and, on the other, in replacing the Gibbs-state condition by a more algebraic characterization of them which could be used for infinite systems. This algebraic characterization is known as the KMS condition [11]. In this section we will derive it from the supersymmetry that we have seen as being responsible for the Gibbs nature of the states. To this end we consider the time-evolution [1]

^{*} This is due to the fact that in $\tilde{L}$ we have only first-order derivatives.

[†] Invariant under the Hamiltonian flow.

of two observables $A_1(\phi^a)$ and $A_2(\phi^a)$ which are independent of the ghosts:

$$A_{1,2}(\phi^a, t) = e^{i\tilde{H}t} A_{1,2}(\phi^a, 0) e^{-i\tilde{H}t} \quad (4.1)$$

The condition $\tilde{Q}_H \tilde{\rho} = 0$ implies that the following expression is zero:

$$\int d^{2n} \phi d^{2n} c A_1(\phi^a, 0) Q_H A_2(\phi^a, t) \tilde{Q}_H \tilde{\rho}(\phi, c) = 0 \quad (4.2)$$

The above expression can be written explicitly as

$$\int d^{2n} \phi d^{2n} c A_1 c^a (\partial_a + \beta \partial_a H) A_2(t) \cdot \frac{\partial}{\partial c^e} \omega^e f(\partial_f - \beta \partial_f H) \tilde{\rho}(\phi, c) = 0 \quad (4.3)$$

Doing integration by parts in $\frac{\partial}{\partial c^e}$ and in ∂_f and remembering the form of the Poisson brackets, the expression (4.3) reduces to

$$\left\langle \dot{A}_1(0) A_2(t) \right\rangle_{\tilde{\rho}} = \frac{1}{\beta} \left\langle \{A_1(0), A_2(t)\} \right\rangle_{\tilde{\rho}} \quad (4.4)$$

This equation is known as the classical KMS condition [13]. From the eqs. (4.2), (4.3) and (4.4) it is clear that, in deriving the KMS condition from the supersymmetry, we did not use the explicit form of the ground state which usually is not known for complicated (infinite-dimensional) systems. Our derivation of (4.4) only embodies the algebraic character of the supersymmetry.

ACKNOWLEDGEMENTS

E.G. wishes to thank the organizers of the workshop for having given him the opportunity to present this material.

REFERENCES

1. B.O.Koopman, Operatorenbehandlung der Klassischen Mechanik, Proc.Nat.Acad.Sc.USA 17 :315 (1931)
2. J. von Neumann, Zur Operatorenmethode in der Klassischen Mechanik, Ann.Math. 33: 587 (1932)
3. E. Gozzi, Hidden BRS invariance in classical mechanics, Phys.Lett. B 201: 525 (1988)
4. E.Gozzi, M. Reuter and W.D.Thacker, Hidden BRS invariance in classical mechanics:II. CERN-TH.5271/89 Phys.Rev.D40,(10)(1989)3363
5. B.S.Schulman, "Techniques and applications of path-integration ", J.Wiley, New York (1981).
6. V.I.Arnold and A.Avez,"Ergodic problems of classical mechanics", W.A.Benjamin Inc. (1968).
7. E.Witten, Supersymmetry and Morse theory, J.Diff.Geom. 17:661 (1982).
8. V.I.Arnold, "Mathematical methods of classical mechanics", Springer Verlag, New York, 1978.
9. M.De Crombrughe and V.Rittenberg, Supersymmetric quantum mechanics, Ann.Phys. 151:99 (1983).
10. E.Witten, Constraints on supersymmetry breaking, Nucl.Phys. B202:253 (1982).
11. R.Haag et al.,On the equilibrium states in quantum statistical mechanics, Comm.Math.Phys. 5:215 (1967) ;
R.Haag et al., Stability and equilibrium states, Comm.Math.Phys. 38:173 (1974) ;
M.Aizenman et al., Stability and equilibrium of infinite classical systems, Comm.Math.Phys. 48:1 (1976) ;
D.Kastler, Equilibrium states of matter and operator algebra, Symposia Mathematica Vol XX (1976),
Istituto Nazionale di Alta Matematica.
12. E.Gozzi and M.Reuter, Algebraic Characterization of ergodicity, CERN-TH.5421/89 Phys.Lett. B vol.233 (3,4)1989
13. G.Gallavotti and E. Verboven, On the KMS boundary condition , Nuovo Cimento, 28:274 (1975).